Infinite Dimensionality and Unboundedness for Sets of Planar Convex Bodies with a Common Directed X-ray

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August 15, 2002

Abstract

This paper discusses properties of sets of planar convex bodies with a common directed X-ray. We show that most sets are locally infinite dimensional. Also, we characterize when some sets are unbounded and attempt to extend this result more generally.

1 Introduction

The motivation behind this paper was to contribute work toward the growing field of Geometric Tomography, which was spawned by Hammer's question in 1961 of how many point sources where required to uniquely determine a convex body [3]. The main results of this paper are that given a smooth body $\mathcal{K}$ there exist an infinitely dimensional family of smooth bodies with the same directed X-ray. We also study the question of whether the set of all convex bodies with the same directed X-Ray as a given convex body K is bounded or unbounded. We find both sufficient conditions and necessary conditions for this set to be unbounded. Nontrivial examples are given where both boundedness and unboundedness occur.

1.1 Notation and Definitions

Definition 1 *A set in the plane is* ***convex*** *if given any two points a, b in the set, the line segment joining a and b is contained entirely within the set.*

Definition 2 *A* ***convex body*** *is a compact, convex subset of the plane with a non-empty interior.*

Definition 3 *A* ***directed X-ray*** *(or fan-beam X-ray) measures the chord length $X(\varphi)$ of a convex body along a particular ray originating from the point source O with the angle of inclination φ.See Figure 1.*

Throughout this paper a convex body will be denoted by K and we will always assume the source is not in K. With polar coordinates centered at the source, K is uniquely determined by a pair of positive continuous polar functions $r(\varphi), R(\varphi)$ satisfying

$$K = \{(s, \varphi) : 0 < r(\varphi) \leq s \leq R(\varphi)\}.$$

It will be assumed unless stated differently that the convex body K is smooth, meaning that the boundary of K is C^2.

It has become conventional to orient the point source at the origin and place the convex body in the upper half plane. If this is not true one can make this true with a translation and/or rotation of the body and source. See Kimble [5] for more information.

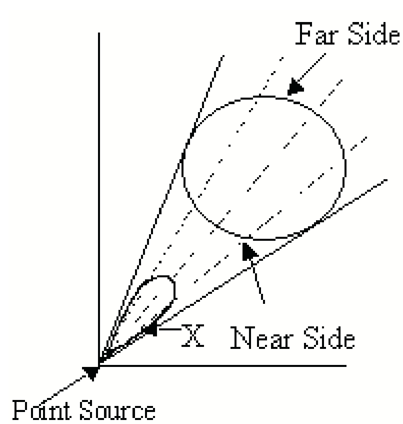
Definition 4 *We refer to the points with polar coordinates $(r(\varphi), \varphi)$ as points on the* ***near side*** *of K and the function $r(\varphi)$ as the near side function of K. Similarly, $(R(\varphi), \varphi)$ are points on the* ***far side*** *and $R(\varphi)$ is the far side function. See Figure 1.*

Since the X-ray $(X(\varphi))$, the far side $(R(\varphi))$, and the near side $(r(\varphi))$, are all functions of φ we will from this point on drop the φ denoting the X-ray body, far side, and near side as X, R, and r, respectively. Notice that $X = R - r$. We consider r to be a positive function, so the special case when $r = 0$ and $X = R$ is usually a triviality left for the reader.

Definition 5 *Let K be a convex body with an associated directed X-ray from source O. Then the* ***supporting cone*** *$\mathcal{C}[\alpha, \beta]$ of K is the largest cone with vertex at O such that for all φ, $\alpha < \varphi < \beta$, $X(\varphi) > 0$. The* ***supporting rays*** *of K are $\varphi = \alpha$, and $\varphi = \beta$.*

From this point on the supporting ray will be denoted α and β where $\alpha < \beta$.

Definition 6 *$\mathcal{C}(K)$ is defined to be the set of all convex bodies with the same directed X-ray as K.*



2 The Differential Operator $\mathcal{K}$

We would like a formula to determine if a function is concave toward or away from the origin. A differential operator that does this for smooth functions is defined next.

Definition 7 *If x is a curve that is twice differentiable on an interval I, the differential operator, $\mathcal{K}(x)$ is defined as:*

$$\mathcal{K}(x) = x^2 + 2(x')^2 - x''x.$$

As one might logically assume, $\mathcal{K}$ is related to curvature, denoted κ. In polar coordinates, the curvature $\kappa(x) = \dfrac{x^2 + 2(x')^2 - x''x}{(x^2 + (x')^2)^{3/2}}$. The denominator is always positive, since the sum of two squares is greater than or equal to zero. Thus, the sign of the curvature is determined solely by the numerator, which is equal to $\mathcal{K}(x)$. For the remainder of the section we will discuss both geometric and analytical properties of $\mathcal{K}$

Theorem 8 *Let $g(\varphi)$ be a C^2 polar function on (α, β) and continuous on $[\alpha, \beta]$, $0 < \beta - \alpha < \pi$. Define the parametric curve $\Gamma =< g(\varphi)\cos\varphi, g(\varphi)\sin\varphi >$. Then,*

1. *Γ is concave toward the origin on (α, β) if and only if $\mathcal{K}(g)(\varphi) \geq 0$ for all $\varphi \in (\alpha, \beta)$.*
2. *Γ is concave away from the origin on (α, β) if and only if $\mathcal{K}(g)(\varphi) \leq 0$ for all $\varphi \in (\alpha, \beta)$.*

The proof of this is omitted but can be found in [1] if the reader so desires.

Claim 9 *If x is a positive C^2 function such that $\mathcal{K}(x) \leq 0$ then $x'' \geq 0$ and hence x' is strictly increasing.*

Proof. We know that $\mathcal{K}(x) = x^2 + 2(x')^2 - x''x \leq 0$. Thus, by adding $x''x$ to both sides one sees that, $x^2 + 2(x')^2 \leq x''x$. Because the sum of two squares is non-negative the previous expression indicates that $0 \leq xx''$. We know that $x > 0$, thus $x'' \geq 0$, if the inequality $0 \leq xx''$ is to be true. ∎

Theorem 10 *For any two non-zero smooth C^2 functions x and y:*

$$\mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2.$$

Proof. Notice $xy'' = -\dfrac{x}{y}\left[-y''y + y^2 + 2(y')^2\right] + \dfrac{2(y')^2x}{y} + xy.$

Thus, $xy'' = -\dfrac{x}{y}\mathcal{K}(y) + xy + \dfrac{2(y')^2x}{y}.$

Similarly, $x''y = -\dfrac{y}{x}\mathcal{K}(x) + xy + \dfrac{2(x')^2y}{x}.$

By combining these two results we see that:

$$-xy'' - x''y = -(xy'' + x''y) = \frac{x}{y}\mathcal{K}(y) - 2xy - \frac{2(y')^2 x}{y} + \frac{y}{x}\mathcal{K}(x) - \frac{2(x')^2 y}{x}.$$

Placing the right hand side of the expression into $\mathcal{K}(x+y) = \mathcal{K}(x) + 2xy + 4x'y' - (x''y + y''x) + \mathcal{K}(y)$, yields:

$$\Longrightarrow \mathcal{K}(x+y) = \mathcal{K}(x) + 2xy + 4x'y' + \frac{x}{y}\mathcal{K}(y) - 2xy - \frac{2(y')^2 x}{y} + \frac{y}{x}\mathcal{K}(x) - \frac{2(x')^2 y}{x} + \mathcal{K}(y)$$

$$\Longrightarrow \mathcal{K}(x+y) = \frac{x}{x}\mathcal{K}(x) + \frac{y}{x}\mathcal{K}(x) + \frac{y}{y}\mathcal{K}(y) + \frac{x}{y}\mathcal{K}(y) - \left[\frac{2(y')^2 x}{y} - 4x'y' + \frac{2(x')^2 y}{x}\right]$$

$$\Longrightarrow \mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\left(\frac{x'}{x}\right)^2 - 2\frac{x'y'}{xy} + \left(\frac{y'}{y}\right)^2\right]$$

$$\Longrightarrow \mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2, \text{ as desired. } \blacksquare$$

Corollary 11 $\mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \mathcal{K}(y) + xy\left[\frac{\mathcal{K}(y)}{y^2} - 2\left[\frac{x'}{x} - \frac{y'}{y}\right]^2\right].$

Proof. From theorem 10 we know that $\mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2$

$$\Longrightarrow \mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \mathcal{K}(y) + \frac{x}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2$$

$$\Longrightarrow \mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \mathcal{K}(y) + xy\left[\frac{\mathcal{K}(y)}{y^2} - 2\left[\frac{x'}{x} - \frac{y'}{y}\right]^2\right], \text{ as desired. } \blacksquare$$

Lemma 12 *If y is a function of φ such that $\mathcal{K}(y) \leq 0$ then for any real number t, $K(ty) \leq 0$*

Proof. From the definition one can see that $\mathcal{K}(ty) = t^2\mathcal{K}(y)$. $\blacksquare$

Lemma 13 *If x and y are positive function of φ such that $\mathcal{K}(x) \leq 0$ and $\mathcal{K}(y) \leq 0$ then $\mathcal{K}(x+y) \leq 0$*

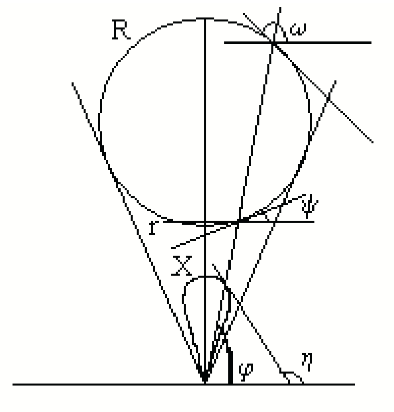
Proof. By Theorem 10 we know that $\mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2$,

By our hypothesis we have $K(x) \leq 0$, and $K(y) \leq 0$. Also, we know that $\frac{x+y}{x} > 0$, $\frac{x+y}{y} > 0$, and $2xy > 0$, since x and y are positive functions. Thus $\mathcal{K}(x+y) = \frac{x+y}{x}\mathcal{K}(x) + \frac{x+y}{y}\mathcal{K}(y) - 2xy\left[\frac{x'}{x} - \frac{y'}{y}\right]^2 \leq 0$, as desired. $\blacksquare$

The form of $\mathcal{K}(x+y)$ from Theorem 10 is pleasing to use because of its geometrical interpretations. We know that $\mathcal{K}(x)$ relates to the curvature of x and likewise $\mathcal{K}(y)$ relates to curvature of y. We will try to further explore the geometric interpretation of $\mathcal{K}$ with the next few lemmas. In the following lemma we will show that $\left[\frac{x'}{x} - \frac{y'}{y}\right]$ is expressible in terms of the angles of inclination of the tangent lines to the graph of x and y at φ.

Lemma 14 $\frac{r'}{r} = \cot(\psi - \varphi)$, *where* $\psi = \psi(\varphi)$ *is the angle of inclination of the tangent line to the graph of* r *at the point* $(r(\varphi), r)$.

Proof. Refer to Figure 2.



In polar coordinates we know that $(x, y) = (r\cos(\varphi), r\sin(\varphi))$. Since $\frac{dy}{dx}$ represents the slope of a line, and that for all line $\frac{dy}{dx} = \tan(\psi) = -\tan(\pi - \psi)$ measures the slope of a line at angle ψ we know that $\frac{dy}{dx}|_\varphi = \tan(\psi)$. Since both x and y are functions of φ, $\frac{dy}{dx}|_\varphi = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}}$. Because r is a function of φ we may use the chain rule to evaluate this derivative, thus $\tan(\psi) = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{r\cos(\varphi) + r'\sin(\varphi)}{r'\cos(\varphi) - r\sin(\varphi)} = \frac{r(\cos(\varphi) + \frac{r'}{r}\sin(\varphi))}{r(\frac{r'}{r}\cos(\varphi) - \sin(\varphi))}$. Through a little algebra we see:

$$\Longrightarrow [\frac{r'}{r}\cos(\varphi) - \sin(\varphi)]\ \tan(\psi) = (\cos(\varphi) + \frac{r'}{r}\sin(\varphi))$$

$$\Longrightarrow \frac{r'}{r}[\cos(\varphi)\tan(\psi) - \sin(\varphi)] = \cos(\varphi) + \tan(\psi)\sin(\varphi)$$

$$\Longrightarrow \frac{r'}{r} = \frac{\frac{\cos(\psi)\cos(\varphi)}{\cos(\psi)} + \frac{\sin(\psi)\sin(\varphi)}{\cos(\psi)}}{\frac{\cos(\varphi)\sin(\psi)}{\cos(\psi)} - \frac{\cos(\psi)\sin(\varphi)}{\cos(\psi)}}$$

$$\Longrightarrow \frac{r'}{r} = \frac{\cos(\psi)\cos(\varphi) + \sin(\psi)\sin(\varphi)}{\cos(\varphi)\sin(\psi) - \cos(\psi)\sin(\varphi)} = \frac{\cos(\psi - \varphi)}{\sin(\psi - \varphi)} = \cot(\psi - \varphi), \text{ as desired. } \blacksquare$$

Similarly it can be proven that $\frac{R'}{R} = \cot(\omega - \varphi)$ and $\frac{X'}{X} = \cot(\eta - \varphi)$ as shown in Figure 2 where $\omega = \omega(\varphi)$ and $\eta = \eta(\varphi)$ are the slopes of the angles of inclination at the points $(R(\varphi), R)$ and $(X(\varphi), X)$ respectively. Before we continue our geometric analysis of the differential operator $\mathcal{K}$, consider the following corollary of Theorem 10 which will lead into a new interpretation of the differential operator $\mathcal{K}$.

Corollary 15 *Given a convex body K with near side r and far side R,* $\mathcal{K}(X) = \dfrac{X}{R}\mathcal{K}(R) - \dfrac{X}{r}\mathcal{K}(r) - 2Rr\left[\dfrac{R'}{R} - \dfrac{r'}{r}\right]^2$.

Proof. Apply Theorem 10 with R-r = X. ■

Lemma 16 *In polar coordinates with curve X, $\mathcal{K}(X) = \kappa(X)X^3\left|\csc^3(\eta-\varphi)\right|$, where $\mathcal{K}$ is the differential operator, κ is the curvature, and η is the angle of inclination of the tangent line to X at a angle φ.*

Proof. From the curvature formula $\kappa(X) = \dfrac{X^2 + 2(X')^2 - X''X}{(X^2+(X')^2)^{3/2}} = \dfrac{\mathcal{K}(X)}{(X^2+(X')^2)^{3/2}}$

$\implies \mathcal{K}(X) = \kappa(X)\left[X^2+(X')^2\right]^{3/2}$

$\implies \mathcal{K}(X) = \kappa(X)X^3\left[1+\left(\dfrac{X'}{X}\right)^2\right]^{3/2}$ by Lemma 14 we can replace $\dfrac{X'}{X}$ with $\cot(\eta-\varphi)$,

$\implies \mathcal{K}(X) = \kappa(X)X^3\left[1+\cot^2(\eta-\varphi)\right]^{3/2}$

$\implies \mathcal{K}(X) = \kappa(X)X^3\left|\csc^3(\eta-\varphi)\right|$. ■

Lemma 17 $\mathcal{K}(X)$

$$= \frac{X}{R}\kappa(R)R^3\left|\csc^3(\omega-\varphi)\right| - \frac{X}{r}\kappa(r)r^3\left|\csc^3(\psi-\varphi)\right| + 2Rr\left(\frac{\sin^2(\omega-\psi)}{\sin^2(\omega-\varphi)\sin^2(\psi-\varphi)}\right)$$

Proof. By Corollary 15, $\mathcal{K}(X) = \dfrac{X}{R}\mathcal{K}(R) - \dfrac{X}{r}\mathcal{K}(r) - 2Rr\left[\dfrac{R'}{R} - \dfrac{r'}{r}\right]^2$.

Now applying Lemma 16 to both $\mathcal{K}(R)$ and $\mathcal{K}(r)$ we see that

$\mathcal{K}(X)=\dfrac{X}{R}\kappa(R)R^3\left|\csc^3(\omega-\varphi)\right| - \dfrac{X}{r}\kappa(r)r^3\left|\csc^3(\psi-\varphi)\right| - 2Rr\left[\dfrac{R'}{R} - \dfrac{r'}{r}\right]$.

By Lemma 14, $\left[\dfrac{R'}{R} - \dfrac{r'}{r}\right]^2 = \left[\cot(\omega-\varphi) - \cot(\psi-\varphi)\right]^2 =$

$\left[\dfrac{\cos(\omega-\varphi)\sin(\psi-\varphi) - \cos(\psi-\varphi)\sin(\omega-\varphi)}{\sin(\omega-\varphi)\sin(\psi-\varphi)}\right]^2 = -\dfrac{\sin^2(\omega-\psi)}{\sin^2(\omega-\varphi)\sin^2(\psi-\varphi)}$. Combining this with the previous result yields

$\mathcal{K}(X) = \dfrac{X}{R}\kappa(R)R^3\left|\csc^3(\omega-\varphi)\right| - \dfrac{X}{r}\kappa(r)r^3\left|\csc^3(\psi-\varphi)\right| + 2Rr\left(\dfrac{\sin^2(\omega-\psi)}{\sin^2(\omega-\varphi)\sin^2(\psi-\varphi)}\right)$.

■

This is one more way to think geometrically about the differential operator K and its properties.

3 Working with a Smooth Body K

Now that we have established several geometric results with the differential operator $\mathcal{K}$ we would like to apply this to a smooth body K. More specifically, we want to know how a smooth body K behaves at the boundaries of the supporting cone. After exploring the boundaries we would like to know what can be added to the convex body to form new convex bodies with the same direct X-ray, and under what conditions such functions can be added. We will start off by discussing what happens at the boundaries of $\frac{R'}{R}$. Then we will analyze how the differential operator behaves at the boundaries of a far side to which has been added a C^2 function. This result will later be helpful in understanding the curvature of a far side that has been altered in some fashion. Once we have developed potential candidates for the far side we will consider what conditions must be placed on the near side in order to form a proper new near side. We will end the section by looking at solutions to a differential equation that appear to play an important role in many of the equations throughout this section.

Lemma 18 *If K is a smooth body with far side R then $\lim_{\varphi \to \alpha^-} \frac{R'}{R} = +\infty$ and the $\lim_{\varphi \to \beta^+} \frac{R'}{R} = -\infty$.*

Proof. We will handle the case for $\varphi \to \alpha^-$ and leave the other case, which can be proven in a similar fashion, for the reader. By Lemma 14 we know that $\frac{R'}{R} = \cot(\omega - \varphi)$, where ω is the angle of inclination of the tangent line to R at a particular point angle φ. Clearly as $\varphi \to \alpha^-$, ω will also approach α. Thus as $\varphi \to \alpha^-$, $\omega - \varphi \to 0$. This implies that $\cot(\omega - \varphi) = \frac{R'}{R}$ will become $\pm\infty$ at the boundaries. However, at α we know that R is an increasing function of φ, because it is concave toward the origin. With this and the fact that for all values of φ, $R > 0$ we can deduce that $\frac{R'}{R} = +\infty$ as $\varphi \to \alpha^-$, as desired ■

Lemma 19 *If K is a smooth body with near side r then $\lim_{\varphi \to \alpha^-} \frac{r'}{r} = -\infty$ and the $\lim_{\varphi \to \beta^+} \frac{r'}{r} = +\infty$.*

Proof. The proof of this Lemma is nearly identical to Lemma 18, simply use the fact that by Lemma 14 one can write $\frac{r'}{r} = \cot(\psi - \varphi)$, where is the angle of inclination of the tangent line to r at a particular angle φ. ■

Lemma 20 *Let K be a smooth body with far side R, whose second derivative R'' is bounded above. Then if s is any positive C^2 function on $[\alpha, \beta]$, $\lim_{\varphi \to \alpha^-, \beta^+} \mathcal{K}(R + s) = +\infty$.*

Proof. We know from our Corollary 11

$$\mathcal{K}(R+s)=\frac{R+s}{s}\mathcal{K}(s)+\mathcal{K}(R)+Rs\left[\frac{\mathcal{K}(R)}{R^2}-2\left[\frac{R'}{R}-\frac{s'}{s}\right]^2\right].$$

By the definition of $\mathcal{K}(R)$ we can replace $\dfrac{\mathcal{K}(R)}{R^2}$ with $1+2\left(\dfrac{R'}{R}\right)^2-\dfrac{R''}{R}$.

$$\Longrightarrow \mathcal{K}(R+s)=\frac{R+s}{s}\mathcal{K}(s)+\mathcal{K}(R)+Rs\left[1+2\left(\frac{R'}{R}\right)^2-\frac{R''}{R}-2\left[\frac{R'}{R}-\frac{s'}{s}\right]^2\right]$$

$$\Longrightarrow \mathcal{K}(R+s)=\frac{R+s}{s}\mathcal{K}(s)+\mathcal{K}(R)$$

$$+Rs\left[1+2\left(\frac{R'}{R}\right)^2-\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2+4\frac{R'}{R}\frac{s'}{s}-2\left(\frac{s'}{s}\right)^2\right]$$

$$\Longrightarrow \mathcal{K}(R+s)=\frac{R+s}{s}\mathcal{K}(s)+\mathcal{K}(R)+Rs\left[1-\frac{R''}{R}+4\frac{R'}{R}\frac{s'}{s}-2\left(\frac{s'}{s}\right)^2\right]$$

$$\Longrightarrow \mathcal{K}(R+s)=\frac{R+s}{s}\mathcal{K}(s)+\mathcal{K}(R)+\left[s(R-R'')+4R's'-2Rs\left(\frac{s'}{s}\right)^2\right]$$

Since s is a twice differentiable continuous function on $[\alpha,\beta]$, we know that $\mathcal{K}(s)=s^2+2(s')^2-ss''$ is a bounded function. Also, the expression $\dfrac{R+s}{s}$ is bounded because R and s are both bounded functions for all values of φ. By the extreme value theorem, the positive function $s(\varphi)$ is bounded away from zero on the closed interval $[\alpha,\beta]$. Thus, $\dfrac{R+s}{s}\mathcal{K}(s)$ is bounded on $[\alpha,\beta]$. At the boundaries we know that $\mathcal{K}(R)=R^2+2(R')^2-RR''\longrightarrow+\infty$ since R'' is bounded above and Lemma 18 implies $(R')^2\longrightarrow+\infty$ at both α and β. In the expression $\left[s(R-R'')+4R's'-2Rs\left(\dfrac{s'}{s}\right)^2\right]$ all terms are bounded below except perhaps $4R's'$. We know that $4R's'\longrightarrow\pm\infty$ at the boundaries once again by Lemma 18. In the case that $4R's'\to+\infty$ the proof would be complete. So let us suppose that $4R's'\geq -C\,|R'|$ were C is some positive constant. In other words, $4R's'\to-\infty$. If this is true, then near α and β, $\mathcal{K}(R+s)$ behaves like $2(R')^2-C\,|R'|$, since all of the other terms are bounded below. Notice $2(R')^2-C\,|R'|=|R'|\,(2\,|R'|-C)\longrightarrow+\infty$, as desired. ■

The above theorem indicates that at the boundaries if we add a C^2 function to R we are concave the correct way for a new proper far side. In the following theorem we will check the rest of the interval.

Theorem 21 *Let K be a smooth body with far side R, such that $\mathcal{K}(R)\geq\epsilon>0$ on $[\alpha,\beta]$ and R'' is bounded above on (α,β). Then if s is a positive C^2 function on $[\alpha,\beta]$, $\mathcal{K}(R+ts)\geq\dfrac{\epsilon}{2}>0$ on the interval $[\alpha,\beta]$ for sufficiently small $t\in\mathbb{R}^+$.*

Proof. By our hypothesis we are able to apply Lemma 20. Thus, we know that at least near α and β, $\mathcal{K}(R+ts)\geq\dfrac{\epsilon}{2}>0$. This allows us to focus our attention on a fixed compact subinterval of (α,β), say $[\gamma,\delta]$.

By Corollary 11 we know that

$$\mathcal{K}(R+ts)=\mathcal{K}(R)+\frac{\mathcal{K}(s)}{s}t(R+ts)+Rts\left[\frac{\mathcal{K}(R)}{R^2}-2\left[\frac{R'}{R}-\frac{s'}{s}\right]^2\right].$$

We have assumed that $\mathcal{K}(R)\geq\epsilon>0$ and that s is a positive C^2 function. The latter implies that $\dfrac{\mathcal{K}(s)}{s}$ is a bounded function, in other words there exist real numbers b and B such that $b\leq\dfrac{\mathcal{K}(s)}{s}\leq B$. Thus $\mathcal{K}(R+ts)\geq\epsilon-bt(R+ts)+Rts\left[\dfrac{\mathcal{K}(R)}{R^2}-2\left[\dfrac{R'}{R}-\dfrac{s'}{s}\right]^2\right]$. If we replace $\mathcal{K}(R)$ with $R^2+2\left(R'\right)^2-RR''$ and expand out $-2\left[\dfrac{R'}{R}-\dfrac{s'}{s}\right]^2$ we see the following:

$$\mathcal{K}(R+ts)\geq\epsilon-bt(R+ts)+Rts\left[1+2\left(\frac{R'}{R}\right)^2-\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2+4\frac{R'}{R}\frac{s'}{s}-2\left(\frac{s'}{s}\right)^2\right]$$

$$\geq\epsilon-bt(R+ts)+t\left[s(R-R'')+4R's'-2Rs\left(\frac{s'}{s}\right)^2\right].$$

Notice $\left[s(R-R'')+4R's'-2Rs\left(\dfrac{s'}{s}\right)^2\right]$ is bounded below on $[\gamma,\delta]$, since all functions are continuous on $[\gamma,\delta]$ and R'' is bounded above. Thus, let M be the lower bound of $\left[s(R-R'')+4R's'-2Rs\left(\dfrac{s'}{s}\right)^2\right]$. Then $\mathcal{K}(R+ts)\geq\epsilon-bt(R+ts)+tM=\epsilon-t\left[b(R+ts)+M\right]$. Since R, t, and s are all bounded function on $[\gamma,\delta]$, we can simply say that $\mathcal{K}(R+ts)\geq\epsilon-tM'$ where M' is the lower bound of $\left[b(R+ts)+M\right]$. The sign of M' is not significant because we are able to control t. We can make it small enough in either case such that $\mathcal{K}(R+ts)\geq\dfrac{\epsilon}{2}>0$, as desired. ■

Thus, we see from Lemma 20 and Theorem 21 that we can create a properly defined far side by simply taking the existing R with non-zero curvature and adding to it a suitable multiple of a C^2 function. Now let us turn our attention toward what what happens to the near side on the interval $[\alpha,\beta]$ when we add some positive multiple of a C^2 function to it. We will see that this time we need to either restrict s to being concave away from the origin or define the near side, r, such that $\mathcal{K}(r)\leq-\xi<0$.

Corollary 22 *Let K be a smooth body with near side r and let s be a positive C^2 function such that $\mathcal{K}(s)\leq 0$. Then for any $t\in\mathbb{R}^+$, $\mathcal{K}(r+ts)\leq 0$.*

Proof. This result follows from combining Lemma 12 with Lemma 13. ■

Now suppose that s did not have the property that $\mathcal{K}(s)\leq 0$, but instead r had a stronger condition of having non-zero curvature.

Claim 23 *Let K be a smooth body with near side r, such that $\mathcal{K}(r)\leq-\xi<0$. If s is a positive C^2 function on $[\alpha,\beta]$, then $\mathcal{K}(r+ts)\leq\frac{-\xi}{2}<0$ where $t\in\mathbb{R}^+$ and t is sufficiently small.*

Proof. Recall that $\mathcal{K}(r) = r^2 + 2(r')^2 - rr'' < 0$, thus $r - r'' < -2\frac{(r')^2}{r}$.

Also, by Theorem 10 we know that $\mathcal{K}(r+ts) = \frac{r+ts}{r}\mathcal{K}(r) + \frac{r+ts}{ts}t^2\mathcal{K}(s) - 2rts\left[\frac{r'}{r} - \frac{s'}{s}\right]^2$
Through similar algebraic technics as seen in earlier proofs we obtain the following:
$\Longrightarrow \mathcal{K}(r+ts) = K(r) + ts\left[r - r'' + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right]$.

By the above argument that $r - r'' < -2\frac{(r')^2}{r}$ we know that
$\mathcal{K}(r+ts) = K(r) + ts\left[r - r'' + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right]$
$< -\xi + ts\left[-2\frac{(r')^2}{r} + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right]$. Also since r, t, s, and s' are all bounded on $[\alpha, \beta]$, we will turn our attention toward $-2\frac{(r')^2}{r} + 4r'\frac{s'}{s}$. Let δ be some small positive number, Consider the following partition of on $[\alpha, \beta]$ into $[\alpha, \alpha+\delta)$, $[\alpha+\delta, \beta-\delta]$, and $(\beta-\delta, \beta]$

Case 24 $\varphi \in [\alpha, \alpha+\delta)\bigcup(\beta-\delta, \beta]$

We know that at the end point $r' \to \pm\infty$ by Lemma 19. If $r' \to -\infty$ then by the above inequality $\mathcal{K}(r+ts) < -\infty$ and the proof is complete. However, if $r' \to +\infty$ then the behavior of $\mathcal{K}(r+ts)$ depends on $-2\frac{(r')^2}{r} + 4r'\frac{s'}{s} \geq -2\frac{|r'|^2}{r} + 4|r'|\frac{s'}{s} \geq -2|r'|(\ \frac{|r'|}{r} + C)$ where C is a constant. It is obvious for the last expression that we can say $\mathcal{K}(r+ts) < -\infty$. In either case we see that near the boundaries $\lim_{\varphi \to \alpha^-, \beta^+} \mathcal{K}(r+ts) = -\infty$, thus implying that we can make $\mathcal{K}(r+ts) \leq \frac{-\xi}{2} < 0$, as desired.

Case 25 $\varphi \in [\alpha+\delta, \beta-\delta]$

Simply observe that on this interval $\left[-2\frac{(r')^2}{r} + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right]$ is bounded above, since all function are continues on $[\alpha+\delta, \beta-\delta]$.

Thus, we can let M be the upper bound of $\left[r - r'' + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right]$, so $\left[r - r'' + (r+ts)\frac{K(s)}{s^2} + 4r'\frac{s'}{s} - 2r\left(\frac{s'}{s}\right)^2\right] \leq M$. From this we see $\mathcal{K}(r+ts) \leq -\xi + tsM$. Because s is a positive C^2 function on $[\alpha, \beta]$, it is clear that $\mathcal{K}(r+ts) \leq -\frac{\xi}{2} < 0$ for $t > 0$ sufficiently small, as desired. ■

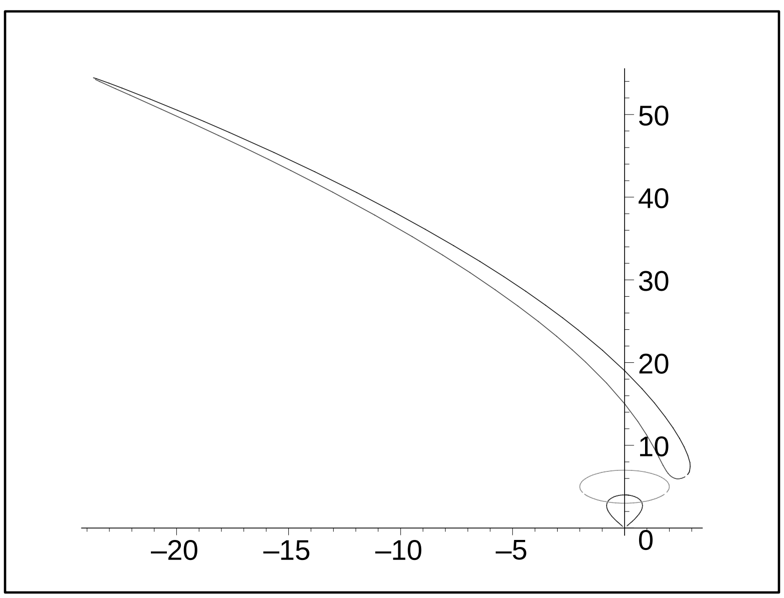
Conclusion 26 *Given a smooth body K with near side r and far side R, such that on $[\alpha, \beta]$ $\mathcal{K}(r) \leq -\xi < 0$ and $\mathcal{K}(R) \geq \epsilon > 0$, and R'' is bounded above on (α, β), then the following is true:*

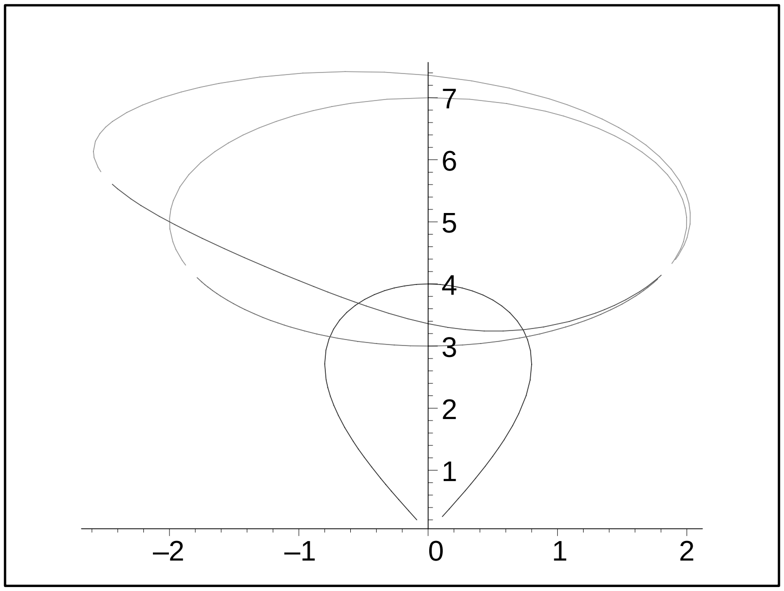
If s is any positive C^2 function, then for sufficiently small t, K' with near side $r + ts$ and far side $R + ts$ is a smooth convex body with the same directed X-ray as K.

Proof. By Theorem 21 $\mathcal{K}(R + ts) > 0$ and by Theorem 23 $\mathcal{K}(r + st) < 0$, thus we know that K' is a convex body by Theorem 8. Note that the X-ray of K' is $R + ts - (r + ts) = R + ts - r - ts = R - r = X$, as desired ■

Observe that because s can be any positive C^2 function we are able to create a set $\{K'\}$ of convex bodies that are infinite dimensional with the same directed X-ray as K.

The following is an illustration of the above conclusion. The smooth body K is the circle centered at $(5, 0)$ with radius 2 far side $R = 5\sin(\varphi) + \sqrt{4 - 25\cos^2(\varphi)}$ and near side $r = 5\sin(\varphi) - \sqrt{4 - 25\cos^2(\varphi)}$. Suppose that $s(\varphi) = \varphi^6 - 7\varphi + 8$, then Figure 3 is a graph of X, K, and $R + s$ and $r + s$. Observe that the new body is not convex. However, multiply s by .03 and we obtain a new near side of $r + .03s$ and a new far side of $R + .03s$. It is clear from Figure 4 that this creates a body that is convex and smooth. From the above conclusion and example we are able to see theory in action. Here we were given a smooth body with no points of zero curvature, and were able to add a proper multiple of a positive C^2 function to this convex body to create a new convex body with the same directed X-ray. In the following section we will consider a smooth body which may or may not have points of zero curvature. From the work accomplished to this point it is clear that the expression $\dfrac{\mathcal{K}(y)}{y^2}$ plays a large role in this set of conditions. Thus, we make a transition from our previous discussion of the differential operator $\mathcal{K}$ to the expression $\dfrac{\mathcal{K}(y)}{y^2}$.





4 A Differential Equation and its Solution

Theorem 27 *Given a smooth body K with near side r and far side R, such that on $[\alpha, \beta]$ $\mathcal{K}(R) \geq \epsilon > 0$, and R'' is bounded above on (α, β), then the following is true:*

If s is a positive C^2 function which is concave away from the origin, then for sufficiently small t, K' with near side $r + ts$ and far side $R + ts$ is a smooth convex body with the same directed X-ray as K.

Proof. Apply Theorems 21 and 22. ■

It is not completely trivial that the set of positive C^2 functions which are concave away from the origin is infinite dimensional. Therefore, we will devote the next two section to determining this factor. We will explicitly exhibit an infinite dimensional family of such functions whose curvature we can control. This family is provided by solutions of the differential equation of $\frac{\mathcal{K}(y)}{y^2} = a, a \in \mathbb{R}$. First, it will be of benefit to rewrite $\frac{\mathcal{K}(y)}{y^2}$ as a function of $\frac{y'}{y}$.

Proposition 28 *For any function y: $\frac{\mathcal{K}(y)}{y^2} = 1 + \left(\frac{y'}{y}\right)^2 - \left(\frac{y'}{y}\right)'$.*

Proof. We know $\mathcal{K}(y) = y^2 + 2(y')^2 - y''y$. Thus $\frac{\mathcal{K}(y)}{y^2} = 1 + 2\left(\frac{y'}{y}\right)^2 + \frac{y''y}{y^2}$. Expanding the equation yields $1 + \left(\frac{y'}{y}\right)^2 + \frac{y'y' - y''y}{y^2}$, which simplifies to $1 + \left(\frac{y'}{y}\right)^2 - \left(\frac{y'}{y}\right)'$. ■

Now, letting $u = \frac{y'}{y}$ we obtain $1 + u^2 - u' = a$, or simply $(1-a) + u^2 = u'$. It is clear to see that since $u = \frac{y'}{y}$, $y = C_0 e^{\int u d\varphi}$. So now the objective is to obtain u and integrate in order to properly define y. Since y is to be thought of as a function of φ, $\frac{du}{d\varphi} = (1-a) + u^2$, or $d\varphi = \frac{du}{(1-a)+u^2}$. It will be convenient to look at a in two separate cases: first when y is a near side, and second when y is a far side.

Case1 Suppose that $a \le 0$. This case contains the near side.

Then $1 - a > 0$. So let $b^2 = 1 - a$, for some $b \in \mathbb{R}^+ \bigcup \{0\}$.

Then $\int \frac{du}{b^2 + u^2}. = \int d\varphi$

$\Longrightarrow \frac{1}{b} \arctan(\frac{u}{b}) = \varphi + C_1$

$\Longrightarrow \arctan(\frac{u}{b}) = b\varphi + C_1$

$\Longrightarrow \frac{u}{b} = \tan(b\varphi + C_1)$

$\Longrightarrow u = b\tan(b\varphi + C_1)$.

Now that we have determine the form for u we can solve for y

$\int u d\varphi = \int b\tan(b\varphi + C_1) = -\ln|\cos(b\varphi + C_1)| + C_4$. So $y = C_0' e^{-\ln|\cos(b\varphi+C_1)|} = \frac{C_0'}{\cos(b\varphi + C_1)}$.

If we let $C_1 = \frac{\pi}{2} + C$ our solution has the form $y = \frac{C_0'}{\sin(b\varphi + C)}$

Case 2 Suppose that $a > 0$. This case contains the far side.

Sub Case 1: If $a = 1$, then $1 - a = 0$

$\Longrightarrow \int \frac{du}{u^2} = \int d\varphi$

$\Longrightarrow -\frac{1}{u} = \varphi + C_2$

$\Longrightarrow u = \frac{-1}{\varphi + C_2}$.

$\Longrightarrow \int u d\varphi = \int \frac{-d\varphi}{\varphi + C_2} = -\ln(\varphi + C_2) + C_3$. So $y = C_0' e^{-\ln(\varphi + C_2)} = \frac{C_0''}{\varphi + C_2}$.

Sub Case 2: If $a > 1$, then $1 - a < 0$. So let $-b^2 = 1 - a$, for some $b \in \mathbb{R}^+$.

$\Longrightarrow \int \frac{du}{-b^2 + u^2} = \int d\varphi$

$\Longrightarrow \int \frac{1}{2b}\left[\frac{1}{u-b} - \frac{1}{u+b}\right] du = \int d\varphi$

$\Longrightarrow \frac{1}{2b} \ln\left|\frac{u-b}{u+b}\right| = \varphi + C_3$

$\Longrightarrow \frac{1}{2b} \ln\left|\frac{u-b}{u+b}\right| = \varphi + C_3$

$\Longrightarrow$by solving for u in the above equation one obtains $u = \frac{b(\pm Ce^{2b\varphi} + 1)}{1 \mp Ce^{2b\varphi}}$.

We will integrate $u = \frac{b(Ce^{2b\varphi} + 1)}{1 - Ce^{2b\varphi}}$ and leave the other case for the reader.

$\int u d\varphi = \int \frac{b(Ce^{2b\varphi}+1)}{1-Ce^{2b\varphi}} d\varphi$

$\Longrightarrow \int u d\varphi = bC \int \frac{e^{2b\varphi} d\varphi}{1-Ce^{2b\varphi}} + b \int \frac{d\varphi}{1-Ce^{2b\varphi}}.$

Observe that for the *second expression* $b \int \frac{d\varphi}{1-Ce^{2b\varphi}} = b \int \frac{d\varphi}{1-Ce^{2b\varphi}} * \frac{2bCe^{2b\varphi}}{2bCe^{2b\varphi}}.$

If we let $m = Ce^{2b\varphi}$ then $\frac{dm}{d\varphi} = 2bCe^{2b\varphi}$, or $dm = 2bCe^{2b\varphi} d\varphi$. This transforms the above equation into:

$b \int \frac{dm}{(1-m)(2bm)} = \frac{1}{2} \int \frac{dm}{(1-m)(m)} = \frac{1}{2} \int \frac{dm}{m-m^2} = \frac{1}{2} \ln m - \frac{1}{2} \ln |(m-1)| + C_4 = \frac{1}{2} \ln \left| (\frac{m}{m-1}) \right| + C_4 = \frac{1}{2} \ln \left| (\frac{Ce^{2b\varphi}}{Ce^{2b\varphi}-1}) \right| + C_4.$

In order to integrate the *first expression* let $m = 1 - Ce^{2b\varphi}$. Then $dm = -2bCe^{2b\varphi} d\varphi$, and substituting into $bC \int \frac{e^{2b\varphi} d\varphi}{1-Ce^{2b\varphi}}$, yields:

$\frac{-1}{2} \int \frac{dm}{m} - \frac{1}{2} \ln m = \frac{-1}{2} \ln \left| (1-Ce^{2b\varphi}) \right| + C_4$. Thus, we are able to see that:

$\int \frac{b(Ce^{2b\varphi}+1)}{1-Ce^{2b\varphi}} d\varphi = \frac{-1}{2} \ln \left| 1-Ce^{2b\varphi} \right| + \frac{1}{2} \ln \left| \frac{Ce^{2b\varphi}}{Ce^{2b\varphi}-1} \right| + C_4$

$\Longrightarrow \int u d\varphi = \frac{1}{2} [\ln \left| \frac{Ce^{2b\varphi}}{Ce^{2b\varphi}-1} \right| - \ln \left| 1-Ce^{2b\varphi} \right| + C_4$

$\Longrightarrow \int u d\varphi = \frac{1}{2} \ln \left| \frac{\frac{Ce^{2b\varphi}}{Ce^{2b\varphi}-1}}{1-Ce^{2b\varphi}} \right| + C_4$

$\Longrightarrow \int u d\varphi = \frac{1}{2} \ln \left| \frac{Ce^{2b\varphi}}{(Ce^{2b\varphi}-1)(1-Ce^{2b\varphi})} \right| + C_4.$

So, $y = C_0 e^{\frac{1}{2} \ln \left| \frac{Ce^{2b\varphi}}{(Ce^{2b\varphi}-1)(1-Ce^{2b\varphi})} \right| + C_4}$

$\Longrightarrow y = C_0''' \left| \frac{Ce^{2b\varphi}}{(Ce^{2b\varphi}-1)(1-Ce^{2b\varphi})} \right|^{1/2}$

$\Longrightarrow y = C_0''' \sqrt[2]{\frac{\left| Ce^{b\varphi} \right|^2}{|(Ce^{2b\varphi}-1)|^2}}$

$\Longrightarrow y = C_0''' \left| \frac{Ce^{b\varphi}}{(Ce^{2b\varphi}-1)} \right|.$

Similarly, it can be show that if $u = \frac{b(-Ce^{2b\varphi}+1)}{1+Ce^{2b\varphi}}$, then $y = C_0''' \left| \frac{Ce^{b\varphi}}{(Ce^{2b\varphi}+1)} \right|.$

With a little bit of algebra, which will be left to the read, it can be show that $\frac{Ce^{b\varphi}}{(Ce^{2b\varphi}-1)}$ can be written in the form $C_1 \operatorname{sech}(b + C_2)$.

Conclusion 29 *The equation* $\frac{\mathcal{K}(y)}{y^2} = a$, $a \in \mathbb{R}$ *has solutions:*

1. $y = C_1 \csc(b\varphi + C)$, *when* $b^2 = 1 - a > 0$

2. $y = \dfrac{C_1}{\varphi + C}$, *when* $a = 1$

3. $y = C_1 \operatorname{csch}(b\varphi + C)$, *when* $-b^2 = 1 - a < 0$

Lemma 30 *The function* $s = \dfrac{1}{\sin(b\varphi + C)}$, *is positive and bounded on compact subsets of the interval* $0 < b\varphi + C < \pi \Longrightarrow \dfrac{-C}{b} < \varphi < \dfrac{\pi - C}{b}$. *Moreoever, all derivaties of* s *are bounded on compact subsets of the same interval.*

Proof. Notice that by the quotient rule any derivative of s will be bounded on compact subest of the same interval because the denominator of the derivative will simply be a power of the denominator of s. Thus, a derivative will go to infinity if and only if $\sin(b\varphi + C) \longrightarrow 0$. ■

For simplicity, let $C_0' = 1$ and $C = 0$. Thus, unless otherwise stated we will consider $s = \dfrac{1}{\sin(b\varphi)}$. With this form of s we show in the following lemma that one can generate an infinite dimensional family of linearly independent functions by controlling the parameter b.

Lemma 31 *For each* $n \in N$, *the set of functions* $s_i = \csc(b_i\varphi)$, *where* $1 < b_1 < b_2 < ... < b_n$, *are linearly independent on* $0 < \varphi < \dfrac{\pi}{b_n}$. *Consequently, for any positive increasing sequence* $\{b_i\}$ *with* $b_i \longrightarrow b$, *the span of the set of functions* $\{s_i = \csc(b_i\varphi), i \in N\}$ *is infinite dimensional on* $0 < \varphi < \dfrac{\pi}{b}$.

Proof. Let $D_1 \csc(b_1\varphi) = 0$. Since $1 < b_1$, and $0 < \dfrac{\pi}{b_1} < \pi$ we know that $\csc(b_1\varphi) \neq 0$. So $D_1 = 0$.

Now suppose the lemma is true for the set of functions $s_i = \csc(b_i\varphi)$, $1 \leq i \leq n-1$. So if $A_1 \csc(b_1\varphi) + A_2 \csc(b_2\varphi) + ... + A_{n-1} \csc(b_{n-1}\varphi) = 0$, then $A_1 = A_2 = ... = A_{n-1} = 0$.

Let $C_1 \csc(b_1\varphi) + C_2 \csc(b_2\varphi) + ... + C_{n-1} \csc(b_{n-1}\varphi) + C_n \csc(b_n\varphi) = 0$. If $C_n = 0$, we are left with $C_1 \csc(b_1\varphi) + C_2 \csc(b_2\varphi) + ... + C_{n-1} \csc(b_{n-1}\varphi) = 0$, and the proof is complete by the induction hypothesis.

So suppose that $C_n \neq 0$. Then $\csc(b_n\varphi)$ can be written as a linear combination of $\csc(b_1\varphi)$ through $\csc(b_{n-1}\varphi)$.

So $\csc(b_n\varphi) = C_1' \csc(b_1\varphi) + C_2' \csc(b_2\varphi) + ... + C_{n-1}' \csc(b_{n-1}\varphi)$.

Note that $0 < \dfrac{b_i\pi}{b_n} < \pi$, for all $i \in [1, n-1]$. Thus, by taking the limit as $\varphi \longrightarrow \dfrac{\pi}{b_n}$, we obtain from the previous equation that $\infty = M$ where M is a constant, a contradiction. Thus, $C_n = 0$. ■

As alluded to earlier we would like to keep a small and negative, which will allow us to obtain an infinite number of properly defined near side functions (s). To be true to our definition of s as a continuous C^2 bounded function on $[\alpha, \beta]$, we need $0 < b\varphi < \pi$ for all values of φ on $[\alpha, \beta]$. Thus, $b < \frac{\pi}{\beta}$. Since we want a to be small and negative and b^2 was defined to equal $1 - a$, we have have a lower bound on b, namely $b > 1$. So a more precise restraint on b will be the following: $1 < b < \frac{\pi}{\beta}$. We will assume this restraint on b from this point forward.

5 The Dimension of $\mathcal{C}(K)$

In this section we will show that for most convex bodies K, $\mathcal{C}(K)$ is infinite dimensional. In fact our proof will show that for each natural number n, K is a boundary point of an n-dimensional subset of $\mathcal{C}(K)$. These notions require a topology on $\mathcal{C}(K)$.

Suppose $K_1, K_2 \in \mathcal{C}(K)$. Since these convex bodies have the same directed X-ray, say X, they are uniquely determined by their near side functions r_1, r_2. We define a metric d on $\mathcal{C}(K)$ by:

$$d(K_1, K_2) = \max_{a \leq \varphi \leq b} |r_1(\varphi) - r_2(\varphi)| .$$

It can be shown that $\mathcal{C}(K)$ is complete in this metric [2].

Definition 32 *A subset U of a metric space $(\mathcal{M}, d)$ has **dimension** n, if U is homeomorphic to an open subset of $\mathbb{R}^n$. $\mathcal{M}$ is **infinite dimensional** if $\mathcal{M}$ contains an n-dimensional subset for all natural numbers n.*

There are only two types of examples that we are aware of where $\mathcal{C}(K)$ is finite dimensional. First, if K is a parallel wedge, then $\mathcal{C}(K)$ is one dimensional, in fact in this case $\mathcal{C}(K)$ is homeomorphic to $[0, \infty)$. (A parallel wedge is a quadrilateral with all its vertices lying on the supporting rays and its sides r and R parallel [5].) Only one function and its multiples are needed to make all near and far sides in $\mathcal{C}(K)$, so it is homeomorphic to an open subset of $\mathbb{R}$.The second case involves convex bodies K with near side $r = 0$ and thus far sides $R = X$. Black and Koop [1] showed if the set $Z = \{(\alpha, \beta) : \mathcal{K}(X) = 0\}$ is not an interval, then $\mathcal{C}(K) = \{K\}$ and hence $\mathcal{C}(K)$ has dimension 0.

We will show that $\mathcal{C}(K)$ is usually infinite dimensional. In other words or all $n \in N$, we claim we have a continuous one-to-one map from $\mathcal{C}(K)$ to an open subset of $\mathbb{R}^n$, whose inverse is also continuous. If we can find such homeomorphisms, by definition $\mathcal{C}(K)$ is infinite dimensional. Furthermore, $\mathcal{C}(K)$ is infinite dimensional near most points.

Let K be a smooth convex body, with near side r and far side R. The procedure we will follow is for each $n \in N$ perturb the near side r to $r + \sum_{i=1}^{n} t_i s_i$, where the s_i are positive linearly independent functions concave away from the origin. (Solutions to $\mathcal{K}(y) = ay^2, a \leq 0$, provide us with such a family.) First we show that $r + \sum_{i=1}^{n} t_i s_i$ is concave away from the origin for all $t_i \geq 0$. Then we show that for t_i sufficiently small $R + \sum_{i=1}^{n} t_i s_i$ is concave toward the origin.

Theorem 33 *Suppose K is a convex body and s_i, $1 \leq i \leq n$, are positive C^2 functions on $[\alpha, \beta]$ which are concave away from the origin. Then $\mathcal{K}(r + \sum_{i=1}^{n} t_i s_i) \leq 0$, where $1 \leq n < \infty$, for all $t_i \in \mathbb{R}^+$.*

Proof. Notice we have shown in Theorem 23 that the conclusion is true for $i = 1$; $\mathcal{K}(r + t_1 s_1) \leq 0$. Now we will assume it is true for $i = n - 1$, and show it is true for $i = n$.

Let $d = r + \sum_{i=1}^{n-1} t_i s_i$.Then by assumption, $\mathcal{K}(d) \leq 0$, so k is a proper near side. Then using Theorem 23 again, we can say $\mathcal{K}(d + t_n s_n) \leq 0$.Therefore $\mathcal{K}(r + \sum_{i=1}^{n} t_i s_i) \leq 0$, where $1 \leq n < \infty$, for all $t_i \in \mathbb{R}^+$. ■

Next we will check that we have a proper infinite dimensional far side.

Theorem 34 *Suppose K is a convex body with $\mathcal{K}(R) \geq \epsilon > 0$ and R'' bounded above. Then for each $n \in N$ there exists a set of functions $s_1, s_2, ..., s_n$ that are linearly independent and convave away from the origin such that $\mathcal{K}(R + \sum_{i=1}^{n} t_i s_i) > 0$, for all t_i sufficiently small.*

Proof. Choose the s_i to be solutions of the differential equation so that we have the set $\{s_i = \csc(b_i \varphi), i \in N\}$. These are concave away from the origin by choice, since $\frac{\mathcal{K}(s_i)}{s_i^2} = -\gamma_i$ where $\gamma_i \geq 0$. We know they are linearly independent from Lemma 31. Notice we have shown in Theorem 21 that the conclusion is true for $i = 1$; $\mathcal{K}(R + t_1 s_1) \geq \frac{\epsilon}{2} = \epsilon_1 > 0$. Now we will assume it is true for $i = n - 1$, and show it is true for $i = n$.

Let $D = R + \sum_{i=1}^{n-1} t_i s_i$. Then by assumption, $\mathcal{K}(D) \geq \epsilon_{n-1} > 0$, so D is a proper far side. Clearly D'' is bounded because all s_i'' are bounded and R'' is bounded above by assumption. Then using Theorem 21 again, we can say $\mathcal{K}(D + t_n s_n) \geq \frac{\epsilon_{n-1}}{2} = \epsilon_n > 0$, for all t_n sufficiently small. Then $\mathcal{K}(R + \sum_{i=1}^{n} t_i s_i) \geq \frac{\epsilon_{i-1}}{2} = \epsilon_i > 0$, where $1 \leq n < \infty$, for all t_i sufficiently small. ■

$\mathcal{C}(K)$ contains at least one subset that is homeomorphic to an open subset of $\mathbb{R}^n$ for all natural numbers n. Therefore $\mathcal{C}(K)$ is locally infinite dimensional.

6 Conditions for Boundedness and Unboundedness of $\mathcal{C}(K)$

We now turn to the question of when $\mathcal{C}(K)$ is unbounded. More precisely, given a K we would like to know whether for each $M > 0$ there exists a convex body K_M which is distance $\geq M$ from the origin and has the same directed X-ray as K. The distance of a convex body K from the origin is, of course, $\inf_{x\in K} \|x\|$, where $\|x\|$ is the standard Euclidean distance in the plane. We will give both sufficient conditions and necessary conditions for $\mathcal{C}(K)$ to be unbounded. These conditions lead us to nontrivial examples of both boundedness and unboundedness of $\mathcal{C}(K)$.

In light of Theorem 10 and Corollary 11, the simplest cases where $\mathcal{C}(K)$ might be unbounded occur when $\mathcal{K}(R) = 0$, K has a flat far side, or $\mathcal{K}(r) = 0$, K has a flat near side. Properties of convex bodies with flat near and far sides were also studied by Black and Koop [1]. So now we investigate when one can have $\mathcal{K}(R) = 0$ and $\mathcal{K}(tR-X) \leq 0$ for all sufficiently large t and when one can have $\mathcal{K}(r) = 0$ and $\mathcal{K}(X + tr) \geq 0$ for all sufficiently large t. We assume throughout that $\mathcal{K}(X) \geq 0$.

Theorem 35 *If $\mathcal{K}(R) = 0$, and $\mathcal{K}(t_1R - X) \leq 0$ for any $t_1 \in \mathbb{R}^+$ and the same φ, then $\mathcal{K}(tR - X) \leq 0$ for all $t \geq t_1, t \in \mathbb{R}^+$.*

Proof. By Corollary 11, we know

$$\mathcal{K}(tR - X) = \frac{tR - X}{R} t\mathcal{K}(R) + \mathcal{K}(X) - tRX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right).$$

Since $\mathcal{K}(R) = 0$, we have

$$\mathcal{K}(tR - X) = \mathcal{K}(X) - tRX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right).$$

Likewise, $\mathcal{K}(t_1R - X) = \mathcal{K}(X) - t_1RX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right)$.

So $\mathcal{K}(X) - t_1RX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right) \leq 0$

$\implies \mathcal{K}(X) \leq t_1RX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right)$.

Since $t \geq t_1$, $tRX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right) \geq t_1RX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right)$.

Thus $tRX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right) \geq \mathcal{K}(X)$.

Then we see that $\mathcal{K}(X) - tRX \left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \right) \leq 0$, so $\mathcal{K}(tR - X) \leq 0$ for all $t \geq t_1, t \in \mathbb{R}^+$. ∎

The previous result show that if there exists a convex body in $\mathcal{C}(K)$ with a flat far side, then $\mathcal{C}(K)$ is unbounded. Also, the proof shows that a necessary condition that a flat far side exists is that

$$\frac{\mathcal{K}(X)}{X^2} > 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2 \text{ where ever } \mathcal{K}(X) > 0.$$

In the next result will will give another sufficient condition for $\mathcal{C}(K)$ to be unbounded with a flat near side. This one deals with a flat near side.

Theorem 36 *If* $\mathcal{K}(r) = 0$, *then* $\mathcal{K}(X + tr) \geq 0$ *for all* $t \in \mathbb{R}^+$ *if and only if* $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2 \geq 0$ *where ever* $\mathcal{K}(X) > 0$.

Proof. Since $\mathcal{K}(r) = 0$, Corollary 11 gives $\mathcal{K}(X+tr) = \mathcal{K}(X)+trX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2\right)$.

Likewise, $\mathcal{K}(X + t_1r) = \mathcal{K}(X) + t_1rX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2\right)$.

So we know $\mathcal{K}(X) + t_1rX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2\right) \geq 0$.

If $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2 < 0$, then when $t \longrightarrow \infty$, $trX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2\right) \longrightarrow -\infty$, so here the $\mathcal{K}(X)$ term will be overwhelmed.

If $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2 \geq 0$, then $\mathcal{K}(X+tr) = \mathcal{K}(X)+trX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2\right) \geq 0$ and then $\mathcal{K}(X + tr) \geq 0$ for all $t \in \mathbb{R}^+$. ■

We have necessary conditions for $\mathcal{C}(K)$ to be unbounded for a body K with one side flat. Now we would like to find necessary conditions for $\mathcal{C}(K)$ to be unbounded for a more general K.

Theorem 37 *Let* K *be a convex body with near side* r, *far side* R, *and directed X-ray* X. *Then*

$$\frac{\mathcal{K}(X)}{X^2} \geq \frac{2r}{R}\left[\frac{X'}{X} - \frac{r'}{r}\right]^2.$$

Proof. By Theorem 10 we know that $\mathcal{K}(X) = \mathcal{K}(R - r) = \frac{R-r}{R}\mathcal{K}(R) - \frac{R-r}{r}\mathcal{K}(r) + 2rR\left[\frac{R'}{R} - \frac{r'}{r}\right]^2$. Since $\mathcal{K}(R) \geq 0$ and $\mathcal{K}(r) \leq 0$, we can say $\mathcal{K}(X) \geq 2rR\left[\frac{R'}{R} - \frac{r'}{r}\right]^2$. Since $R = r + X$ and $R' = r' + X'$, substitution gives:

$$2rR\left[\frac{R'}{R} - \frac{r'}{r}\right]^2 = 2rR\left[\frac{r'+X'}{r+X} - \frac{r'}{r}\right]^2 = 2rR\left[\frac{rr' + rX' - rr' - Xr'}{r(r+X)}\right]^2$$

$$= \frac{2}{rR}(rX' - Xr')^2 = \frac{2}{rR}\left[rX\left(\frac{X'}{X} - \frac{r'}{r}\right)\right]^2 = \frac{2rX^2}{R}\left[\frac{X'}{X} - \frac{r'}{r}\right]^2.$$

Therefore $\frac{\mathcal{K}(X)}{X^2} \geq \frac{2r}{R}\left[\frac{X'}{X} - \frac{r'}{r}\right]^2$. ■

Corollary 38 *If $\mathcal{C}(K)$ is unbounded, then for each $\epsilon, 0 < \epsilon < 1$ there exists a continuous nonincreasing function $\psi = \psi(\varphi)$ such that:*

$$\frac{\mathcal{K}(X)}{X^2} \geq 2(1-\epsilon)\left[\frac{X'}{X} - \cot(\psi - \varphi)\right]^2.$$

Proof. From the previous theorem, we know that $\frac{\mathcal{K}(X)}{X^2} \geq \frac{2r}{R}\left[\frac{X'}{X} - \frac{r'}{r}\right]^2$. Recall from Lemma 14 the expression $\frac{r'}{r} = \cot(\psi - \varphi)$ where ψ is the angle of inclination of the tangent line to the graph of r at angle φ. Thus $\frac{r'}{r}$ depends only on the angle of inclination and not the distance from the origin. To see that ψ is a nonincreasing function of φ, note that from Proposition 28 $0 \geq \frac{\mathcal{K}(y)}{y^2} = 1 + \left(\frac{r'}{r}\right)^2 - \left(\frac{r'}{r}\right)'$

$= 1 + \cot^2(\psi - \varphi) + \csc^2(\psi - \varphi)\left(\frac{d\psi}{d\varphi} - 1\right)$

$= \csc^2(\psi - \varphi)\left(\frac{d\psi}{d\varphi}\right)$. Hence $\frac{d\psi}{d\varphi} \leq 0$.

If $\mathcal{C}(K)$ is unbounded, then we may choose r and R such that $\frac{r}{R}$ is as close as desired to 1, because as $r \longrightarrow \infty$, $\frac{r}{R} = \frac{r}{r+X} \longrightarrow 1$. Thus for each $\epsilon, 0 < \epsilon < 1$ there exists a continuous nonincreasing function $\psi = \psi(\varphi)$ such that $\frac{\mathcal{K}(X)}{X^2} \geq 2(1-\epsilon)\left[\frac{X'}{X} - \cot(\psi - \varphi)\right]^2$. ■

This is our necessary condition for $\mathcal{C}(K)$ to be unbounded.

7 An Illustration with Circles

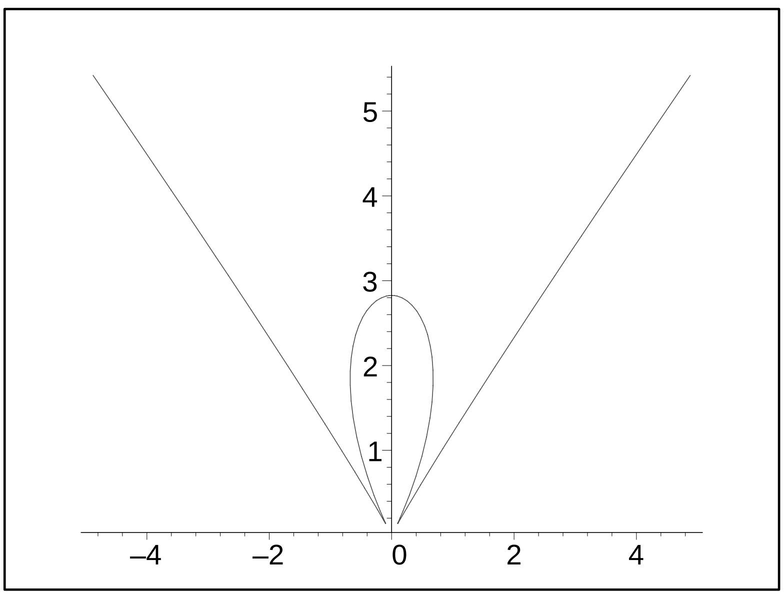
To see what our condition means, we now simplify the problem by seeing what happens when our convex body is a circle. We will take K centered at $(0, j)$ and have radius p, $0 < p < j$, so that K is in the open upper half plane. With this K we can do explicit computations and make graphs.

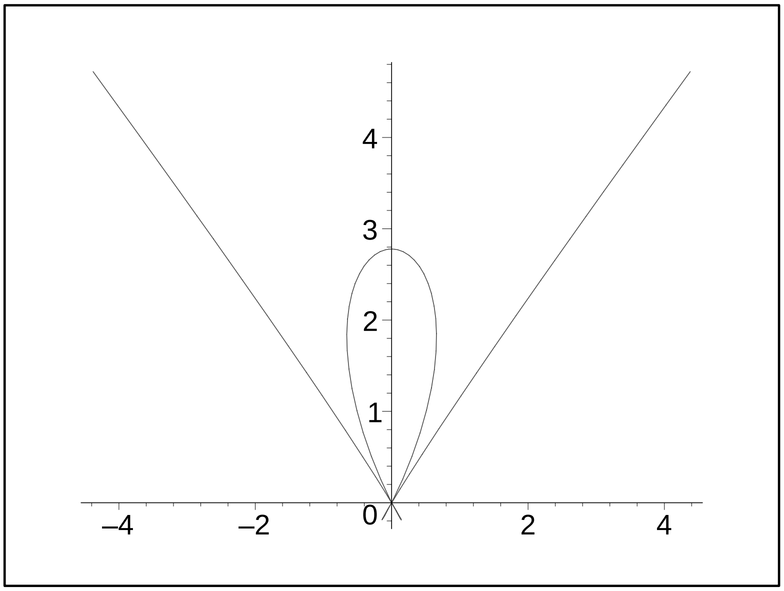
First, we need a few equations. The equation of a flat horizontal side is $t \csc\varphi$, where $t \in \mathbb{R}^+$. The equation of our circle in rectangular coordinates is $x^2 + (y-j)^2 = p^2$, so in polar coordinates this is $r^2 - 2jr\sin\varphi + j^2 - p^2 = 0$, where r represents the distance from the origin. We can simplify this to $r^2 - 2jr\sin\varphi + j^2 - p^2 = 0 \implies r = \frac{2j\sin\varphi \pm \sqrt{4j^2\sin^2\varphi - 4(j^2 - p^2)}}{2} \implies j\sin\varphi \pm \sqrt{p^2 - j^2\cos^2\varphi}$.

So the far side has the equation $R = j\sin\varphi + \sqrt{p^2 - j^2\cos^2\varphi}$ and the near side has the equation $r = j\sin\varphi - \sqrt{p^2 - j^2\cos^2\varphi}$. Then

$X = R - r = 2\sqrt{p^2 - j^2\cos^2\varphi} = 2j\sqrt{\left(\frac{p}{j}\right)^2 - \cos^2\varphi}$. We put X in terms of $\frac{p}{j}$ to see how this ratio affects K. Notice we now have a constraint on $\frac{p}{j}$, namely $\frac{p}{j} \geq \cos\alpha$, because we have oriented K such that $\cos\alpha = -\cos\beta$ and $\cos\beta \leq \cos\varphi \leq \cos\alpha$.

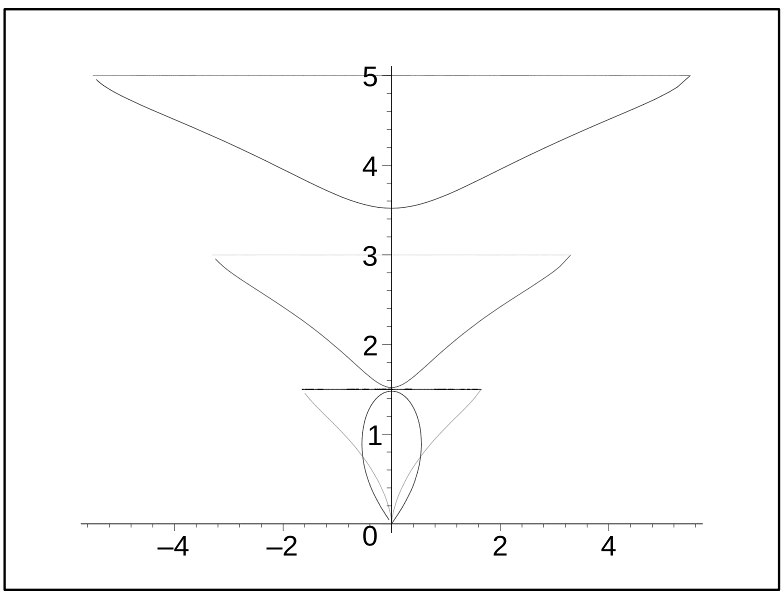
As we did in the above section, we will make one side of the circle flat and see what its other side looks like. The goal is to determine for what ratios of $\frac{p}{j}$ we can push the new body to infinity and maintain convexity. We know we want the function $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2 > 0$, following from Theorem 35 and Theorem 36. Notice we will have the same condition on $\frac{p}{j}$ whether we try to make the far side or near side flat. After graphing the function at various values of $\frac{p}{j}$, we found that for approximately $\frac{p}{j} \leq 0.74$ this set $\mathcal{C}(K)$ is unbounded. See Figure 5 for the graph of $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2$ at $\frac{p}{j} = 0.74$ and Figure 6 for the same function graphed at $\frac{p}{j} = 0.75$, where it becomes less than 0.





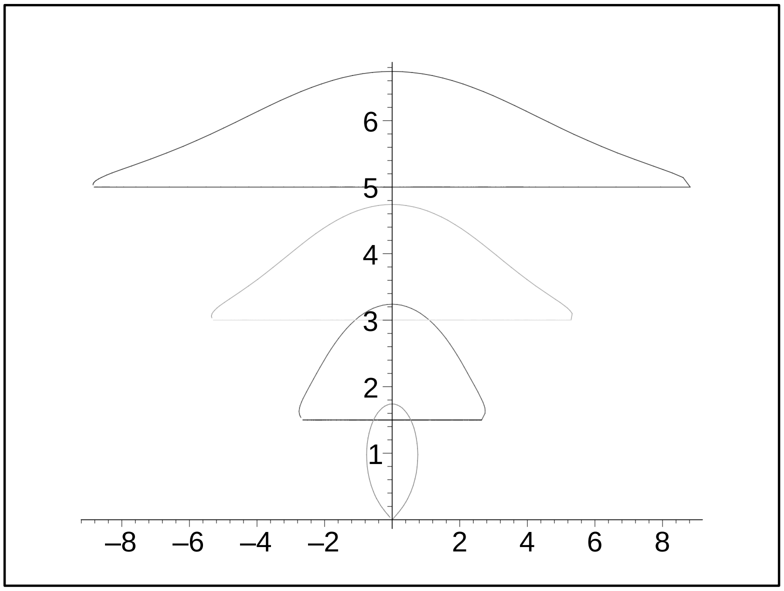
We can show the details of our theorems more precisely by graphing examples of convex bodies and their X-rays for different values of $\frac{p}{j}$. Theorem 35 can be illustrated by graphing the functions X, $R-X$, and R. For a flat far side, we want

$\mathcal{K}(X) - tRX\left(\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{R'}{R}\right]^2\right) \leq 0$ for t sufficiently large. In Figure 7 we have a body that can be pushed out to infinity with a flat side and maintain convexity because $\frac{p}{j} = 0.74$. However, with small values of t we see the body is not convex.



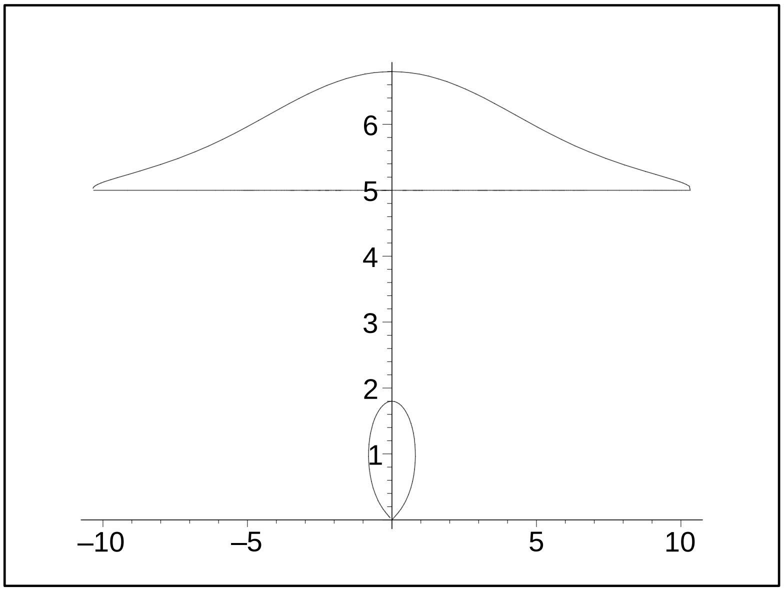
In Theorem 36 we saw that the condition for being able to push a flat near side to infinity is simpler than that of the flat far side. If $\frac{\mathcal{K}(X)}{X^2} - 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2 \geq 0$, it does not matter what t is; this is the only way to get a proper far side when pushed out to infinity. The $\mathcal{C}(K)$

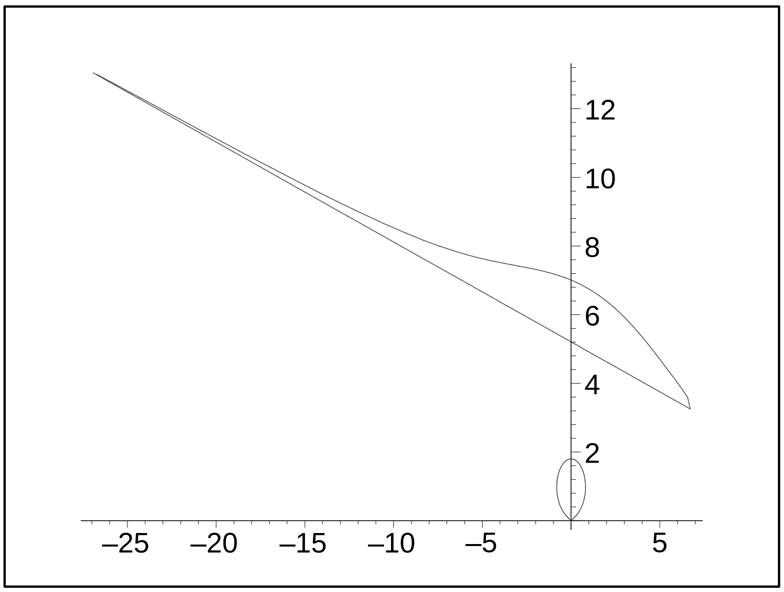
in Figure 8 cannot be pushed to infinity with a flat side because $\frac{p}{j} = 0.87$, even though for values of t that are small the body is clearly convex.



Now we would like to consider the circles who have ratio of $\frac{p}{j} \geq 0.75$. For simplicity, let us only think about adding functions to alter the near side and see how it affects the far side. Remember that we can not flatten one side of the circles with $\frac{p}{j} \geq 0.75$ and push it out to infinity by multiplying by t and letting $t \longrightarrow \infty$. But perhaps we can check whether $\mathcal{C}(K)$ is unbounded in another way, by making one of these curved using the solution to the differential equation in Section 5. Upon trying this, we observed that even with $\frac{p}{j} \leq 0.74$, the more a near side was concave away from the origin, the less we could push out the body.

Next we decided to add a slanted flat side such as $r = t\csc(\varphi + C)$ to a circle with $\frac{p}{j} \geq 0.75$. With a horizontal flat side, the far side of this body is not concave toward the origin near $\varphi = \alpha$ and $\varphi = \beta$. With a slanted flat side, we find that the far side of the body can either be proper near α and more incorrect than the horizontal flat side version near β or the opposite scenario. Thus we see that a circle K with a slanted flat near side will not be unbounded for $\frac{p}{j} \geq 0.75$ either. In fact, this method probably makes the body bounded at values even lower than 0.75. See Figure 9 and Figure 10 for an illustration. Figure 9 is the body with a horizontal flat side added and Figure 10 is the same body with a slanted flat side added instead. $\frac{p}{j} = 0.9$ in these graphs.





So it appears near α we have to add a line with angle of inclination of the tangent line, ψ, such that $\psi > \frac{\pi}{2}$, and near β we need $\psi < \frac{\pi}{2}$. Such a near side would clearly not be concave away from the origin. So it seems as though $\mathcal{C}(K)$ is bounded unless we can give it a flat horizontal side, meaning for each $\epsilon, 0 < \epsilon < 1$ there exists a continuous decreasing function $\psi = \psi(\varphi)$ such that $\frac{\mathcal{K}(X)}{X^2} \geq 2(1-\epsilon)\left[\frac{X'}{X} - \cot(\psi - \varphi)\right]^2$. This is a necessary condition for all K by Corollary 38. We would like to show that this is a necessary and sufficient condition for $\mathcal{C}(K)$ to be unbounded when K is a circle. We need to find out with more analytical evidence, instead of merely conjecturing with geometric evidence, what values of ψ produce acceptable far sides at particular values of φ.First we need confirm the following claim.

Claim 39 *If* $\frac{\mathcal{K}(X)}{X^2} \geq 2\left[\frac{X'}{X} - \cot(\psi - \varphi)\right]^2$, *we can write:*

Theorem 40 $\varphi - \operatorname{arccot}\left(-\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right) \le \psi \le \varphi - \operatorname{arccot}\left(-\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right)$.

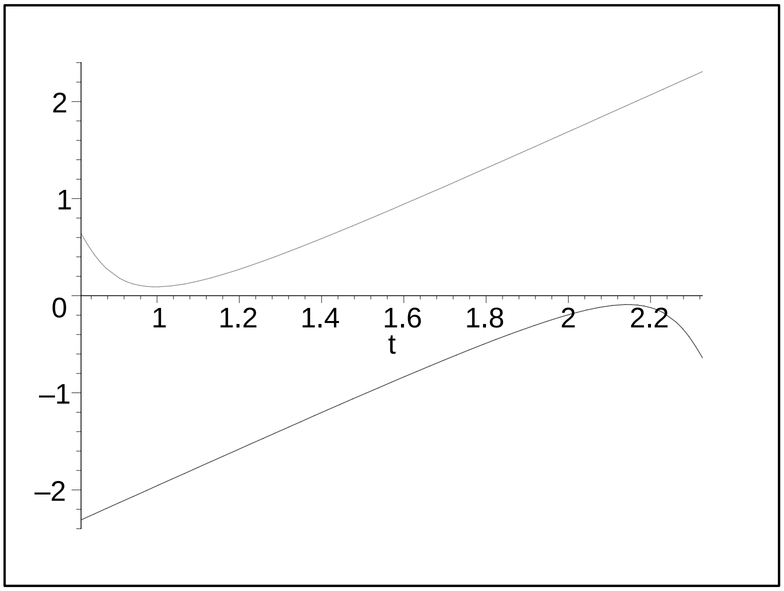
Proof. $-\sqrt{\frac{\mathcal{K}(X)}{2X^2}} \le \frac{X'}{X} - \cot(\psi-\varphi) \le \sqrt{\frac{\mathcal{K}(X)}{2X^2}} \Longrightarrow -\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}} \le -\cot(\psi-\varphi) \le -\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}$.

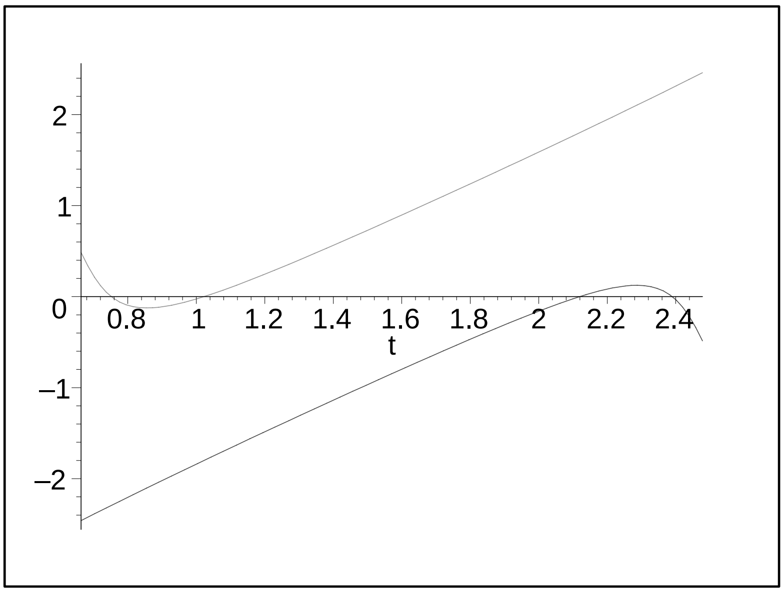
Since cotangent is an odd function, we have $-\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}} \le \cot(\varphi-\psi) \le -\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}$.

Notice that $\cot(\varphi-\psi) = -\cot(\psi-\varphi) = -\frac{r'}{r}$ is always a decreasing function from α to β. We showed in Lemma 19 the $\lim_{\varphi\to\alpha^-} \frac{r'}{r} = -\infty$ and the $\lim_{\varphi\to\beta^+} \frac{r'}{r} = +\infty$. The inverse cotangent function will always be decreasing on an interval that does not cross any of its singularities and (α,β) is such an interval. So applying arccot to the inequality reverses it. Therefore we have $\operatorname{arccot}\left(-\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right) \le \varphi - \psi \le \operatorname{arccot}\left(-\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right)$

$\Longrightarrow \varphi - \operatorname{arccot}\left(-\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right) \le \psi \le \varphi - \operatorname{arccot}\left(-\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right)$. ∎

It is easy to see from Lemma 14 that the condition $\frac{\mathcal{K}(X)}{X^2} \ge 2\left[\frac{X'}{X} - \frac{r'}{r}\right]^2$ is equal to the condition $\frac{\mathcal{K}(X)}{X^2} \ge 2\left[\frac{X'}{X} - \cot(\psi-\varphi)\right]^2$. Thus if $\mathcal{C}(K)$ is unbounded, we should be able to find values of ψ between $\varphi - \operatorname{arccot}\left(-\frac{X'}{X} - \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right)$ and $\varphi - \operatorname{arccot}\left(-\frac{X'}{X} + \sqrt{\frac{\mathcal{K}(X)}{2X^2}}\right)$. This means we should be able fit a nonincreasing function between the graphs of the two functions, since $\psi = \psi(\varphi)$ is a decreasing function. Standard rectangular coordinate graphs show that if $\frac{p}{j} \le 0.74$ we can fit the nonincreasing function $y = 0$ in between at the very least. If $\frac{p}{j} \ge 0.75$, we cannot fit in any decreasing function. Figures 11 and 12 show what these graphs look like for $\frac{p}{j} = 0.7$ and $\frac{p}{j} = 0.8$ respectively. Therefore $\mathcal{C}(K)$ is bounded if $\frac{p}{j} \ge 0.75$. We conclude that there exists a real number $u, 0.74 < u < 0.75$ such that for these circles, $\mathcal{C}(K)$ is unbounded when $\frac{p}{j} < u$ and bounded when $\frac{p}{j} > u$.





Further research could hopefully determine necessary and sufficient conditions on the X-ray data X that will imply boundedness or unboundedness for the general sets $\mathcal{C}(K)$. For the sets that turn out to be bounded, it would be interesting to explore if it is possible to estimate what the bounds are simply from looking X.

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