Almost Tree-Like Plane Curves

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Abstract

First we investigate the number of extremal n-curves. We also investigate and classify a new class of curves which we call almost tree-like curves. Almost tree-like curves extend the set of curves for which Arnold's invariants may be quickly and easily computed. Finally, we demonstrate that for almost tree-like curves the genus is equal to the unknotting number of the corresponding knot.

1 Background

We begin by giving the basic definitions needed to understand almost tree-like curves and their alternating knots. Most of the information in this section can be found in [2], the reader is encouraged to make themselves familiar with the material presented there. For examples, and further details we refer to [2].

Definition 1.1 *A regular **plane curve** is an immersion of a circle into the plane having only regular double points.*

Our plane curves as above have no self-tangencies. We also study the basic invariants St, J^+, J^- as first introduced by Arnold, defined in [3]. For an example of moves which change the values of Arnold's invariants see Figure 1.

Definition 1.2 *The **index (or Whitney index)** of an immersion of an oriented curve into the oriented plane is the rotation number of the tangent vector.*

The Whitney index of a plane curve is essentially the rotation angle of the tangent vector with fixed base point divided by 2π.

Theorem 1.3 (Whitney) *There exists a homotopy between plane curves if and only if the plane curves have the same index.*

By Whitney's Theorem we choose one curve of each index to serve as the representatives under the equivalence relation of homotopy.

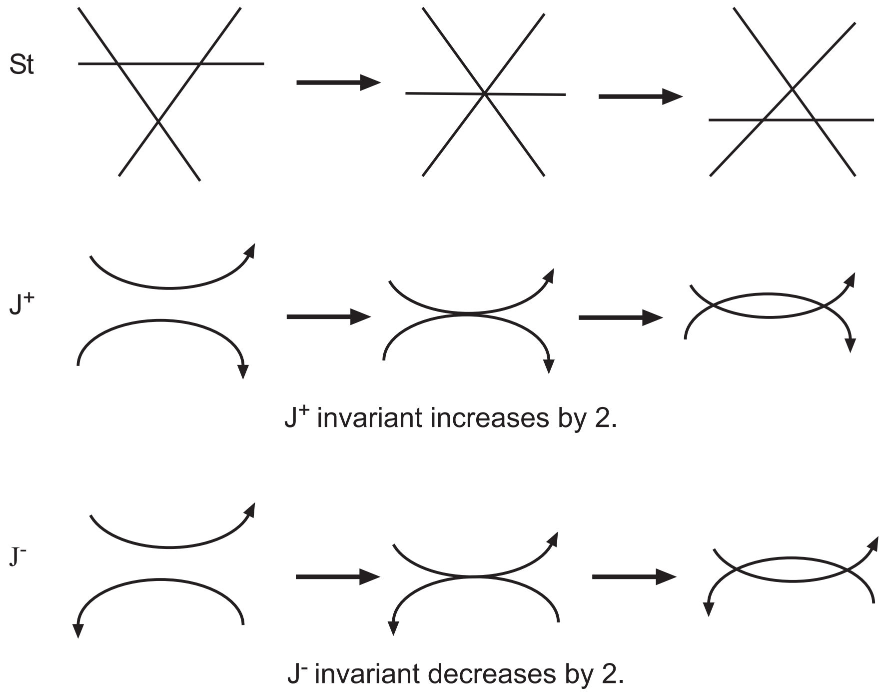
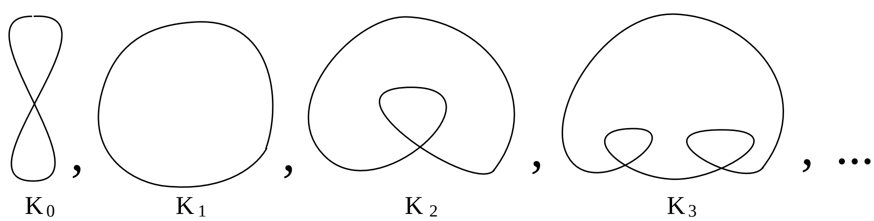


Figure 1: Moves which change Arnold's invariants.

Definition 1.4 *The following sequence consists of the standard curves K_i where i is the index of the curve.*



	K_0	K_1	K_2	K_3
St	*0*	*0*	*1*	*2*
J^+	*0*	*0*	*-2*	*-4*
J^-	*-1*	*0*	*-3*	*-6*

It follows from the local formula in [3] that for a plane curve with n double points the maximal value of the index is $n+1$. We call a plane curve **extremal** if its index has the value $n+1$, where n is the number of double points.

Definition 1.5 *A plane curve is **tree-like** if every double point divides it into two loops having no other common points.*

All extremal curves are tree-like, for details see [3]. Tree-like curves have a special tree structure that uniquely determines them up to diffeomorphims of the plane. See Definition 1.11 for details. We will refer to a tree-like curve with n ordinary double points is an n-curve.

Consider a tree with a vertex for each region of the plane that is bounded by a tree-like curve and an edge connecting vertices which correspond to loops that are connected at a double point. Each vertex will then have an **associated disc** which corresponds to the region in the plane that is bounded by part of the curve that corresponds to the vertex. The reader should note that the associated disc of one vertex may bound the associated disc of another vertex.

Definition 1.6 *The* ***root*** *r of a tree is a vertex such that all other vertices of the tree may be reached by traversing a connected sequence of edges which starts at r.*

Definition 1.7 *A* ***cycle*** *in a graph is a set of connected edges such that there exists a closed connected path which visits each vertex along the path exactly once.*

Definition 1.8 *For a tree T the* ***father*** *of a vertex $v_i \in T$ is the vertex closest to v_i on the path from the root to v_i. We often denote this father vertex as $f(v_i)$.*

Proposition 1.9 *Let c be the character function defined on the vertices of the tree of a tree-like curve with v_0 as the root, uniquely defined by the properties:*

(a) $c(v_0) = -1$

(b) $c(v_i) = 1$ *if the associated disc of v_i lies inside the associated disc of its father,*

(c) $c(v_i) = -1$ *if the associated disc of v_i lies outside the associated disc of its father.*

Proof. See [2]. ■

Definition 1.10 *Let T be the tree of an A-structure, then F is a subtree of T if each vertex on F corresponds to an associated disc that touches the unbounded region.*

Definition 1.11 *An* ***A-structure*** *of a tree-like curve consists of the following objects:*

(a) *T - the tree of the curve.*

(b) *F - the subtree, sometimes refered to as the* ***frontier.***

(c) *c - the character function.*

When choosing the root of an A-structure, we may choose any arbitrary vertex on the subtree F.

Definition 1.12 *A* ***planar structure*** *on a finite tree is an ordering of edges and vertices such that no two edges intersect.*

Definition 1.13 *An A-structure is called* ***planar*** *if the tree T is endowed with planar structure.*

Theorem 1.14 *There exists a bijection between the sets of planar A-structures having n edges and of classes of oriented tree-like curves having n double points.*

Proof. See [2]. ■

Definition 1.15 *Define a function t on the vertices of the A-structure as follows:*

(a) $t(v_0) = 0$,

(b) $t(v_i) = t(f(v_i)) + 1 \quad \text{if } c(v_i) = 1$,

(c) $t(v_i) = -t(f(v_i)) \quad \text{if } c(v_i) = -1$,

where v_0 is the root of the tree.

Theorem 1.16 *Let γ be a tree-like curve with n double points, then Arnold's invariants for γ are given by,*

$$St = \sum_{i=0}^{n} t(v_i), \qquad J^+ = -2St, \qquad J^- = -2St - n,$$

where the function t is as above.

Proof. See [2]. ■

Proposition 1.17 *For extremal curves, $t(v_i)$ is simply the distance from the root vertex v_0 to v_i.*

Proof. See [2]. ■

Theorem 1.18 *The maximal value of the invariant St on the set of all n-curve is equal to*

$$St_{max}(n) = \frac{n(n+1)}{2}$$

and it is attained at one unique n-curve for each n.

Proof. See [2]. ■

We give examples of curves with the maximal value of *St* and the A-structures for each, see Figure 2.

Definition 1.19 *The degree of a vertex on a graph is the number of edges touching that vertex.*

A knot is an embedding of the circle into $\mathbb{R}^3$. Given an oriented plane curve in $\mathbb{R}^3$, at each double point we cross over and under alternately. In [6] this algorithm is proven to give an alternating knot. This algorithm is well defined up to the first choice of crossing. We define the **unknotting number** of a knot to be the minimum number of crossings that must be switched so that the knot can be deformed into a circle by diffeomorphism of the ambient space. Furthermore, a knot is trivial if the unknotting number is equal to zero. The **Seifert splitting** at a double point is the splitting of the curve so that the direction is changed locally, as in Figure 3. If we perform the Seifert splitting at every double point, then we get a collection of cycles. We refer to this collection of cycles as **Seifert cycles.**

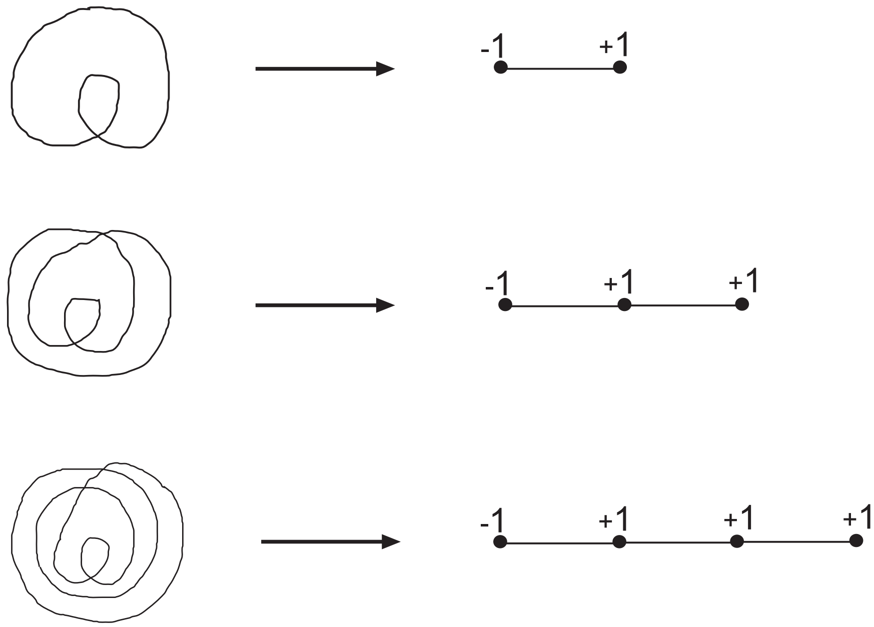


Figure 2: Examples of tree-like curves with maximal St.

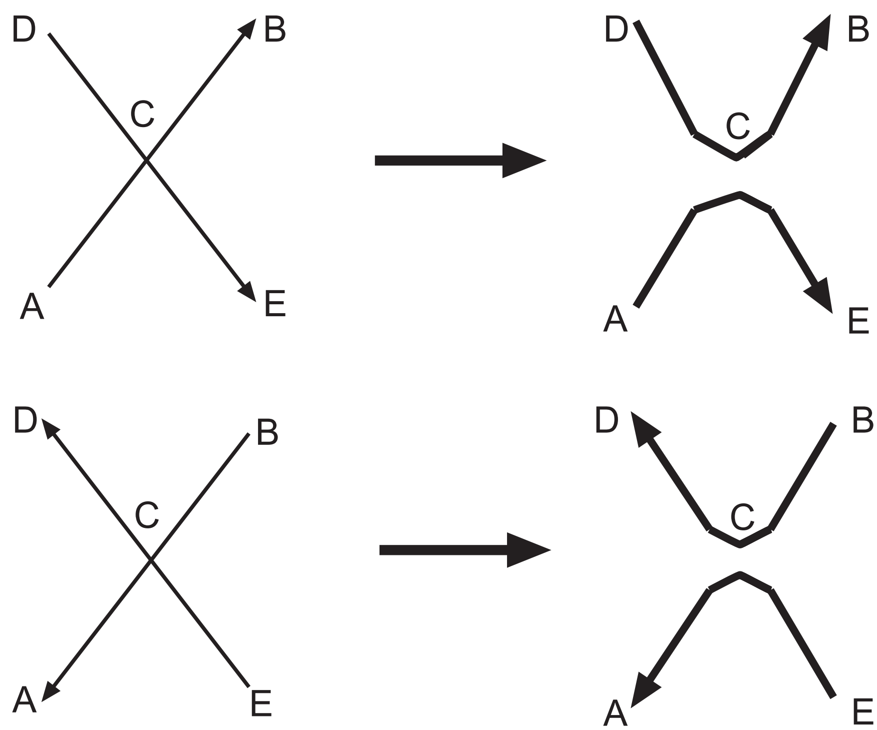


Figure 3: Example of Seifert splitting as in [6]

Definition 1.20 ***Seifert's algorithm:***

(a) *Perform the Seifert splitting at each double point in the plane curve.*

(b) *In the case of nested Seifert cycles, lift the inner disk according to the level of nesting.*

(c) *Glue the disks with twisted bands at the points which correspond to double points according to the direction of the crossing in the curve.*

Definition 1.21 *A* ***Seifert surface*** *of a knot is an orientable surface with the knot as its boundary.*

Remark 1.22 *The Seifert surface obtained from Seifert's algorithm is minimal for alternating knot, for details see [6]*

Lemma 1.23 *The Euler characteristic of a Seifert surface from Seifert's algorithm is equal to $1+s-n$, where s is the number of Seifert cycles and n the number of double points of a plane curve.*

Proof. See [6]. ■

The proofs of the following two fundamental results can be found in [6].

Theorem 1.24 *A plane curve is tree-like if and only if it has $n+1$ Seifert cycles, where n is the number of double points of the curve.*

Lemma 1.25 *A knot is trivial if and only if its genus is equal to 0.*

We refer to the genus of Seifert surface as the genus of the plane curve.

Theorem 1.26 *A plane curve is tree-like if and only if the corresponding alternating knot is trivial.*

Proof. Follows from the above. ■

2 Preliminaries

In this section we present a theorem for counting extremal curves.

Definition 2.1 *Define $St(n,k) = St_{max}(n) - k$.*

We note that for $k \geq 0$ and $n > k$, any tree-like curve with St=$St(n,k)$ is extremal. The reason is simple if we consider the A-structure of the curve. To satisfy the requirement that St=$St(n,k)$ none of the vertices except the root may have negative character. If all the vertices (except the root) have positive character then the orientation of each loop is the same and thus the maximum value of the index is obtained. For $k \geq 0$ and $n > k$, let $\varphi(n,k)$ denote the number of extremal n-curves with St=$St(n,k)$.

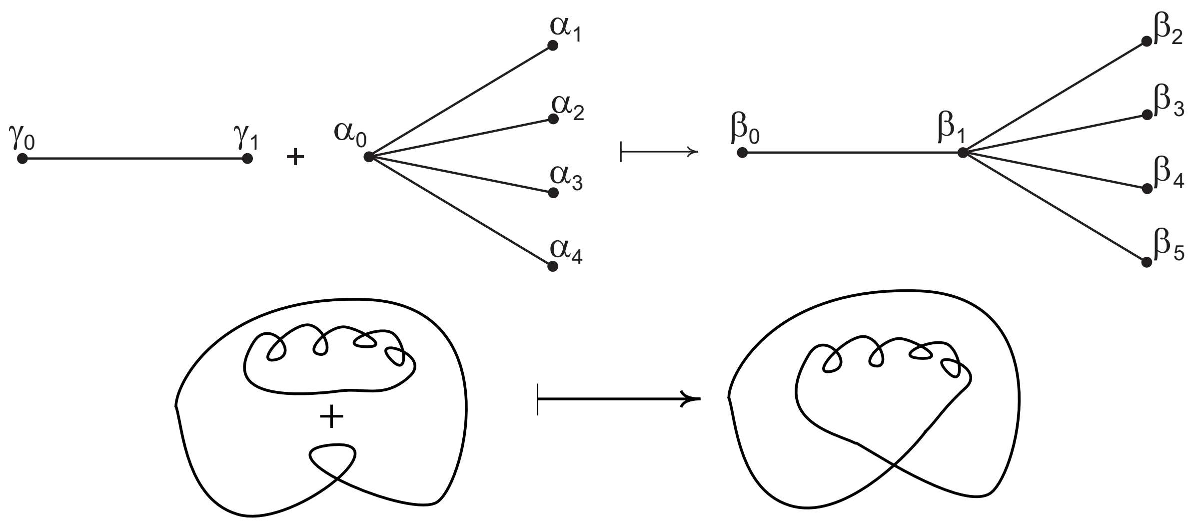
Definition 2.2 *Let γ be the A-structure of an n-curve with maximal St. Let α be the A-structure of an extremal k-curve. Choose an orientation of the curves corresponding to α and γ such that they may be connected. Define the* ***Extremal Sum*** *of α and γ to be the process which combines the root vertex of α and the terminal vertex of γ. Let the root vertex of γ be the root vertex of the resulting A-structure, β, and call this root β_0. Now β represents an extremal $(n+k)$-curve.*

The extremal sum as above is well defined in terms of the A-structures. By Theorem 1.14 this means that the extremal sum of two curves is well defined on the classes of tree-like curves. Let $n > k+1$ and let α be an extremal $k+1$-curve, and let γ be an extremal $(n-k-1)$-curve with $St = St_{max}(n-k-1)$. Let β be the extremal sum of α and γ, then

$$St(\beta) = St(\alpha) + St(\gamma) + (k+1)(n-k-1).$$

The equation above for $St(\beta)$ is derived directly from the A-structures.

Example 2.3 *Below we see the extremal sum of K_2 with K_5.*



	γ_0	γ_1	α_0	α_1	α_2	α_3	α_4
character	*-1*	*+1*	*-1*	*+1*	*+1*	*+1*	*+1*
distance	*0*	*1*	*0*	*1*	*1*	*1*	*1*

	β_0	β_1	β_2	β_3	β_4	β_5
character	*-1*	*+1*	*+1*	*+1*	*+1*	*+1*
distance	*0*	*1*	*2*	*2*	*2*	*2*

Theorem 2.4 *Let $k \geq 0$ then $\varphi(n,k) = \varphi(k+1,k)$ for all $n > k$.*

Proof. Fix $k \geq 0$ and let $n > k+1$. Let

$$K = \{\beta : \beta \text{ an A-structure}, St(\beta) = St(n,k)\}$$

and

$$\widehat{K} = \{\alpha : \alpha \text{ an A-structure}, St(\alpha) = St(k+1,k)\}.$$

We note that K and $\widehat{K}$ are sets of A-structures corresponding to extremal curves. Let γ be the A-structure of the $(n-k-1)$-curve with $St = St_{max}(n-k-1)$. By Theorem 1.18, γ is unique. Now, define

$$\begin{aligned} g : \widehat{K} &\to K \\ \alpha &\mapsto \gamma + \alpha \end{aligned}$$

where $+$ represents the extremal sum. To show that $\#(K) = \#(\widehat{K})$ we need to prove that g is a bijection.

First, we show that g is a surjection. That is, for each $\beta \in K$ there exists an $\alpha \in \widehat{K}$ such that $\beta = g(\alpha)$. We need to show that β has a path, p, from the root, v_0, to a vertex, v_i, such that $t(v_i) = n-k-1$. Moreover, we need to show that each vertex on the path p must have degree at most 2 except the root v_0 which has degree 1 and the last vertex which may have degree greater than 2. Instead, suppose there exists some vertex v_m such that $t(v_m) < n-k-1$ and $v_m \notin p$. The maximal St attainable by β is

$$\frac{(n-1)n}{2} + t(v_m) < \frac{n^2+n-2k}{2}.$$

But this contradicts,

$$St(\beta) = \frac{n^2+n-2k}{2}.$$

From the above contradiction we see both that the path p exists, and that no vertex in p except the terminal vertex has degree greater than 2.

To construct γ, let the tree of γ be p and assign positive character to each vertex except the root of p. We may reverse the extremal sum by doing the following: give the $n-k+1$-th vertex of β a character of -1 and remove the first $n-k-1$ vertices and first $n-k-1$ edges from β. Call this new A-structure α. To prove that $\alpha \in \widehat{K}$ it suffices to prove that $St(\alpha) = St(k+1,k)$. We know that,

$$St(\beta) = St(\alpha) + St(\gamma) + (k-1)(n-k-1).$$

Substitute for $St(\beta)$ and $St(\gamma)$, then solve for $St(\alpha)$. We get,

$$\begin{aligned} St(\alpha) &= \frac{k^2+k+2}{2} \\ &= St(k+1,k). \end{aligned}$$

We conclude that $\alpha \in \widehat{K}$ and thus $\beta = g(\alpha)$ which implies g is a surjection.

To show that g is an injection, let $\alpha_1, \alpha_2 \in \widehat{K}$ and $\beta \in K$ such that $g(\alpha_1) = \beta$ and $g(\alpha_2) = \beta$. This tells us,

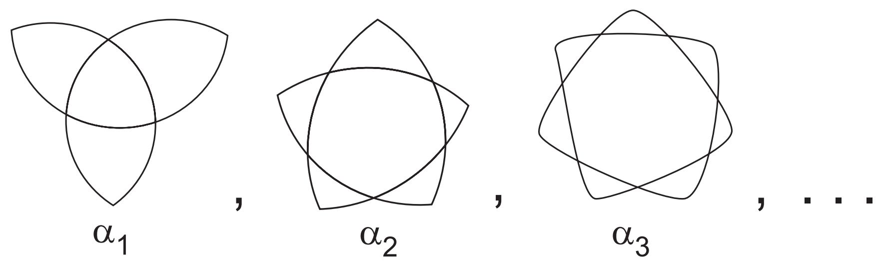
$$\gamma + \alpha_1 = \gamma + \alpha_2,$$

where $+$ is the extremal sum. We already know that the extremal sum is well defined on A-structures and the corresponding curves, so we conclude that $\alpha_1 = \alpha_2$. Thus, g is an injection. We may now conclude that g is a bijection. Since there exists a bijection between K and $\widehat{K}$ we know that $\#(K) = \#(\widehat{K})$. ■

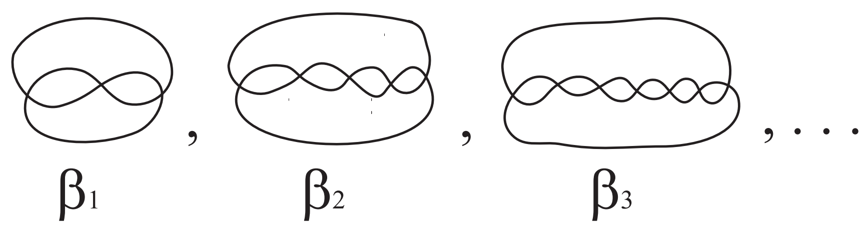
3 Almost Tree-Like Curves

In this section we develop a new classification of curves which is an extension to the tree-like curves described in [2]. We also give a combinatorial structure known as a DP-structure which allows for easy enumeration of almost tree-like curves and an easy computation of Arnold's invariants.

Definition 3.1 *Let* α_i *be the sequence of the following non-tree-like curves.*



Definition 3.2 *Let* β_i *be the sequence of the following non-tree-like curves.*



We note that the Whitney index is constant for each sequence. Furthermore, the unknotting number of the alternating knot of α_i and β_i is equal to i.

Definition 3.3 *The curve* γ *is* ***Almost Tree-Like*** *if one of the following is true for each double point* d_k*:*

(a) γ *may be disconnected into two disjoint curves at* d_k*.*

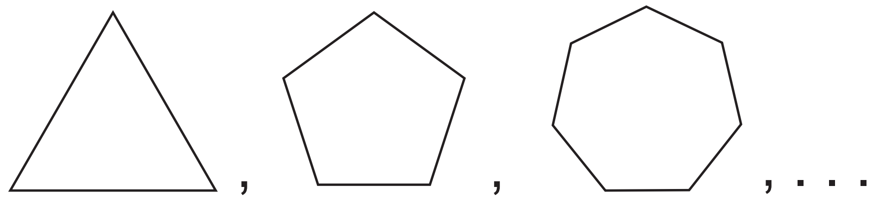
(b) d_k *lies on one of* α_i *(or* β_i*) and* α_i *(or* β_i*) is the whole curve.*

(c) d_k *is one of exactly* $(2i+2)$ *double points which lies on a part of the curve such that the curve may be disconnected into two disjoint curves at a unique double point* d_j*. Furthermore, one of the resulting curves is* α_i *(or* β_i*) where* d_j *is on the exterior boundary of* α_i *(or* β_i*) and* d_k *is a double point on* α_i *(or* β_i*).*

Definition 3.4 *The almost tree-like curve* γ *is called* ***Simple Almost Tree-Like*** *if* γ *has exactly one double point which may be disconnected into two curves with one being* α_i *(or* β_i*) and the other being tree-like.*

Definition 3.5 *A* ***DP-graph*** *is a connected planar graph such that:*

(a) *The only cycles are of the following form which all have an odd number of vertices,*



We refer to a cycle with k vertices as a k-cycle.

(b) *Each cycle connects to the rest of the DP-graph at exactly one vertex of the cycle.*

Definition 3.6 *A **Negative Cycle** is a k-cycle as in Definition 3.5 in which each vertex is assigned a value of* -1*. Similarly a **Positive Cycle** is a k-cycle in which each vertex is assigned a value of* $+1$.

Remark 3.7 *Let* $k > 2$ *be odd and let* $i = (k-1)/2$*. A positive k-cycle in a DP-graph corresponds to* β_i*. Similarly a negative k-cycle corresponds to* α_i*.*

Definition 3.8 *The **Maximal Subtree of a DP-Graph** is the largest connected subgraph which contains only one vertex from each cycle.*

Remark 3.9 *The maximal subtree of a DP-graph is a connected planar acyclic graph or simply a tree.*

Definition 3.10 *The **Sub A-structure** of a DP-graph G is the maximal subtree of G endowed with an A-structure.*

Definition 3.11 *A **DP-structure** consists of the following:*

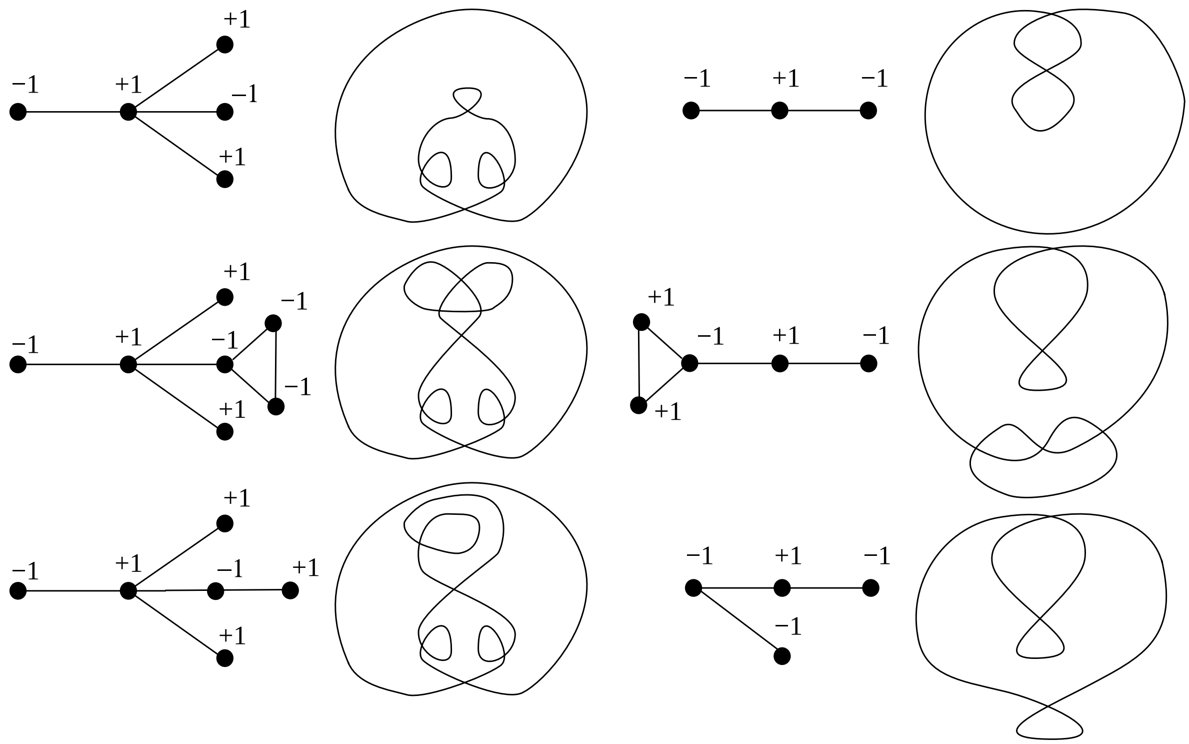
(a) *A non-empty DP-graph, G, where each cycle is either positive or negative.*

(b) *A character function c defined for each vertex in the following manner:*

  (i) If the vertex lies on the sub A-structure then use the character of the sub A-structure.

  (ii) If the vertex does not lie on the sub A-structure then the character is $+1$ *if the vertex is on a positive cycle and* -1 *if the vertex is on a negative cycle.*

Example 3.12 *Examples of tree-like curves and simple almost tree-like curves. The last curve in each column is the result of deforming the simple almost tree-like curve to a tree-like curve as in Lemma 3.14 and Lemma 3.15.*



Theorem 3.13 *There is a bijection between the equivalence classes of oriented almost tree-like curves and the set of distinct DP-structures.*

Proof. We know from Theorem 1.14 that each A-structure corresponds to one equivalence class of curves. It suffices to show that when we extend an A-structure α corresponding to the curve $\widehat{\alpha}$ to get a DP-structure γ corresponding to the curve $\widehat{\gamma}$ that there is only one way to modify $\widehat{\alpha}$ so that it will correspond to the new curve $\widehat{\gamma}$. Due to the symmetry of each of the α_i and β_i and the restriction that they connect to the rest of the curve by at most one double point we see that there is only one equivalence class of curves corresponding to each DP-structure. ■

Lemma 3.14 *Let γ be the DP-structure of a simple almost tree-like curve, $\widehat{\gamma}$, having a negative k-cycle, and let $i = (k-1)/2$. Then $\widehat{\gamma}$ may be deformed to a tree-like curve η in which α_i has been replaced by K_2. Furthermore,*

$$J^+(\widehat{\gamma}) = J^+(\eta) + (k-1).$$

Proof. Same context as above. First we show that it is possible to deform K_2 to α_i for any $i > 0$. We proceed inductively on i. We easily see this is true when $i = 1$. To go from α_i to α_{i+1} we simply deform part of the interior portion of α_i over the exterior boundary part of α_i. This move creates two double points and increases the value of J^+ by 2. This process can be reversed allowing us to always return to K_2. If we need to deform from α_i to K_2 we must perform i of these J^+ moves. Hence, the value of J^+ increases by $2i = k-1$.

We may restrict our attention to α_i where it appears in $\widehat{\gamma}$. As above it is possible to deform α_i to K_2 locally. Note that when we deform the curve in this manner then the J^+ invariant of $\widehat{\gamma}$ only changes on the part of the curve corresponding to α_i. Thus,

$$J^+(\widehat{\gamma}) = J^+(\eta) + (k-1).$$

This gives the desired result. ■

Lemma 3.15 *Let γ be the DP-structure of a simple almost tree-like curve, $\widehat{\gamma}$, having a positive k-cycle, and let $i = (k-1)/2$. Then $\widehat{\gamma}$ may be deformed to a tree-like curve η in which β_i has been replaced by K_0. Furthermore,*

$$J^+(\widehat{\gamma}) = J^+(\eta) + (k-1).$$

Proof. Similarly as Lemma 3.14. ■

Lemma 3.16 *Let η be the DP-structure of an almost tree-like curve, $\widehat{\eta}$, having cycles $\gamma_1, \dots, \gamma_p$ then $\widehat{\eta}$ may be deformed to a tree-like curve, δ, in which each portion of $\widehat{\eta}$ corresponding to γ_i is replaced by K_0 or K_2 as appropriate. Furthermore,*

$$J^+(\widehat{\eta}) = J^+(\delta) + q - p,$$

where q is the total number of vertices which are on the cycles.

Proof. Evident from Lemmas 3.14 and 3.15. ■

Definition 3.17 *We now define cf the contribution function. Let A be the sub A-structure of a DP-graph G. For each vertex v_i,*

(a) *if v_i is part of the maximal subtree of G and v_i does not lie on a cycle then $cf(v_i) = t(v_i)$, where t is defined by the sub A-structure.*

(b) *if $v_i \in A$ and v_i lies on a negative cycle then*

$$cf(v_i) = 2t(v_i) + 1.$$

(c) *In all other cases,*

$$cf(v_i) = 0.$$

Theorem 3.18 *Let γ be an almost tree-like curve with m double points and $v_0, \dots, v_n$ to be the vertices of its DP-structure, then*

$$St(\gamma) = \sum_{i=0}^{n} cf(v_i).$$

Moreover, let p be the number of cycles and q be the number of vertices which lie on a cycle then

$$J^+ = -2St + q - p,$$

and

$$J^- = J^+ - m.$$

Proof. Same context as above. For the vertices that are not on the sub A-structure we see that $cf(v_i) = 0$. By definition of cf, $cf(v_i) = t(v_i)$ for all v_i which lie on the sub A-structure but do not lie on a cycle. For each v_i which lies on the sub A-structure and on a cycle there are two cases to consider. The cases are:

(a) The vertex v_i lies on a negative k-cycle. Let $\ell = (k-1)/2$, by Lemma 3.14 we may deform γ locally at the part of the curve which corresponds to the negative k-cycle to a curve which has K_2 instead of α_ℓ. Let δ be the DP-structure of the deformed curve with ω_1 in δ corresponding to v_i in the DP-structure of γ. Also, let ω_2 be the child of ω_1. Since the deformation of γ as above does not change the St invariant, we need only check that for the part of the DP-structure which has been modified that the invariant St agrees. See Figure 4.

$$\begin{aligned} t(\omega_1) + t(\omega_2) &= t(\omega_1) + t(f(\omega_2)) \\ &= t(\omega_1) + t(\omega_1) + 1 \\ &= 2t(v_i) + 1 \\ &= cf(v_i). \end{aligned}$$

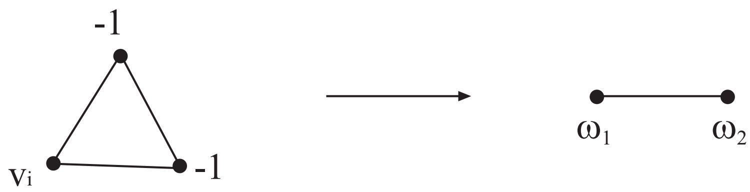
(b) The vertex v_i lies on a positive k-cycle. Let $\ell = (k-1)/2$, by Lemma 3.15 we may deform γ locally at the part of the curve which corresponds to the positive k-cycle to a curve which has K_0 instead of β_ℓ. Let δ be the DP-structure of the deformed curve with ω_1 in δ corresponding to v_i in the DP-structure of γ. Also, let ω_2 be the child of ω_1. Since the deformation of γ as above does not change the St invariant, we need only check that for the part of the DP-structure which has been modified that the invariant St agrees. See Figure 5.

$$\begin{aligned} t(\omega_1) + t(\omega_2) &= t(\omega_1) - t(f(\omega_2)) \\ &= t(v_i) - t(v_i) \\ &= 0 \\ &= cf(v_i). \end{aligned}$$

Deform γ as above to a tree-like curve, η. Label the vertices of the A-structure corresponding to η as $w_0, \ldots, w_m$. Now we see

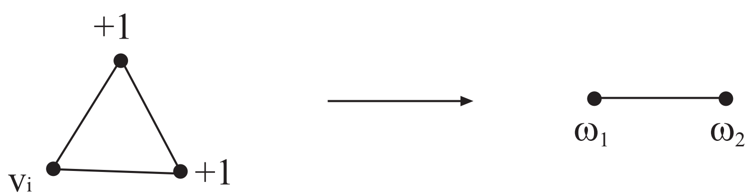
$$\begin{aligned} St(\gamma) &= \sum^{n} cf(v_i) \\ &= \sum_{i=0}^{m} t(w_i) \\ &= St(\eta). \end{aligned}$$

The value of the invariant J^+ follows from Lemma 3.16. From [3] we see that in general for a curve with m double points $J^- = J^+ - m$. ■



vertex	c	t
v_i	± 1	$t(v_i)$
ω_1	$c(v_i)$	$t(v_i)$
ω_2	$+1$	$t(v_i)+1$

Figure 4: Computing St locally for K_2.

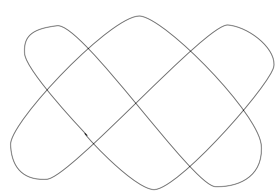


vertex	c	t
v_i	± 1	$t(v_i)$
ω_1	$c(v_i)$	$t(v_i)$
ω_2	-1	$-t(v_i)$

Figure 5: Computing St locally or K_0.

4 Knot Theoretic Connections

One may be tempted to make the following conjecture: "The genus of the minimal Seifert surface of a plane curve is equal to the unknotting number of the alternating knot." This conjecture is false, consider the following plane curve:



As shown in [6] the genus of the minimal Seifert surface of a plane curve is $(1-s+n)/2$, where s is the number of Seifert cycles and n is the number of double points. In this case the genus is 1. The unknotting number of the alternating knot for this curve is 2.

Definition 4.1 *Given a curve γ if it is possible to disconnect γ at a double point into two curves γ_1 and γ_2, then we say γ_1 and γ_2 are* ***connected at a double point*** *in γ.*

Lemma 4.2 *Let γ_1 and γ_2 be plane curves and let g_1 and g_2 be the genera of γ_1 and γ_2 respectively. If γ is a curve such γ_1 and γ_2 are connected at a double point in γ then the genus of γ is equal to g_1+g_2.*

Proof. Let s_1 and n_1 be the number of Seifert cycles and double points respectively of γ_1. Similarly define s_2 and n_2 for γ_2. We see that when we connect γ_1 to γ_2 at a double point in γ that the number of Seifert cycles in γ is just s_1+s_2. The number of double points is n_1+n_2+1. The genus of γ_1 and γ_2 is $g_1=(1-s_1+n_1)/2$ and $g_2=(1-s_2+n_2)/2$ respectively. The genus of γ is,

$$\begin{aligned}\frac{1-s_1-s_2+n_1+n_2+1}{2} &= \frac{1-s_1+n_1}{2}+\frac{1-s_2+n_2}{2}\\ &= g_1+g_2.\end{aligned}$$

Thus the genus is additive for curves that are connected at a double point. ■

The number of Seifert cycles for any α_i (or β_i) is two, this can be easily verified. The number of double points increases by two as i increases, so the genus of α_i is equal to i. The unknotting number of the alternating knot corresponding to α_i (or β_i) is i, consider the following argument. Fix $i>1$ then find the alternating knot corresponding to α_i (or β_i). If any crossing in the knot is switched then the resulting knot may be deformed and projected so that the projection is α_{i-1} (or β_{i-1}); but no simpler projection is possible. Here we mean simpler in the sense of the unknotting number. In the case of α_1 and β_1 it is easy to show that the unknotting number is 1.

Lemma 4.3 *If γ_1 and γ_2 are almost tree-like curves connected at a double point in γ then the unknotting number of the alternating knot corresponding to γ is the sum of the unknotting numbers of the alternating knots corresponding to γ_1 and γ_2.*

Proof. Since it has been shown that alternating knots corresponding to tree-like curves have an unknotting number equal to zero, it suffices to prove that if α_i and α_j, β_i and β_j, or α_i and β_j are connected at a double point that the unknotting number is additive. Since the genus is additive and the result of connecting the these curves will have the minimal genus for the corresponding knot. Since the genus will be minimal this implies that the knot is no simpler than sum of the knots components. Thus, in this setting the unknotting number is additive. ■

Theorem 4.4 *Let γ be an almost tree-like curve, then the genus of γ is equal to the unknotting number of the alternating knot corresponding to γ.*

Proof. Follows from above. ■

5 Conclusion

We studied tree-like curves because we wanted to generalize the results of tree-like curves. In our project, we have shown that given an almost tree-like curve we can always find a corresponding DP-structure. Furthermore, Theorem 3.18 gives a formula to calculate Arnold's invariants.

We think there is a lot of work still to be done with DP-structures. It may be possible to extend the definition of almost tree-like curves to include the case where α_i (or β_i) has more than one tree-like curve attached at a double point on the exterior boundary. There are possible other sequences of curves that could be included in the definition of a generalization of almost tree-like curves.

Some of the necessary properties would be that the sequence is symmetric and may be deformed to a curve which is tree-like and symmetric. We also feel that there are several insightful corollaries of Theorem 3.18 to be explored.

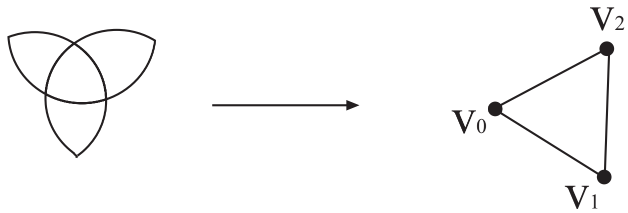
Given a set, K, of non-tree like curves with n double points corresponding to a given Gauss diagram and index, then we can split some non-tree like curves in K at a single double point to get a new set, $\widehat{K}$, of $(n-1)$-curves. When $n \leq 4$ the trees that correspond to the tree-like curves in K consists of all the trees that generate all tree-like $(n-1)$-curves. However, this does not hold for $n \geq 5$.

We would like to make the following conjecture. The genus of a plane curve is always less than or equal to the unknotting number of the alternating knot corresponding to the plane curve.

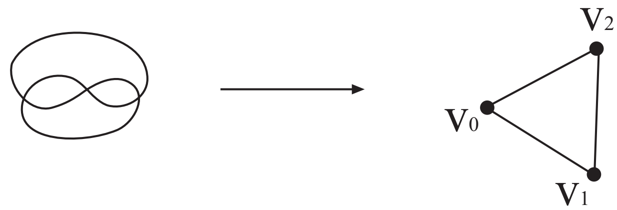
6 Appendix: DP-structures of almost tree-like curves up to 5 double points

We give a listing for reference of the DP-structures for the almost tree-like curves up to 5 double points, not including tree-like curves.

$n = 3$

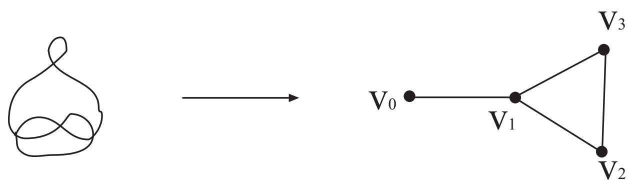


v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	1
v_1	-1		0
v_2	-1		0

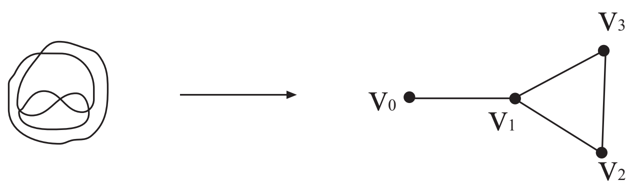


v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1		0
v_2	+1		0

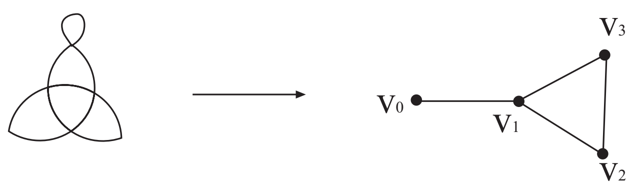
$n = 4$



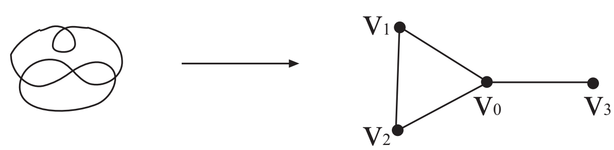
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1	0	0
v_2	+1		0
v_3	+1		0



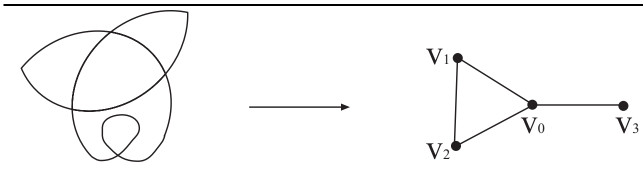
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1	1	0
v_2	+1		0
v_3	+1		0



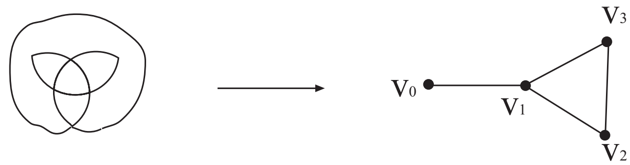
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1	0	1
v_2	-1		0
v_3	-1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1		0
v_2	+1		0
v_3	+1	1	1

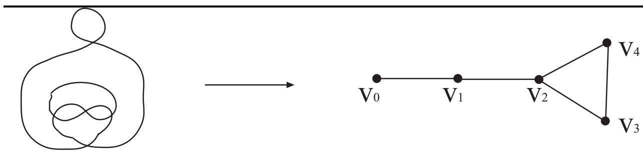


v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1		0
v_2	-1		0
v_3	+1	1	1

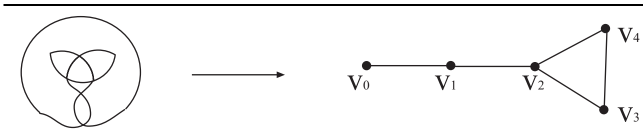


v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1	1	3
v_2	-1		0
v_3	-1		0

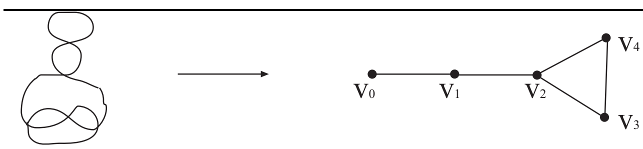
$n = 5$



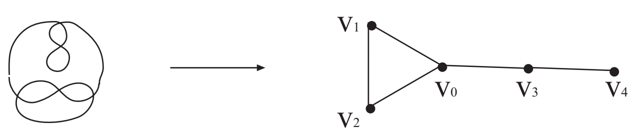
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1	0	0
v_2	+1	1	0
v_3	+1		0
v_4	+1		0



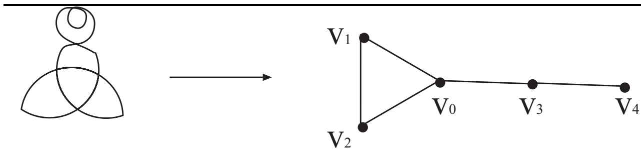
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1	1	1
v_2	-1	-1	-1
v_3	-1		0
v_4	-1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1	0	0
v_2	-1	0	0
v_3	+1		0
v_4	+1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1		0
v_2	+1		0
v_3	+1	1	1
v_4	-1	-1	-1



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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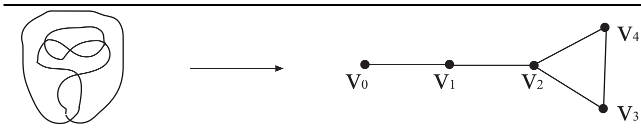
v_1	-1		0
v_2	-1		0
v_3	-1	0	0
v_4	+1	1	1



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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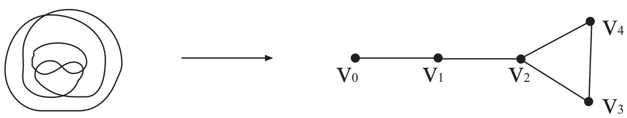
v_1	+1	1	1
v_2	-1	-1	0
v_3	+1		0
v_4	+1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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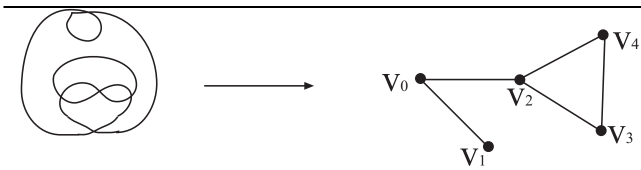
v_1	+1	1	1
v_2	+1	2	0
v_3	+1		0
v_4	+1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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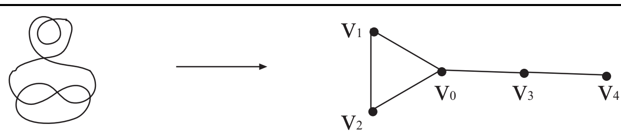
v_1	+1	1	1
v_2	+1	1	0
v_3	+1		0
v_4	+1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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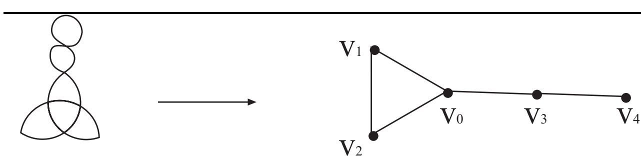
v_1	+1		0
v_2	+1		0
v_3	-1	0	0
v_4	+1	1	1



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
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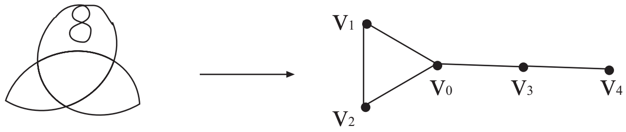
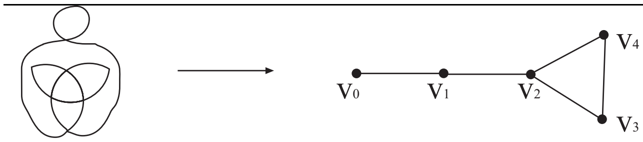
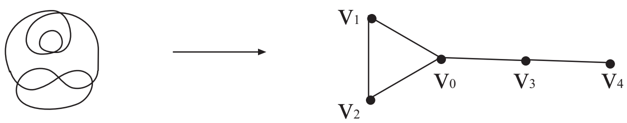
v_1	-1		0
v_2	-1		0
v_3	-1	0	0
v_4	-1	0	0



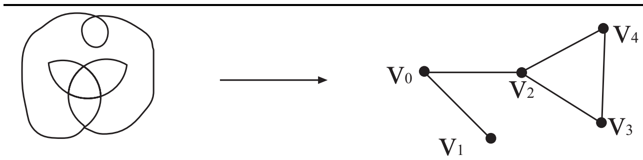
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	1
v_1	-1		0
v_2	-1		0
v_3	+1	+1	1
v_4	-1	-1	-1



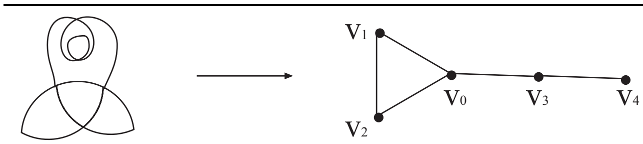
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	-1	0	0
v_2	+1	1	3
v_3	-1		0
v_4	-1		0



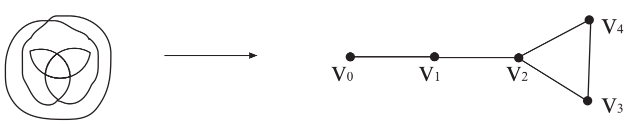
v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1		0
v_2	+1		0
v_3	+1	1	1
v_4	+1	2	2



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1	1	1
v_2	+1	1	3
v_3	-1		0
v_4	-1		0



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	1
v_1	-1		0
v_2	+1		0
v_3	+1	1	1
v_4	+1	2	2



v_i	$c(v_i)$	$t(v_i)$	$cf(v_i)$
v_0	-1	0	0
v_1	+1	1	1
v_2	+1	2	3
v_3	-1		0
v_4	-1		0

References

[1] C. Adams. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* W. H. Freeman and Company, New York 1994. pp. 50-55.

[2] F. Aicardi. *Tree-like Curves.* Advances in Soviet Mathematics. Volume 21, 1994. pp. 1-31.

[3] V. I. Arnold. *Plane Curves, Their Invariants, Perentroikas and Classifications.* Advances in Soviet Mathematics. Volume 21, 1994. pp. 33-91.

[4] V. I. Arnold. *Remarks on the Enumeration of Plane Curves.* American Mathematical Society Translation. (3) Vol. 173, 1996. pp. 17-32.

[5] Louis H. Kauffman. *New Invariants in the Theory of Knots.* The American Mathematical Monthly. Volume 95, Issue 3. March 1988. pp. 195-242.

[6] Chenghui Luo. *Proof of Arnold's Conjectures About Plane Curves.* Preprint PhD Thesis. Brown University 1997 pp. 1-24.

[7] Michael Polyak. *New Whitney-type Formulas for Plane Curves.* American Mathematical Society Translation. (2) Vol. 190, 1999. pp. 103-111.

[8] Boris Shapiro. *Tree-Like Curves and Their Number of Inflection Points.* American Mathematical Society Translation. (2) Vol. 190, 1999. pp. 113-129.

[9] A. N. Shumakovich. *Explicit Formulas for the Strangeness of a Plane Curve.* St. Petersburg Math. J. Vol. 7 No. 3, 1996. pp. 445-472.

[10] Oleg Viro. *Generic Immersions of the Circle to Surfaces and the Complex Topology of Real Algebraic Curves.* American Mathematical Society Translation. (2) Vol. 173, 1996. pp. 231-251.