Verifying a Triangle From Two Directed X-rays

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Abstract

We exhibit some results relating the directed x-rays of a convex polygon with the location of vertices of the polygon. We also establish a useful lemma comparing x-rays of related triangles. These results are applied to verify triangles from certain special pairs of points, and we attempt to extend their application enough to handle determination.

1 Introduction

Hammer [5] posed his general x-ray problem in 1961, and so spawned the field of geometric tomography, on which Gardner's book [3] is the chief comprehensive work. His question was to discover how many x-rays would be required, in general, to uniquely determine a convex body in the plane. The statement of his problem was general enough that both parallel and directed x-rays were considered. We will be focusing on the directed x-ray problem, about which some powerful results have been discovered. It can easily be shown that one directed x-ray is insufficient for determination. In 1983, Falconer [2] and Gardner [4] independently discovered that two x-rays were sufficient, but only if the line through the point sources intersected the convex body in question. In 1986, Volčič [9] showed that three directed x-rays are sufficient to uniquely determine a convex body, so long as the three point sources are not collinear. It is thus of interest to discover whether two x-rays are sufficient in general. Indeed, it is not known if there is any convex body that can be uniquely determined by two directed x-rays without the intersection condition. Convex polygons are among the simplest of planar convex bodies, and triangles are the simplest of these, so it is reasonable to begin attack of this problem by considering whether or not a triangle can be uniquely determined by directed x-rays from any two point sources. This problem, in turn, can be broken down into cases, depending on where the point sources are located relative to the triangle.

2 Notation and Definitions

We will always be working in the plane $\mathbb{R}^2$.

Definition 2.1 *A subset S of the plane is called* ***star-shaped with respect to*** *p if for every ray l based at p, $S \cap l$ is either a line segment, a point, or empty.*

Definition 2.2 *A* ***convex body*** *is a compact, convex subset of the plane with nonempty interior.*

Definition 2.3 *The* ***directed x-ray*** *of a set K with respect to a* ***point source*** *(or* ***x-ray source****) p is the function $X_pK : [0, 2\pi] \to \mathbb{R}$ given by*

$$X_pK(\theta) = \int_0^\infty \chi_K(t(\cos\theta, \sin\theta) + p)dt$$

for $0 \leq \theta \leq 2\pi$, where χ_K is the characteristic function of K.

For sets K star-shaped with respect to p, the directed x-ray measures the length of the line segment defined by the intersection of K and the ray based at p with angle of inclination θ. Convex bodies are star-shaped with respect to any point, so this interpretation holds for convex bodies.

As Black & Koop [1] noted, given a convex body K and some fixed p not in K, K can be described uniquely as a pair of functions $r_pK(\varphi)$, $R_pK(\varphi)$ such that $K = \{(t, \varphi) : r_pK(\varphi) \leq t \leq R_pK(\varphi)\}$ where (t, φ) are polar coordinates with respect to p and a horizontal axis through p. In this context, we call the points $(r_pK(\varphi), \varphi)$ the **near side** of K and the points $(R_pK(\varphi), \varphi)$ the **far side** of K. Then by our assertions above, we actually have $X_pK = R_pK - r_pK$ for appropriate φ. Note that through this relationship, knowing any two of X_pK, R_pK, and r_pK is equivalent to knowing K.

Definition 2.4 *The* ***convex hull*** *of a set S in the plane, $\mathrm{ch}(S)$, is the intersection of all convex sets containing S.*

Definition 2.5 *A* ***convex polygon*** *is the convex hull of at least 3 points in the plane, such that not all of the points are collinear. In particular, a* ***triangle*** *is the convex hull of three noncollinear points in the plane.*

Remark 2.6 *The convex hull of any S is itself convex; in particular, a convex polygon is a convex body. In general, if $S = \{p_1, \ldots, p_k\}$ are the vertices on the boundary of a convex polygon P, then $\mathrm{ch}(S) = P$.*

We make the above definition in order to avoid confusion with the intuitive notion of a convex polygon, which includes only its boundary.

Definition 2.7 *A convex body K is* ***determined (from all other convex bodies) by the directed x-rays at the points*** *P if whenever M is a convex body and $X_pK = X_pM$ for all $p \in P$, $K = M$.*

Definition 2.8 *A convex body K can be* ***verified (from all other convex bodies) by the directed x-rays at the points*** *P if P can be chosen such that if M is a convex body and $X_pK = X_pM$ for all $p \in P$, $K = M$.*

These definitions are quite similar, the chief difference being that with verification, the set of points P depends on K and is often chosen based on a priori knowledge about K. Note that Volčič's result [9] above is a result on determination, with P being any set of three noncollinear points in $\mathbb{R}^2$, where Falconer's [2] and Gardener's [4] result is on verification, with P a pair of points chosen such that the line through them intersects K. As we shall see later, verification results can be melded together to form determination results.

Let ∂K denote the boundary of a convex body K. Without delving into topological definitions, we understand that ∂K is the simple closed curve separating the interior of K from the rest of the plane. A point $p \in \partial K$ will be called **smooth** if there is a unique tangent line to ∂K at p; otherwise, it will be called **nonsmooth**. The simplest example of a nonsmooth point is a vertex on the boundary of a convex polygon. By convexity, we have for any convex body K that all but countably many points on ∂K are smooth points. That is, only countably many points on ∂K can be nonsmooth.

We are interested in verification of convex bodies from two point x-rays such that the line containing both x-ray sources misses the convex body itself. So, via rigid motions of the plane (which do not alter relative values of x-rays), we assume - unless otherwise stated - that convex bodies are in the open upper half plane, hereafter denoted by H, and point sources are on the x axis, denoted by ∂H. Consequently, near side and far side functions will only be defined somewhere in $(0, \pi)$, so we avoid trouble with negative angles. Since all of our x-ray sources are on the x axis, we identify the x axis with $\mathbb{R}$ (including order properties). Thus instead of writing $(p, 0)$, we write p, and we write $p < q$ when the x coordinate of p is less than the x coordinate of q. Accordingly, let $X_pK = X_{(p,0)}K$ be the directed x-ray of K from $p \in \mathbb{R}$.

Let $p \in \mathbb{R}$. For $\varphi \in (0, \pi)$, let $l_p(\varphi)$ be the ray from p with angle of inclination φ. For $I \subseteq (0, \pi)$, let $C_pI = (\bigcup_{\varphi \in I} l_p(\varphi)) \backslash \{p\}$; i.e., let C_pI be the cone in H "spanned" by p and the angles of inclination I. For a convex body K, let the **support** of K with respect to source p, denoted by $\operatorname{spt}_p K$, be the closure of $\{\varphi \in (0, \pi) : X_pK(\varphi) > 0\}$. Then $C_p(\operatorname{spt}_p K)$ is the **supporting cone** of K with respect to p. When the context is clear, we will write this as C_pK. If $\operatorname{spt}_p K = [\alpha, \beta]$, then the rays $l_p(\alpha)$ and $l_p(\beta)$ will be called, respectively, the **lower supporting ray** and **upper supporting ray** of K.

Finally, denote the length of a line segment s by $\operatorname{len}(s)$, denote the interior of a set S in $\mathbb{R}$ or $\mathbb{R}^2$ as $\operatorname{int}(S)$, and let $d(p, q)$ denote the Euclidean distance between two points p and q in the plane.

3 Developments

Now we exhibit some results that allow us to reduce determination of polygons down to determination of vertices and locate said vertices. Following that, we develop an interesting lemma that proves critical in handling verification.

3.1 Preliminary Results

Theorem 3.1 *Suppose $K \subset H$ is a convex polygon with vertices $S = \{p_1, \ldots, p_k\}$, and p is an x-ray source. If $M \subset H$ is a convex body with $X_pK = X_pM$ and $S \subset \partial M$, then $K = M$.*

Proof. $S \subset \partial M \subset M$ so $K = \text{ch}(S) \subset M$ as M is convex. Hence $X_pK \leq X_pM$, with $X_pK < X_pM$ if and only if $K \neq M$. But $X_pK = X_pM$, and therefore $K = M$. ∎

This reduces determining a convex polygon down to finding all of its vertices via its x-ray data. Before continuing, we derive the polar form of the formula for a line. In Euclidean coordinates, this is $y - y_0 = m(x - x_0)$ where m is the slope of the line and (x_0, y_0) is a point on the line. Substituting $(x, y) = (r\cos\theta, r\sin\theta)$, we obtain $r\sin\theta - y_0 = m(r\cos\theta - x_0)$ or

$$r = r(\theta) = \frac{y_0 - mx_0}{\sin\theta - m\cos\theta}. \tag{1}$$

Of course only valid values of θ are considered, and (1) does not work for vertical lines. We can, however, extend (1) to handle vertical lines. Let $\psi \neq \pi/2$ be the angle of inclination of the line, so that $m = \tan\psi$. Then as $\cos\psi \neq 0$,

$$r(\theta) = \frac{y_0 - (\tan\psi)x_0}{\sin\theta - (\tan\psi)\cos\theta} = \frac{y_0\cos\psi - x_0\sin\psi}{\sin(\theta - \psi)} \tag{1'}$$

for appropriate θ. Now note that if we let $\psi = \pi/2$ in (1′), we get $r(\theta) = -x_0/\sin(\theta - \pi/2) = x_0/\cos\theta$, which is the polar equation for a vertical line through (x_0, y_0). Thus (1′) extends (continuously) to all values of ψ, and holds for all lines.

Theorem 3.2 *For any line segment defined as a positive polar function $l(\varphi)$ on an interval $[\alpha, \beta]$, $0 < \alpha < \beta < \pi$, the limits of $l'(\varphi)$ as $\varphi \to \alpha^+$ and as $\varphi \to \beta^-$ are finite.*

Proof. Let $p = (p_1, p_2)$ be a point on l, and $\psi \in [0, \pi)$ be the angle of inclination of l. Let $k = p_2\cos\psi - p_1\sin\psi$. Then by (1′), $l(\varphi) = k\csc(\varphi - \psi)$ for all $\varphi \in [\alpha, \beta]$. $l(\varphi)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β) as it is defined everywhere on $[\alpha, \beta]$. Differentiating, we obtain

$$l'(\varphi) = -k\csc(\varphi - \psi)\cot(\varphi - \psi) = -\frac{k\cos(\varphi - \psi)}{\sin^2(\varphi - \psi)} \tag{2}$$

which implies immediately that

$$|l'(\varphi)| \leq \frac{k}{\sin^2(\varphi - \psi)} \tag{3}$$

for all $\varphi \in (\alpha, \beta)$. The numerator in (3) is constant so as $\varphi \to \alpha^+$, $|l'(\varphi)| \to \infty \Rightarrow \lim_{\varphi \to \alpha^+} \sin^2(\varphi - \psi) = 0 \Rightarrow \sin(\alpha - \psi) = 0 \Rightarrow \alpha = \psi$. Similarly, as $\varphi \to \beta^-$, $|l'(\varphi)| \to \infty \Rightarrow \beta = \psi$. But since $l(\varphi)$ is defined and positive on $[\alpha, \beta]$, l intersects both of the rays $\varphi = \alpha$ and $\varphi = \beta$ in H, so l is parallel to neither of those rays, which implies ψ is neither α nor β. Thus $|l'(\varphi)| \nrightarrow \infty$ for $\varphi \to \alpha^+$, $\varphi \to \beta^-$. Therefore $l'(\varphi)$ is actually continuous on $[\alpha, \beta]$, so $\lim_{\varphi \to \alpha^+} l'(\varphi)$ and $\lim_{\varphi \to \beta^-} l'(\varphi)$ exist and are finite. ∎

Corollary 3.3 *If $T \subset H$ is a convex polygon, p is a point source, and $[\alpha, \beta]$ supports X_pT, then $(X_pT)'_+(\alpha)$ and $(X_pT)'_-(\beta)$ exist and are finite.*

Proof. This follows immediately from the previous theorem by translation of the plane and noting that ∂T is a finite union of line segments. Near supporting rays, r_pT and R_pT are lines, and $(X_pT)' = (R_pT)' - (r_pT)'$, which is finite valued at α and β. ■

In particular, this holds for triangles. This is, of course, not true for smooth convex bodies (i.e., those with boundary consisting entirely of smooth points), by Kimble [7].

Theorem 3.4 ([8] Thm 2.4) *Let K be a convex body, p a point source. Suppose $l_p(\varphi_0)$ intersects the interior of K. Then X_pK is not differentiable at $\varphi = \varphi_0$ if and only if at least one of the intersection points of $l_p(\varphi_0)$ and ∂K is a nonsmooth point of ∂K.*

Lam and Solmon's preceding theorem allows us to identify possibilities for locations of nonsmooth points on boundaries of convex bodies as long as we avoid supporting rays. The next results allow us to consider nonsmooth points on supporting rays.

Theorem 3.5 *Let K be a convex body, p be a point source, and $[\alpha, \beta]$ be the support of $X = X_pK$. If there is a smooth point on the near side or far side of ∂K on the ray $l_p(\alpha)$ ($l_p(\beta)$), then $X'_+(\alpha)$ ($X'_-(\beta)$) is infinite.*

Proof. Only consider the ray $l_p(\alpha)$ for now. Suppose the near side point of ∂K on $l_p(\alpha)$ is smooth. Then $l_p(\alpha)$ is tangent to ∂K at this point. By convexity, ∂K has a tangent line except at countably many points. In particular, since the angle if inclination of $l_p(\alpha)$ is α, continuity says there exists an $\epsilon > 0$ such that all lines tangent to the near side polar points $(r_pK(\varphi), \varphi) \in \partial K$ between the angles $\varphi = \alpha$ and $\varphi = \alpha + \epsilon$ have positive slope (i.e., angle of inclination between 0 and α). Call this angle of inclination $\psi(\varphi)$. Letting $r = r_pK$, we then have $r'(\varphi) = r(\varphi)\cot(\psi(\varphi) - \varphi)$ for all but countably many $\varphi \in (\alpha, \alpha + \epsilon)$ by computations in [7] Theorem 5.1. By continuity, $\psi(\varphi) \to \alpha$ as $\varphi \to \alpha^+$. Since $\psi(\varphi) < \varphi$ for φ near α by definition, $\psi(\varphi) - \varphi \to 0^-$ as $\varphi \to \alpha^+$, which implies $\cot(\psi(\varphi) - \varphi) \to -\infty$ as $\varphi \to \alpha^+$. Since r is continuous and $r(\alpha) > 0$, we then get $r'_+(\alpha) = -\infty$. By convexity, we cannot have $\psi(\varphi) > \varphi$ near α, so the only infinite value $r'_+(\alpha)$ can take on is $-\infty$. By an analogous argument, if the far side point of ∂K on $l_p(\alpha)$ is smooth, $R'_+(\alpha) = +\infty$, where $R = R_pK$; otherwise, $R'_+(\alpha)$ is finite. If either of these is true, then $X'_+(\alpha) = R'_+(\alpha) - r'_+(\alpha)$ $\Rightarrow X'_+(\alpha) = \infty$. Similarly, if either of the near side or far side points of ∂K on $l_p(\beta)$ are smooth, then $X'_-(\beta) = -\infty$. ■

Corollary 3.6 *Let K be a convex body, p be a point source, and $[\alpha, \beta]$ be the support of $X = X_pK$. If $X'_+(\alpha)$ ($X'_-(\beta)$) is finite then:*

1. If $X(\alpha)$ ($X(\beta)$) is zero, then there is exactly one nonsmooth point of ∂K on $l_p(\alpha)$ ($l_p(\beta)$).

2. If $X(\alpha)$ ($X(\beta)$) is positive, then there are exactly two nonsmooth points of ∂K on $l_p(\alpha)$ ($l_p(\beta)$).

Proof. This follows as the contrapositive of the preceding theorem. If the derivative in question is finite, then neither the near side nor the far side points of ∂K on the corresponding supporting ray is smooth. If the corresponding x-ray is zero, then the near side and far side coincide on the supporting ray and there is exactly one nonsmooth point. Otherwise, the near side and far side points do not coincide, and thus those two points are both nonsmooth. By convexity, the line segment connecting those two points is in ∂K and on the supporting ray in question. As line segments are smooth, there are no additional nonsmooth points on said supporting ray; hence, there are exactly two nonsmooth points there. ■

3.2 The Triangle Lemma

For all $S \subset \overline{H}$, define $\mu(S) = \iint_S y^{-1} dA$, allowing the value $+\infty$. Through change of variables to polar coordinates, for convex $S \subset \overline{H}$,

$$\mu(S) = \int_{\mathrm{spt}_p S} \frac{X_p S(\varphi)}{\sin(\varphi)} d\varphi \qquad (*)$$

for all point sources $p \in \partial H$ ([3] Lemma 5.2.5).

Proposition 3.7 *μ is an outer measure on $\overline{H}$ and all Borel sets in H are μ-measurable.*

All of the sets we consider with μ will be Borel, so we need only use the fact that $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint Borel sets A, B.

Remark 3.8 *By definition, $\mu(s) = 0$ for any line segment s, so $\mu(\mathrm{int}(T)) = \mu(T \backslash \partial T) = \mu(T)$ for all convex polygons T. Thus $\mu(T \backslash B) = \mu(T)$ for any $B \subset \partial T$.*

Proposition 3.9 *If $T \subset \overline{H}$ is a triangle with exactly one vertex on ∂H (the x axis), then $\mu(T) < \infty$.*

Proof. See [9] Lemma 3.1 or [3] Lemma 5.2.4. ■

Lemma 3.10 *Fix $h > 0$. Fix two distinct points p_1, p_2 on the x axis and two distinct points q_1, q_2 on the line $y = h$ such that the x coordinates of p_1, q_1 are less than those of p_2, q_2, respectively. Let $v = \overline{p_1 q_2} \cap \overline{p_2 q_1}$, $T_1 = \mathrm{ch}(p_1, q_1, v)$, $T_2 = \mathrm{ch}(p_2, q_2, v)$. Then $\mu(T_1) = \mu(T_2)$. (see Figure 1.)*

Proof. For $0 \le t \le h$, define the width of convex body K by $w(K, t) = \mathrm{len}(L_t \cap K)$, where L_t is the line $y = t$. Let $U_1 = \mathrm{ch}(p_1, q_1, q_2)$, $U_2 = \mathrm{ch}(p_2, q_1, q_2)$. Note that $w(U_1, 0) = w(U_2, 0) = 0$ and $w(U_1, h) = w(U_2, h) = \mathrm{len}(\overline{q_1 q_2})$. Since the U_i are triangles, similarity gives $w(U_i, t)$ linear in t on $[0, h]$. Both of these functions share values at the points 0 and h, so $w(U_1, t) = w(U_2, t)$ for all $t \in [0, h]$. In particular, at the y coordinate y_0 of v, $w(T_1, y_0) = w(U_1, y_0) = w(U_2, y_0) = w(T_2, y_0)$ by construction of T_i. Thus, by definition of T_i, we now have $w(T_1, t) = w(T_2, t)$ for $t = 0, y_0, h$. Since T_1 and T_2 are triangles, both of the functions $w(T_i, t)$, $i = 1, 2$, are linear in t on the intervals $[0, y_0]$ and $[y_0, h]$. By equality at the endpoints of these intervals ($t = 0, y_0, h$), we then have $w(T_1, t) = w(T_2, t)$ for all $t \in [0, h]$. Finally, the definition of μ yields

$$\mu(T_i) = \iint_{T_i} \frac{1}{y} dA = \int_0^h \frac{w(T_i, t)}{t} dt$$

for $i = 1, 2$, which implies $\mu(T_1) = \mu(T_2)$ by equality of $w(T_1, t)$ and $w(T_2, t)$. ■

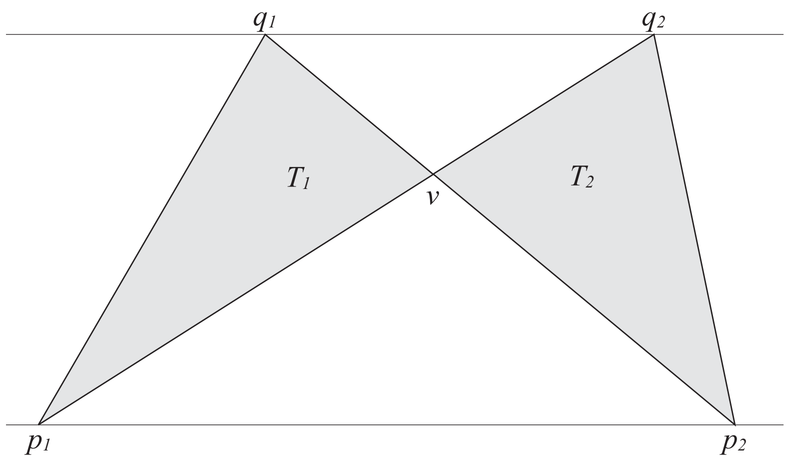

Figure 1: lemma 3.10

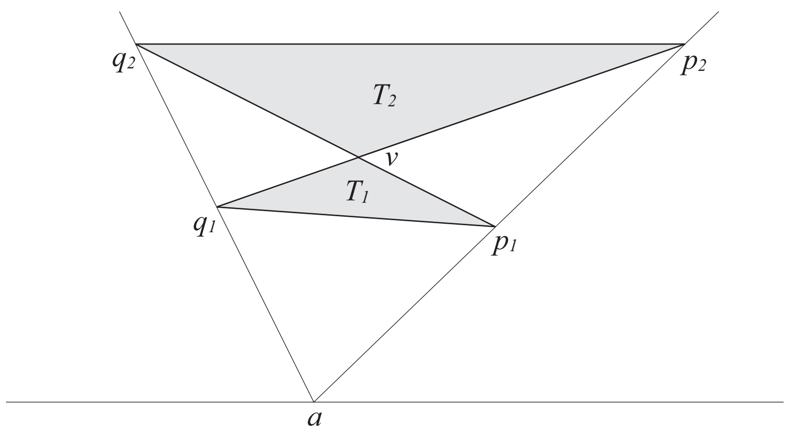

Figure 2: theorem 3.11

Theorem 3.11 *Let a be a point source, $\alpha < \beta$ in $[0, \pi]$, $p_1, p_2 \in l_a(\alpha)$, $q_1, q_2 \in l_a(\beta)$ such that $d(a, p_1) < d(a, p_2)$ and $d(a, q_1) < d(a, q_2)$. Let $v = \overline{p_1 q_2} \cap \overline{p_2 q_1}$, $T_1 = \mathrm{ch}(p_1, q_1, v)$, $T_2 = \mathrm{ch}(p_2, q_2, v)$. Then there exists a nonempty open interval $I \subset (\alpha, \beta)$ such that $X_a T_1(\varphi) < X_a T_2(\varphi)$ for all $\varphi \in I$. (see Figure 2.)*

Proof. Without loss of generality, we can rotate the plane about a so that $\alpha = 0$; this does not alter the relative values of the directed x-rays. Let l be the line parallel to the x axis (i.e., parallel to $\overline{p_1 p_2}$) and containing q_1, and let $v' = l \cap \overline{p_1 q_2}$. Let $T_1' = \mathrm{ch}(p_2, v, v')$. Since $v' \in \overline{v q_2} \backslash \{v, q_2\}$ by construction, $T_1' \subset T_2$. As l is parallel to $\overline{p_1 p_2}$, Lemma 3.10 applies and $\mu(T_1) = \mu(T_1')$. Then by Remark 3.8 $\mu(T_2) = \mu(T_1') + \mu(\mathrm{ch}(p_2, q_2, v')) > \mu(T_1') = \mu(T_1)$. Consequently, by (*),

$$\int_\alpha^\beta \frac{X_a T_2(\varphi)}{\sin(\varphi)} d\varphi > \int_\alpha^\beta \frac{X_a T_1(\varphi)}{\sin(\varphi)} d\varphi$$

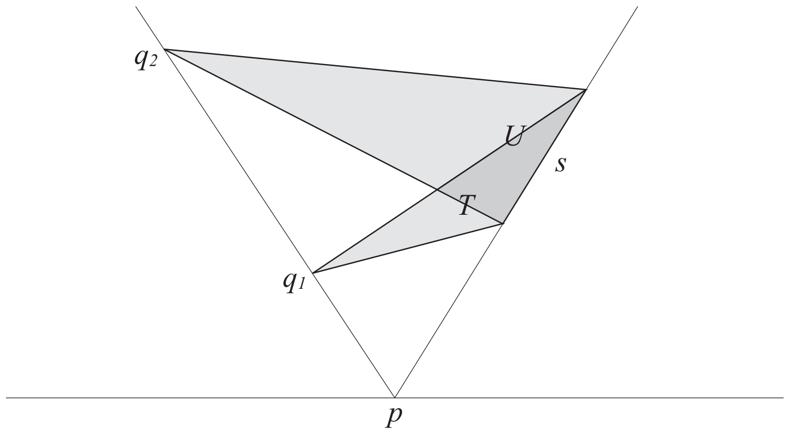

Figure 3: the triangle lemma

so that

$$\int_{\alpha}^{\beta} \frac{X_a T_2(\varphi) - X_a T_1(\varphi)}{\sin(\varphi)} d\varphi > 0$$

This integral exists by Propositions 3.7 and 3.9, and since $\sin(\varphi) > 0$ and the integrand is continuous on (α, β), there exists a nonempty open interval $I \subset (\alpha, \beta)$ such that $X_a T_2(\varphi) > X_a T_1(\varphi)$ for all $\varphi \in I$, as desired. ∎

Lemma 3.12 (Triangle Lemma) *Let p be a point source, and fix distinct values α, β in $(0, \pi)$. Fix a line segment s on $l_p(\alpha)$, and fix two distinct points q_1 and q_2 on $l_p(\beta)$ such $d(p, q_1) < d(p, q_2)$. Let the triangles T, U be given by $T = ch(s \cup q_1)$, $U = ch(s \cup q_2)$. Then there exists $\varphi \in I$ such that $X_p T(\varphi) < X_p U(\varphi)$, where I is the nondegenerate open interval between α and β. (see Figure 3.)*

Proof. Let $S = T \cap U$, $T_1 = T \backslash U$, $T_2 = U \backslash T$. By previous theorem, there exists $\varphi \in I$ such that $X_p T_1(\varphi) < X_p T_2(\varphi)$. Note that since T_1 and S intersect only on a line segment, the directed x-ray of $T_1 \cup S$ from p equals the sum of each of the directed x-rays of T_1 and S from p, by Proposition 3.7 and Remark 3.8. Such holds similarly for T_2. But note that $T = T_1 \cup S$ and $U = T_2 \cup S$, so that $X_p U(\varphi) - X_p T(\varphi) = X_p T_2(\varphi) - X_p T_1(\varphi) > 0$ for some $\varphi \in I$. ∎

Remark 3.13 *A stronger result is available: given the context of this lemma, $X_p T(\varphi) < X_p U(\varphi)$ for all $\varphi \in \mathrm{int}(I)$. The result is not necessary in this exposition and the proof involves very lengthy (but elementary) calculations, so it is omitted.*

4 Verification of a Triangle

We now apply what we know about relations between x-ray data and nonsmooth boundary points to verify triangles. Recall that, without loss of generality, we are only considering triangles in the open upper half plane H and x-ray sources on the x axis.

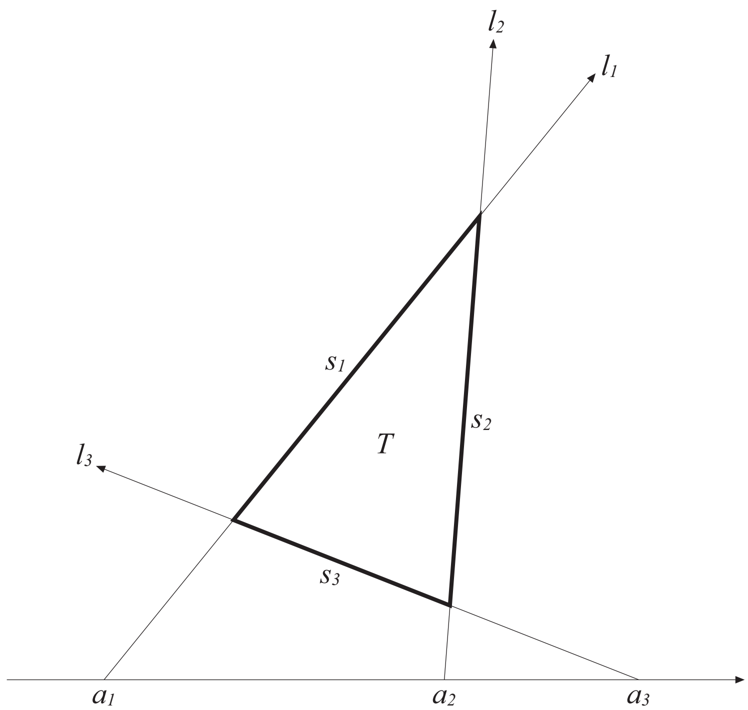

Figure 4: the triangle context

4.1 The Triangle Context

Let $T \subset H$ be a triangle. Borrowing notation and labels from Johnson [6], label the sides of T s_1, s_2, s_3, and let l_i be the unique ray in $\overline{H}$ containing s_i, for $i = 1, 2, 3$, s_i not horizontal. Let $a_i = (a_i, 0)$ be the intersection of l_i with the x axis for $i = 1, 2, 3$. Relabel if necessary so that $a_1 < a_2 < a_3$. If one of the l_i is horizontal, one of the a_i will not exist; in this case, relabel so that $a_1 < a_2$ and l_3 is horizontal. Call the a_i the **base points** of T. Let $[\alpha_i, \beta_i] = \mathrm{spt}_{a_i} T$. Note that for all base points of T, exactly one of $X_{a_i}(\alpha_i)$ and $X_{a_i}(\beta_i)$ is nonzero. We call a base point a_i of T **positively oriented** if $X_{a_i}(\beta_i) > 0$, and we call a_i **negatively oriented** if $X_{a_i}(\alpha_i) > 0$; we may at times refer to this property as the **orientation** of a base point. When necessary, we will use this as the context for triangles in the upper half plane, as shown in Figure 4. In this figure, a_1 and a_3 are positively oriented, while a_2 is negatively oriented.

Given a fixed triangle in the triangle context (and, as necessary, generalizing to the case where a_3 does not exist), we note that the base points a_i partition ∂H into four intervals. Given two point sources on ∂H, we may have two base points, one base point, or no base points. Any source that is not a base point will be in the interior of one of the aforementioned intervals. As we will show, it is not necessary to consider each combination of base point and interval as a separate case. Indeed, in some instances, cases are given by orientation of a base point.

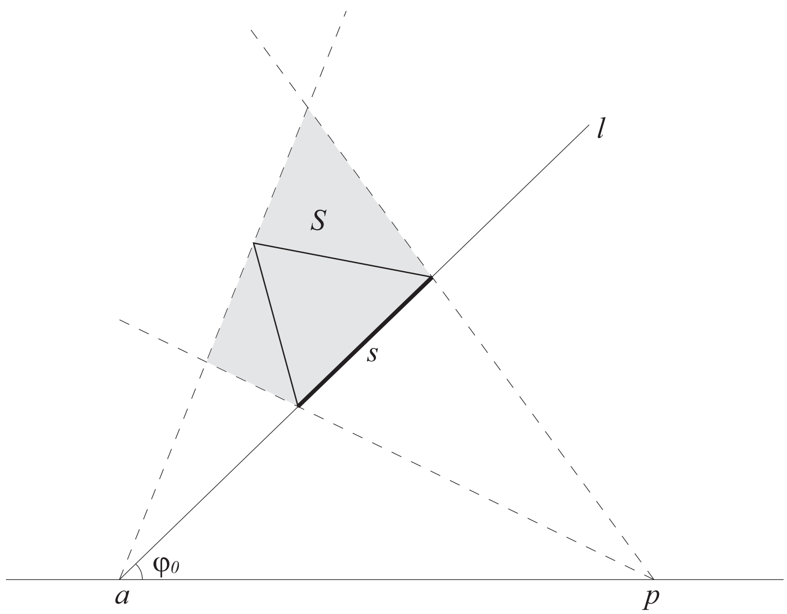

Figure 5: theorem 4.2

4.2 Verification With Base Points

The following theorem by Johnson [6] handled the case where both point sources are base points. Johnson's proof involved methods of "counting" nonsmooth points similar to our own.

Theorem 4.1 ([6] Thm 8) *Any triangle in H can be verified from all other convex bodies by any two of its base points.*

We can begin to tackle the one base point case with this theorem.

Theorem 4.2 *Let $T \subset H$ be a triangle in the context given above and fix a side $s = s_i$ of T. Fix a point source p such that s is either the far side of T or the near side of T with respect to p. Then T can be verified from all other convex bodies by $a = a_i$ and p. (see Figure 5.)*

Proof. Suppose $K \subset H$ is a convex body such that $X_aT = X_aK$ and $X_pT = X_pK$. Let φ_0 be the angle of inclination of $l = l_i$, so that $l_i = l_{a_i}(\varphi_0)$. Let $S = C_aT \cap C_pT$. Since T and K share the same x-rays from a and p, $K \subset S$. $X_aK(\varphi_0) = X_aT(\varphi_0) = \text{len}(s)$, so K contains a line segment in $l \cap S$ with the same length as s. Since s is either a far side or a near side of T with respect to p, each endpoint of s is on a supporting ray of T from p. Hence $\text{len}(l \cap S) = \text{len}(s)$, so that K must contain s. Since s itself is in a supporting ray (namely l), no point of s is in $\text{int}(S)$. $K \subset S$ so, consequently, no point of s is in $\text{int}(K) \subset \text{int}(S)$. Thus $s \subset \partial K$. T and K share supporting rays, so s intersects both supporting rays of K with respect to p. Hence, since $s \subset \partial K$ and K is convex, s is either the near side or the far side of K with respect to p. The orientation of a determines which side of l S lies on, and thus determines which side of s K lies on. In turn, this tells us whether s is the near or far

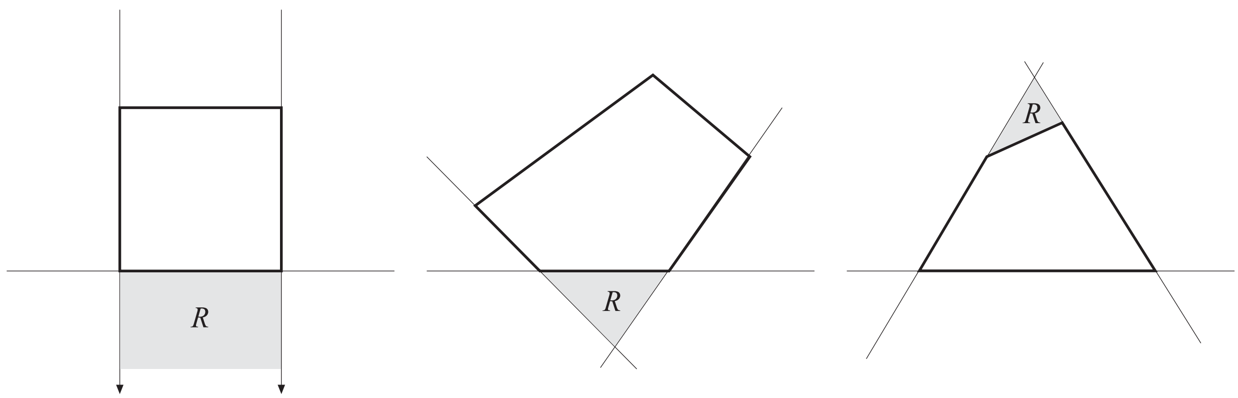

Figure 6: corollary 4.5 cases I, II, III

side of K with respect to p (i.e., if $p > a$ and a is negatively oriented, s is the near side). In particular, s is the near (far) side of K with respect to p iff s is the near (far) side of T with respect to p. Now T and K share a near (or far) side with respect to p, and $X_pT = X_pK$, so $T = K$. ■

Remark 4.3 *Johnson's theorem above is a special case of this one, if we set p as a base point distinct from a.*

Remark 4.4 *The proof of Theorem 4.2 does not rely on T being a triangle. Indeed, if T is any convex body containing a line segment s, l is the line containing s, a is the intersection of l with the x axis, and p is a point as above, then T can be verified from p and a. Note that if ∂T is everywhere smooth, no such p can be found.*

By Remark 4.4, we get the following corollary.

Corollary 4.5 *If P is a convex polygon, then there exist two points p and a such that P can be verified by the directed x-rays at p and a and such that the line through p and a does not intersect P.*

Proof. If P has three sides, then Theorems 4.1 or 4.2 suffice. Assume P has at least four sides. Fix a side s of P, and let l be the line containing s. Let m and n be the lines containing the sides of P adjacent to s, and let $v = m \cap n$ if it exists. First, we construct a desirable p. By convexity of P, we can let D be the open half-plane on the side of l opposite P.

Case I: v does not exist. Let R be the intersection of D and the closed region between (and containing) m and n.

Case II: $v \in D$. Let $R = \mathrm{ch}(s \cup v) \backslash s$.

Case III: $v \notin D$. Let $R = \mathrm{ch}(s \cup v) \backslash P$. Here R is nonempty because P has at least four sides; i.e. $v \notin P$.

Figure 6 shows the R in each of these cases. Note that $R \cap P = \emptyset$ and for any $p \in R$, s is the near side of P with respect to p. Now fix $p \in R$. Since P is closed and $p \notin P$, we can find a line l' through p such that $l' \cap P = \emptyset$ and l' is not parallel to l. Let $a = l \cap l'$.

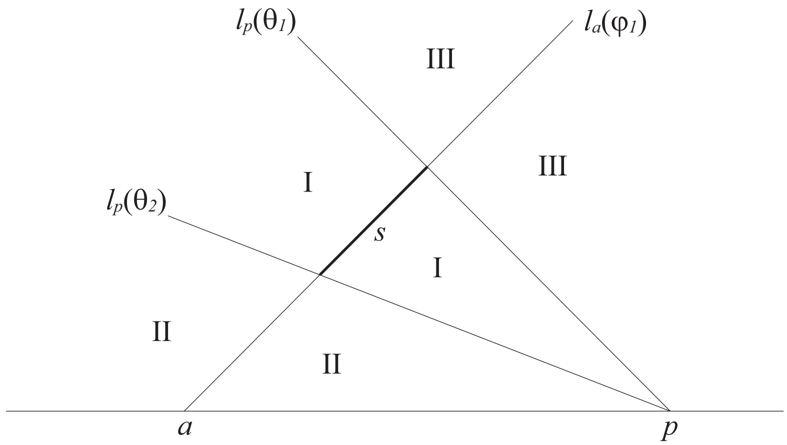

Figure 7: cases of theorem 4.7

Now we rotate the plane so that $l' = \partial H$, and are reduced to the setting in Remark 4.4, thus completing the proof. ■

Claim 4.6 *For two sources $p > a$ and angles $\theta, \varphi \in (0, \pi)$, $l_p(\theta) \cap l_a(\varphi)$ exists if and only if $\varphi < \theta$.*

The following theorem is the legwork in verification with one base point. In the diagrams below, the points $p_i \in H$ are labeled i.

Theorem 4.7 *Let $T \subset H$ be a triangle and let a be any base point of T. Then T can be verified from all other convex bodies by the two point sources a, p for any $p > a$.*

Proof. Fix $p > a$. Let s be the side of T corresponding to the base point a; let φ_1 be such that $s \subset l_a(\varphi_1)$. Let $\theta_1 < \theta_2$ be such that $l_p(\theta_1)$ and $l_p(\theta_2)$ each contain an endpoint of s. Let v be the vertex of T not on $l_a(\varphi_1)$. Suppose $K \subset H$ is a convex body such that $X_aK = X_aT$ and $X_pK = X_pT$. As $C_p(0, \pi) = H$, we have three cases, as shown in Figure 7.

Case I: $v \in C_p[\theta_1, \theta_2]$. This would imply $T \subset C_p[\theta_1, \theta_2]$, so that s is either a far side or a near side of T. Then $K = T$ by Theorem 4.2 above.

Case II: $v \in C_p(\theta_2, \pi)$. Let φ_2, θ_3 be the unique angles such that $v \in l_a(\varphi_2) \cap l_p(\theta_3)$. Let $I = \mathrm{spt}_a T$ ($I = [\varphi_1, \varphi_2]$ or $[\varphi_2, \varphi_1]$, whichever is nondegenerate). $s \subset l_a(\varphi_1)$ so $X_aT(\varphi_1) > 0$. $l_a(\varphi_2) \cap T = \{v\}$ by construction so $X_aT(\varphi_2) = 0$. By Corollary 3.3, the limits of $(X_aT)'(\varphi)$ as φ approaches either endpoint of I exist and are finite. Thus by equality of x-rays for T and K we have:

$$\begin{array}{cc} X_aK(\varphi_1) > 0 & X_aK(\varphi_2) = 0 \\ \left| \lim_{\varphi \in I, \varphi \to \varphi_1} (X_aK)'(\varphi) \right| < \infty & \left| \lim_{\varphi \in I, \varphi \to \varphi_2} (X_aK)'(\varphi) \right| < \infty \end{array} \tag{4}$$

By (4) and Corollary 3.6, ∂K must have exactly one nonsmooth point on $l_a(\varphi_2)$ and two nonsmooth points on $l_a(\varphi_1)$. Since X_aK is smooth on $\mathrm{int}(I)$, ∂K has no nonsmooth points in $C_a(\mathrm{int}(I))$ by Theorem 3.4. Hence ∂K has exactly three nonsmooth points.

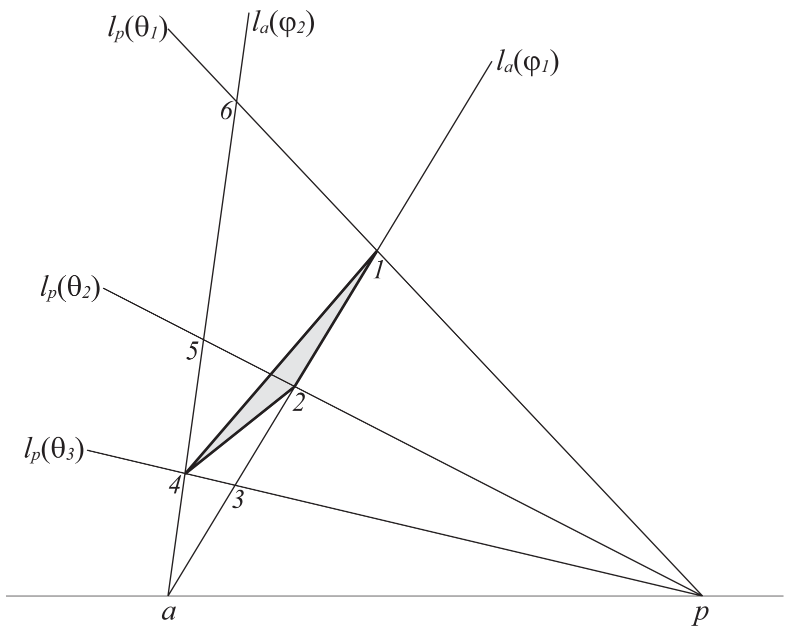

Figure 8: theorem 4.7 case II

Now since $v \in l_p(\theta_3)$ with $\theta_3 > \theta_2$, each of the rays $l_p(\theta_i)$, $i = 1, 2, 3$, contains exactly one nonsmooth point of ∂T. So $X_pT(\theta_1) = X_pT(\theta_3) = 0$, $\left|(X_pT)'_+(\theta_1)\right| < \infty$, and $\left|(X_pT)'_-(\theta_3)\right| < \infty$. Again by equality of x-rays we get:

$$\begin{array}{cc} X_pK(\theta_1) = 0 & X_pK(\theta_3) = 0 \\ \left|(X_pK)'_+(\theta_1)\right| < \infty & \left|(X_pK)'_-(\theta_3)\right| < \infty \end{array} \tag{5}$$

Consequently, by Corollary 3.6, ∂K has exactly one nonsmooth point on each of the rays $l_p(\theta_1)$ and $l_p(\theta_3)$. As $l_p(\theta_2)$ contains a nonsmooth point of ∂T, X_pT is not differentiable at θ_2. Therefore X_pK is not differentiable at θ_2, so ∂K has at least one nonsmooth point on $l_p(\theta_2)$ by Theorem 3.4. If there were two nonsmooth points of ∂K on $l_p(\theta_2)$, then ∂K would have at least four nonsmooth points, a contradiction as ∂K has exactly three such points. Thus there is exactly one nonsmooth point of ∂K on $l_p(\theta_2)$.

We now know, in summary, that ∂K has exactly three nonsmooth points, two on $l_a(\varphi_1)$, one on $l_a(\varphi_2)$, and one on each $l_p(\theta_i)$, $i = 1, 2, 3$. Thus the vertices can only be on intersections of these rays from p and a. (see Figure 8.) Define the following points if they exist: $p_i = l_p(\theta_i) \cap l_a(\varphi_1)$ for $i = 1, 2, 3$, $p_4 = v = l_p(\theta_3) \cap l_a(\varphi_2)$, $p_5 = l_p(\theta_2) \cap l_a(\varphi_2)$, and $p_6 = l_p(\theta_1) \cap l_a(\varphi_2)$, so $T = \mathrm{ch}(p_1, p_2, p_4)$. By construction, $\varphi_1 < \theta_1 < \theta_2 < \theta_3$ and $\varphi_2 < \theta_3$, so p_i always exists for $1 \le i \le 4$ according to our claim above. Also, p_5 exists iff $\varphi_2 < \theta_2$ and p_6 exists iff $\varphi_2 < \theta_1$. All of the intersections of the rays on which nonsmooth points of ∂K lie are these p_i, so the set of nonsmooth points of ∂K is a subset of $\{p_i\}_{i=1}^6$. Now we rephrase what we know:

$$\begin{array}{c} 2 \text{ of } p_1, p_2, p_3 \text{ are in } \partial K,\ 1 \text{ of } p_4, p_5, p_6 \text{ is in } \partial K \\ 1 \text{ of } p_1, p_6 \text{ is in } \partial K,\ 1 \text{ of } p_2, p_5 \text{ is in } \partial K,\ 1 \text{ of } p_3, p_4 \text{ is in } \partial K \end{array} \tag{**}$$

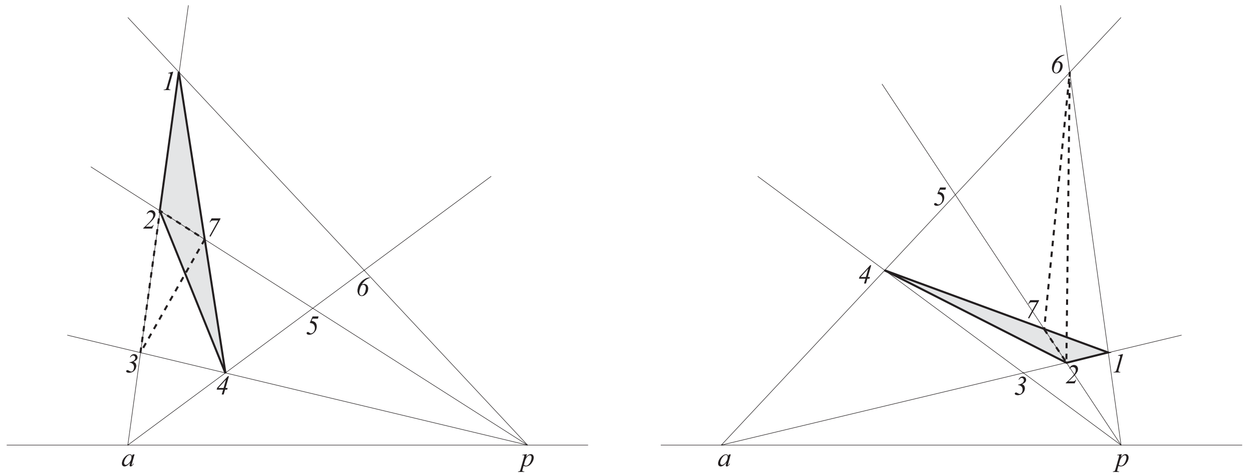

Figure 9: theorem 4.7 cases II(a), II(b)

Suppose $p_2 \notin \partial K$. Then $p_1, p_3 \in \partial K$ by (**), so convexity yields $\overline{p_1 p_3} \subset K$, which implies $X_a K(\varphi_1) \geq \text{len}(\overline{p_1 p_3}) > \text{len}(\overline{p_1 p_2}) = X_a T(\varphi_1)$, a contradiction as K and T share the same x-ray from a. Thus $p_2 \in \partial K$. Now either p_1 or p_3 is in ∂K as well.

If $p_3 \in \partial K$, then by (**) we have $\{p_2, p_3, p_6\} \subset \partial K$. If p_6 does not exist ($\varphi_2 \geq \theta_1$), this cannot occur. If $\text{len}(\overline{p_1 p_2}) \neq \text{len}(\overline{p_2 p_3})$, then we again arrive at a contradiction as these are, respectively, $X_a T(\varphi_1)$ and $X_a K(\varphi_1)$. Assume p_6 exists and $\text{len}(\overline{p_1 p_2}) = \text{len}(\overline{p_2 p_3})$. From here, there are two subcases to consider. (see Figure 9.)

Case II(a): $\varphi_2 < \varphi_1$, i.e., a is positively oriented. Let $p_7 = \overline{p_1 p_4} \cap l_p(\theta_2) \in \partial T$. Since p_2 is on both far sides of K and T with respect to p (by orientation of a) and $X_p T(\theta_2) = X_p K(\theta_2)$, $p_7 \in \partial K$. Thus $\{p_2, p_3, p_7\} \subset K \Rightarrow \text{ch}(p_2, p_3, p_7) \subset K$. As $d(p, p_3) > d(p, p_4)$ (by orientation of a), the triangle lemma 3.12 says there exists $\theta \in (\theta_2, \theta_3)$ such that $X_p(\text{ch}(p_2, p_3, p_7)) > X_p(\text{ch}(p_2, p_4, p_7))$ at θ. Thus $X_p K \geq X_p(\text{ch}(p_2, p_3, p_7)) > X_p(\text{ch}(p_2, p_4, p_7)) = X_p T$ at θ, which implies $X_p K \neq X_p T$.

Case II(b): $\varphi_1 < \varphi_2$, i.e., a is negatively oriented. Let $p_7 = \overline{p_1 p_4} \cap l_p(\theta_2) \in \partial T$ as before. Since p_2 is on both near sides of K and T with respect to p and $X_p T(\theta_2) = X_p K(\theta_2)$, $p_7 \in \partial K$. Thus $\{p_2, p_6, p_7\} \subset K \Rightarrow \text{ch}(p_2, p_6, p_7) \subset K$. As $d(p, p_6) > d(p, p_1)$, Lemma 3.12 says there exists $\theta \in (\theta_1, \theta_2)$ such that $X_p(\text{ch}(p_2, p_6, p_7)) > X_p(\text{ch}(p_1, p_2, p_7))$ at θ. Thus $X_p K \geq X_p(\text{ch}(p_2, p_6, p_7)) > X_p(\text{ch}(p_1, p_2, p_7)) = X_p T$ at θ, which implies $X_p K \neq X_p T$.

Either way, we arrive at a contradiction since K and T share the same directed x-rays from p. Thus $p_3 \notin \partial K$, so $p_1 \in \partial K$. Then by (**) we must have $p_4 \in \partial K$. Noting that $T = \text{ch}(p_1, p_2, p_4)$, we have $\{p_1, p_2, p_4\} \subset \partial K$, which implies $K = T$ by Theorem 3.1.

Case III: $v \in C_p(0, \theta_1)$. Let φ_2, θ_0 be the unique angles such that $v \in l_a(\varphi_2) \cap l_p(\theta_0)$. Let $I = \text{spt}_a T$. $s \subset l_a(\varphi_1)$ so $X_a T(\varphi_1) > 0$. $l_a(\varphi_2) \cap T = \{v\}$ by construction so $X_a T(\varphi_2) = 0$. By Corollary 3.3, the limits of $(X_a T)'(\varphi)$ as φ approaches either endpoint of I exist and are finite. Thus by equality of x-rays for T and K we have conditions (4) again. As above, ∂K has three nonsmooth points: one nonsmooth point on $l_a(\varphi_2)$ and two nonsmooth points on $l_a(\varphi_1)$.

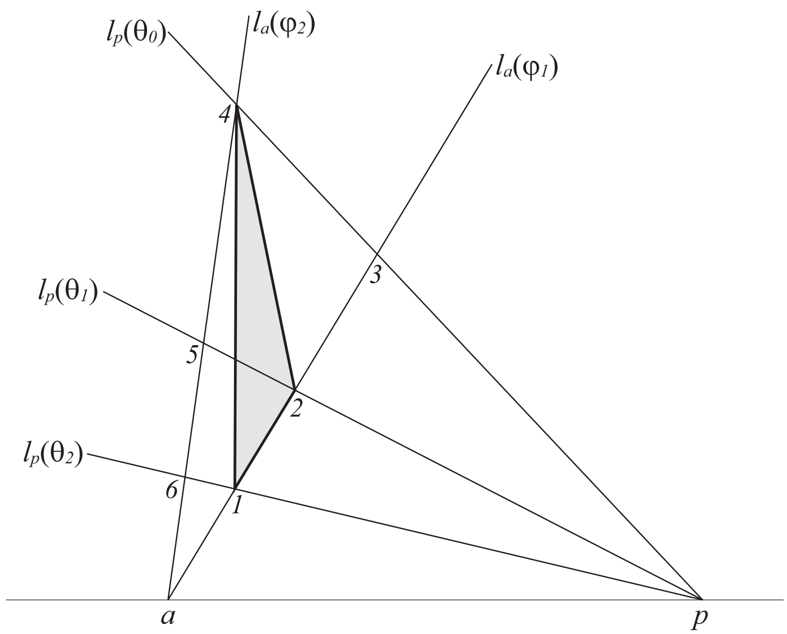

Figure 10: theorem 4.7 case III

Now since $v \in l_p(\theta_0)$ with $\theta_0 > \theta_1$, each of the rays $l_p(\theta_i)$, $i = 0, 1, 2$, contains exactly one nonsmooth point of ∂T. So $X_pT(\theta_0) = X_pT(\theta_2) = 0$, $\left|(X_pT)'_+(\theta_0)\right| < \infty$, and $\left|(X_pT)'_-(\theta_2)\right| < \infty$. By equality of x-rays we get conditions analogous to (5):

$$\begin{array}{cc} X_pK(\theta_0) = 0 & X_pK(\theta_2) = 0 \\ \left|(X_pK)'_+(\theta_0)\right| < \infty & \left|(X_pK)'_-(\theta_2)\right| < \infty \end{array} \tag{6}$$

By Corollary 3.6, ∂K has exactly one nonsmooth point on each ray $l_p(\theta_0)$, $l_p(\theta_2)$. $l_p(\theta_1)$ contains a nonsmooth point of ∂T, so $X_pK = X_pT$ is not differentiable at θ_1, which implies ∂K has at least one nonsmooth point on $l_p(\theta_1)$ by Theorem 3.4. If there were two nonsmooth points of ∂K on $l_p(\theta_1)$, then ∂K would have at least four nonsmooth points, a contradiction as ∂K has only three such points. Thus there is exactly one nonsmooth point of ∂K on $l_p(\theta_1)$.

We now know that ∂K has exactly three nonsmooth points, two on $l_a(\varphi_1)$, one on $l_a(\varphi_2)$, and one on each $l_p(\theta_i)$, $i = 0, 1, 2$; the vertices can only be on intersections of these rays from p and a. (see Figure 10.) Define the following points if they exist: $p_1 = l_p(\theta_2) \cap l_a(\varphi_1)$, $p_2 = l_p(\theta_1) \cap l_a(\varphi_1)$, $p_3 = l_p(\theta_0) \cap l_a(\varphi_1)$, $v = p_4 = l_p(\theta_0) \cap l_a(\varphi_2)$, $p_5 = l_p(\theta_1) \cap l_a(\varphi_2)$, and $p_6 = l_p(\theta_2) \cap l_a(\varphi_2)$. Again, $T = \mathrm{ch}(p_1, p_2, p_4)$. By construction, $\varphi_1 < \theta_1 < \theta_2$ and $\varphi_2 < \theta_0 < \theta_1 < \theta_2$, so p_i always exists for $i \neq 3$, and p_3 exists iff $\varphi_1 < \theta_0$. The set of nonsmooth points of ∂K is a subset of $\{p_i\}_{i=1}^6$, and we have the conditions (**) on the p_i once again. By the same argument as above, $p_2 \in \partial K$, and one of p_1 and p_3 is in ∂K.

If $p_3 \in \partial K$, then by (**) we have $\{p_2, p_3, p_6\} \subset \partial K$. Of course, if p_3 does not exist this cannot occur. If p_3 exists and $\mathrm{len}(\overline{p_1p_2}) \neq \mathrm{len}(\overline{p_2p_3})$, then we arrive at a contradiction as above. Assume p_3 exists and $\mathrm{len}(\overline{p_1p_2}) = \mathrm{len}(\overline{p_2p_3})$, so that we can consider the subcases shown in Figure 11:

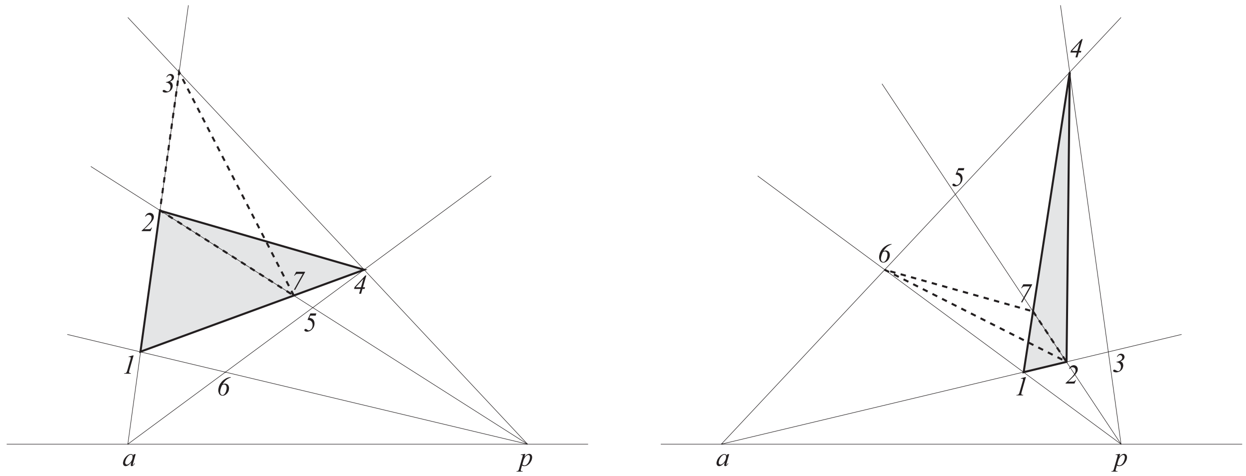

Figure 11: theorem 4.7 cases III(a), III(b)

Case III(a): $\varphi_2 < \varphi_1$, i.e., a is positively oriented. Let $p_7 = \overline{p_1 p_4} \cap l_p(\theta_1) \in \partial T$. Since p_2 is on both far sides of K and T with respect to p and $X_pT(\theta_1) = X_pK(\theta_1)$, $p_7 \in \partial K$. Thus $\{p_2, p_3, p_7\} \subset K \Rightarrow \text{ch}(p_2, p_3, p_7) \subset K$. As $d(p, p_3) > d(p, p_4)$, Lemma 3.12 says there exists $\theta \in (\theta_0, \theta_1)$ such that $X_p(\text{ch}(p_2, p_3, p_7)) > X_p(\text{ch}(p_2, p_4, p_7))$ at θ. Thus $X_pK \geq X_p(\text{ch}(p_2, p_3, p_7)) > X_p(\text{ch}(p_2, p_4, p_7)) = X_pT$ at θ, which implies $X_pK \neq X_pT$.

Case III(b): $\varphi_1 < \varphi_2$, i.e., a is negatively oriented. Let $p_7 = \overline{p_1 p_4} \cap l_p(\theta_2) \in \partial T$. Since p_2 is on both near sides of K and T with respect to p and $X_pT(\theta_1) = X_pK(\theta_1)$, $p_7 \in \partial K$. Thus $\{p_2, p_6, p_7\} \subset K \Rightarrow \text{ch}(p_2, p_6, p_7) \subset K$. As $d(p, p_6) > d(p, p_1)$, the triangle lemma says there exists $\theta \in (\theta_1, \theta_2)$ such that $X_p(\text{ch}(p_2, p_6, p_7)) > X_p(\text{ch}(p_1, p_2, p_7))$ at θ. Thus $X_pK \geq X_p(\text{ch}(p_2, p_6, p_7)) > X_p(\text{ch}(p_1, p_2, p_7)) = X_pT$ at θ, which implies $X_pK \neq X_pT$.

Thus we arrive at a contradiction since K and T share the same directed x-rays from p. So $p_3 \notin \partial K$ and $p_1 \in \partial K$. Then by (**) we must have $p_4 \in \partial K$. Noting that $T = \text{ch}(p_1, p_2, p_4)$, we have $\{p_1, p_2, p_4\} \subset \partial K$, which implies $K = T$ by Theorem 3.1. This completes the proof. ■

By an easy symmetry, we can now settle the one base point case.

Corollary 4.8 *Let $T \subset H$ be a triangle and let a be any base point of T. Then T can be verified from all other convex bodies by the two point sources a, p for any p distinct from a.*

Proof. Suppose $K \subset H$ is a convex body with the same directed x-rays as T from p and a. If $p > a$, then $K = T$ by previous theorem. Suppose $p < a$. Let a', p', T', and K' be the reflections about the y axis of a, p, T, and K, respectively. Then $p' > a'$, a' is a base point of triangle $T' \subset H$, and $K' \subset H$ is a convex body that shares the same directed x-rays as T' from a' and p'. Thus $K' = T'$ by the previous theorem, so that $K = T$. ■

4.3 Verification Without Base Points

This case appears to be the most complicated and is not completely solved here. Before continuing, we establish a general setting for handling this case.

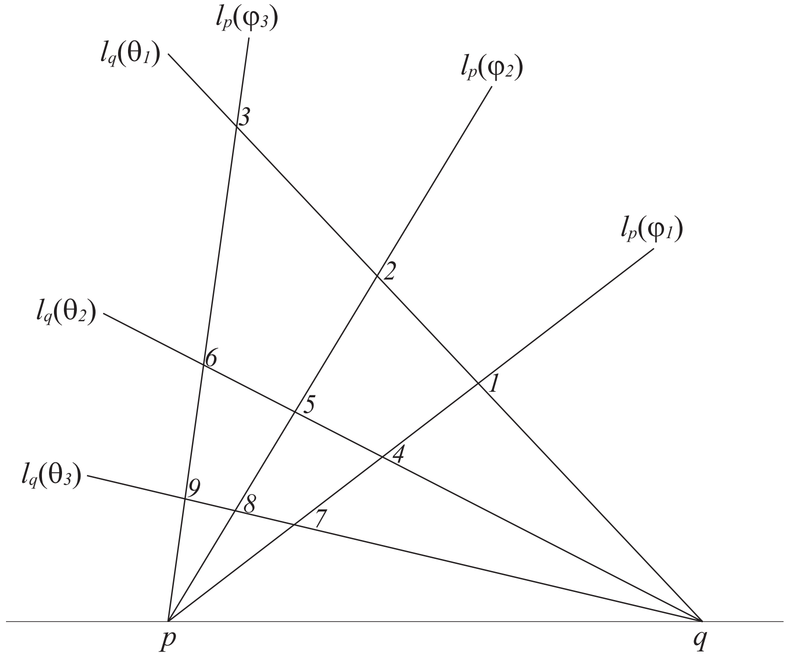

Figure 12: intersections of rays, no base point case

Let $T \subset H$ be a triangle and choose two distinct p, $q \in \partial H$ that are not base points. Since p is not a base point, each vertex of T lies on its own ray $l_p(\varphi)$; similarly each vertex of T lies on exactly one $l_q(\theta)$. Accordingly, define $\varphi_1 < \varphi_2 < \varphi_3$ so that each vertex of T lies on one $l_p(\varphi_i)$ and define $\theta_1 < \theta_2 < \theta_3$ such that each vertex of T lies on one $l_q(\theta_j)$. Suppose $K \subset H$ is a convex body with the same directed x-rays as T from p and q.

By Corollary 3.3, the limits of $(X_pT)'(\varphi)$ as $\varphi \to \varphi_1^+$ and as $\varphi \to \varphi_3^-$ are finite. By definition, $X_pT(\varphi_i) = 0$ for $i = 1, 3$. Then by equality of x-rays of K and T, we get that ∂K has exactly one nonsmooth point on each of the rays $l_p(\varphi_1)$ and $l_p(\varphi_3)$ through Corollary 3.6. Similarly, replacing p with q and φ_i with θ_j, we see ∂K has exactly one nonsmooth point on each of the rays $l_q(\theta_1)$ and $l_q(\theta_3)$. Since ∂T has a vertex on $l_p(\varphi_2)$, $X_pT = X_pK$ is not differentiable at φ_2, so $l_p(\varphi_2)$ contains either one or two nonsmooth points of ∂K by Theorem 3.4. Analogously, using the directed x-ray of T from q, we see that $l_q(\theta_2)$ contains one or two nonsmooth points of ∂K. Since ∂T is smooth everywhere except at its three vertices, $X_pT = X_pK$ is smooth in $(\varphi_1, \varphi_2) \cup (\varphi_2, \varphi_3)$, so ∂K contains no nonsmooth points outside of the rays $l_p(\varphi_i)$. It follows that ∂K either has three or four vertices, and these all occur on the intersections of the $l_p(\varphi_i)$ and $l_q(\theta_j)$. If ∂K has three vertices, then $l_p(\varphi_2)$ and $l_q(\theta_2)$ each contain exactly one nonsmooth point of ∂K; otherwise, they each contain two nonsmooth points of ∂K. In either case, we have reduced verification to looking at possible configurations of vertices on intersections of rays, in the same manner as in the proof of Theorem 4.7.

Now label the intersection points p_i as follows. Say $p_1 = l_p(\varphi_1) \cap l_q(\theta_1)$, $p_2 = l_p(\varphi_2) \cap l_q(\theta_1)$, $p_3 = l_p(\varphi_3) \cap l_q(\theta_1)$, $p_4 = l_p(\varphi_1) \cap l_q(\theta_2)$, and so on, to $p_9 = l_p(\varphi_3) \cap l_q(\theta_3)$. (See Figure 12.) Similarly to what happened in the one base point case, p_2, p_3, and p_6 may not exist due to relative values of the angles of inclination of the rays. We will hereafter assume that they

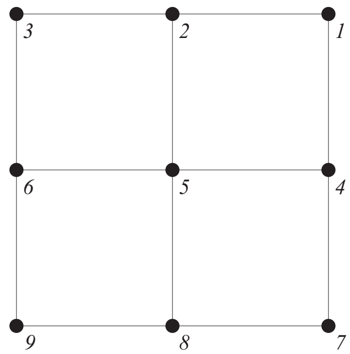

Figure 13: grid of p_i

do exist when considering selections of vertices. If nonexistence of some p_i does occur for some particular T, then any cases that use the nonexistent vertices obviously cannot occur; regardless, our assertions will cover all possible cases.

From our assertions above, we get conditions on possible locations of the nonsmooth points on ∂K, which must be selected from the p_i. If ∂K has four nonsmooth points, then:

$$\begin{gathered} 1 \text{ of } p_1, p_2, p_3 \text{ is in } \partial K,\ 1 \text{ of } p_7, p_8, p_9 \text{ is in } \partial K \\ 1 \text{ of } p_1, p_4, p_7 \text{ is in } \partial K,\ 1 \text{ of } p_3, p_6, p_9 \text{ is in } \partial K \\ 2 \text{ of } p_2, p_5, p_8 \text{ are in } \partial K,\ 2 \text{ of } p_4, p_5, p_6 \text{ are in } \partial K \end{gathered} \tag{*4}$$

Otherwise, ∂K has three points and this holds:

$$\begin{gathered} 1 \text{ of } p_1, p_2, p_3 \text{ is in } \partial K,\ 1 \text{ of } p_7, p_8, p_9 \text{ is in } \partial K \\ 1 \text{ of } p_1, p_4, p_7 \text{ is in } \partial K,\ 1 \text{ of } p_3, p_6, p_9 \text{ is in } \partial K \\ 1 \text{ of } p_2, p_5, p_8 \text{ is in } \partial K,\ 1 \text{ of } p_4, p_5, p_6 \text{ is in } \partial K \end{gathered} \tag{*3}$$

For the sake of expressing the possible configurations of nonsmooth points on ∂K, we arrange the points p_i in a rectangular fashion as in Figure 13. Using the figure and the conditions (*3), (*4), it can easily be seen that the set of vertices of ∂K is one of the following, as shown in Figure 14.

$$\begin{gathered} S_1 = \{p_2, p_4, p_6, p_8\},\ S_2 = \{p_1, p_5, p_6, p_8\},\ S_3 = \{p_3, p_4, p_5, p_8\},\ S_4 = \{p_2, p_4, p_5, p_9\}, \\ S_5 = \{p_2, p_5, p_6, p_7\},\ S_6 = \{p_3, p_5, p_7\},\ S_7 = \{p_1, p_5, p_9\},\ S_8 = \{p_3, p_4, p_8\}, \\ S_9 = \{p_1, p_6, p_8\},\ S_{10} = \{p_2, p_6, p_7\},\ S_{11} = \{p_2, p_4, p_9\} \end{gathered} \tag{S}$$

We can actually eliminate the (*4) case with some intuitively clear geometrical arguments.

Lemma 4.9 *Let distinct lines l, l' in the plane intersect at a point o. Let $p \in l$, $q \in l'$ such that $p \neq o$ and $q \neq o$. Let r be a point in the plane such that l' lies between p and r and l lies between q and r. Then $o \in \operatorname{int}(\operatorname{ch}(p, q, r))$.*

Proof. Without loss of generality, let l be horizontal, p with a greater x coordinate than o, and q with a greater y coordinate than o. Since r lies to the left of l', o is in the open half-plane with boundary $\overleftrightarrow{qr}$ containing p. Since r lies below l, o is in the open

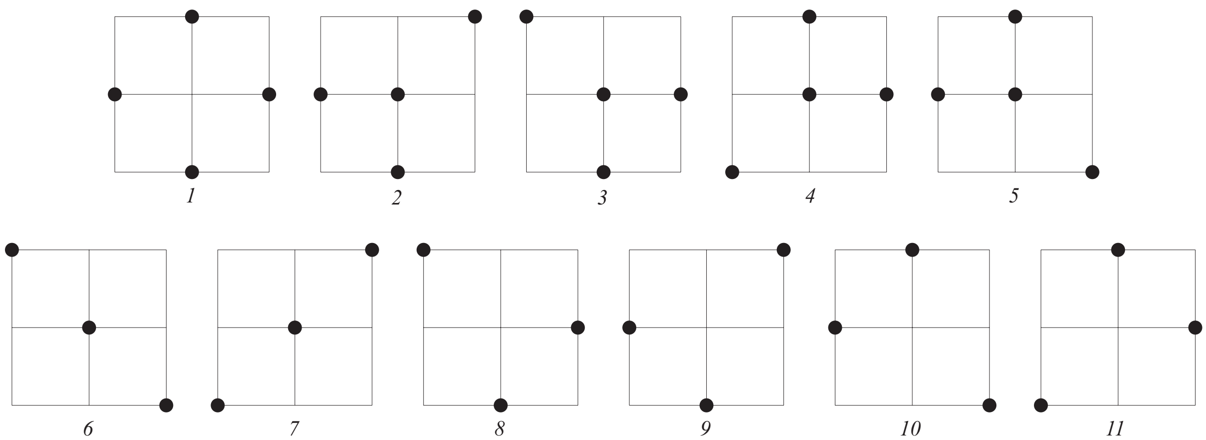

Figure 14: grid representation of (S): i denotes S_i

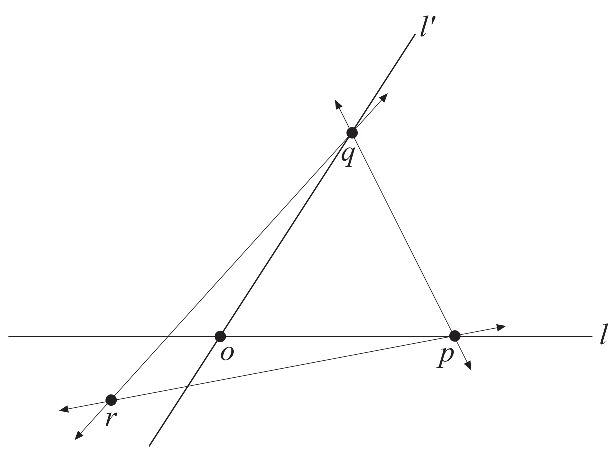

Figure 15: lemma 4.9

half-plane with boundary $\overleftrightarrow{rp}$ containing q. By definition of r, o lies in the open half-plane with boundary $\overleftrightarrow{pq}$ containing r. The intersection of these three half-planes is exactly the interior of $\mathrm{ch}(p, q, r)$, so $o \in \mathrm{ch}(p, q, r)$. (See Figure 15.) ■

Theorem 4.10 *There is no convex body K with $S_i \subset \partial K$ for any $i \in \{2, 3, 4, 5\}$.*

Proof. Consider the case $i = 2$. Note that $S_2 = \{p_1, p_5, p_6, p_8\}$. Let l be the line containing $l_p(\varphi_2)$ and l' the line containing $l_q(\theta_2)$. Note that $l \cap l' = \{p_5\}$, $p_8 \in l$, and $p_6 \in l'$. By definition, both p_6 and p_8 are distinct from p_5. Additionally, by definition of the p_i, l lies between p_6 and p_1 and l' lies between p_8 and p_1 (consult Figure 12). Hence Lemma 4.9 implies $p_5 \in \mathrm{int}(\mathrm{ch}(p_1, p_6, p_8))$, so that $p_5 \in \mathrm{int}(K)$. Thus $p_5 \notin \partial K$, a contradiction. We conclude that S_2 is not contained in ∂K. Similar arguments establish the result for $i = 3, 4, 5$. ■

Theorem 4.11 *There is no convex body K with $S_1 \subset \partial K$ such that $X_pK = X_pT$ and $X_qK = X_qT$, given the setting above.*

Proof. If there is K such that $S_1 \subset \partial K$, then both p_2 and p_8 are in ∂K. Both of these points are on $l_p(\varphi_2)$, so $X_pK(\varphi_2) \geq \mathrm{len}(\overline{p_2p_8})$. By equality of x-rays, we then get $X_pT(\varphi_2) \geq \mathrm{len}(\overline{p_2p_8})$. However, note that $l_p(\varphi_2) \cap C_q[\theta_1, \theta_3] = \overline{p_2p_8}$. Since $T \subset C_q[\theta_1, \theta_3]$ by definition, $T \cap l_p(\varphi_2) \subseteq \overline{p_2p_8}$. Hence $X_pT(\varphi_2) \leq \mathrm{len}(\overline{p_2p_8})$. It follows that $X_pT(\varphi_2) = \mathrm{len}(\overline{p_2p_8})$, and consequently, $T \cap l_p(\varphi_2) = \overline{p_2p_8} \Rightarrow \overline{p_2p_8} \subset T$. Now p_2 and p_8 are each on supporting rays of T from q, so that by our construction, p_2 and p_8 are both nonsmooth points (vertices in this case) on ∂T. Both of these points are on $l_p(\varphi_2)$, so T has two vertices on a single ray from p, a contradiction by our construction. Hence S_1 is not contained in ∂K. (The same argument may be applied using p_4, p_6, and the directed x-ray from q, if we so desire.) ■

Corollary 4.12 *If $T \subset H$ is a triangle, p, $q \in \partial H$ are not base points of T, and $K \subset H$ is a convex body sharing the directed x-rays of T from both p and q, then ∂K has exactly three nonsmooth points.*

Proof. If ∂K has four nonsmooth points, then $S_i \subset \partial K$ for some $1 \leq i \leq 5$, which cannot happen by the previous two theorems. ■

Now we know that if we are given a triangle T and distinct p, q that are not base points, then any convex body K sharing x-rays with T from p and q must have three vertices in one of the six configurations S_6, S_7, ..., S_{11}. Note that throughout our arguments, we never explicitly specified a triangle T, because the setting above holds in general if we avoid base points. Indeed, by our construction, the vertices on some ∂T must be in one of the configurations S_6, S_7, ..., S_{11}.

So, to complete the solution to verification without base points, we need to show that for every configuration $S(T)$ of vertices on ∂T and for every configuration $S(K)$ of nonsmooth points on ∂K, the setting above yields $K = T$. At first glance there are 36 cases (pairs of $(S(K), S(T))$ to consider, but we can reduce the number of cases, and some of them seem to be relatively easy to attack. With Figure 13 in mind, call p_2, p_4, p_6, p_8 **edge points**, p_1, p_3, p_7, p_9 **corner points**, and p_5 the **center point**.

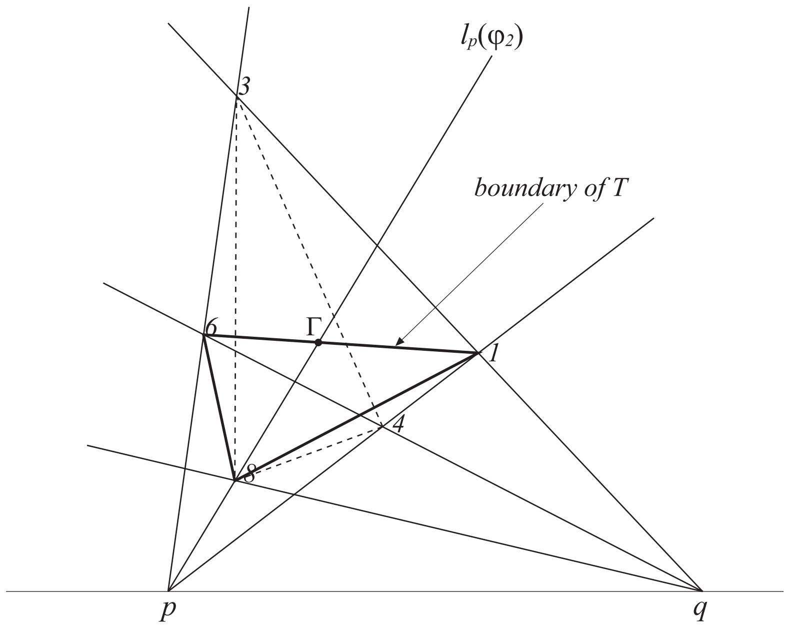

Figure 16: case II, no base points

Case I: (6 pairs of configurations) $S(K) = S(T)$. This is the most trivial case. If the nonsmooth points on ∂K are exactly the vertices of T, then $K = T$ by Theorem 3.1. For the remaining cases, assume $S(K) \neq S(T)$.

Case II: (8 pairs of configurations) $S(K)$ and $S(T)$ share an edge point. Take, for example, $S(K) = S_8$ and $S(T) = S_9$, as shown in Figure 16. Then K and T both have p_8 on their near side. By equality of x-rays, there is a point Γ on $l_p(\varphi_2)$ where ∂K and ∂T intersect. Now the triangle lemma comes in handy. By definition and convexity, $\text{ch}(\Gamma, p_6, p_8) \subset T$ and $\text{ch}(\Gamma, p_3, p_8) \subset K$. By the triangle lemma there exists a $\varphi_0 \in (\varphi_2, \varphi_3)$ such that $X_p(\text{ch}(\Gamma, p_6, p_8)) < X_p(\text{ch}(\Gamma, p_3, p_8))$ at φ_0. By construction of Γ, $X_p(\text{ch}(\Gamma, p_6, p_8)) = X_pT$ at φ_0; since $\text{ch}(\Gamma, p_3, p_8) \subset K$, we have $X_p(\text{ch}(\Gamma, p_3, p_8)) < X_pK$ at φ_0. Then by transitivity, $X_pT < X_pK$ at φ_0, a contradiction by equality of x-rays. If $S(K) = S_9$ and $S(T) = S_8$ (swapped), we get the same conclusion by using $\text{ch}(\Gamma, p_1, p_8)$ and $\text{ch}(\Gamma, p_4, p_8)$. Intuitively, this argument should extend to other possibilities where $S(K)$ and $S(T)$ share an edge point, since our choices of $S(K)$, $S(T)$ seem representative of this whole case; observe the symmetries in Figure 14.

Case III: (8 pairs) $S(K)$ and $S(T)$ share a corner point. Figure 17 is a representative of this case. Take e.g., $S(K) = S_9$ and $S(T) = S_7$. From this figure, we see that $X_pK(\varphi_2) > X_pT(\varphi_2)$, which would contradict equality of x-rays. Since we are dealing with the center point and two edge points opposite one corner point, it seems that this simple argument would hold in general. Suppose we exchange assignments so that $S(K) = S_7$ and $S(T) = S_9$. If K is going to share x-rays with T from p, K must contain a line segment of length $\text{len}(\overline{p_8\Gamma})$, where $\Gamma = \overline{p_1p_6} \cap l_p(\varphi_2)$. The figure - and the construction in general - would apparently imply that p_5 cannot be a far side point of K with respect to p; if this was the case, then the near side point of K with respect to p on $l_p(\varphi_2)$ would be outside the supporting cone of K from q. Thus p_5 would have to be a near side point of K. At least in

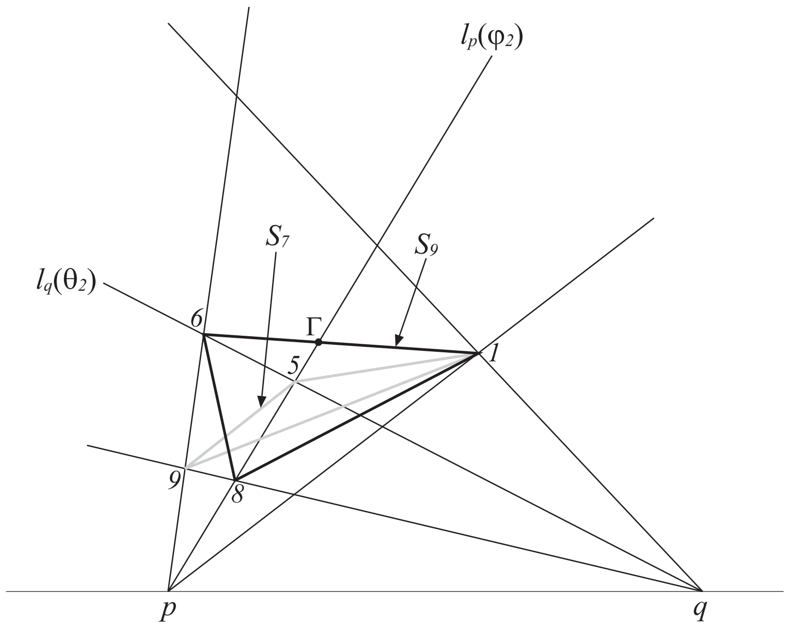

Figure 17: case III, no base points

Figure 17, this would violate convexity since p_5 is on the opposite side of $\overline{p_1 p_9}$ from p. It is unclear whether this would hold in general (either given our $S(K)$, $S(T)$ or using a different pair), since varying the angle of φ_2 changes the location of p_5 relative to p_1 and p_9.

Case IV: (2 pairs) $S(K)$ and $S(T)$ share a center point. Here, we must use the configurations S_6 and S_7, since these are the only three-point configurations with center points. Figure 18 then fully represents this case. It would appear that elements from cases II and III are present here. We would want to look at intersection points of boundaries with $l_p(\varphi_2)$ and $l_q(\theta_2)$. Using these we could compare lengths of line segments along the same ray, as in case III, or we could apply the triangle lemma, as in case II. Of course, just as in case III, we are dealing with the center point p_5, and so generalization appears difficult, since varying φ_2 or θ_2 changes the relative location between the center point and other points. We suspect the arguments suggested would work, so long as care was taken to address this issue.

Case V: (8 pairs) $S(K)$ and $S(T)$ share no points, but are not rotations of one another in Figure 14. A representative example would be to take $S(K) = S_7$ and $S(T) = S_{10}$; see Figure 19. Here we begin to see difficulty. There is some hope for an argument as - in this picture at least - the segment $\overline{p_2 \Gamma}$ is longer than both $\overline{p_2 p_5}$ and $\overline{p_5 p_8}$. This is critical to note since p_5 is a boundary point on one of our convex bodies. However, determining whether this holds in general looks difficult, and the trouble is of course compounded by the fact that we are once again dealing with a center point, as in cases III and IV. Additionally, the triangle lemma looks to be of no use here.

Case VI: (4 pairs) $S(K)$ and $S(T)$ share no points, but *are* rotations of one another in Figure 14. Take, e.g., S_8 and S_{10}, as in Figure 20. As in case V, there is difficulty comparing lengths of line segments on the same ray. Take, for example, $\overline{p_2 \Gamma}$ and $\overline{p_8 \Gamma'}$. In cases VI and V, we felt we had at least the possibility of constructing an argument with subcases based on

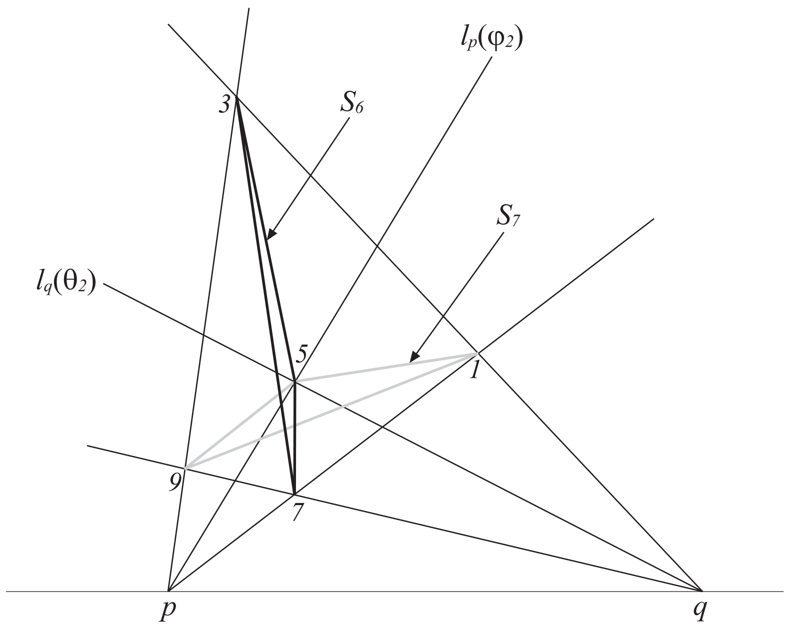

Figure 18: case IV, no base points

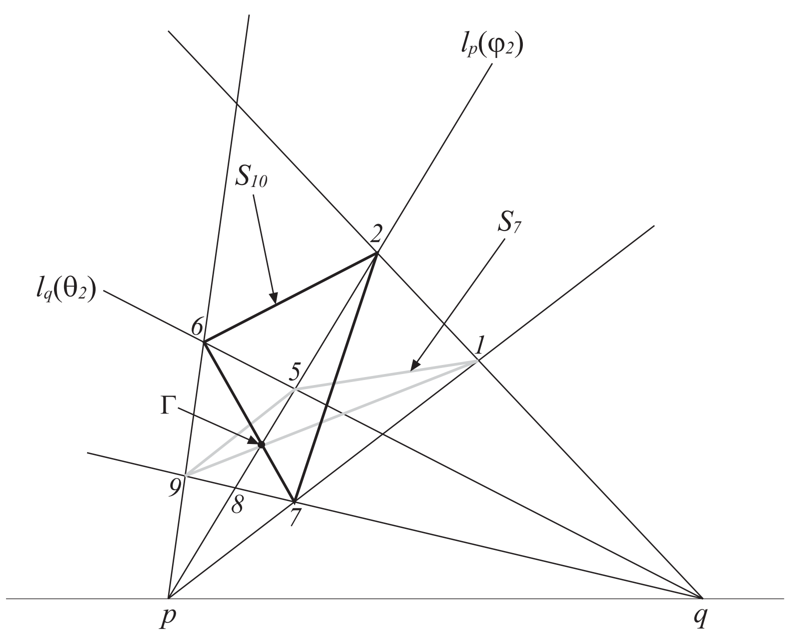

Figure 19: case V, no base points

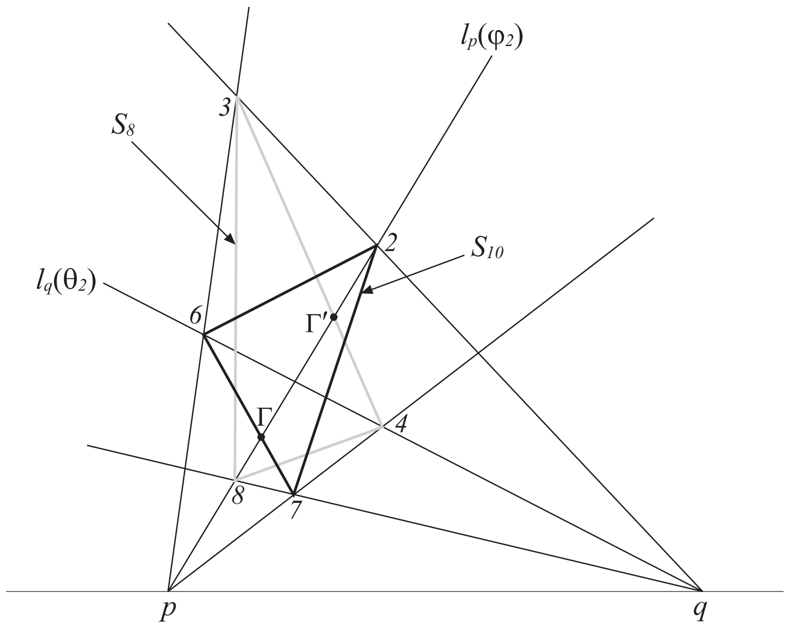

Figure 20: case VI, no base points

location of the center point. Here, we have no such luxury (however small), since the center point is not on the boundary of any convex body we consider. Again, the triangle lemma looks inapplicable. Per Figure 20, there are apparently no convenient relations between the configurations that would allow us to apply any of our results here.

These six cases account for all 36 possibilities, and we see our favored methods fail for (at least) the 12 possible pairs of configurations coming from the last two cases.

5 Conclusion

Through results relating directed x-rays with nonsmooth points on boundaries of convex bodies (both on and off supporting rays) we were able to verify the triangle from two points, as long as at least one was a base point. Additionally, in this regard, the triangle lemma proved critical in eliminating some special cases. Extending to verification without base points, we could at least say that any convex body sharing x-rays with the triangle must have exactly three nonsmooth points on its boundary. These nonsmooth points could only occur in six configurations with respect to the setting we outlined. Thus, it seemed that by looking at these configurations and their relation to the triangle, we could achieve verification without base points. This would have yielded the conclusion that all triangles can be determined by any two points in the plane. However, upon closer inspection, we saw that there were cases where verification apparently cannot be done through our methods. We believe that the cases I through IV in the previous section are tenable via our methods, while cases V and VI are not.

From here, we note two other tools that may be useful in completing these cases, or perhaps finding a counterexample. The first of these is the curvature operator $\mathcal{K}$ introduced in [1]. As we are dealing with triangles, we are automatically working with convex bodies whose boundaries have zero curvature (and hence whose near and far sides have zero $\mathcal{K}$) except at three points. Perhaps, under the assumption $\mathcal{K} = 0$, some results on the curvature operator would become simple enough to apply to our problem. The other tool in mind is the μ outer measure introduced in section 3.2. This outer measure was of critical importance in the results on verification and determination by Falconer [2], Gardner [4], and Volčič [9]. While we did use it here to prove the triangle lemma, we did not employ its power in our later results on verification. This is certainly worth considering in future work on this problem, so that we can approach the goal of determining a triangle.

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