Reconstructing Convex Polyhedrons in $\mathbb{R}^3$

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Abstract

This paper concerns the theory and methods of the reconstruction of convex polyhedrons from a single directed x-ray. A uniqueness theorem for a wedge using specific configurations of seven or eight rays is proven. An example is given of a tetrahedron which was reconstructed successfully. A uniqueness theorem for a wedge using six rays remains to be proven.

1 Introduction

In this paper, we address the problem of reconstructing convex polyhedrons in $\mathbb{R}^3$ from a single directed X-ray from a source outside of the polyhedron. Much of the inspiration for this paper has been a product of Lam and Solmon [2], who developed methods for reconstructing convex polygons in the plane from one directed X-ray. We wanted do things analogously in $\mathbb{R}^3$. The area of mathematics in which we are working is called geometric tomography. A comprehensive reference to the subject is Gardner [1].

The first section is used to establish notation and to give a few preliminary results. We introduce a ternary form that is used throughout. We also define a wedge and show that four rays, with some conditions, are sufficient to distinguish a parallel wedge from other convex bodies. The normal vector to the parallel planes can be determined, but not the precise location of the planes in the wedge.

After section one, we focus only on convex polyhedrons. In the second section, we show that almost all of the rays along which nonsmooth points of the edges of a polyhedron can be detected from a single directed X-ray and derive geometric information about the nonsmooth points from the left and right partial derivative with respect to θ of the directed X-ray function through such a point. Next, we prove that a nonparallel wedge is uniquely determined with specific configurations of seven or eight rays. This comes from taking four rays each along two different planes and applying the results of Lam and Solmon [2]. In section three, we give an algebraic method for reconstructing a nonparallel wedge from the eight rays which is based on the reconstruction formulas in [2]. Section five gives examples of polyhedrons to be reconstructed. Finally, in section six, we show attempts that were made to uniquely determine a nonparallel wedge using only six rays. While the work is incomplete at this time, we still feel that future work on the subject might be able to successfully determine a nonparallel wedge using only six rays.

1.1 Preliminaries

We use the spherical coordinates (ρ,θ,ϕ) with the following conversion to rectangular coordinates (x,y,z):

$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$, and $\rho = \sqrt{x^2+y^2+z^2}$

Throughout this paper, K refers to a convex polyhedron in the "upper half space" $\{(x,y,z) : z > 0\}$, or, equivalently, $\{(\rho,\theta,\phi) : \rho > 0,\ 0 \le \phi < \pi/2\}$. We use $\vec{\omega}$ to refer to a direction, that is, a point on the unit sphere. A direction is uniquely determined by θ and ϕ, and we sometimes refer to the direction of (θ,ϕ). The relationship is shown here: $\vec{\omega} = [\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi] = \dfrac{[x,y,z]}{\sqrt{x^2+y^2+z^2}}$.

By an edge of a polyhedron, we mean a line segment intersection of two faces of the polyhedron. A vertex is a point intersection of three or more faces of the polyhedron.

Definition 1.1 *The directed x-ray function, $X(\vec{\omega}) = X(\theta,\phi)$, is the function that for each $\vec{\omega}$ gives the length of the intersection of the ray from the origin in the direction $\vec{\omega}$ with the polyhedron K.*

Since the boundary of K consists of plane sections, $X(\vec{\omega})$ is the distance between two planes along a ray from the origin in the direction $\vec{\omega}$.

Definition 1.2 *By a supporting ray of a convex polyhedron, we mean a ray from the origin that intersects the polyhedron only on the boundary.*

The intersection of a supporting ray with a convex polyhedron is either a point on the intersection of two or more faces of the polyhedron or a closed line segment on one or two faces of the polyhedron. Let S be the set of all supporting rays of a polyhedron K. $S \cap K$ is a surface that divides the boundary of K into two sets, N and F, where $N = \{(\rho,\theta,\phi) : \rho = r(\theta,\phi)\}$ and $F = \{(\rho,\theta,\phi) : \rho = R(\theta,\phi)\}$. Any ray from the origin that intersects the interior of the polyhedron will intersect the boundary of the polyhedron at exactly two points. The point closest to the origin is on N, and the other point is on F. We call $r(\theta,\phi)$ the *near boundary function*, $R(\theta,\phi)$ the *far boundary function*, N the near boundary, and F the far boundary. A supporting ray from the origin also intersects both F and N. For supporting rays that intersect the polyhedron at one point, the point of intersection is on F and N. For supporting rays that intersect the polyhedron in a closed line segment, the point of the line segment closest to the origin is on N, and the point of the line segment farthest from the origin is on F. The other points of the line segment are not on F or N. $R(\theta,\phi)$ is concave toward the origin, and $r(\theta,\phi)$ is concave away from the origin.

From the explanation above, the far and near boundaries, F and N, each intersect any ray from the origin in at most one point. This implies that each face of the polyhedron in F or N must be a section of a plane that does not intersect the origin.

Lemma 1.3 *The value of the x-ray function in the direction of the ray defined by all positive scalar multiples of the vector* $[\alpha,\beta,\gamma]$, $\gamma > 0$, *where the plane on the near boundary is* $a_n x + b_n y + c_n z = d_n$ *and the plane on the far boundary is* $a_f x + b_f y + c_f z = d_f$ *is*

$$X = \left[\frac{d_f}{a_f\alpha + b_f\beta + c_f\gamma} - \frac{d_n}{a_n\alpha + b_n\beta + c_n\gamma}\right]\sqrt{\alpha^2+\beta^2+\gamma^2}.$$

Proof. The ray $\{[x,y,z] = t[\alpha,\beta,\gamma] : \gamma, t > 0]\}$ intersects the far boundary plane at $[t_f\alpha, t_f\beta, t_f\gamma]$, and it intersects the near boundary plane at $[t_n\alpha, t_n\beta, t_n\gamma]$. That is, $a_f t_f \alpha + b_f t_f \beta + c_f t_f \gamma = d_f$ and $a_n t_n \alpha + b_n t_n \beta + c_n t_n \gamma = d_n$. Solving for t_f and t_n, we get

$$t_f = \frac{d_f}{a_f\alpha + b_f\beta + c_f\gamma} \text{ and } t_n = \frac{d_n}{a_n\alpha + b_n\beta + c_n\gamma}.$$

The distance between the two points is the value of the x-ray function,

$$X = \sqrt{(t_f - t_n)^2\alpha^2 + (t_f - t_n)^2\beta^2 + (t_f - t_n)^2\gamma^2} = (t_f - t_n)\sqrt{\alpha^2+\beta^2+\gamma^2}$$

$$= \left[\frac{d_f}{a_f\alpha + b_f\beta + c_f\gamma} - \frac{d_n}{a_n\alpha + b_n\beta + c_n\gamma}\right]\sqrt{\alpha^2+\beta^2+\gamma^2}. \blacksquare$$

A general form of a plane that does not intersect the origin is $\{(x,y,z) : ax + by + cz = 1\}$. Using Lemma 1.3, along with known values of the x-ray function for six distinct directions defined by the points on the unit sphere $\{[x_i, y_i, z_i] : \sqrt{x_i^2 + y_i^2 + z_i^2} = 1, i = 1, \dots, 6\}$ and far and near boundary planes

$$\pi_f = \{(x,y,z) : a_f x + b_f y + c_f z = 1\}$$

$$\pi_n = \{(x,y,z) : a_n x + b_n y + c_n z = 1\},$$

one can obtain six nonlinear equations with six unknowns, a_f, b_f, c_f, a_n, b_n, and c_n. The equations have the form

$$X_i = \frac{1}{a_f x_i + b_f y_i + c_f z_i} - \frac{1}{a_n x_i + b_n y_i + c_n z_i},\ i = 1, \dots, 6.$$

(x_i, y_i, z_i) can be obtained from the spherical coordinates of each direction, θ_i and ϕ_i.

Now, suppose that for $j = 1,2,3,4$ rays from the origin with directions $\overrightarrow{\omega}_j$ such that $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$ are linearly independent with $\overrightarrow{\omega}_4 = \dfrac{t_1\overrightarrow{\omega}_1 + t_2\overrightarrow{\omega}_2 + t_3\overrightarrow{\omega}_3}{\|(t_1,t_2,t_3)\|}$, where $t_i > 0$, meet the convex body K in points $p_j = r_j\overrightarrow{\omega}_j$ and $P_j = R_j\overrightarrow{\omega}_j$ where $0 < r_j \le R_j$. For our purposes, assume that $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$ are aligned as in Figure 1. Since K is convex, the point P_4 either lies on the plane joining $P_1, P_{2,} P_3$ or the plane separates P_4 from the origin. We determine this by doing a volume comparison. There are four volumes of different tetrahedrons to be calculated: $V_1 = V(o,P_1,P_{2,}P_3)$, $V_2 = V(o,P_1,P_{2,}P_4)$, $V_3 = V(o,P_2,P_{3,}P_4)$, $V_4 = V(o,P_1,P_{3,}P_4)$ where o stands for the origin. From vector calculus, we find that $V_1 = \frac{1}{6}(P_1 \times P_2)\cdot P_3$, $V_2 = \frac{1}{6}(P_1 \times P_2)\cdot P_4$, $V_3 = \frac{1}{6}(P_2 \times P_3)\cdot P_4$, $V_4 = \frac{1}{6}(P_1 \times P_3)\cdot P_4$. Now, we get the inequality $V_1 \le V_2 + V_3 + V_4$ where equality only holds when P_4 lies on the plane joining $P_1, P_{2,} P_3$. Thus,

$$0 \le R_1R_2R_4(\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2)\cdot\overrightarrow{\omega}_4 + R_2R_3R_4(\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)\cdot\overrightarrow{\omega}_4 + R_1R_3R_4(\overrightarrow{\omega}_3 \times \overrightarrow{\omega}_1)\cdot\overrightarrow{\omega}_4 - R_1R_2R_3(\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2)\cdot\overrightarrow{\omega}_3.$$

Similar considerations with the points p_j give that

$$0 \ge r_1r_2r_4(\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2)\cdot\overrightarrow{\omega}_4 + r_2r_3r_4(\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)\cdot\overrightarrow{\omega}_4 + r_1r_3r_4(\overrightarrow{\omega}_3 \times \overrightarrow{\omega}_1)\cdot\overrightarrow{\omega}_4 - r_1r_2r_3(\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2)\cdot\overrightarrow{\omega}_3.$$

These computations lead us to define the following ternary form.

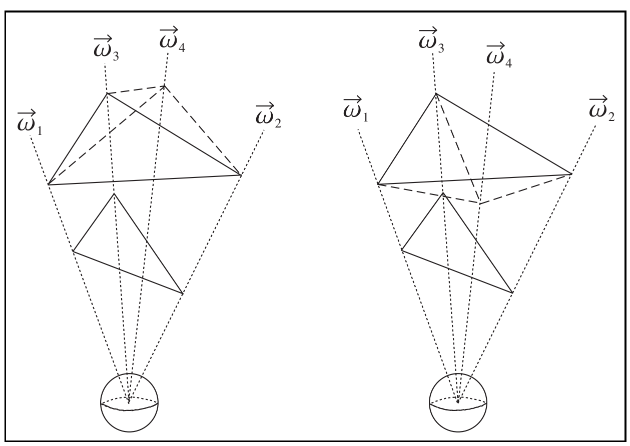

Figure 1

Definition 1.4 *Given vector* $r = [r_1, r_2, r_3, r_4] \in \mathbb{R}^4$, *and* 4×3 *matrix*

$$\overrightarrow{\omega} = \begin{bmatrix} \overrightarrow{\omega}_1 \\ \overrightarrow{\omega}_2 \\ \overrightarrow{\omega}_3 \\ \overrightarrow{\omega}_4 \end{bmatrix} \in \mathbb{R}^4 \times \mathbb{R}^3,$$ *where* $\overrightarrow{\omega}_j$ *are directions, we define*

$T(r) = T(r\overrightarrow{\omega}) = r_1 r_2 r_4 (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2) \cdot \overrightarrow{\omega}_4 + r_2 r_3 r_4 (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3) \cdot \overrightarrow{\omega}_4$
$+ r_1 r_3 r_4 (\overrightarrow{\omega}_3 \times \overrightarrow{\omega}_1) \cdot \overrightarrow{\omega}_4 - r_1 r_2 r_3 (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2) \cdot \overrightarrow{\omega}_3.$

The lemma below follows easily from the discussion above and properties of the cross product.

Lemma 1.5 *Suppose that directions $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$ are linearly independent and oriented as in Figure 1, with*
$\overrightarrow{\omega}_4 = \dfrac{t_1\overrightarrow{\omega}_1 + t_2\overrightarrow{\omega}_2 + t_3\overrightarrow{\omega}_3}{\|(t_1,t_2,t_3)\|}$, *where $0 < t_i$, and that $r_j \geq 0$, $j = 1,2,3,4$.*

(a) $T(r) = 0$ if and only if the points $r_j\overrightarrow{\omega}_j$ are coplanar.

(b) If $T(r) > 0$, then the plane passing through $r_1\overrightarrow{\omega}_1$, $r_2\overrightarrow{\omega}_2$, and $r_3\overrightarrow{\omega}_3$ separates $r_4\overrightarrow{\omega}_4$ from the origin. If $T(r) < 0$, then the origin and $(r_4, \overrightarrow{\omega}_4)$ lie on the same side of the plane passing through $r_1\overrightarrow{\omega}_1$, $r_2\overrightarrow{\omega}_2$, and $r_3\overrightarrow{\omega}_3$.

(c) If for $j = 1,2,3,4$ the rays along directions $\overrightarrow{\omega}_j$ meet the boundary of the convex body K in points $r_j\overrightarrow{\omega}_j$, and $R_j\overrightarrow{\omega}_j$ where $0 < r_j \leq R_j$, then $T(R) \geq 0$ and $T(r) \leq 0$.

Lemma 1.6 *Let K be a convex body. Suppose that $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$ are linearly independent, and oriented as in Figure 1, with $\overrightarrow{\omega}_4 = \dfrac{t_1\overrightarrow{\omega}_1 + t_2\overrightarrow{\omega}_2 + t_3\overrightarrow{\omega}_3}{\|(t_1,t_2,t_3)\|}$, where $0 < t_i$, and $X(\overrightarrow{\omega}_j) = X_j$, $j = 1,2,3,4$. If $X = [X_1, X_2, X_3, X_4]$, then $T(X) \geq 0$. Consequently, the set $\{r\overrightarrow{\omega} : 0 \leq r \leq X(\overrightarrow{\omega}_j)\}$ is a convex body.*

Proof. In light of Lemma 1.4 (a) and (b), it suffices to show that $T(X) \geq 0$. Since K is convex, we may write $X = R - r$, where $r_j\overrightarrow{\omega}_j$, and $R_j\overrightarrow{\omega}_j$, $0 < r_j \leq R_j$, are the points of intersection of the four rays with K. By Lemma 1.4 (c), $T(R) \geq 0$, and $T(r) \leq 0$. Consequently,

$$(1) \qquad R_4 \geq \frac{R_1 R_2 R_3 e_1}{R_1 R_2 e_2 + R_2 R_3 e_3 + R_1 R_3 e_4}$$

$$r_4 \leq \frac{r_1 r_2 r_3 e_1}{r_1 r_2 e_2 + r_2 r_3 e_3 + r_1 r_3 e_4},$$

where $e_1 = (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2) \cdot \overrightarrow{\omega}_3$, $e_2 = (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2) \cdot \overrightarrow{\omega}_4$, $e_3 = (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3) \cdot \overrightarrow{\omega}_4$, and $e_4 = (\overrightarrow{\omega}_3 \times \overrightarrow{\omega}_1) \cdot$

$\overrightarrow{\omega}_4$. Now,

$$(2) \qquad \begin{aligned} T(X) &= T(R-r) \\ &= (R_1-r_1)(R_2-r_2)(R_4-r_4)e_2+(R_2-r_2)(R_3-r_3)(R_4-r_4)e_3 \\ &\quad +(R_1-r_1)(R_3-r_3)(R_4-r_4)e_4-(R_1-r_1)(R_2-r_2)(R_3-r_3)e_1. \end{aligned}$$

From the inequalities (1) we have that

$$R_4-r_4 \geq \frac{R_1R_2R_3e_1}{R_1R_2e_2+R_2R_3e_3+R_1R_3e_4} - \frac{r_1r_2r_3e_1}{r_1r_2e_2+r_2r_3e_3+r_1r_3e_4}$$

$$= \frac{\begin{array}{c}[e_1(R_1R_2R_3r_1r_2e_2+R_1R_2R_3r_2r_3e_3+R_1R_2R_3r_1r_3e_4 \\ -r_1r_2r_3R_1R_2e_2-r_1r_2r_3R_2R_3e_3-r_1r_2r_3R_1R_3e_4)]\end{array}}{(R_1R_2e_2+R_2R_3e_3+R_1R_3e_4)(r_1r_2e_2+r_2r_3e_3+r_1r_3e_4)}.$$

Replace R_4-r_4 in (2) by the last expression. Some elementary algebra (which we omit) results in

$$T(X) = T(R-r)$$

$$\geq \frac{\begin{array}{c}[R_2r_2X_2(R_3r_1-R_1r_2)^2e_1e_2e_3+R_1r_1X_1(R_3r_2-R_2r_3)^2e_1e_2e_4 \\ +R_3r_2X_3(R_1r_2-R_2r_1)^2e_1e_3e_4]\end{array}}{(R_1R_2e_2+R_2R_3e_3+R_1R_3e_4)(r_1r_2e_2+r_2r_3e_3+r_1r_3e_4)} \geq 0.$$

■

Lemma 1.7 *Assume that* r, R, *and* X *are as in* (2). *Then* $T(X) = T(R-r) = 0$ *if and only if* $T(R) = T(r) = 0$, $R_3r_1 - R_1r_3 = 0$, $R_3r_2 - R_2r_3 = 0$, *and* $R_1r_2 - R_2r_1 = 0$.

Proof. From the proof of Lemma 1.8, $T(X) = T(R-r) = 0$ if and only if $R_3r_1 - R_1r_3 = 0$, $R_3r_2 - R_2r_3 = 0$, and $R_1r_2 - R_2r_1 = 0$ and equality holds in both inequalities in (1). The inequalities in (1) hold if and only if $T(R) = T(r) = 0$. ■

Definition 1.8 *A wedge,* W, *is a convex polyhedron, whose near face is a subset of a single plane* π_n *and whose far face is a subset of a single plane* π_f *and whose remaining faces lie on cutting planes. [Cutting planes of a convex polyhedron* K *(with respect to the origin) are those planes that are uniquely determined by the origin and an edge of* K*.] We always assume that* W *is a positive distance from the origin. The planes* π_n *and* π_f *may intersect along only one cutting plane or at only one point along the intersection of two adjacent cutting planes. The planes* π_n *and* π_f *do not intersect at all in the interior of the wedge, and thus a wedge contains no non-smooth points in its interior. If the planes* π_n *and* π_f *are parallel,* W *is called a parallel wedge.*

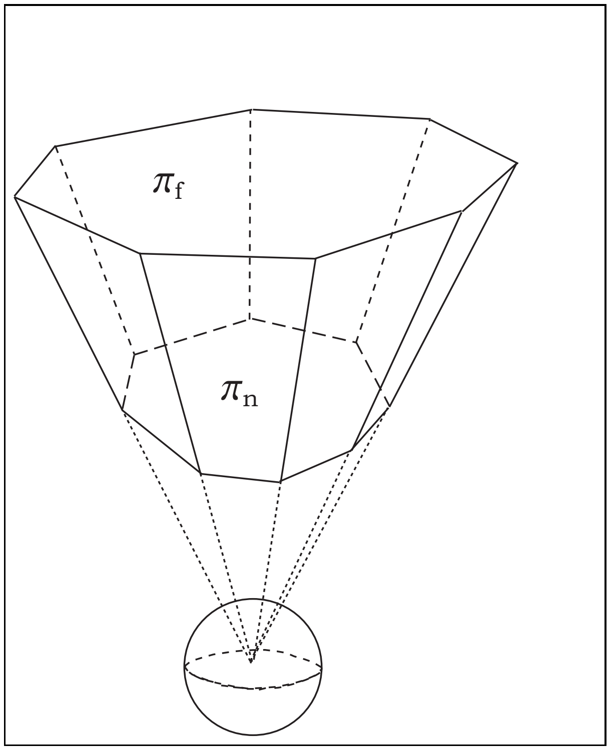

Figure 2

It is known that a parallel wedge cannot be reconstructed from one directed X-ray. Suppose that a parallel wedge W has vertices $r_1\overrightarrow{\omega}_1, R_1\overrightarrow{\omega}_1, \ldots,$ $r_n\overrightarrow{\omega}_n, R_n\overrightarrow{\omega}_n$ with $0 < r_j < R_j,\ j = 1,\ldots,n$. If $a_j \geq -r_j$, and $(r_j + a_j)r_{j+1} = (r_{j+1} + a_{j+1})r_j$ for $j = 1,\ldots,n-1$ and $(r_n + a_n)r_1 = (r_1 + a_1)r_n$ then the wedge W with vertices $(r_1 + a_1)\overrightarrow{\omega}_1, (R_1 + a_1)\overrightarrow{\omega}_1, \ldots, (r_n + a_n)\overrightarrow{\omega}_n, (R_n + a_n)\overrightarrow{\omega}_n$ is a parallel wedge with the same directed X-rays as W for any angle $\overrightarrow{\omega}$ in the convex cone generated by the $\overrightarrow{\omega}_j$*'s*.

The next result shows that four rays suffice to determine whether a wedge is parallel. The symbols X, r, R stand for vectors in $\mathbb{R}^4$. Use is also made of the ternary form T as defined in Definition (1.3).

Theorem 1.9 *Suppose K is a convex body. Suppose that directions* $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2,$ $\overrightarrow{\omega}_3$ *are linearly independent with direction* $\overrightarrow{\omega}_4 = \dfrac{t_1\overrightarrow{\omega}_1 + t_2\overrightarrow{\omega}_2 + t_3\overrightarrow{\omega}_3}{\|(t_1,t_2,t_3)\|}$, *where* $0 < t_i$, *and* $X(\overrightarrow{\omega}_j) = X_j,\ j = 1,2,3,4$. *Let C be the convex cone generated by* $\overrightarrow{\omega}_1,\ \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$. *Let* $X = [X_1, X_2, X_3, X_4]$. *If* $T(X) = 0$, *then the intersection of K with the cone C is a parallel wedge and the planes* π_n *and* π_f *bounding the parallel wedge have common normal vectors* $(X_2\overrightarrow{\omega}_2 - X_1\overrightarrow{\omega}_1) \times (X_3\overrightarrow{\omega}_3 - X_2\overrightarrow{\omega}_2)$. *Nothing can be determined about the distance of the parallel wedge from the origin even if additional rays are included. In particular, if K is a wedge, it is a parallel wedge.*

Proof. For $j = 1,2,3,4$ let $r = [r_1, r_2, r_3, r_4]$ and $R = [R_1, R_2, R_3, R_4]$, where $0 < r_j \leq R_j$, be the points of intersection of the ray $\overrightarrow{\omega} = \overrightarrow{\omega}_j$ with K. Then $X = R - r$. Suppose that $T(X) = 0$. By Lemma 1.6 $T(R) = T(r) = 0$, $R_3r_1 - R_1r_3 = 0$, $R_3r_2 - R_2r_3 = 0$, and $R_1r_2 - R_2r_1 = 0$. By Lemma 1.4 (a), the points r_j are coplanar as are the points R_j. Hence the intersection of K with the cone C is a wedge. It remains to show that the intersection is a parallel wedge. Since $R_3r_1 - R_1r_3 = 0$, $r1/R1 = r3/R3$. Hence the triangles with vertices $o, r_1\overrightarrow{\omega}_1, r_3\overrightarrow{\omega}_3$ and $o, R_1\overrightarrow{\omega}_1, R_3\overrightarrow{\omega}_3$ are similar. So, the rays from $r_1\overrightarrow{\omega}_1$ to $r_3\overrightarrow{\omega}_3$ and from $R_1\overrightarrow{\omega}_1$ to $R_3\overrightarrow{\omega}_3$ are parallel. Likewise, the rays from $r_1\overrightarrow{\omega}_1$ to $r_2\overrightarrow{\omega}_2$ and from $R_1\overrightarrow{\omega}_1$ to $R_2\overrightarrow{\omega}_2$ are parallel as are the rays from $r_2\overrightarrow{\omega}_2$ to $r_3\overrightarrow{\omega}_3$ and from $R_2\overrightarrow{\omega}_2$ to $R_3\overrightarrow{\omega}_3$. Thus K is a parallel wedge. Since $X_j = R_j - r_j, X_1/X_3 = r_1/r_3 = R_1/R_3$, $X_1/X_2 = r_1/r_2 = R_1/R_2$, and $X_2/X_3 = r_2/r_3 = R_2/R_3$. Hence, the common normal vector of P_N and P_F is $(r_2\overrightarrow{\omega}_2 - r_1\overrightarrow{\omega}_1) \times (r_3\overrightarrow{\omega}_3 - r_2\overrightarrow{\omega}_2) = (X_2\overrightarrow{\omega}_2 - X_1\overrightarrow{\omega}_1) \times (X_3\overrightarrow{\omega}_3 - X_2\overrightarrow{\omega}_2)$. ■

2 Detecting Nonsmooth Points

Definition 2.1 *A nonsmooth point on a polyhedron is a point on the boundary of the polyhedron where there is not a unique tangent plane. This is equivalent to a point on a polyhedron that is on the intersection of two or more faces.*

Proposition 2.2 *With the possible exception of a finite set, directions in which there are nonsmooth points on edges of polyhedrons can be found by detecting discontinuities in the derivative of the x-ray function with respect to* θ, $\frac{\partial X}{\partial \theta}(\theta, \phi)$, *if nonsmooth points are not found in the same directions on the near and far boundary functions, and if the directions are not directions of supporting rays.*

Proof. We are concerned with the intersection of two faces of a polyhedron. We can assume that both faces consist of sections of planes that do not intersect the origin. Otherwise, one of the faces would be a subset of the intersection of K and the set of all supporting rays. Let our planes π_1 and π_2 be defined as follows:

$\pi_1 = \{(x,y,z) : a_1x + b_1y + c_1z = 1\}$ and
$\pi_2 = \{(x,y,z) : a_2x + b_2y + c_2z = 1\}$.

The point (x_o, y_o, z_o) lies in $\pi_1 \cap \pi_2$. That is,

$a_1x_o + b_1y_o + c_1z_o = 1$, and $a_2x_o + b_2y_o + c_2z_o = 1$.

Using the equations for conversion from rectangular to spherical,
$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$, we can write the previous equations instead as

$a_1\rho_o\cos\theta_o\sin\phi_o + b_1\rho_o\sin\theta_o\sin\phi_o + c_1\rho_o\cos\phi_o = 1$, and

$a_2\rho_o\cos\theta_o\sin\phi_o + b_2\rho_o\sin\theta_o\sin\phi_o + c_2\rho_o\cos\phi_o = 1$.

Analogous equations also describe π_1 and π_2 in general. Solving for ρ in each of these equations yields

$$\rho_o = \frac{1}{a_1 \cos\theta_o \sin\phi_o + b_1 \sin\theta_o \sin\phi_o + c_1 \cos\phi_o}$$

$$= \frac{1}{a_2 \cos\theta_o \sin\phi_o + b_2 \sin\theta_o \sin\phi_o + c_2 \cos\phi_o},$$

$$\rho_1(\theta,\phi) = \frac{1}{a_1 \cos\theta \sin\phi + b_1 \sin\theta \sin\phi + c_1 \cos\phi},$$

$$\rho_2(\theta,\phi) = \frac{1}{a_2 \cos\theta \sin\phi + b_2 \sin\theta \sin\phi + c_2 \cos\phi}.$$

Differentiating with respect to θ at (θ_o, ϕ_o) gives us

$$\frac{\partial\rho_1}{\partial\theta}(\theta_o,\phi_o) = \frac{-(-a_1 \sin\theta_o \sin\phi_o + b_1 \cos\theta_o \sin\phi_o)}{(a_1 \cos\theta_o \sin\phi_o + b_1 \sin\theta_o \sin\phi_o + c_1 \cos\phi_o)^2}$$

$$= (a_1 \sin\theta_o \sin\phi_o - b_1 \cos\theta_o \sin\phi_o)\rho_o^2.$$

Analogously, $\frac{\partial\rho_2}{\partial\theta}(\theta_o,\phi_o) = (a_2 \sin\theta_o \sin\phi_o - b_2 \cos\theta_o \sin\phi_o)\rho_o^2$. Taking the difference of the two derivatives yields

$$\left[\frac{\partial\rho_1}{\partial\theta} - \frac{\partial\rho_2}{\partial\theta}\right](\theta_o,\phi_o) = \left[(a_1 - a_2)\sin\theta_o \sin\phi_o + (b_2 - b_1)\cos\theta_o \sin\phi_o\right]\rho_o^2$$
$$= \left[(a_1 - a_2)y_o + (b_2 - b_1)x_o\right]\rho_o.$$

Since ρ_o will always be positive, the sign of the above difference equals the sign of the expression $(a_1 - a_2)y_o + (b_2 - b_1)x_o$. A nonsmooth point at (x_o, y_o, z_o) can be detected by differentiating with respect to θ whenever $(a_1 - a_2)y_o + (b_2 - b_1)x_o \neq 0$.

If $a_1 = a_2$ and $b_1 = b_2$, then either $\pi_1 = \pi_2$ or the planes intersect in the x-y plane. Since we are only concerned with intersections of planes in line segments where $z > 0$, we can disregard this case. The difference will be of no use on the z-axis, where $x = y = 0$; this is not surprising, since changes in θ are meaningless on the z-axis. If $(a_1 - a_2)y + (b_2 - b_1)x = 0$ for more than one point on the intersection of π_1 and π_2, it is implied that $(a_1 - a_2)y + (b_2 - b_1)x = 0$ for infinitely many points.

Suppose there are infinitely many solutions to the system of linear equations

$$a_1x + b_1y + c_1z = 1,\ a_2x + b_2y + c_2z = 1,\ (b_2 - b_1)x + (a_1 - a_2)y = 0.$$

Consider the augmented matrix of the system:

$$\begin{bmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ b_2 - b_1 & a_1 - a_2 & 0 & 0 \end{bmatrix}.$$

The first two rows are linearly independent, because $\pi_1 \neq \pi_2$, and hence the rank of the matrix is at least two. The system has infinitely many solutions, and hence the rank of the matrix is exactly two. The third row can be written as a linear combination of the first two rows. Then $s[a_1, b_1, c_1, 1] + t[a_2, b_2, c_2, 1] = [b_2 - b_1, a_1 - a_2, 0, 0]$, $s, t \in \mathbb{R}$. From the last term, $s + t = 0 \Longrightarrow s = -t$. Substituting $-s$ for t, the first two terms give us $s(a_1 - a_2) = b_2 - b_1$ and $s(b_1 - b_2) = a_1 - a_2$. From either equation, we see that $a_1 = a_2 \Leftrightarrow b_1 = b_2$, so we know that $a_1 \neq a_2$ and $b_1 \neq b_2$. Dividing the equations and cross multiplying yields $(a_1 - a_2)^2 = -(b_1 - b_2)^2$. This is impossible, since we assumed that a_1, a_2, b_1, and b_2 are real. Therefore, if there is a point on the intersection of the two planes where the difference in the derivative with respect to θ cannot be detected, then it is unique.

The intersection in question occurs either on $F = \{(\rho, \theta, \phi) : \rho = R(\theta, \phi)\}$ or on $N = \{(\rho, \theta, \phi) : \rho = r(\theta, \phi)\}$. These are related to the x-ray function, $X(\theta, \phi)$, by the following equation:

$$X(\theta, \phi) = R(\theta, \phi) - r(\theta, \phi).$$

Differentiating with respect to θ, we get

$$\frac{\partial}{\partial\theta}X = \frac{\partial}{\partial\theta}R - \frac{\partial}{\partial\theta}r.$$

For a function of two real variables, we adopt the notation

$$\frac{\partial^-}{\partial x} f(x,y) = \lim_{h \to 0^-} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and}$$

$$\frac{\partial^+}{\partial x} f(x,y) = \lim_{h \to 0^+} \frac{f(x+h,y) - f(x,y)}{h}.$$

To find discontinuities in the derivative of the x-ray function with respect to theta, we must evaluate the difference

$$\frac{\partial^-}{\partial\theta}X - \frac{\partial^+}{\partial\theta}X = \frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R + \frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r.$$

From what we have shown above, $\frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R$ or $\frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r$ will be nonzero only in directions where there are nonsmooth points on the far or near boundaries. Moreover, we have shown that $\frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R$ or $\frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r$ will be nonzero, with the possible exception of singular points on edges, for all directions where there are nonsmooth points on the far or near boundaries. Therefore, $\frac{\partial^-}{\partial\theta}X - \frac{\partial^+}{\partial\theta}X$ will be nonzero, with the possible exception of singular points on edges, for all directions where there are nonsmooth points on the far or near boundaries (exclusive), and it will be zero for all directions where there are not nonsmooth points on the far or near boundaries. ■

Remark 2.3 *The problem remaining is the case when there are directions along which there are nonsmooth points on the far and near boundaries. If* $\frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R$ *and* $\frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r$ *are always the same sign (when both of them are non-zero), then* $\frac{\partial^-}{\partial\theta}X - \frac{\partial^+}{\partial\theta}X$ *will also have this sign, and the direction where the difference is taken will be known to have at least one nonsmooth point. However, if it is possible for* $\frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R$ *and* $\frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r$ *to have opposite signs, then they may add to zero. All directions where this happened would be directions with nonsmooth points that are not detected by discontinuity in the derivative of the x-ray function with respect to* θ.

Conjecture 2.4 *Directions in which there are nonsmooth points on edges of polyhedrons can be found by detecting discontinuities in the derivative of the x-ray function with respect to* θ, $\frac{\partial X}{\partial\theta}(\theta,\phi)$, *if the directions are not directions of supporting rays of the polyhedron.* $\frac{\partial^-}{\partial\theta}X - \frac{\partial^+}{\partial\theta}X > 0$ *in these directions.*

We attempted, but could not prove this conjecture. One approach we took was to look at curves on the boundary of the polyhedron at constant spherical coordinate ϕ_o. Let $\rho(\theta,\phi_o)$ be such a curve. We then considered the projection of this curve onto the *x*-*y* plane. We describe the curve in the *x*-*y* plane in polar coordinates, $r(\theta) = \rho(\theta,\phi_o)\sin\phi_o$. Then

$$\frac{d}{d\theta}r(\theta) = \frac{\partial}{\partial\theta}\rho(\theta,\phi_o)\sin\phi_o,$$
$$\frac{d^-}{d\theta}r - \frac{d^+}{d\theta}r = \left[\frac{\partial^-}{\partial\theta}\rho - \frac{\partial^+}{\partial\theta}\rho\right]\sin\phi_o.$$

Except on the *z*-axis, where $\sin\phi_o = 0$, $\sin\phi_o$ is positive because $\phi_o < \frac{\pi}{2}$. Therefore, $\frac{d^-}{d\theta}r - \frac{d^+}{d\theta}r$ has the same sign as $\frac{\partial^-}{\partial\theta}\rho - \frac{\partial^+}{\partial\theta}\rho$. Lam and Solmon [2] showed that when a function $r(\theta)$ in the

x-y plane is locally concave toward the origin at a nonsmooth point, $\frac{d^-}{d\theta}r - \frac{d^+}{d\theta}r > 0$ at that point, and when $r(\theta)$ is locally concave away from the origin at a nonsmooth point, $\frac{d^-}{d\theta}r - \frac{d^+}{d\theta}r < 0$ at that point. The far boundary function $R(\theta,\phi)$ is concave toward the origin, and the near boundary function $r(\theta,\phi)$ is concave away from the origin. What we need to show in order to prove the conjecture is that nonsmooth points in the direction of (θ,ϕ_o) where the boundary function ρ is locally concave toward (away) from the origin will be mapped to points where $r(\theta)$ is locally concave toward (away) from the origin. If that is the case, then $\frac{\partial^-}{\partial\theta}X - \frac{\partial^+}{\partial\theta}X = \frac{\partial^-}{\partial\theta}R - \frac{\partial^+}{\partial\theta}R + \frac{\partial^+}{\partial\theta}r - \frac{\partial^-}{\partial\theta}r > 0$ in directions where there are nonsmooth points on the near and far boundary functions.

3 Determining Uniqueness

Suppose we are given two wedges, W and W^*, that have equal x-rays along a given set of rays. We want to know when and if this implies that $W = W^*$. At first it seemed that six rays would be sufficed to uniquely determine a wedge, as there are six unknowns. However, as of yet, we have not been able to find an appropriate configuration of six rays through a wedge which we can prove uniqueness.

3.1 Uniqueness from Eight Rays

In this section we show that a nonparallel wedge is uniquely determined by the values of the x-ray function over particular configurations of seven or eight rays through the wedge. The following proof also leads to a reconstruction algorithm described in the following section.

Theorem 3.1 *A nonparallel wedge is uniquely determined by its integrals over eight rays when the set of rays can be divided into two sets of four rays and the sets of four lie in two distinct planes.*

Proof. Each set of four rays, $\Omega_j,\ j = 1,2$, determines a two-dimensional cross-section of the wedge. This cross section is a two-dimensional wedge as defined in [2]. Thus, the set of eight rays determines two distinct 2-D wedges that lie in nonparallel planes. Since the 3-D wedge is nonparallel, at most one of the 2-D wedges can be a parallel wedge. It has been shown by Lam and Solmon that a nonparallel 2-D wedge is uniquely determined by its integrals over four rays (that meet the wedge in segments of finite length) Thm 4.1 [2]. Thus, for each boundary plane, at least one unique line that lies in the plane can be determined. If both of the 2-D wedges are nonparallel, then for each boundary plane we can determine two nonparallel lines that lie in each plane. Thus the planes are both uniquely determined. Suppose one set of rays, say Ω_1, produces a parallel 2-D wedge. Let the plane determined by the span of Ω_1 be called π_p. By Thm 3.1 [2], the common slope of the near and far sides of the wedge in plane π_p is uniquely determined. Hence, the direction of the line of intersection of the near and far side planes is uniquely determined. The set Ω_2 uniquely determines a point on the near and far side planes, by Thm 4.1 [2]. These two pieces of information uniquely determine the near and far side planes, and hence the wedge. ■

Remark 3.2 *Uniqueness also holds for seven rays given a configuration of rays in which one of the rays lies in the intersection of the two planes and thus belongs to both sets of four*

Corollary 3.3 *A convex polyhedron in $\mathbb{R}^3$ is uniquely determined by a single directed x-ray with the exception of parallel wedges.*

4 Reconstruction Algorithms

In this section we describe a method of reconstruction of convex polyhedron that follows from the two-dimensional reconstruction techniques in Lam and Solmon [2] as well as from the discussion in the previous section. We also describe attempts to use Newton's method to solve systems of six nonlinear equations in six unknowns to solve for the plane constants of the far and near boundary.

We first take x-ray data over a wide range of θ and ϕ. Depending on how many zero valued x-rays this produces, the maximum and minimum values of θ and ϕ for which we took data can be adjusted to produce more non-zero data values. We then take the divided difference with respect to θ to approximate $\frac{\partial^-}{\partial\theta}X(\theta,\phi)-\frac{\partial^+}{\partial\theta}X(\theta,\phi)$ along each ray where data was taken using the formula

$$\frac{\partial^-}{\partial\theta}X(\theta,\phi)-\frac{\partial^+}{\partial\theta}X(\theta,\phi)\approx\frac{-X(\theta+\Delta\theta,\phi)+2X(\theta,\phi)-X(\theta-\Delta\theta,\phi)}{\Delta\theta}.$$

Although it is interesting to plot the values of this divided difference with respect to θ in three dimensions as a function of theta and phi, it proves more useful to plot in two dimensions the values of θ and ϕ for which the absolute values of the divided differences were greater than some small cut-off value. An appropriate value for the cut-off can be found through trial and error. Once this plot is made with θ on the horizontal axis and ϕ on the verticle axis, the points (θ,ϕ) which corresponded to a ray along which there is at least one nonsmooth point can be seen to lie on curves in the θ-ϕ plane. The graph helps to visualize the approximate location of supporting rays, as a jump in the divided difference also occurs when moving from directions which pass through the polyhedron to rays which don't pass through the polyhedron and thus have an x-ray value of zero. In addition, from the graph, we can see how many wedges there are and approximately where the boundaries of each wedge lie. By staying a reasonable distance from the apparent boundaries of each wedge, we can then choose rays that we know come from a single wedge. The next step is to apply the ternary form to four appropriate directions within each wedge to determine if any of them is a parallel wedge.

Then, for each nonparallel wedge, we pick $\Omega_j = \{\vec{\omega}_{j1}, \vec{\omega}_{j2}, \vec{\omega}_{j3}, \vec{\omega}_{j4}\}$ $j = 1,2$, and for each parallel wedge we pick $\Omega_j = \{\vec{\omega}_{j1}, \vec{\omega}_{j2}, \vec{\omega}_{j3}\}$ $j = 1,2$. Let $\vec{u}_j = [u_{jx}, u_{jy}, u_{jz}]$ be a unit vector that lies in the intersection of the plane spanned by Ω_j and the *x*-*y* plane. Let $\vec{n}_j = [n_{jx}, n_{jy}, n_{jz}]$ be a unit normal vector to the plane spanned by Ω_j. Let $\vec{v}_j = [v_{jx}, v_{jy}, v_{jz}]$ be a unit vector perpendicular to $\vec{u}_j$ and $\vec{n}_j$. We can define the plane, π_j, spanned by each Ω_j by taking the rectangular components of three of the rays in Ω_j. Let these components be labeled (x_i, y_i, z_i) for $i = 1,2,3$. Then the equation of π_j is

$$Ax + By + Cz = 0 \text{ where}$$

$$A = y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2),$$

$$B = z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2), \text{ and}$$

$$C = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2).$$

Then,

$$\vec{n}_j = (A^2 + B^2 + C^2)^{-1/2}[A, B, C],$$

$$\vec{u}_j = (A^2 + B^2)^{-1/2}[B, -A, 0], \text{ and}$$

$$\vec{v}_j = (A^2 + B^2 + A^2C^2 + B^2C^2 + 2AB)^{-1/2}[-CA, -BC, A^2 + B^2].$$

For each set we must then determine the angle β each direction $\vec{\omega}$ makes with the line of intersection of the *x*-*y* plane and the plane determined by the span of the set of four rays. Let $\beta_{ji} = \cos^{-1}(\vec{u}_j \cdot \vec{\omega}_{ji})$ for all $\vec{\omega}_{ji} \in \Omega_j$. Sets must be ordered $\Omega_j = \{\vec{\omega}_{j1}, .., \vec{\omega}_{jn}\}$ s.t. $0 < \beta_{j1} < \ldots < \beta_{jn} < \pi$.

If the 3-D wedge is a nonparallel wedge, then for each Ω_j, we can use the techniques in Lam & Solmon Thm 5.10 [2] to reconstruct any two-dimensional nonparallel wedge that is determined by Ω_j by the following. For convenience, let β_{ji} be called β_i and $X_i = X(\vec{\omega}_{ji})$ for $i = 1,2,3,4$, and $r_i\vec{\omega}_{ji}$ and $R_i\vec{\omega}_{ji}$ are the intersections of the ray in direction $\vec{\omega}_{ji}$ with the near and far planes, respectively. We solve for the slopes m and M of the near and far planes, respectively, by solving the system below.

$$k_1 mM + k_2(m + M) = k_3$$
$$k_4 mM + k_5(m + M) = k_6 \text{ where}$$

$$k_1 = -[\sin(\beta_3 - \beta_2)\cos^2\beta_1 X_1 - \sin(\beta_3 - \beta_1)\cos^2\beta_2 X_2 + \sin(\beta_2 - \beta_1)\cos^2\beta_3 X_3]$$

$$
\begin{aligned}
k_2 = {} & [\sin(\beta_3-\beta_2)\sin(2\beta_1)X_1 - \sin(\beta_3-\beta_1)\sin(2\beta_2)X_2 \\
& + \sin(\beta_2-\beta_1)\sin(2\beta_3)X_3]/2 \\
k_3 = {} & \sin(\beta_3-\beta_2)\sin^2\beta_1 X_1 - \sin(\beta_3-\beta_1)\sin^2\beta_2 X_2 \\
& + \sin(\beta_2-\beta_1)\sin^2\beta_3 X_3 \\
k_4 = {} & -[\sin(\beta_4-\beta_2)\cos^2\beta_1 X_1 - \sin(\beta_4-\beta_1)\cos^2\beta_2 X_2 \\
& + \sin(\beta_2-\beta_1)\cos^2\beta_4 X_4] \\
k_5 = {} & [\sin(\beta_4-\beta_2)\sin(2\beta_1)X_1 - \sin(\beta_4-\beta_1)\sin(2\beta_2)X_2 \\
& + \sin(\beta_2-\beta_1)\sin(2\beta_4)X_4]/2 \\
k_6 = {} & \sin(\beta_4-\beta_2)\sin^2\beta_1 X_1 - \sin(\beta_4-\beta_1)\sin^2\beta_2 X_2 \\
& + \sin(\beta_2-\beta_1)\sin^2\beta_4 X_4
\end{aligned}
$$

Then $r_1 = \sin(\beta_i - \tan^{-1}(m))[X_i \sin(\beta_i - \tan^{-1}(M)) - X_1 \sin(\beta_1 - \tan^{-1}(M))]$
$/\sin(\tan^{-1}(m) - \tan^{-1}(M))\sin(\beta_i - \beta_1)$.

If the above equation for r_1 produces a negative answer, then it is really the negative of R_1,and we need to switch m and M, and let $r_1 = R_1 - X_1$. If the original value of r_1 is positive, then $R_1 = r_1 + X_1$. Then

$$
r_i = \frac{\sin(\beta_1 - \tan^{-1}(m))}{\sin(\beta_i - \tan^{-1}(m))} r_1 \quad \text{and}
$$

$$
R_i = \frac{\sin(\beta_1 - \tan^{-1}(M))}{\sin(\beta_i - \tan^{-1}(M))} R_1 \text{ for } j = 2,3,4.
$$

If both 2-D wedges determined by Ω_1 and Ω_2 are nonparallel, then we can use the equations above for both wedges. Now let $r_{ji}\vec{\omega}_{ji}$ and $R_{ji}\vec{\omega}_{ji}$ be the intersections of the ray in direction $\vec{\omega}_{ji}$ with the near and far planes, respectively. If one set, say Ω_1, results in a parallel 2-D wedge, then we can determine $m_1 = \dfrac{X(\vec{\omega}_{1k})\sin(\beta_{1k}) - X(\vec{\omega}_{1h})\sin(\beta_{1h})}{X(\vec{\omega}_{1k})\cos(\beta_{1k}) - X(\vec{\omega}_{1h})\cos(\beta_{1h})}$ using some $k,h \in \{1,2,3,4\} \quad j \neq k$. Then, if p_f and p_n are points determined to be on the far and near boundary planes of the wedge from the lines found from the nonparallel 2D wedge, then $p'_n = p_n + (u_{jx} + mv_{jx}, u_{jy} + mv_{jy}, u_{jz} + mv_{jz})$ $p'_f = p_f + (u_{jx} + mv_{jx}, u_{jy} + mv_{jy}, u_{jz} + mv_{jz})$ are also on the far and near boundary planes respectively. Then, we need to determine at least three non-collinear points on each plane, and each plane can be determined from these two sets of three points. In our program, we chose to compute all of the $r_{ji}\vec{\omega}_{ji}$ and $R_{ji}\vec{\omega}_{ji}$ that we were able to and p'_n and p'_f, if necessary, and then use a least square approximation in Maple 8 to fit a plane to the far and near boundary plane points.

If the 3–D wedge is a parallel wedge, we can determine the common slope of the boundary planes in two distinct directions as shown by Lam and Solmon [2] by the formula

$$
m_j = \frac{X(\vec{\omega}_{j2})\sin(\beta_{j2}) - X(\vec{\omega}_{j1})\sin(\beta_{j1})}{X(\vec{\omega}_{j2})\cos(\beta_{j2}) - X(\vec{\omega}_{j1})\cos(\beta_{j1})} \text{ for } j = 1,2.
$$

Since only one parallel wedge can exist, once all non-parallel wedges are constructed, the location of any parallel wedge is evident. However, the following procedure can help to define the parallel wedge more precisely.

First, it can be shown that the perpendicular distance between the boundary planes of a parallel wedge is $d = X_{ji} \sin(tan^{-1}(m_j) - \beta_{ji})$ for any ω_{ji} in our set of three rays. Then, given slopes m_1 for Ω_1 and m_2 for Ω_2 common to both boundary planes, we know that the vectors $\vec{\alpha}_j = [u_{jx} + m_j v_{jx}, u_{jy} + m_j v_{jy}, u_{jz} + m_j v_{jz}]$, $j = 1,2$ are parallel to both planes. Thus, the cross product $\vec{N}' = \vec{\alpha}_1 \times \vec{\alpha}_2$ is a normal vector to both planes. Let $\vec{N} = \frac{\vec{N}'}{||\vec{N}'||} = [N_x, N_y, N_z]$ and (x_o, y_o, z_o) be a point on the near boundary plane. The vector sum of $(x_o, y_o, z_o) + d\vec{N}$ will give us a point on the far boundary plane. We can write the equations for the near and far planes, respectively, as

$$N_x(x - x_o) + N_y(y - y_o) + N_z(z - z_o) = 0 \text{ and}$$

$$N_x(x - (x_o + dN_x)) + N_y(y - (y_o + dN_y)) + N_z(z - (z_o + dN_z)) = 0.$$

Thus, if we can determine (x_o, y_o, z_o), we can find the equations of both planes. We can find an approximate value for (x_o, y_o, z_o) by taking the approximated cutting plane equations along two adjacent sides of the parallel wedge where at least one cutting plane borders another wedge. Then, we can solve for the point of intersection of the two cutting planes along with a near plane from a wedge which shares a side with the parallel wedge that lies along one of the aforementioned cutting planes, and let this point be (x_o, y_o, z_o).

An alternate way of solving for the equations to the boundary planes of a non-parallel wedge involves solving a system of six nonlinear equations in six unknowns. Because we need to determine six pieces of information, this led us to believe that perhaps six is the minimum number of rays along which non-zero x-ray data can be taken in order to uniquely determine the wedge. So far, our attempts to prove that some configuration of six rays could uniquely determine a non-parallel wedge have not been successful. However, we also have attempted to do some numerical analysis for determining the boundary planes of a non-parallel wedge. The system of equations arises from the formula that is used to take X-ray data. We let $X_i = \frac{1}{a_f x_i + b_f y_i + c_f z_i} - \frac{1}{a_n x_i + b_n y_i + c_n z_i}$ for $i = 1,..,6$ where X_i is the value of the x-ray along the direction whose coordinates on the unit sphere are (x_i, y_i, z_i). Then, we want to solve for a_f, b_f, c_f, a_n, b_n, and c_n where $a_f x + b_f y + c_f z = 1$ is the equation for the far boundary plane and $a_n x + b_n y + c_n z = 1$ is the equation for the near boundary plane. We used the "FindRoot" function in Mathematica 4.1 to perform a Newton's Method algorithm on the equations for different initial values. We found that when the function did converge to roots, it was very accurate. However, for many sets of initial guesses for the values of the plane constants, the program did not converge. Thus, the success of this method is greatly dependent on having good initial guesses for the values of the plane constants.

Once the plane equations have been found for all non-parallel wedges of the polyhedron, we then want to determine the vertices. One way to do this is to find the vertices of every wedge of the polyhedron given the equations of the far and near boundary planes found by one of the procedures described above. For each wedge, find the points of intersection of two adjacent cutting planes with the far and near plane. Repeat this process for all pairs of adjacent cutting planes of the wedge. For adjacent wedges, this process may yield slightly different values for the same vertices. However, we know that along a given intersection of cutting planes, vertices on the far planes of wedges should match up and vertices on the near planes of wedges should match up. Thus, if the values of such vertices do not agree exactly, then we can take the average.

5 Reconstruction Examples

For all reconstructions, we used one C++ program which we wrote to take data and approximate differences in ”left” and ”right” derivatives with respect to θ in each direction. We determined the cutting planes for each wedge by using the divided differences described in the last section. After we figured out a range of directions for a wedge, we used another C++ program which we wrote to use x-ray function values for eight directions in the wedge to find eight points on the far plane and eight points on the near plane. Of the eight directions, four are at one constant θ and the other four were at another distinct constant θ. None of the wedges in the examples we completed were parallel wedges. We used a least squares fit in Maple 8 to find the plane on which each set of eight points approximately lie.

Example 1 Our first example is a tetrahedron with vertices $(0,1,1)$, $(1,1,1)$, $(8/5,11/5,6/5)$, $(0,2,0.5)$. The plane equations for each side are given by $y-6z=-5$, $y+2z=3$, $4x-4y-8z=-12$, and $8x-y-18z=-11$.

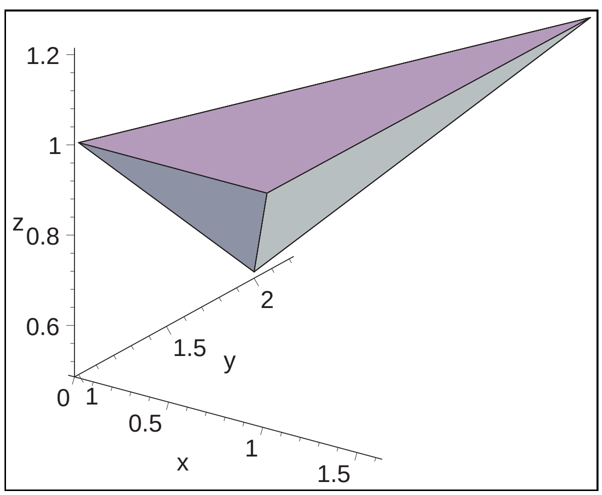

Figure 3

Figure 3 is a view of the tetrahedron from $x>0, y<0, z>0$. Note that the x and y scales shown are not the x-, y-, and z-axes, since the origin is not in the picture. Figure 4 (below) shows a plot of points (θ,ϕ) where our program determined that there were nonsmooth points. The ”triangular” regions with no points plotted inside are wedges. In this example there are four distinct wedges to reconstruct.

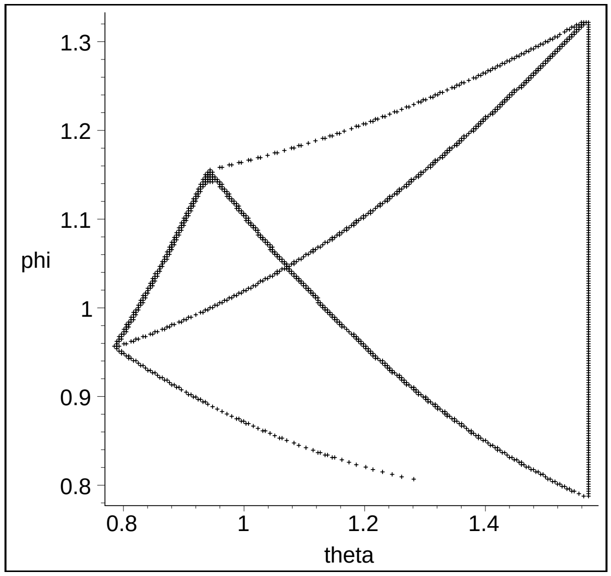

Figure 4

Using the method described above, we obtained equations for a near and far plane for each wedge. Of these eight equations, there were only four distinct planes represented, i.e., the planes on which the faces of the tetrahedron lie. In this example, each plane was represented by a reconstructed plane twice. We could tell, because there were four sets of two equations whose coefficients agreed to seven to nine digits of precision. To get the equations for our four planes, we added each of these sets of two together. For the special case of a tetrahedron, it is especially easy to find the vertices when the equations of the planes are known. We took each combination of three planes and solved for the point of intersection. We got the vertices:

$(-5.244998721 \cdot 10^{-7}, .9999998600, 1.000000011)$,
$(.9999996621, .9999998616, .9999999721)$,
$(1.600000158, 2.200000246, 1.200000036)$,
$(8.429996632 \cdot 10^{-7}, 2.000000129, .5000001030)$.

Example 2 Our second example is not an example of a polyhedron which we reconstructed, but of a challenging example which we did not have time to reconstruct. It is a two wedge polyhedron with one parallel wedge and one nonparallel wedge. It has seven faces, and it has vertices $(1,1,2)$, $(-1,1,2)$, $(-1,-1,2)$, $(1,-1,2)$, $(0.5,0.5,1)$, $(-0.5,0.5,1)$, $(-0.5,-0.5,1)$, $(0.5,-0.5,1)$, $(1.5,0.75,1.5)$, $(1.5,-0.75,1.5)$. It is symmetric in the x-z plane. Below is a view of the polyhedron from $x,y,z > 0$.

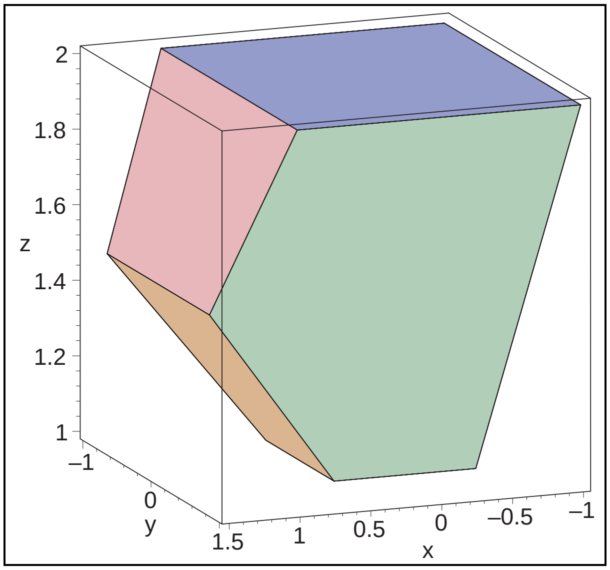

Figure 5

This example is difficult, because we can only reconstruct the parallel wedge by using points from the non-parallel wedge on the cutting plane between them along with a normal vector to both planes in the parallel wedge. Figure 6 (below) shows a plot of points (θ,ϕ) where our program determined that there were nonsmooth points. Note that the vertical axis does not start at zero.

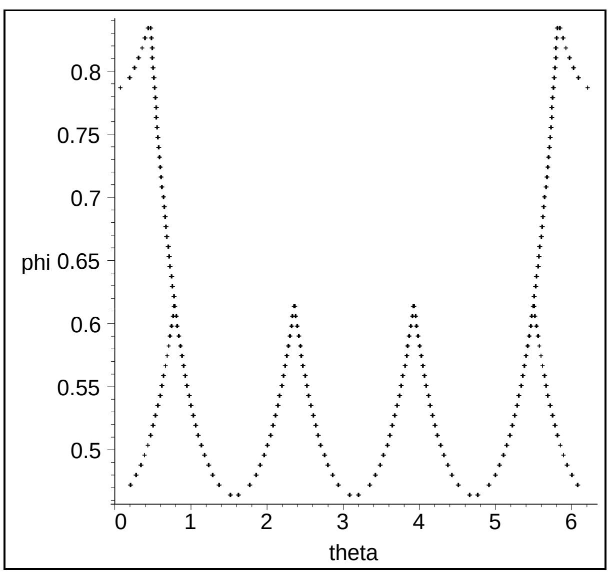

Figure 6

When looking at the above plot, it is important to note that $\theta = 0$ and $\theta = 2\pi \approx 6.28$ are identified. Thus there are two wedges: one is on the left and right sides of the plot, and the other is on the bottom, underneath the wave-like curves.

6 Incomplete Work

In this section, we describe attempts to prove that a nonparallel wedge is uniquely determined by six rays. There are six unknowns to be determined, therefore a system of six equations with six unknowns may be able to uniquely determine a wedge.

6.1 The Four + Two Uniqueness Idea

One of the ways to find uniquely determine a plane is to find two lines in the plane. Therefore, the problem of uniqueness can be broken down into the 2-D case. First, take three coplanar rays intersecting the 3-D wedge. Either the lines that contain the intersections of the rays with the planes in the wedge are parallel or they are not. If they are parallel, then we know the slope of the lines in the two planes along the first three rays. Then, take the other three rays to be coplanar with only one of the first three rays. If these new lines that contain the intersections of the rays with the planes in the wedge are parallel, then the 3-D wedge would be parallel. If the lines are not parallel, then the four rays in the 2-D wedge defined by them uniquely determine two lines, one for the near side plane, and one for the far side plane. In this case, we can uniquely determine the planes. We have the slopes of two lines and hence, we have the normal vector for each of the planes. The normal vector together with any point along the uniquely determined line determines the wedge.

If the 2-D wedge defined by the first three rays are not parallel, then take a fourth ray that is coplanar with the first three, and we have a uniquely determined line on each of the near and far side. Now, take two more rays coplanar with only one of the first four. If the lines that contain the intersections of the rays with the planes in the wedge are parallel, we are done by reasons similar to those above. If they are not, then the thought was that we could uniquely determine the wedge. This is the question of the Four + Two Uniqueness Theorem: While one normally needs four rays to uniquely determine a 2-D wedge, is it possible to uniquely determine a 2-D wedge with three rays if we know the exact location of one of the outside rays? In general no, as we shall show.

Consider a wedge W in the plane. Suppose that the cone $C(\theta_3, \theta_1)$, where $\theta_3 < \theta_1 < \theta_3 + \pi$, intersects the wedge W in the four points r_3, R_3, r_1, R_1 with $0 < r_3 \leq R_3$ the points along the ray θ_3, and $0 < r_1 \leq R_1$ the points along the ray θ_1. Of course, equality may hold along only one of the rays. Assume we know r_1 and R_1 and fix θ_1. Then if we take another ray θ_2, $\theta_3 \leq \theta_2 \leq \theta_1$, the points of intersection of the ray with angle θ_2 with the near and far boundary lines of W are respectively

$$
\begin{aligned}
r_2 &= r_2(\theta_2) = \frac{r_1 r_3 \sin(\theta_1 - \theta_3)}{r_3 \sin(\theta_2 - \theta_3) + r_1 \sin(\theta_1 - \theta_2)}, \\
R_2 &= R_2(\theta_2) = X_2 + r_2 = \frac{R_1 (X_3 + r_3) \sin(\theta_1 - \theta_3)}{(X_3 + r_3) \sin(\theta_2 - \theta_3) + R_1 \sin(\theta_1 - \theta_2)}, \\
X_1 &= R_1 - r_1,\ X_2 = R_2 - r_2,\ X_3 = R_3 - r_3.
\end{aligned}
$$

After subtracting X_2 from the second equation, and setting the two equations equal to each other, we obtain a quadratic equation where every value is known, except r_3.

(3) $$a_1 r_3^2 + a_2 r_3 + a_3 = 0,$$
where

$$a_1 = X_2 \sin^2(\theta_2 - \theta_3) - X_1 \sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3)$$

$$\begin{aligned} a_2 = {} & X_3 r_1 \sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3) + X_2 X_3 \sin^2(\theta_2 - \theta_3) \\ & + X_2 r_1 \sin(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2) + X_2 R_1 \sin(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2) \\ & - X_3 R_1 \sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3), \end{aligned}$$

$$\begin{aligned} a_3 = {} & X_2 X_3 r_1 \sin(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2) - X_3 R_1 r_1 \sin(\theta_1 - \theta_3)\sin(\theta_1 - \theta_2) \\ & + X_2 R_1 r_1 \sin^2(\theta_1 - \theta_2). \end{aligned}$$

We know that there are only two possible answers for r_3, one being the corresponding value to W, the second being another wedge provided $r_3 > 0$. So, suppose W^* is another wedge with points of intersection $r_1, R_1, r_2 + b, R_2 + b, r_3 + c, R_3 + c$ with the rays θ_1, θ_2, and θ_3 respectively, where necessarily $b > -r_2$ and $c > -r_3$. The wedges W, W^* have the same directed X-rays from the angles θ_1, θ_2, and θ_3. Since this is true, we get a similar quadratic equation to (3) and we know that the two roots to the quadratic equation are r_3 and $r_3 + c$.

$$a_1(r_3 + c)^2 + a_2(r_3 + c) + a_3 = 0$$

Setting the two equations equal to each other, we obtain the two possible values for c.

$$c = 0,\ c = -2r_3 - X_3 - \frac{X_2(X_1 + 2r_1)\sin(\theta_1 - \theta_2)}{X_2 \sin(\theta_2 - \theta_3) - X_1 \sin(\theta_1 - \theta_3)}.$$

Originally, the hope was to show that from the quadratic equation, one answer was always positive, while one was negative, and since $r_3 > 0$, we would be done at that point. This would be true if and only if $a_1 a_3 < 0$ (from above). To simplify computations, we took θ_2 to be the midangle between θ_1 and θ_3. So, $\sin(\theta_1 - \theta_2) = \sin(\theta_2 - \theta_3)$, and $\sin(\theta_1 - \theta_3) = 2\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2)$. Then

(4) $$\begin{aligned} a_1 a_3 = {} & r_1 \sin^4(\theta_1 - \theta_2)[X_2 - 2X_1 \cos(\theta_1 - \theta_2)] \\ & * [(X_3 + R_1)X_2 - 2R_1 X_3 \cos(\theta_1 - \theta_2)]. \end{aligned}$$

When the angle $(\theta_1 - \theta_3)$ is sufficiently small, then the sign of (4) is positive. In other words, it is not always the case that W is unique in this instance because there would be two possible wedges with the same X-ray data. This becomes a little more clear when one looks at c again when it isn't zero.

$$c = -2r_3 - X_3 - \frac{X_2(X_1 + 2r_1)}{X_2 - 2X_1\cos(\theta_1 - \theta_2)}.$$

As $(\theta_1 - \theta_3)$ gets closer to zero, X_2 and X_3 get closer to X_1, and r_3 also gets closer to r_1. Hence, c converges to zero and it makes sense why we get two positive answers since we knew our roots were r_3 and $r_3 + c$ in the first place.

To conclude, while this uniqueness theorem did not quite work out in every case, at least we know what goes wrong.

6.2 Another Six Ray Uniqueness Idea

The following is another experiment in the quest for a 6 ray uniqueness theorem.

Consider a wedge W in the plane. Suppose that there are three linearly independent directions $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$, that intersect the near and far planes of the wedge with the six points $r_1, R_1, r_2, R_2, r_3, R_3$ with $0 < r_1 \leq R_1$ along the direction $\overrightarrow{\omega}_1$, $0 < r_2 \leq R_2$ along the direction $\overrightarrow{\omega}_2$, and $0 < r_3 \leq R_3$ along the direction $\overrightarrow{\omega}_3$. Of course, equality may hold only along one of the rays. Then for any other direction $\overrightarrow{\omega}$, with the points r, R that intersect the near and far side planes of the wedge, then $(R_2\overrightarrow{\omega}_2 - R_1\overrightarrow{\omega}_1) \times (R_3\overrightarrow{\omega}_3 - R_1\overrightarrow{\omega}_1) \cdot (R\overrightarrow{\omega} - R_1\overrightarrow{\omega}_1) = 0$. After some manipulation, we find that:

$$R = \frac{R_1R_2R_3\overrightarrow{\omega}_1 \cdot (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)}{R_1R_2\overrightarrow{\omega} \cdot (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2) + R_2R_3\overrightarrow{\omega} \cdot (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3) + R_1R_3\overrightarrow{\omega} \cdot (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_3)}.$$

We obtain a similar result by replacing the $R_i's$ with $r_i's$.To simplify computation, we let $e_1 = \overrightarrow{\omega} \cdot (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_2)$, $e_2 = \overrightarrow{\omega} \cdot (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)$, and $e_3 = \overrightarrow{\omega} \cdot (\overrightarrow{\omega}_1 \times \overrightarrow{\omega}_3)$. So, we now find that the equation for the X-ray along vector $\overrightarrow{\omega}$ is

$$X = X(\overrightarrow{\omega}) = \frac{R_1R_2R_3\overrightarrow{\omega}_1 \cdot (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)}{R_1R_2e_1 + R_2R_3e_2 + R_1R_3e_3} - \frac{r_1r_2r_3\overrightarrow{\omega}_1 \cdot (\overrightarrow{\omega}_2 \times \overrightarrow{\omega}_3)}{r_1r_2e_1 + r_2r_3e_2 + r_1r_3e_3}.$$

Now, if there is another wedge W^* with points of intersection $r_1 + a, R_1 + a, r_2 + b, R_2 + b, r_3 + c, R_3 + c$ along the directions $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3$ respectively, where necessarily $a > -r_1, b > -r_2, c > -r_3$ then W and W^* have the same directed X-ray from another angle $\overrightarrow{\omega}$ if and only if

$$\frac{R_1R_2R_3}{R_1R_2e_1 + R_2R_3e_2 + R_1R_3e_3} - \frac{r_1r_2r_3}{r_1r_2e_1 + r_2r_3e_2 + r_1r_3e_3}$$

$$= \frac{(R_1 + a)(R_2 + b)(R_3 + c)}{(R_1 + a)(R_2 + b)e_1 + (R_2 + b)(R_3 + c)e_2 + (R_1 + a)(R_3 + c)e_3}$$

$$-\frac{(r_1+a)(r_2+b)(r_3+c)}{(r_1+a)(r_2+b)e_1+(r_2+b)(r_3+c)e_2+(r_1+a)(r_3+c)e_3}.$$

Much of the computation from this point on was done using Maple 8, and since we did not seem to get the desired results, much of the work will be omitted. After cross multiplying and other necessary steps, we obtained a formula of the form

$$\begin{aligned}(5)\qquad & h_1e_1^3+h_2e_1^2e_2+h_3e_1^2e_3+h_4e_2^3+h_5e_2^2e_1\\ &+h_6e_2^2e_3+h_7e_3^3+h_8e_3^2e_1+h_9e_3^2e_2+h_{10}e_1e_2e_3=0.\end{aligned}$$

As it turns out, h_1, h_4, and h_7 are each zero. Then, we choose our other three directions $\overrightarrow{\omega}_4$, $\overrightarrow{\omega}_5$, $\overrightarrow{\omega}_6$ with r_i and R_i along those directions. We chose $\overrightarrow{\omega}_4$ to be the direction lying on the plane formed by $\overrightarrow{\omega}_1$ and $\overrightarrow{\omega}_2$ where $\overrightarrow{\omega}_1\cdot\overrightarrow{\omega}_4=\overrightarrow{\omega}_2\cdot\overrightarrow{\omega}_4$, or in other words, that $\overrightarrow{\omega}_4$ is the angle bisector of $\overrightarrow{\omega}_1$ and $\overrightarrow{\omega}_2$. That means that $e_1=0$, $e_2=e_3$, and equation (5) is $h_6+h_9=0$, because e_2, e_3 are not zero. Choosing $\overrightarrow{\omega}_5$ and $\overrightarrow{\omega}_6$ in a similar way, we get three equations.

$$\begin{aligned}F(a,b,c)=(R_1r_2-R_2r_1)&[(2R_2r_2+R_1r_2+R_2r_1)a+(2R_1r_1+R_1r_2+R_2r_1)b]\\ &+(R_1+R_2)(r_2a-r_1b)^2-(r_1+r_2)(R_2a-R_1b)^2=0\end{aligned}$$

$$\begin{aligned}G(a,b,c)=(R_2r_3-R_3r_2)&[(2R_3r_3+R_2r_3+R_3r_2)b+(2R_2r_2+R_2r_3+R_3r_2)c]\\ &+(R_2+R_3)(r_3b-r_2c)^2-(r_2+r_3)(R_3b-R_2c)^2=0\end{aligned}$$

$$\begin{aligned}H(a,b,c)=(R_1r_3-R_3r_1)&[(2R_3r_3+R_1r_3+R_3r_1)a+(2R_1r_1+R_1r_3+R_3r_1)c]\\ &+(R_1+R_3)(r_3a-r_1c)^2-(r_1+r_3)(R_3a-R_1c)^2=0.\end{aligned}$$

Perhaps further study of these equations will yield positive results, but in this REU experience, nothing great has risen from them yet. The hope is that either $a=b=c=0$, or that W is a parallel wedge. One possibility is to use the Inverse Function Theorem, thinking of H, G, and F as being functions of a,b,c and the r_i's and R_i's as being constant. One can easily verify that these functions are continuous, with continuous first partial derivatives. Then, when the Jacobian $J=\dfrac{\partial(F,G,H)}{\partial(a,b,c)}$ is not zero at $(0,0,0)$, then F, G, H each have an inverse for a suitably restricted neighborhood with center $(0,0,0)$. What this means for our purposes is that the map $(a,b,c)\to(F(a,b,c),G(a,b,c),H(a,b,c))$ is one-to-one in a neighborhood at $(0,0,0)$. Hence, there are no wedges "near" W with the same X-rays in six directions. The equation for the Jacobian is as follows.

$$\begin{aligned}J=2(R_1r_2-R_2r_1)&(R_1r_3-R_3r_1)(R_2r_3-R_3r_2)\\ &[(R_1r_2-R_2r_1)(R_2r_3-R_3r_2)(R_3r_3-R_1r_1)\\ &+(R_1r_3-R_3r_1)(R_2r_3-R_3r_2)(R_3r_3-R_1r_1)\\ &+(R_1r_2-R_2r_1)(R_1r_3-R_3r_1)(R_3r_3-R_2r_2)].\end{aligned}$$

The zero set of this Jacobian has not been explored fully, but some things are known about it. If only one of $(R_1r_2 - R_2r_1), (R_1r_3 - R_3r_1)$, or $(R_2r_3 - R_3r_2)$ equals zero, say $(R_1r_2 - R_2r_1) = 0$ for example, the triangles formed by r_1, R_1, r_2, R_2 along directional vectors $\overrightarrow{\omega}_1, \overrightarrow{\omega}_2$ are similar which means there is a 2d parallel wedge. If any two of $(R_1r_2 - R_2r_1), (R_1r_3 - R_3r_1)$, or $(R_2r_3 - R_3r_2)$ equal zero, then the wedge W is a parallel wedge. The other obvious case where $J = 0$ is when $R_1r_1 = R_2r_2 = R_3r_3$. Again, this has not been explored to it's full extent yet, but in all cases we've tried thus far, it implies that W is a parallel wedge.

To conclude, while one cannot abandon this method yet, it would still be good to know one way or another. We did at least seem to produce some nice looking equations in H, G, and F. If it is the case that six rays uniquely determine a wedge, it would be good to know exactly what restrictions one needs to put on the six chosen rays. Further study along these lines should be done.

References

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