Self-Similar Constructions of the Rigid Antoine Necklace in Euclidean Three-Space

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Abstract

Malešič and Repovš [2] have proven that there exists a wild Cantor set in $\mathbb{R}^3$ which is Lipschitz homogenous in $\mathbb{R}^3$. They constructed such a wild Cantor set in the Antoine Necklace. We will generalize their construction, to arrive at determined classes of the Antoine Necklace. It is believed that their methods may be altered to prove that our more general constructions are also Lipschitz homogenous in $\mathbb{R}^3$.

1 Introduction

The standard Ternary Cantor Set is constructed by starting with the unit interval and deleting the open middle third. At each successive step in the construction, the open middle third is removed from each remaining interval, so that after n steps in the construction, there are 2^n closed intervals, each of length $\frac{1}{3^n}$. We define a topological space to be a Cantor Set if it is homeomorphic to the Ternary Cantor Set. The following result is known.

Theorem 1 *A topological space X is a Cantor Set if and only if X is a compact, perfect, totally disconnected metric space.*

We will concern ourselves in this paper with the Antoine Necklace, a *wild* Cantor Set. We will consider the Antoine Necklace only in $\mathbb{R}^3$, and so the following definitions will be sufficient to arrive at our results.

Definition 1 *The Cantor Set C is wild if there does not exist a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ taking C onto a standard Cantor Set.*

We will need the following definitions to construct a wild Cantor Set.

Definition 2 *Let* J_1, J_2 *be disjoint, simple clsoed curves in* $\mathbb{R}^3$*.* J_1 *and* J_2 *are of* simple linking type *if there exists a homeomorphism* $h : \mathbb{R}^3 \to \mathbb{R}^3$ *st.*

$$h(J_1) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1,\ z = 0\}$$
$$h(J_2) = \{(x, y, z) \in \mathbb{R}^3 | (x-1)^2 + z^2 = 1,\ y = 0\}$$

J_1 *and* J_2 *are "unlinked" if* J_1 *is null-homotopic in* $\mathbb{R}^3 - J_2$*.*

Definition 3 *The set of* N *tori,* $\{T_i\}_{i=1}^N$*, is said to be* wild *if*

1. *For* $i = 2, 3, \ldots, N-1$*, the cores of the tori* T_i *and* T_{i-1} *are of simple linking type. (also for* T_i *and* T_{i+1}*).*
2. T_i *and* T_j *are unlinked for* $j \notin \{i-1, i, i+1\}$*.*
3. T_1 *and* T_N *are of simple linking type.*
4. T_1 *and* T_j *are unlinked for* $j \notin \{1, 2, N\}$*.*
5. T_N *and* T_j *are unlinked for* $j \notin \{N-1, N, 1\}$*.*

The Antoine Necklace is a wild Cantor Set, constructed with an infinite, recursive process. Starting with a one-holed torus $A_0 = B^2 \times S^1$, one inserts a closed, wild linked chain of tori into its interior. Call this chain A_1. At the second stage, an analogous chain is inserted into each of the tori of A_1. The union of all these chains of the second step, we call A_2. Carrying this process out to infinity, the limiting space is called the Antoine Necklace $A = \cap_{i=1}^{\infty} A_i$. This set is also referred to as a Blankinship Cantor Set, when constructed in higher dimensions, using generalized tori. The sequence $\{A_i\}$ is said to be a defining sequence for the Cantor Set A.

Theorem 2 (Sher, 1966) *Suppose* M *and* N *are Antoine's necklaces in* $\mathbb{R}^3$ *with canonical defining sequences* $\{M_i\}$ *and* $\{N_i\}$ *respectively. Then* M *and* N *are equivalently imbedded in* $\mathbb{R}^3$ *if and only if* $\{M_i\} \sim \{N_i\}$*.*

Note: Two such sequences $M_1, M_2, M_3, \ldots$ and $N_1, N_2, N_3, \ldots$ are said to be equivalent if for each positive integer i there is a homeomorphism h_i of $\mathbb{R}^3$ on to itself such that (1) for each positive integer i, $h_{i+1}|(\mathbb{R}^3 - M_i) = h_i|(\mathbb{R}^3 - M_i)$, and (2) for each positive integer i, $h_i(M_i) = N_i$.

Theorem 2 leads to a method by which Antoine Cantor Sets may be distinguished. The following definition is due to Wright [4].

Definition 4 *An* Antoine Graph G *is a graph* G *so that* G *is the countable union of nested subgraphs.* $\phi = G_{-1} \subset G_0 \subset G_1 \subset \cdots$*. The graph* G_0 *is a single vertex. For each vertex* v *of* $G_i - G_{i-1}$ *there is a polygonal simple closed curve* $P(v)$ *with at least four vertices st.* $P(v) \cap P(w) = \phi$*. The graph* G_{i+1} *consists of* G_o *plus the union of* $P(v)$*,* v *a vertex of* $G_i - G_{i-1}$*, plus edges running between* v *and the vertices of* $P(v)$*. For an Antoine graph* G*, the subgraphs* $G_0, G_1, G_2, \ldots$ *are uniquely determined, since the vertex in* G_0 *is the only vertex that does not separate* G*. (See Figure* 1*)*

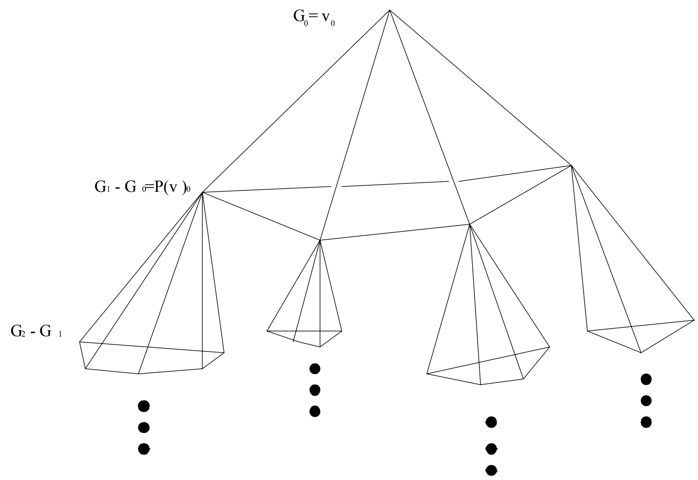

Figure 1: Figure 1: The Antoine Graph

Given a solid torus T in $\mathbb{R}^3$ and an Antoine graph G, we think of the graph as a set of instructions for constructing a canonical defining sequence A_i for an Antoine Cantor set in $\mathbb{R}^3$. The vertex in G_0 corresponds to $A_0 = T$. The vertices of $G_i - G_{i-1}$ correspond to components of A_i embedded in A_{i-1}. If a vertex v of $G_i - G_{i-1}$ corresponds to the component K of A_i, then the vertices of $P(v)$ correspond to the components contained in $A_{i+1} \cap K$. Furthermore, the components of $A_{i+1} \cap K$ are adjacent and linked if and only if the corresponding vertices share an edge in the graph.

On the other hand, given a canonical defining sequence A_i for an Antoine Cantor set X, we may associate an Antoine graph $G(X)$ so that the defining sequence follows the instructions of the graph as in the previous paragraph.

It follows from Wright's paper that any two defining sequences for the Cantor Set X correspond to isomorphic graphs. This result, together with Sher's theorem above give the following result.

Theorem 3 *If X and Y are Antoine Cantor sets so that $G(X)$ is not isomorphic to $G(Y)$, then X and Y are not equivalently embedded in $\mathbb{R}^3$.*

And so we can characterize Antoine Cantor Sets by their corresponding Antoine graphs. One can think of the vertices of an Antoine graph as labeled with a k-tuple. For a vertex $v \in G_i$, there will be i coordinates tracking which component of G_j the vertex is contained in for each $j < i$. Then there will be a coordinate of the k-tuple denoting the number of components in $P(v)$. Finally, we will label each vertex of the graph with a twisting number $\mathcal{T}$. This last coordinate will be discussed later.

2 The Problem

Now, it is clear that from a topological viewpoint, any number of tori may be inserted into a particular chain of the construction. We have limited our investigations to rigid constructions in $\mathbb{R}^3$, with a perfectly recursive construction. That is, our tori will be geometric tori rather than just topological tori, and the penultimate coordinate in the k-tuple of each vertex of the Antoine graph will be fixed. The dimensions of such Antoine Cantor Sets are trivial to find, with sufficient adjustments. For example, take the model where A_0 is a torus with core of radius 1, and the radius of the rotated disc is equal to $\frac{1}{4}$. Then let A_1 consist of 32 evenly spaced tori with the corresponding dimensions $\frac{99}{800}$ and $\frac{1}{800}$, and such that adjacent, linked tori lie in perpendicular planes. Now continue the construction, so as to preserve the ratio of the dimensions of the tori in the $i^{\underline{\text{th}}}$ and $(i+1)^{\underline{\text{st}}}$ levels. Uncountably many such examples can be found without method or system, but instead by miniscule adjustments.

We wish to characterize different classes of Antoine Cantor Sets in very specific ways. In order to find the solutions to our characterization problem, we have narrowed our search through additional restrictions. We also wish the tori contained in A_{i+1} to be related to those in A_i by a simple scaling followed by an isometry. That is, every torus in the construction is similar to every other. We also wish our construction to be self-similar. So the last two coordinates of each vertex in the associated Antoine Graph ($|P(V)|$ and $\mathcal{T}$) will be constant over the entire graph. Finally, we will only consider those constructions where the chains inside adjacent, linked tori are themselves of simple linking type; i.e. every closed chain only circles through its corresponding torus once.

These additional restrictions will allow us to determine and characterize possible constructions of the Antoine Cantor Set in $\mathbb{R}^3$.

Now, the constructions we will be considering in this paper will be more general from the example above in that adjacent, linked tori T_i, T_{i+1} will not necessarily lie in perpendicular planes. The difference between φ', the angle between the planes of consecutive tori, and the perpendicular, we call φ. In order to preserve a high degree of symmetry, we force φ to be the same for any pair of adjacent, linked tori in the construction. This restriction naturally leads to the question of how many "full twists" any particular chain undergoes in its circumference. It is believed that for our rigid, self-similar constructions with chains of simple linking type, any Antoine Necklace is characterized up to homeomorphism by its Antoine Graph, labelled as described above, with the coordinate $\mathcal{T}$ at each vertex denoting the number of rotations of the chain embedded in the component corresponding to that vertex. Again, our self-similiarity restriction will hold this last coordinate of each vertex constant over the whole graph, as well as the penultimate coordinate, denoting the number of components associated with $P(v)$.

In this paper, we have begun to find the minimum number N of tori in each chain that allows for a non-self-intersecting construction of the Antoine Necklace, with the above restrictions, for each value of n, the number of "full twists" in the chain. [1] It seems clear that once this minimum is achieved, one can simply add more tori, in specific quantities derived below, to obtain another viable construction of the Antoine Necklace. In this sense, we are searching for primitive, self-similar, rigid constructions of the Antoine Necklace in $\mathbb{R}^3$.

3 Preliminaries

Malešič and Repovš, [2], propose a method to build the linked chain at each stage of the construction of the Antoine Necklace. In order to equally space all the tori in the chain, they place them at the vertices of a regular N-gon circumscribed in the core of the original torus. They call the vertices c_g for $g \in \mathbb{Z}/N$.

In their construction, they first scale the original torus by a factor of $\lambda < 1$. They then translate the torus to the appropriate location c_g. Finally, they rotate the torus by an angle of $\frac{\pi}{2}$ about the axis of symmetry of the N-gon which is perpendicular to the vector c_g.

This is by no means a canonical construction of the linked chain, and it is not a priori clear that it will be the easiest construction with which to work when checking if the construction, for particular values of λ, is geometrically legitimate. For example, we will have to check that two adjacent tori do not intersect. For this condition, it seems easier to have a relation between each scaled torus and the torus adjacent to it.

Without loss of generality, we can assume that the original torus has as its core the unit circle, centered at the origin, and lying in the xy-plane. We can then scale the torus and translate it by the vector $(1, 0, 0)$. From this smaller torus, we can generate the other tori of the chain by simply shifting it along a side of our N-gon to the next vertex, and then twisting it around a chosen axis (either the one in the above construction, or perhpas an incident side of the N-gon). Alternately, instead of a translation, we could instead arrive at the next torus through a rotation of $\mathbb{R}^3$ by an angle of $\frac{2\pi}{N}$. Either of these seems as natural as the "cookie-cutter" approach, constructing each torus in the chain from the original larger torus.

After some investigation, both these latter constructions were found to involve too many complexities. There is no computationally practical way to check if the surfaces of two general tori are touching, with either rectangular, toroidal, or spherical coordinates. Our method for ensuring they do not is included in sections 6 and 7. The method's compatibility with the standard rectangular basis of $\mathbb{R}^3$ made the original cookie-cutter construction most easily computable for concrete examples.

[1] Although $\mathcal{T}$ will be the last coordinate of each vertex in the Antoine graph, we will deal primarily with n throughout the paper, as it seems a more intuitive concept. The relations between $\mathcal{T}$ and n will be made at the end, in orderto characterize our solutions more exactly.

4 Notation

A torus in $\mathbb{R}^3$ is uniquely determined by the radius of its core, the radius of its rotated disc, and a located vector v at the center of the torus, perpendicular to the plane of the core. The radius of the core we call r. The radius of the rotated disc we call ρ. The scaling factor by which we will multiply each dimension of the large torus is $\lambda < 1$. The angle $\frac{2\pi}{N}$ by which each torus in the chain is related to its neighbor we will call θ.

Finally, we have a twist variable, the angle φ. In [2], the angle between the planes of consecutive cores in the chain was always $\frac{\pi}{2}$. Our constructions will be more general. We take this angle of $\frac{\pi}{2}$ to be natural, however, and so we measure φ as the angle away from this perpendicular. Keeping our constructions symmetric and perfectly iterative, however, φ will be equal for each torus, and in fact, constant throughout the construction of the necklace. For the purposes of this paper, we have identified the torus T with T', where T' is the torus, rotated by an angle of π about any axis perpendicular to v. That is, an upside-down doughnut is a doughnut.

5 Choices and Consequences of φ

With our more general φ, there are some considerations that must be taken into account. Most obviously, since we demand, in fact define φ to be equal for each pair of consecutive tori, we must check that in fact φ is equal for each pair of consecutive tori. That is, we must guarantee through some restriction that the core of the last torus lies in a plane that defines an angle of φ relative to the xy-plane. And so we need the torus to be eventually twisted by a total angle of $0 \bmod \pi$. This in fact gives us restrictions on φ as well as on N, the number of tori in the chain, through the following results.

Claim 1 *For $\gamma \notin \mathbb{Q}$, $[n\gamma]$, the fractional part of γ, never repeats a value for $n \in \mathbb{Z}_+$.*

Proof: Assume the contrary.

$$\Rightarrow n_1\gamma = n_2\gamma + l \text{ for some } n_1, n_2, l \in \mathbb{Z} \Rightarrow (n_1 - n_2)\gamma = l \Rightarrow \gamma = \frac{l}{n_1 - n_2} \Rightarrow \gamma \in \mathbb{Q}$$

Contradiction.

Now, setting $\varphi = \frac{\pi}{m}$, what we are really concerned with is the angle between the planes of consecutive cores; that is we are concerned with the angle

$$\varphi' = \frac{\pi}{2} - \frac{\pi}{m} = \frac{(m-2)\pi}{2m}.$$

Lemma 1 *To guarantee that φ' is in fact the angle between the planes of any two consecutive cores in a chain, in particular, the last core and the first, m must be a rational number.*

Proof: Consider a twist of the torus by π equal to the identity transformation, since it is symmetric across the plane containing its core. So we will consider π as the unit of rotation. Now, we wish φ to be the same at each linking. Suppose $m \notin \mathbb{Q}$. Then, saying we have N tori in a chain, we must have the fractional part of $(m-2)N/m = 0$ (i.e. the fractional part of the twist relative to multiples of π). Assuming m to be an irrational number, we get that $(m-2)/2m$ is irrational, and so $(m-2)N/m \neq 0$ for any $N \in \mathbb{Z}_+$. And so the Lemma follows from the Claim.

So φ must be a rational fraction of π. Call $\varphi' = \frac{\pi}{2} - \varphi = \frac{\pi}{2} - \frac{\pi}{m} = \frac{m-2}{2m}\pi$. Since m is rational, so $\frac{m-2}{2m}$ is too. Look at it in reduced form $\frac{p}{q}$. The first time a number t of these twists equals an integer multiple of π is $t = q$. It is clear that the only possible values for N are multiples of q. And for every q tori, there are p full twists in the chain's circumference ($n = kp$ for some $k \in \mathbb{Z}$).

For various values of possible φ, we have generated the following list of candidate values of N.

The Big List

φ	N (number of tori)	n (full twists)	φ	N (number of tori)	n (full twists)
0	$20, 22, 24, \ldots$	$10, 11, 12, \ldots$	$\frac{\pi}{10}$	$5, 10, 15, \ldots$	$2, 4, 6, \ldots$
$\frac{\pi}{2}$	—	—	$\frac{3\pi}{10}$	$5, 10, 15 \ldots$	$1, 2, 3 \ldots$
$\frac{\pi}{3}$	$6, 12, 18, \ldots$	$1, 2, 3, \ldots$	$\frac{\pi}{11}$	$22, 44, 66, \ldots$	$9, 18, 27, \ldots$
$\frac{\pi}{4}$	$4, 8, 12, \ldots$	$1, 2, 3, \ldots$	$\frac{2\pi}{11}$	$22, 44, 66, \ldots$	$7, 14, 21, \ldots$
$\frac{\pi}{5}$	$10, 20, 30, \ldots$	$3, 6, 9, \ldots$	$\frac{3\pi}{11}$	$22, 44, 66, \ldots$	$5, 10, 15, \ldots$
$\frac{2\pi}{5}$	$10, 20, 30, \ldots$	$1, 2, 3, \ldots$	$\frac{4\pi}{11}$	$22, 44, 66, \ldots$	$3, 6, 9, \ldots$
$\frac{\pi}{6}$	$3, 6, 9, \ldots$	$1, 2, 3, \ldots$	$\frac{5\pi}{11}$	$22, 44, 66, \ldots$	$1, 2, 3, \ldots$
$\frac{\pi}{7}$	$14, 28, 42, \ldots$	$5, 10, 15, \ldots$	$\frac{\pi}{12}$	$12, 24, 36, \ldots$	$5, 10, 15, \ldots$
$\frac{2\pi}{7}$	$14, 28, 42, \ldots$	$3, 6, 9, \ldots$	$\frac{5\pi}{12}$	$12, 24, 36, \ldots$	$1, 2, 3, \ldots$
$\frac{3\pi}{7}$	$14, 28, 42, \ldots$	$1, 2, 3, \ldots$	$\frac{\pi}{13}$	$26, 52, 78, \ldots$	$11, 22, 33, \ldots$
$\frac{\pi}{8}$	$8, 16, 24, \ldots$	$3, 6, 9, \ldots$	$\frac{2\pi}{13}$	$26, 52, 78, \ldots$	$9, 18, 27, \ldots$
$\frac{3\pi}{8}$	$8, 16, 24, \ldots$	$1, 2, 3, \ldots$	$\frac{3\pi}{13}$	$26, 52, 78, \ldots$	$7, 14, 21, \ldots$
$\frac{\pi}{9}$	$18, 36, 54, \ldots$	$7, 14, 21, \ldots$	$\frac{4\pi}{13}$	$26, 52, 78, \ldots$	$5, 10, 15, \ldots$
$\frac{2\pi}{9}$	$18, 36, 54, \ldots$	$5, 10, 15, \ldots$	$\frac{5\pi}{13}$	$26, 52, 78, \ldots$	$3, 6, 9, \ldots$
$\frac{4\pi}{9}$	$18, 36, 54, \ldots$	$1, 2, 3, \ldots$	$\frac{6\pi}{13}$	$26, 52, 78, \ldots$	$1, 2, 3, \ldots$

From the table, it is notable that for every denominator listed, there is a numerator such that every value of n is attainable. Stated more precisely,

Theorem 4 (Rubinstein–Soper) *For every q, there exists $p \in \mathbb{Z}_{>0}$ st. $(p,q) = 1$, $p < \frac{q}{2}$, and setting $\varphi = \frac{p}{q}\pi$ allows one to attain any value of n; i.e. one can set, for different values of N, $n = 1, 2, 3, \ldots$.*

Proof: Since $\varphi = \frac{p}{q}\pi$, so $\frac{\pi}{2} - \varphi = \frac{(q-2p)\pi}{2q}$. So, it is clear from the above method for finding n, we want $(q-2p)|2q$ for some p.
Case 1*:* q is odd.
Take $p = \frac{q-1}{2}$. So certainly, $p < \frac{q}{2}$. We need only check that $(p,q) = 1$.

Suppose not.

$$\Rightarrow \exists\, c \in \mathbb{Z} \text{ st. } c\,|\frac{q-1}{2} = p,\; c\,|\,q, c \neq 1.$$

$$\Rightarrow m_1 c = \frac{q-1}{2}, m_2 c = q \text{ for some } m_1, m_2 \in \mathbb{N}$$

$$\Rightarrow 2m_1 c = q - 1,\; m_2 c = q \Rightarrow 2m_1 c + 1 = m_2 c \Rightarrow c\,|1$$

$$\Rightarrow c = 1 \text{ Contradiction.}$$

So $(p, q) = 1$, so we can get all integer values of n with this choice of p.
Case 2: q is even.
The case $q = 2$ is degenerate, as φ would then be some multiple of $\frac{\pi}{2}$. The odd multiples clearly do not facillitate a chain, since every core in the chain will lie in the xy-plane. And the even multiples will not give a full twist of just one unless the chain has only two links. This is clearly nonsensical in our rigid construction.

So we may assume $q > 2$. There are again two cases; the case where q is divisible by 4, and the case where it is not.

First take q to be divisible by 4. Recall, $q > 2$. So $\frac{q}{2} - 1 \in \mathbb{Z}_{>}0$ Choose this to be p. Clearly, then, $p < \frac{q}{2}$. And $(q - 2p)|2q$ because $q - 2p = 2$ which obviously divides $2q$. So we must only verify that $\left(\frac{q}{2} - 1, q\right) = 1$.

Suppose not.

$$\Rightarrow \exists\, c \in \mathbb{Z}, c \neq 1, \text{ st. } c \mid \left(\frac{q}{2} - 1\right), \text{ and } c \mid q \Rightarrow m_1 c = \frac{q}{2} - 1,\; m_2 c = q$$

$$\Rightarrow 2(m_1 c + 1) = m_2 c \Rightarrow c(m_2 - 2m_1) = 2$$

Now, $4|q \Rightarrow \frac{q}{2} - 1$ is odd. So m_1 and c must both be odd, since $m_1 c = \frac{q}{2} - 1$. But both of the factors $c, (m_2 - 2m_1)$ on the left of the above identity must be integers, so by the Fundamental Theorem of Arithmetic, one of them is equal to 1, and the other equal to 2 (modulo signs). c is odd, and so in particular, $c \neq 2$. Therefore, $c = 1$. Contradiction. So in this case, where $4|q$, p and q as chosen are relatively prime, and so the theorem is satisfied in this case.

Now, suppose 4 does not divide q. Then take $p = \frac{q}{2} - 2$. Clearly, $\frac{q}{2} - 2 < \frac{q}{2}$. Also, $(q - 2p)|2q$ because $q - 2p = q - q + 4 = 4|2q$ since q is even.
Check that $(p, q) = 1$. Suppose not. So

$$\exists\, c \in \mathbb{Z}_{>1} \text{ st. } m_1 c = \frac{q}{2} - 2 \text{ and } m_2 c = q$$

$$\Rightarrow 2m_1 c + 4 = q = m_2 c \Rightarrow c(m_2 - 2m_1) = 4$$

Again, by the Fundamental Theorem of Arithmetic, we know that the two factors are either $1, 4$ or $2, 2$ (again, modulo signs). Now,

$$c = \frac{1}{m_1}\left(\frac{q}{2} - 2\right) = \frac{1}{m_2}q \Rightarrow m_2(q-4) = 2m_1 q \Rightarrow q(m_2 - 2m_1) = 4m_2$$

Now, if m_2 is odd, so $m_2 - 2m_1$ is odd as well. It follows from above that $m_2 - 2m_1 = 1$. This implies $c = 4$. But recall, $q = m_2 c = 4m_2 \Rightarrow 4|q$. Contradiction. By assumption, $4 \not| q$. Now, if m_2 is even, it follows that $(m_2 - 2m_1)$ is also even, so i$(m_2 - 2m_1)$ is either 2, or 4. But $q = m_2 c$ and $4 \not| q$, so m_2 even implies c must be odd. So $c = 1$. Contradiction.

So whenever $4 \not| q$, we have that $(p, q) = 1$ for the chosen value of p. And the theorem is proven.

In our search, we will be primarily interested in the limiting cases of possible constructions. Of course, the case of zero full twists is impossible, as the core of each torus in the chain would have to lie in the same plane. The above theorem, however, can be used to generate a list of candidate cases to achieve only one full twist. The other “extreme" will be explained in Section 9.

Of course, not all these cases represent possible constructions. There are several inequalities with which we may eliminate the geometrically impossible entries from our table.

6 Restrictions

The diameter of the general torus is $2r + 2\rho$, and the diameter of any torus contained within the original torus must be less than 2ρ. So we have the condition that $\lambda(2r + 2\rho) < 2\rho \Rightarrow$

$$\lambda < \frac{\rho}{r + \rho} \tag{1}$$

Also, to link the tori, two tori need to be able to be threaded through the hole of one. So at the very least, assuming $\varphi = 0$ even when the chain is straight and not curving within the torus containing it, the diameter of the hole, $2(r - \rho)$, must be greater than 4ρ. And so we have the added condition

$$\rho < \frac{1}{3} r \tag{2}$$

Now, following Malešič and Repovš, we have centered the small tori at the vertices of a regular N-gon. Calling the length of each side of the N-gon l, to link the tori without intersection, the inner wall of each torus must traverse at least half the length the side of the N-gon. That is, $\lambda(r - \rho) > \frac{1}{2}l$, or

$$\lambda > \frac{1}{2(r - \rho)} l \tag{3}$$

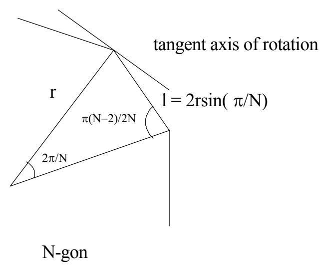

Figure 2: The N-gon with tangent line

In the N-gon, the angle interior to each vertex is $2\alpha = \frac{\pi(N-2)}{N}$. Also note that a single side of the N-gon subtends an angle of $\frac{2\pi}{N}$. So using the law of sines, we have

$$\frac{\sin(\frac{2\pi}{N})}{l} = \frac{\sin\alpha}{r} \Rightarrow \frac{\sin(\frac{2\pi}{N})}{l} = \frac{\sin(\frac{\pi}{2} - \frac{2\pi}{2N})}{r} = \frac{cos(\frac{\pi}{N})}{r}$$

$$\Rightarrow l = r\frac{\sin(\frac{2\pi}{n})}{\cos(\frac{\pi}{N})} = r\frac{\sin(\frac{\pi}{n})\ \cos(\frac{\pi}{N})}{\cos(\frac{\pi}{N})} = 2r\sin\left(\frac{\pi}{N}\right)$$

This transforms inequality (3) into

$$\lambda > \frac{r\sin(\frac{\pi}{N})}{r-\rho} \tag{4}$$

This is the weakest inequality that must be satisfied to ensure that the chain can go all the way around the larger torus in which it is imbedded. Combining this with inequality (1), guaranteeing that the tori of the chain fit inside the larger torus, we have

$$\frac{\rho(r-\rho)}{r(r+\rho)} > \sin\left(\frac{\pi}{N}\right) \tag{5}$$

In fact, we can put a slightly stronger bound on λ from below. Because each torus in the chain is in effect twisted about the line tangent to the core of the larger torus at the $i^{\underline{\text{th}}}$ vertex, so the inner walls of the tori will have to extend, not just beyond $\frac{1}{2}l$, but the beyond the length of the segment of the tangent axis at c_g, up to the point where it intersects the corresponding axis of c_{g+1} (see Figure 2). Now, since the angle interior to the vertices of a regular N-gon are equal to $\pi - \frac{2\pi}{N}$, so the angle between a side of the N-gon and the tangent line is exactly $\frac{\pi}{N}$. So, in fact, the quantity $\lambda(r-\rho)$ must be larger than $\frac{l}{2\cos(\frac{\pi}{N})}$. Making this correction, relation (5) becomes

$$\frac{\rho(r-\rho)}{r(r+_{\rho})} > \tan\left(\frac{\pi}{N}\right) \tag{6}$$

We will use these inequalities to determine which entries on our Big List do not represent possible constructions. It should be noted that we will define r of our model "larger torus" to be 1 (without loss of generality, it will be the unit of our calculations). So we have $\lambda = \lambda r$. This simplification, together with relation (4), $\Rightarrow \lambda > \tan(\frac{\pi}{N})$. Now, the tangent function is monotonic increasing between 0 and $\frac{\pi}{2} > \varphi$. ($\varphi \neq \frac{\pi}{2}$ always or the cores of consecutive tori in the chain would intersect, and so certainly would the tori. $\varphi > \frac{\pi}{2}$ is equivalent to $\varphi < \frac{\pi}{2}$ with a reversed orientation.) And for $N \leq 12, \tan(\frac{\pi}{N}) > \frac{1}{4}$. But by necessity, with $r = 1, \rho < \frac{1}{3}$, so inequality (1) implies $\lambda < \frac{1}{4}$. We therefore need not check any cases from our list where $N \leq 12$.

During the initial stages of the investigation, the equations were greatly simplified by considering the chain as straight, rather than as curved, and strung on the core of a torus. This is because the assumption that the chain is straight lends a great deal of symmetry to the construction. In the straight chain, the plane of one core will intersect the plane of the other core in a radius of that core. And so the relation from one to the other is completely symmetric, in fact equal up to a change of sign (going forwards versus backwards, or viewing the torus as right-side-up versus upside-down). The approximation is also acceptable, since, although the conditions are easier both to visualize as well as to derive, they are weaker. When the tori link in straight chains, there is only more room between them (due to the fact that the planes intersect along radii of the cores) than if they were in a curved chain with otherwise identical dimensions. Of course, in the straight chain, one can also include a spacing variable σ into one's considerations that measures the distance between the outer wall of one torus and the center of a torus with which it is linked. We do not need such a variable because our spacing is declared by fiat from the Malešič-Repovš construction, and our other conditions take the spacing into account already.

Unfortunately, in searching for conditions that would determine when one torus intersected one of the tori with which it shares a neighbor, the straight chain case can not be used. This is because in the circular, closed chain, two tori will enter their common neighbor from different directions as it were, and so might intersect each other immediately in the curved chain given particular dimensions and twist, but never intersect each other in the straight chain, regardless of φ.

This problem is solved, however, by centering two spheres of radius $r + \rho$ at the vertices of our N-gon. The intersection of any two non-consecutive tori implies an intersection of the corresponding spheres, and so we can confidently assert the nonintersections of our chains through inspection of the spheres alone.

We now have a "twist condition", which gave us "The Big List", as well as the above spacing condition. The inequalities above give a "stringiness condition", and so we are ready to proceed. [2]

[2] All of the above conditions can be generalized by substituting r_2 and ρ_2 for λr and $\lambda\rho$ to verify the legitimacy of the motivational example given in the Introduction

7 Distance Formula

Given a twist angle φ and the number of tori embedded in a single torus of the Antoine Necklace we would like to be able to determine bounds on ρ and λ. In this way we will be able to determine which possible combinations of variables will satisfy the conditions of section 6. Since the construction of the Antoine Cantor Set is geometrically very complex, we can simplify the problem by considering only the cores of the tori. The cores of any pair of tori will be two circles in $\mathbb{R}^3$. By determining the minimum distance between any two pair of cores, we can obtain an upper bound on ρ. Clearly ρ must be less than one half the minimum distance between any two cores. For if it was greater than this value the two tori would surely intersect.

More precisely, let T_1 and T_2 be any two tori embedded in a torus with meridional radius ρ in the Antoine Cantor Set with coresponding cores C_1 and C_2. Then

$$\lambda\rho < \frac{1}{2}\min\{\, x \mid x = |p_2 - p_1|, p_1 \in C_1,\ p_2 \in C_2\}$$

Since the cores of tori are rotated and twisted along with the tori during the construction of the Antoine Cantor Set, we may perform all the necessary transformations on the cores alone. Let us give an explicit description of such tori cores. We will consider an arbitrary iteration in the construction of the Antoine Cantor Set. (Our self-similarity restrictions allow us to view it as the second iteration without loss of generality.) We first parametrize a circle in $\mathbb{R}^3$. The chain of tori will lie on the vertices of an N-gon inside a general torus with core of radius r. Hence each embedded torus will have a core of radius λr. Our parametrized core, $c(t)$, is

$$c(t) = \begin{pmatrix} \lambda r\cos(t) \\ \lambda r\sin(t) \\ 0 \end{pmatrix}$$

We first rotate $c(t)$ about the y-axis by an angle of φ'. Call this transformation A. We follow A by a rotation in the xy-plane by an angle of θ. Recall that $\varphi' = \pi/2 - \varphi$ and $\theta = 2\pi/N$. This will allow us to then translate the core to the correct vertex of the embedded N-gon. Call the rotation and translation transformations B and C respectively. Due to the conditions by which we will construct the Antoine Cantor Set, it is clear that the $i^{\underline{\text{th}}}$ core will undergo transformation A $(i-1)$ times and transformation B $(i-1)$ times. (That is, the first torus is only translated to the point $(r, 0, 0)$, the second torus is twisted once by an angle of φ', rotated once by an angle of θ then translated to the next vertex of the N-gon and so on.) Let us consider the commutative properties of A and B. Consider the transformations A and B in matrix representation.

$$A := \begin{pmatrix} \cos(\varphi') & 0 & -\sin(\varphi') \\ 0 & 1 & 0 \\ \sin(\varphi') & 0 & \cos(\varphi') \end{pmatrix}$$

$$B := \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to verify that A and B are multiplicatively noncommutative. How then should one procede with the order in which the transformations are performed? One possibility is to apply the transformation (AB) to $c(t)$ $(i-1)$ times before translating it to the $i^{\underline{\text{th}}}$ vertex. This method would seem to preserve the relationship between one torus and the next since only one operator is being applied $(i-1)$ times for the ith core. This, however, turns out to be computationally complex when finding the transformation $(AB)_\alpha$ for each core $c_\alpha(t)$, where $\alpha = (i-1) \in 0, 1, 2, \ldots, N-1$. The other obvious choice is to apply transformation A $(i-1)$ times and then apply B $(i-1)$ times, to complete all twisting and all rotating separately. This is computationally more feasable, since to obtain A^α and B^α one must simply multiply the trigonometric arguments of A and B by α respectively. Hence for each core, $c_\alpha(t)$, there are unique transformations A_α and B_α which are easy to compute. Similarly, the translation transformation C will depend on which vertex we wish to translate the core. Thus there is a unique C_α for each c_α. It is not clear that one of these methods is more beneficial to our construction of the Antoine Cantor Set for any other reasons. So, we choose the latter method for ease of computaion. The matrix representations for transformations A_α, B_α and C_α are as follows.

$$A_\alpha = \begin{pmatrix} \cos(\alpha\varphi') & 0 & -\sin(\alpha\varphi') \\ 0 & 1 & 0 \\ \sin(\alpha\varphi') & 0 & \cos(\alpha\varphi') \end{pmatrix} B_\alpha = \begin{pmatrix} \cos(\alpha\theta) & -\sin(\alpha\theta) & 0 \\ \sin(\alpha\theta) & \cos(\alpha\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} C_\alpha = \begin{pmatrix} r\cos(\alpha\frac{2\pi}{N}) \\ r\sin(\alpha\frac{2\pi}{N}) \\ 0 \end{pmatrix}$$

As we noted above $\theta = \frac{2\pi}{N}$. We can, therefore, rewrite C_α as follows

$$C_\alpha = \begin{pmatrix} r\cos(\alpha\theta) \\ r\sin(\alpha\theta) \\ 0 \end{pmatrix}$$

Hence, $c_\alpha(t) = (B_\alpha)(A_\alpha)(c(t)) + (C_\alpha)$. That is,

$$c_\alpha(t) = \begin{pmatrix} \cos(\alpha\theta) & -\sin(\alpha\theta) & 0 \\ \sin(\alpha\theta) & \cos(\alpha\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha\varphi') & 0 & -\sin(\alpha\varphi') \\ 0 & 1 & 0 \\ \sin(\alpha\varphi') & 0 & \cos(\alpha\varphi') \end{pmatrix} \begin{pmatrix} \lambda r\cos(t) \\ \lambda r\sin(t) \\ 0 \end{pmatrix} + \begin{pmatrix} r\cos(\alpha\theta) \\ r\sin(\alpha\theta) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda r\cos(\alpha\theta)\cos(\alpha\varphi')\cos(t) - \lambda r\sin(\alpha\theta)\sin(t) + r\cos(\alpha\theta) \\ \lambda r\sin(\alpha\theta)\cos(\alpha\varphi')\cos(t) + \lambda r\cos(\alpha\theta)\sin(t) + r\sin(\alpha\theta) \\ \lambda r\sin(\alpha\varphi')\cos(t) \end{pmatrix}$$

Since we will be considering successive pairs of tori, we wish to have a representation for $c_{\alpha+1}(t)$.

$$c_{\alpha+1}(t) = \begin{pmatrix} (\cos(\alpha+1)\theta\cos(\alpha+1)\varphi')\lambda r\cos t - (\sin(\alpha+1)\theta)\lambda r\sin t + r\cos(\alpha+1)\theta \\ (\sin(\alpha+1)\theta\cos(\alpha+1)\varphi')\lambda r\cos t + (\cos(\alpha+1)\theta)\lambda r\sin t + r\sin(\alpha+1)\theta \\ (\sin(\alpha+1)\varphi')\lambda r\cos t \end{pmatrix}$$

The methods we will use to determine the minimum distance between $c_\alpha(t)$ and $c_{\alpha+1}(t)$ will invlove many linear transformations. It will, therefore, benefit us to consider the first core as lying at the origin of our basis for $\mathbb{R}^3$. We only need consider the cores one pair at a time, hence this can be done without loss of generality. Our new parametrization of c_α and $c_{\alpha+1}$ are as follows.

$$c_\alpha(t) = \begin{pmatrix} \lambda r\cos(\alpha\theta)\cos(\alpha\varphi')\cos(t) - \lambda r\sin(\alpha\theta)\sin(t) \\ \lambda r\sin(\alpha\theta)\cos(\alpha\varphi')\cos(t) + \lambda r\cos(\alpha\theta)\sin(t) \\ \lambda r(\sin\alpha\varphi')\cos(t) \end{pmatrix}$$

$$c_{\alpha+1}(t) = \begin{pmatrix} \lambda r(\cos((\alpha+1)\theta)\cos((\alpha+1)\varphi'))\cos(t) - \lambda r(\sin((\alpha+1)\theta))\sin(t) + r\cos(2(\alpha+1)\frac{\pi}{N}) - r\cos(2\alpha\frac{\pi}{N}) \\ \lambda r(\sin((\alpha+1)\theta)\cos((\alpha+1)\varphi'))\cos(t) + \lambda r(\cos((\alpha+1)\theta))\sin(t) + r\sin(2(\alpha+1)\frac{\pi}{N}) - r\sin(2\alpha\frac{\pi}{N}) \\ \lambda r\sin((\alpha+1)\varphi')\cos(t) \end{pmatrix}$$

We are now concerned with finding the minimum distance between any two cores in the first stage of the Antoine Cantor Set. Again, it clearly suffices to show that the first stage works, as our constructions is perfectly self-similar. It is not, however, obvious that the minimum distance between one pair of successive cores will be identical to the minimum distance between all other pairs of successive cores. As an intuitive justification for not making this assumption, we consider the angles between vectors orthogonal to the planes in which two successive cores lie. Our motivation for doing this is to show that there is not a sufficient degree of symmetry among the linked cores to justify the above assumption. A simple selection of values for the above variables shows that the angles between such vectors do not coincide among all pairs of torus cores. Let $\varphi' = \pi/3$ and $N = 30$. Consider the vector $v = (0, 0, 1)$. The vector v is parallel to the vector orthogonal to the plane of the first torus core (i.e the torus corresponding to $\alpha = 0$). We need only perform transformations A_α and B_α on v to determine the angle between two successive normal vectors. Call the twisted and rotated vector v_α.

$$\begin{aligned} v_\alpha &= \begin{pmatrix} \cos(\alpha\theta) & -\sin(\alpha\theta) & 0 \\ \sin(\alpha\theta) & \cos(\alpha\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha\varphi') & 0 & -\sin(\alpha\varphi') \\ 0 & 1 & 0 \\ \sin(\alpha\varphi') & 0 & \cos(\alpha\varphi') \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\cos(\alpha\theta)\sin(\alpha\varphi') \\ -\sin(\alpha\theta)\sin(\alpha\varphi') \\ \cos(\alpha\varphi') \end{pmatrix} \end{aligned}$$

Then next normal vector is

$$v_{(\alpha+1)} = \begin{pmatrix} -\cos((\alpha+1)\theta)\sin((\alpha+1)\varphi') \\ -\sin((\alpha+1)\theta)\sin((\alpha+1)\varphi') \\ \cos((\alpha+1)\varphi') \end{pmatrix}$$

We determine the angle between the two vectors via the inner product formula $\langle v, w\rangle = ||v||\, ||w|| \cos\omega$, where ω is the angle between the vectors v and w and $0 \leq \omega < 2\pi$. We use $||v||$ to denote the magnitude of a vector v.

$$\langle v_\alpha, v_{\alpha+1}\rangle = \cos(\alpha\theta)\sin(\alpha\varphi')\cos((\alpha+1)\theta)\sin((\alpha+1)\varphi')$$
$$+\sin(\alpha\theta)\sin(\alpha\varphi')\sin((\alpha+1)\theta)\sin((\alpha+1)\varphi') + \cos(\alpha\varphi')\cos((\alpha+1)\varphi')$$
$$= ||v_\alpha|| ||v_{\alpha+1}|| \cos\omega_\alpha$$

It is easy to check that $||v_\alpha|| = ||v_{\alpha+1}|| = 1$. Now let $\alpha = 0$. Then $\cos\varphi' = \cos\omega_0$. Hence $\varphi' = \omega_0 = \frac{\pi}{3}$ and the angle between v_0 and v_1 is $\pi/3$. Now let $\alpha = 1$. Then

$$\cos\left(\frac{\pi}{15}\right)\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{15}\right)\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{\pi}{15}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{15}\right)\sin\left(\frac{2\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)$$
$$= \cos(\omega_1)$$

Simplifying we get $\frac{3}{2}\cos(\frac{\pi}{15}) - \frac{1}{2} = \cos(\omega_1)$. Hence the angle between v_1 and v_2 is $\omega_1 = \arccos(\frac{3}{2}\cos(\frac{\pi}{15}) - \frac{1}{2})$. Suppose that $\omega_0 = \omega_1$. Then

$$\cos\omega_0 = \cos\omega_1 \Rightarrow \frac{3}{2}\cos\frac{\pi}{15} - \frac{1}{2} = \frac{1}{2} \Rightarrow \cos\frac{\pi}{15} = \frac{2}{3}$$

Since $\cos(x)$ is monotonic decreasing on the interval $[0, \frac{\pi}{2}]$ and $\lim_{n\to\infty} \cos\frac{\pi}{n} = 1$ we have $\cos\frac{\pi}{m} \leq \cos\frac{\pi}{15} \leq \cos\frac{\pi}{n}$ for all $n \geq 15$ and $2 \leq m \leq 15$. Let $m = 4$. Then $\frac{1}{\sqrt{2}} \leq \frac{2}{3} \Rightarrow 9 \leq 8$. Clearly this is a contradiction. Thus $\omega_0 \neq \omega_1$.

We now proceed to construct a function to evaluate the distances between cores of tori. It has been shown, [5], that there is no way to compute the minimum distance between two general circles in $\mathbb{R}^3$ in terms of radicals. However, one can evaluate these distances numerically to a sufficient degree of accuracy. Given a specific value for t, say t_0, for the core $c_{\alpha+1}(t)$ the problem of finding the minimum distance between two cirlces in $\mathbb{R}^3$ is reduced to finding the minimum distance between a single point and a circle in $\mathbb{R}^3$ (see Figure 3). Due to the continuity of our distance function, taking a sufficiently large number of t evenly distributed throughout the interval $[0, 2\pi]$, we can obtain a good approximation for the minimum distance between $c_\alpha(t)$ and $c_{\alpha+1}(t)$.

Using the parametric representations of $c_\alpha(t)$ and $c_{\alpha+1}(t)$ choose a value, t_0, for $c_{\alpha+1}(t)$ and project the resulting vector onto the plane, P_α, containing $c_\alpha(t)$. Call this projected vector $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$. The distance, d_1, between $c_{\alpha+1}(t_0)$ and its projection onto P_α is the minimum distance from $c_{\alpha+1}(t_0)$ to P_α. This clearly follows from Pythagoras. Then one must determine the minimum distance between $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$ to the circle $c_\alpha(t)$. The minimum distance between a circle and a point in the plane of the circle is along the line joining the point to the center of the cirlce. This too follows from Pythagoras.

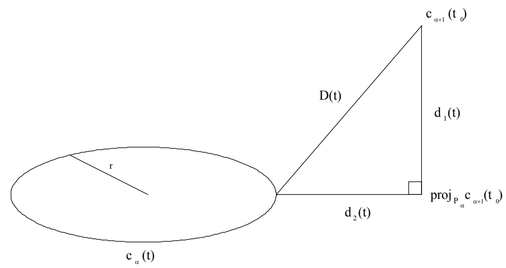

Figure 3: The nearest approximation of a point in $\mathbb{R}^3$ on a circle

[3] One can consider $c_\alpha(t)$ as lying at the origin of its plane, so the distance between a point p in the plane and $c_\alpha(t)$ is just $|p| - |\text{radius of } c_\alpha(t)|$. Hence the minimum distance between the projection of $c_{\alpha+1}(t_0)$ to $c_\alpha(t)$ is simply $|\, ||\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)|| - r\, |$. Call this distance d_2. Once again, by Pythagoras, the distance between $c_{\alpha+1}(t_0)$ and $c_\alpha(t)$ is $D = \sqrt{d_1^2 + d_2^2}$. In fact, we minimize D by minimizing d_1 and d_2.

In order to find $d_1 = ||c_{\alpha+1}(t_0) - \text{proj}_{P_\alpha} c_{\alpha+1}(t_0)||$ and $d_2 = |\, ||\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)|| - r|$ we consider a change of basis in $\mathbb{R}^3$. All vectors are written in terms of the standard basis $\mathcal{S} = \{e_1, e_2, e_3\}$. Consider the orthonormal ordered basis

$$\mathcal{B} = \{\frac{c_\alpha(0)}{||c_\alpha(0)||}, \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||}, \frac{c_\alpha(0)}{||c_\alpha(0)||} \times \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||}\} \text{ where}$$

$$\frac{c_\alpha(0)}{||c_\alpha(0)||} = \begin{pmatrix} \cos\alpha\theta\cos\alpha\varphi' \\ \sin\alpha\theta\cos\alpha\varphi' \\ \sin\alpha\varphi' \end{pmatrix}, \quad \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||} = \begin{pmatrix} -\sin\alpha\theta \\ \cos\alpha\theta \\ 0 \end{pmatrix},$$

$$\frac{c_\alpha(0)}{||c_\alpha(0)||} \times \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||} = \begin{pmatrix} \cos(\alpha\theta)\cos(\alpha\varphi') \\ \sin(\alpha\theta)\cos(\alpha\varphi') \\ \sin(\alpha\varphi') \end{pmatrix} \times \begin{pmatrix} -\sin\alpha\theta \\ \cos\alpha\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\alpha\varphi'\cos\alpha\theta \\ -\sin\alpha\theta\sin\alpha\varphi' \\ \cos\alpha\varphi' \end{pmatrix}$$

Denote the component vector of a vector v written in terms of the basis $\mathcal{B}$ by $[v]_\mathcal{B}$. Hence if

$$v = \beta_1 \frac{c_\alpha(0)}{||c_\alpha(0)||} + \beta_2 \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||} + \beta_2 \left(\frac{c_\alpha(0)}{||c_\alpha(0)||} \times \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||} \right), \text{ then } [v]_\mathcal{B} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

Let

$$[c_{\alpha+1}(t_0)]_\mathcal{B} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

[3]For a more complete proof of this argument, see Appendix 2

for some $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$. The basis vectors $\frac{c_\alpha(0)}{||c_\alpha(0)||}$ and $\frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||}$ span the plane P_α that contains $c_\alpha(t)$. Note also that $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0) \in P_\alpha$. Since $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$ is an orthogonal projection, it is clear that

$$\text{proj}_{P_\alpha} c_{\alpha+1}(t_0) = \left(\gamma_1 \frac{c_\alpha(0)}{||c_\alpha(0)||}, \gamma_2 \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||}\right)$$

A straight forward computation shows that

$$||\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)|| = \sqrt{(\gamma_1)^2 + (\gamma_2)^2}$$

In order to determine $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$, it is therefore sufficient to know the first two components, γ_1 and γ_2, of $[c_{\alpha+1}(t_0)]_{\mathcal{B}}$. Similarly the third component of the vector $[c_{\alpha+1}(t_0)]_{\mathcal{B}}$ will give us the distance from $c_{\alpha+1}(t_0)$ to $\text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$ since the basis vector $\frac{c_\alpha(0)}{||c_\alpha(0)||} \times \frac{c_\alpha(\frac{\pi}{2})}{||c_\alpha(\frac{\pi}{2})||}$ is of unit length and perpendicular to the vector $c_{\alpha+1}(t_0) - \text{proj}_{P_\alpha} c_{\alpha+1}(t_0)$. Therefore, by a change of basis we will get d_1 and d_2, thus obtaining D.

Consider the matrix M which has as its columns the vectors of the basis $\mathcal{B}$.

$$M = \begin{pmatrix} \cos\alpha\theta\cos\alpha\varphi' & -\sin\alpha\theta & -\sin\alpha\varphi'\cos\alpha\theta \\ \sin\alpha\theta\cos\alpha\varphi' & \cos\alpha\theta & -\sin\alpha\theta\sin\alpha\varphi' \\ \sin\alpha\varphi' & 0 & \cos\alpha\varphi' \end{pmatrix}$$

Applying a component vector in terms of the basis $\mathcal{B}$, $[v]_{\mathcal{B}}$, to the matrix M gives us v. $M[v]_{\mathcal{B}} = v$. Therefore, the inverse of M, M^{-1}, takes a vector v and returns its component vector $[v]_{\mathcal{B}}$. $M^{-1}v = [v]_{\mathcal{B}}$. Since the columns of M are linearly independent M^{-1} must exist. By computing M^{-1} we can then determine the vector $[c_{\alpha+1}(t)]_{\mathcal{B}}$. $M^{-1}c_{\alpha+1}(t) = [c_{\alpha+1}(t)]_{\mathcal{B}}$.

$$M^{-1} = \begin{pmatrix} \cos(\alpha\theta)\cos(\alpha\varphi') & \sin(\alpha\theta)\cos(\alpha\varphi') & \sin(\alpha\varphi') \\ -\sin(\alpha\theta) & \cos(\alpha\theta) & 0 \\ -\sin(\alpha\varphi')\cos(\alpha\theta) & -\sin(\alpha\theta)\sin(\alpha\varphi') & \cos(\alpha\varphi') \end{pmatrix}$$

Define the following variable substitutions:

$$u_\alpha = \cos(\alpha\theta) \quad v_\alpha = \sin(\alpha\theta) \quad w_\alpha = \cos(\alpha\varphi') \quad z_\alpha = \sin(\alpha\varphi')$$

Then

$$[c_{\alpha+1}(t)]_{\mathcal{B}} = \begin{pmatrix} u_\alpha w_\alpha & v_\alpha w_\alpha & z_\alpha \\ -v_\alpha & u_\alpha & 0 \\ -z_\alpha u_\alpha & -v_\alpha z_\alpha & w_\alpha \end{pmatrix} \begin{pmatrix} \lambda r[(u_{\alpha+1})(w_{\alpha+1})\cos(t) - (v_{\alpha+1})\sin(t)] + r(u_{\alpha+1} - u_\alpha) \\ \lambda r[(v_{\alpha+1})(w_{\alpha+1})\cos(t) + (u_{\alpha+1})\sin(t)] + r(v_{\alpha+1} - v_\alpha) \\ \lambda r(z_{\alpha+1})\cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \gamma_3(t) \end{pmatrix}$$

where

$$\begin{aligned}
\gamma_1(t) &= u_\alpha w_\alpha \left(\lambda r \left[u_{\alpha+1} w_{\alpha+1} \cos t - v_{\alpha+1} \sin t\right] + r\left(u_{\alpha+1} - u_\alpha\right)\right) \\
&\quad + v_\alpha w_\alpha \left(\lambda r \left[v_{\alpha+1} \cos t + u_{\alpha+1} \sin t\right] + r\left(v_{\alpha+1} - v_\alpha\right)\right) + z_\alpha \lambda r z_{\alpha+1} \cos t \\
\gamma_2(t) &= -v_\alpha \left(\lambda r \left[u_{\alpha+1} w_{\alpha+1} \cos t - v_{\alpha+1} \sin t\right] + r\left(u_{\alpha+1} - u_\alpha\right)\right) \\
&\quad + u_\alpha \left(\lambda r \left[v_{\alpha+1} \cos t + u_{\alpha+1} \sin t\right] + r\left(v_{\alpha+1} - v_\alpha\right)\right) \\
\gamma_3(t) &= -z_\alpha u_\alpha \left(\lambda r \left[u_{\alpha+1} w_{\alpha+1} \cos t - v_{\alpha+1} \sin t\right] + r\left(u_{\alpha+1} - u_\alpha\right)\right) \\
&\quad - v_\alpha z_\alpha \left(\lambda r \left[v_{\alpha+1} \cos t + u_{\alpha+1} \sin t\right] + r\left(v_{\alpha+1} - v_\alpha\right)\right) + w_\alpha \lambda r z_{\alpha+1} \cos t
\end{aligned}$$

Hence $d_1(t) = \gamma_3(t)$ and $d_2(t) = \sqrt{(\gamma_1(t))^2 + (\gamma_2(t))^2} - r$, finally giving us our distance function $D(t) = \sqrt{(d_1(t))^2 + (d_2(t))^2}$.

In order to calculate distances between consecutive cores with this function, a C++ program was implemented (see Appendix 1). The basic method for finding the minimum distance is to take a sufficiently large number of t in the interval $\left[0, \frac{\pi}{2}\right]$, compute $D(t)$ for these values and take the minimum of these distances. We do this for each pair of consecutive tori in the chain and then determine the minimum value of these distances. $\lambda\rho$ is then set at one half this minimum distance. Call this value ρ'. In order to determine whether we have a valid construction of the Antoine Cantor Set, we must check that there exists a possible ρ such that $\lambda\rho < \rho'$, and such that relation (6) of section 6 is satisfied. The program has been given values of φ and N. Call the given value of $N = N_0$. We are now only concerned with values of $\lambda\rho$ that satisfy the relation (6) and are less than ρ'. Take the function

$$F(\rho) := \frac{\rho - \rho^2}{1 + \rho} - \tan\left(\frac{\pi}{N_0}\right)$$

arrived at from relation (6).

We are only interested in those ρ that result in positive values of $F(\rho)$ to guarantee that the inequality is satisfied. Given the two roots of this equation we notice that ρ must fall between them. Denote the two roots by r_1 and r_2 where $r_1 < r_2$. Our program checks that r_1 is less than ρ', verifying that there exists a suitable choice for $\rho < \frac{1}{2}D(t)$, as well as inequality (1) from section 6. In order to become a candidate to be a valid construction, these inequalities must be satisfied.

Now, in order to compute the values of the function $D(t)$, we need to input values for φ, θ, λ, and r. The program takes N, p, and q, where $\varphi = \frac{p\pi}{q}$, as input from the user. With this information the values of θ and φ are computed. The value of r is always assumed to be 1 without loss of generality. Hence, all that is needed to extract numerical values from the function $D(t)$ is a value for λ. Since we do not have an explicit formula for λ in terms of the given variables, we make use of the above inequalites which express upper and lower bounds on λ (recall: $\tan\left(\frac{\pi}{N}\right) < \lambda < \frac{1}{4}$). The program assigns a minimum value to λ then procedes to check if this is an acceptable value. If not, λ is incremented by a sufficiently small amount and the program checks to see if this new value is acceptable. The program repeats this process until a suitable value is found, in which case the process ends and the construction is considered to be a possible candidate. If no suitable value of λ is found, then the construction is not valid. This numerical approximation can be used with an acceptable level of confidence that we do not miss a critical value for λ due again to the sufficient smoothness of the functions and inequalities we are working with.

8 The Big List – Revisited

φ	N	n	$\mathcal{T}$	φ	N	n	$\mathcal{T}$	φ	N	n	$\mathcal{T}$
$\frac{2}{5}\pi$	50	5	20	$\frac{4}{13}\pi$	52	10	16	$\frac{1}{7}\pi$	56	20	8
$\frac{3}{7}\pi$	70	5	30	$\frac{1}{13}\pi$	26	11	2	$\frac{2}{9}\pi$	72	20	16
$\frac{2}{7}\pi$	28	6	8	$\frac{1}{10}\pi$	30	12	3	$\frac{3}{11}\pi$	88	20	24
$\frac{3}{10}\pi$	30	6	9	$\frac{1}{8}\pi$	32	12	4	$\frac{4}{13}\pi$	104	20	32
$\frac{1}{3}\pi$	36	6	12	$\frac{1}{5}\pi$	40	12	8	$\frac{1}{9}\pi$	54	21	6
$\frac{4}{11}\pi$	44	6	16	$\frac{2}{7}\pi$	56	12	16	$\frac{2}{11}\pi$	66	21	12
$\frac{3}{8}\pi$	48	6	18	$\frac{4}{11}\pi$	88	12	32	$\frac{3}{13}\pi$	78	21	18
$\frac{5}{13}\pi$	52	6	20	$\frac{5}{13}\pi$	104	12	40	$\frac{1}{13}\pi$	52	22	4
$\frac{3}{7}\pi$	84	6	36	$\frac{1}{9}\pi$	36	14	4	$\frac{1}{7}\pi$	70	25	10
$\frac{3}{13}\pi$	26	7	6	$\frac{2}{11}\pi$	44	14	8	$\frac{1}{11}\pi$	66	27	6
$\frac{1}{4}\pi$	28	7	7	$\frac{3}{13}\pi$	52	14	12	$\frac{2}{13}\pi$	78	27	12
$\frac{1}{4}\pi$	32	8	8	$\frac{1}{12}\pi$	36	15	3	$\frac{1}{9}\pi$	72	28	8
$\frac{1}{11}\pi$	22	9	2	$\frac{1}{8}\pi$	40	15	5	$\frac{2}{11}\pi$	88	28	16
$\frac{1}{8}\pi$	24	9	3	$\frac{1}{7}\pi$	42	15	6	$\frac{3}{13}\pi$	104	28	24
$\frac{2}{13}\pi$	26	9	4	$\frac{1}{5}\pi$	50	15	10	$\frac{1}{7}\pi$	84	30	12
$\frac{1}{5}\pi$	30	9	6	$\frac{2}{9}\pi$	54	15	12	$\frac{1}{13}\pi$	78	33	6
$\frac{2}{7}\pi$	42	9	12	$\frac{3}{11}\pi$	66	15	18	$\frac{1}{11}\pi$	88	36	8
$\frac{4}{11}\pi$	66	9	24	$\frac{2}{7}\pi$	70	15	20	$\frac{2}{13}\pi$	104	36	16
0	20	10	0	$\frac{4}{13}\pi$	78	15	24	$\frac{1}{13}\pi$	104	44	8
$\frac{1}{12}\pi$	24	10	2	$\frac{1}{11}\pi$	44	18	4	0	22	11	0
$\frac{1}{10}\pi$	25	10	2.5	$\frac{1}{8}\pi$	48	18	6	0	24	12	0
$\frac{1}{7}\pi$	28	10	4	$\frac{2}{13}\pi$	52	18	8	0	26	13	0
$\frac{2}{9}\pi$	36	10	8	$\frac{2}{7}\pi$	84	18	24	0	28	14	0
$\frac{3}{11}\pi$	44	10	12	$\frac{1}{12}\pi$	48	20	4	$\frac{1}{20}\pi$	20	9	1

The above list (sorted in order of n) is all that remains of "The Big List". All cases with $N < 13$ have been eliminated from the list, in accordance with the derived bounds on λ, included in section 6 (Restrictions). All other cases have been eliminated by the program (see Appendix 1).

9 New Directions

In this paper, we have emphasized what we have called n, the number of full twists. From the formulas of our theorem (section 5), we see that when finding a rational fraction of π, primitive in the sense that any integer number of full twists may be achieved, there is a linear relation between the size of the denominator

q of $\varphi = \frac{p\pi}{q}$ and the size of p, keeping φ very close to $\frac{\pi}{2}$. Unfortunately, this forces the tori in the embedded chain to become more and more "stringy", thereby forcing the original ρ to shrink accordingly. It is not known whether there exists a construction of the rigid, self-similar Antoine Necklace in $\mathbb{R}^3$ that achieves only one full twist. The fewest number of twists we have found is 5. Finding a lower number of twists is at this point, an exercise in inspection. And once a lower number of twists is achieved, finding a minimum number of tori, N, is also non-systematic. It seems, however, that both problems may be solved with number theoretic results of a similar order to our theorem above in "Choices and Consequences of φ".

There is also a limit in the other direction, alluded to earlier in this paper. In some sense, a chain with a value of $\varphi = 0$ is in a natural state. Considering for the moment, topological tori that are not rigid, so there is a homeomorphism of any embedded chain to another with equal N, where the cores of all but at most one torus in the chain lie either in the xy-plane, or in a plane perpendicular to it. Of course, with $\varphi = 0$, no adjustments are necessary, which is why we consider it the natural state of the chain. In the general case, however, slicing the torus meridionally through a diameter of each core, we get cylindrical solids, each of which has half a "vertical" torus coming from one end, and half a "horizontal" torus coming from the other, the two linking in the center of the cylinder. Every cylindrical subset of the torus will fit this description save one. Inside the designated last cylinder, half of the last topological torus in the chain will link with the first. All twists accumulated in the original chain through the twisting angle φ will manifest themselves in this last link. Still, the last torus will emerge from one end of the cylinder, its cross-sectional discs either horizontally (if N is odd) or vertically (if N is even) oriented. And it will link with the first torus, so that that part of the core intersecting the plane of the core of the first torus is perpendicular to the plane of the first core; i.e. that segment of the last core is exactly vertical. The number of twists this core undergoes in this last cylinder to complete the chain we call $\mathcal{T}$, and is clearly related to the twisting number n.

For the natural case, when $\varphi = 0$, regardless of N, $\mathcal{T} = 0$. Again, it is the very fact that no correction is needed (i.e. $\mathcal{T} = 0$) to determine the topological twist involved that makes this state natural. For $\varphi = \frac{p\pi}{q}$, $p \neq 0$, and N tori in the chain, $\mathcal{T} = N\frac{p}{q}$, whereas $n = N\frac{q-2p}{2q} = N(\frac{1}{2} - \frac{p}{q})$. So for example, with $N = 20$, and $\varphi = \frac{\pi}{20}$, we have $n = 9$, but $\mathcal{T} = 1$. But for $N = 20, \varphi = 0$, we have $n = 10$, and $\mathcal{T} = 0$. [4]

It turns out that in chains of simple linking type (i.e. they only go around the torus once) the Antoine Cantor Set can be characterized up to homeomorphism by the Antoine Graph with each vertex labeled with a value of $\mathcal{T}$. This results from the one to one relation twixt knot polynomials and the classes of knots they describe. To give $\mathcal{T}$ values of 1 is already done, as shown in the table

[4] For odd N, the natural state is no longer canonical. Clearly, setting φ equal to 0 leaves the last torus in the xy-plane. So $\mathcal{T}$ will no longer be an integer, but instead some integer plus or minus $\frac{1}{2}$.

included in section 8, which is our Big List, each entry also having its value of $\mathcal{T}$ shown.

The value $n = 1$ seems to correspond to the limit as $\mathcal{T}$ goes to infinity. As stated above, whether the value $n = 1$ is achieved sooner is still an open question.

Finally, recall that when we manipulate the tori (more specifically, their cores) in order to construct any given chain, we use matrices to twist the core by φ' and then to rotate the core so that the line described by v intersects the z-axis. Recall that these matrices do not commute. [5] This fact forces us to choose whether to transform the $i^{\underline{\text{th}}}$ core by $A^i B^i$, or by $(AB)^i$. To preserve the relation between consecutive cores in the chain, the latter seems preferable, as then each core differs from its predecessor by AB. Our method of computation, however, makes this choice impractical, and so we applied the former transformation. It is not known at this time whether our choice makes more constructions geometrically possible in the rigid model, or is more or less desirable for any other reason.

In any case, we have constructed classes of the rigid, self-similar Antoine Cantor Set, determined by N, and $\mathcal{T}$. We believe these models can be shown to be Lipschitz homogenous in $\mathbb{R}^3$ by generalizing the methods of Malešič and Repovš.

[5] It should be noted, however, that AB is clearly related to BA, as it is the transpose modulo the fact that all nondiagonal entries have opposite sign.

10 Appendix 1

C++ Code to Eliminate Candidate Constructions

```
#include<math.h>
#include<iostream.h>

//prototype for function that calculates the distance between a point c_alpha+1(t) and
//the core c_alpha. Takes arguments phi'=Pi*p1/p2, N1=# of tori, a=alpha,
// Lam=lambda=scaling factor. Returns distance.
double Distance(double p1, double q1, double N1, double t, double a, double Lam);


int main()
{

 //initialize variables and arrays

 //  phi=Pi*p/q,  N=# of tori, L=lambda

 //**Note** r=radius of first tori in Antoine Cantor Set is assumed to be 1 WLOG

 int array[4][20], listSize;
 int in1=1, in2=1, in3=1;
 double p=0, q=0, N=0, an=1, a, val1, L;
 double T=0, t=0, j=0, Pi, min, rho, D, par;
 Pi=3.14159265358979;

 //user controlled loop. determines how many times one wants to run program
 while(an==1)
 {

  cout<<"\nEnter the number of cases you want to check (less than 20).\n\n";
  cin>>listSize;
  if(listSize>20) listSize=20;

  //fills array with user input
  //ith p, q and N fill spots array[0][i-1], array[1][i-1], and array[2][i-1] resp.
  for(int m=0; m<listSize; m++)
  {

   cout<<"CASE "<<m+1<<endl;
   cout<<"\nEnter p and q\n";
   cin>>array[0][m];
   cin>>array[1][m];
   cout<<"\nEnter the number of tori\n";
```

```
  cin>>array[2][m];
  array[3][m]=0;
 }

 //if listSize < 20 fills rest of array with 0s
 for(int h=listSize; h<20; h++)
 {
  array[0][h]=0;
  array[1][h]=0;
  array[2][h]=0;
  array[3][h]=0;
 }


//loop runs listSize times, checking each pair of phi and N in array
for(int z=0; z<listSize; z++)
{
 p=array[0][z];
 q=array[1][z];
 N=array[2][z];

 //distance formula requires lambda
 //start lambda at lowest possible value
 //and increment until we find a case that works
 //or until lambda surpasses its max value
 for(L=sin(Pi/N); L<.25; L=L+.01)
 {


 min=100;
 double Min=101;


 //vertex loop. checks each pair of consecutitve tori
 for(int k=0; k<N; k++)
 {

  //following two loops numerically compute Distance between the kth and k+1th tori

  for(int i=0; i<=1000; i++)
  {
   j=2*Pi*i/1000.0;
   D=Distance(p, q, N, j, k, L);
   if(D<min){min=D; t=j;}

  }
```

```
  T=t;
  for(i=0; i<=2000; i++)
  {
   j= T - 2*Pi/1000.0 + i*2*Pi/1000000.0;
   D=Distance(p, q, N, j, k, L);
   if(D<min){min=D; t=j;}

  }
  if(min<Min){Min=min, a=k+1; par=t;}
 }


 //sets rho at max value
 rho=0.5*Min/L;


 //checks inequalities to determine validity of variables phi, N, lambda, rho
 val1=((1-tan(Pi/N))-sqrt((1-tan(Pi/N))*(1-tan(Pi/N))-4*tan(Pi/N)))/(2.0);
 if(rho>val1 && L*rho+L<rho)
 {
  array[3][z]=1;//denotes candidate
  L=2;

 }
 else
 {
  array[3][z]=0;//denotes noncandidate

 }
 }
}


cout<<"\nRESULTS: (1=yes 0=no) \n\n";

//outputs results
for(int i=0; i<listSize; i++)
{
 cout<<array[3][i]<<" ";

}

cout<<"\nDo you want to run it again?\n\n"
 <<"Enter 1 for yes 0 for no.\n\n";
cin>>an;
```

```
 cout<<endl;
 cout << endl;
 };


return 0;
};


//Distance function definition
double Distance(double p1, double q1, double N1, double t, double a, double Lam)
{
 double x, y, Pi=3.14159265358979, d1, d2, d3, d4, d5, d6, d7, d8, d9, d10;

 //x=Phi= Pi/2 - phi' = angle of twist off of the vertical
 //y=Rho= angle of rotation to correct vertex

 x=Pi*((0.5)-((1.0*p1)/(1.0*q1)));
 y=2*Pi/N1;

 d2=cos((a+1)*y)*cos((a+1)*x)*Lam*cos(t)-sin((a+1)*y)*Lam*sin(t)+cos((a+1)*y)-cos(a*y);
 d3=sin((a+1)*y)*cos((a+1)*x)*Lam*cos(t)+cos((a+1)*y)*Lam*sin(t)+sin((a+1)*y)-sin(a*y);
 d1=-(1.0)*sin(a*x)*cos(a*y)*(d2)-(1.0)*sin(a*y)*sin(a*x)*(d3)
+(1.0)*cos(a*x)*sin((a+1)*x)*Lam*cos(t);


 d4=cos((a+1)*y)*cos((a+1)*x)*Lam*cos(t)-sin((a+1)*y)*Lam*sin(t)+cos((a+1)*y)-cos(a*y);

 d5=sin((a+1)*y)*cos((a+1)*x)*Lam*cos(t)+cos((a+1)*y)
*Lam*sin(t)+sin((a+1)*y)-sin(a*y);

 d6=(1.0)*cos(a*y)*cos(a*x)*(d4)+(1.0)*sin(a*y)*
cos(a*x)*(d5)+sin(a*x)*sin((a+1)*x)*Lam*cos(t);

 d7=cos((a+1)*y)*cos((a+1)*x)*Lam*cos(t)-sin((a+1)*y)*Lam*sin(t)+cos((a+1)*y)-cos(a*y);

 d8=sin((a+1)*y)*cos((a+1)*x)*Lam*cos(t)+cos((a+1)*y)*Lam*sin(t)+sin((a+1)*y)-sin(a*y);

 d9=-(1.0)*sin(a*y)*(d7)+(1.0)*cos(a*y)*(d8);


 d10=sqrt((d6*d6)+(d9*d9))-Lam;

 return (sqrt((d1*d1)+(d10*d10)));//returns distance
}
```

11 Appendix 2

The nearest approximation of a point b in $\mathbb{R}^3$ on the general circle

Theorem 5 *Take the general circle C in $\mathbb{R}^3$, with centers c, and an arbitrary point $b \in \mathbb{R}^3$ satisfying a particular condition to be stated below. Take the line segment l of minimal length between C and b. Let $l = \overline{ab}$ where $a \in C$. Then the projection of the segment $s = \overline{cb}$, joining the center of C to b, onto the plane $P \supset C$, will intersect the point a.*

Proof: Project b to $b' \in P$. Now, $\overline{bb'} \perp P_1$, and by Pythagoras, $|\overline{bb'}|^2 + |\overline{b'a}|^2 = |\overline{ba}|^2$. If $b' = a$, then the theorem follows trivially. So we may assume $|\overline{b'a}| \neq 0$; i.e. $b' \neq a$. There will therefore be two cases (by the Jordan Curve Theorem).
Case 1: b' is outside of C.

Draw the segment $\overline{b'c}$. Examine the triangle $\triangle b'ca$. Suppose $\overline{b'a}$ is not perpendicular to the vector t_a tangent to C at the point a. Then $\overline{b'a}$ is not the extension of a radius of C, and our triangle is not degenerate. So the triangle inequality tells us that

$$|\overline{b'a}| + |\overline{ac}| > |\overline{b'c}| \tag{7}$$

Let $d = C \cap \overline{b'c}$. The existence of d is guaranteed by the Jordan Separation Theorem. So

$$|\overline{b'd}| + |\overline{dc}| = |\overline{b'c}| \tag{8}$$

Note that $|\overline{dc}| = r$, the radius of C. Since $|\overline{ac}| = r$ as well, so $(1), (2) \Rightarrow |\overline{b'a}| > |\overline{b'd}|$.

$$\Rightarrow |\overline{bd}|^2 = |\overline{bb'}|^2 + |\overline{b'd}|^2 < |\overline{bb'}|^2 + |\overline{b'a}|^2 = |\overline{ba}|^2$$

This contradicts the minimality of $l = |\overline{ba}|$. It therefore follows that in this case, $|\overline{b'a}|$ must be perpendicular to the vector t_a.
Case 2: b' lies inside C.

Again, take $\triangle b'ca$. If b' lies on $\overline{ca}$, then $\overline{b'a} \perp t_a$. So we may assume that $b' \cap \overline{ca} = \phi$. In particular, we may assume that $b' \neq c$. Then

$$|\overline{cb'}| + |\overline{b'a}| > |\overline{ca}| = r$$

Extending the segment $\overline{cb'}$ until it intersects C, call the point of intersection d. By construction, $|\overline{cb'}| + |\overline{b'd}| = |\overline{cd}| = r$. So again, $|\overline{b'd}| < |\overline{b'a}|$. This contradicts the minimality of l as in Case 1. So here too, $\overline{b'a} \perp t_a$.

So the projection of l into P, call it $l_P \perp t_a$. So the extension of l_P includes a radius of C. Extend the segment in this way until it includes c. We will examine the segment $\overline{b'c}$.

Now, $\overline{bb'} \perp P$, so $\overline{b'a}$ is the projection of $\overline{ba}$, as the side of the right triangle $\triangle bb'a$. In the right triangle $\triangle bb'c$, $\overline{b'c} \subset P$ is similarly the projection of $\overline{bc}$ into P. Recall that the projection into P of $\overline{bc}$ intersects C at a, and $\overline{b'c} \perp t_a$. The theorem follows.

References

[1] Blankinship, W.A. (1951) Generalization of a construction of Antoine, *Ann. of Math.* Vol. 53, No. 2, pp. 276-297.

[2] Malešič, Repovš (1996) On Characterization of Lipschitz Manifolds *New developments in differential geometry, Budapest* pp. 265-277, *Kluwer Acad. Publ., Dordrecht* 1999.

[3] Sher, R.B. (1968) Concerning Uncountably Many Wild Cantor Sets in E^3, *Proc. Amer. Math. Soc.* **19** pp. 1195-1200.

[4] Wright, D. (1986) Rigid Sets in E^n, *Pacific Journal of Mathematics* Vol. 121, No. 1, pp. 245-256.

[5] Neff, C.A. (1990) Finding the distance between two circles in three-dimensional space, *IBM J. Res. Develop.* No. 5, pp. 770-775.

[6] Antoine, L. (1929) Sur l'homeomorphie de deux figures et de leur voisinages, *J. Math. Pures Appl.*, **4**, pp. 221-325.

[7] Eaton, W.T. (1973) A generalization of the dog bone space to E^n, *Proc. Amer. Math. Soc.*, **19**, pp. 379-387.