REGULAR HOMOTOPY CLASSES OF CURVES ON THE TORUS

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ABSTRACT. Whitney's Theorem provides a natural classification of regular closed curves in the plane. There is an intuitive analog of this theorem for regular closed curves on the torus T^2, which states that homotopy class and rotation index are sufficient to classify these curves. We present a proof of this fact, as well as a number of attempts to construct the homotopy between curves in the same class.

1. INTRODUCTION

One of the most common themes in all of mathematics is the desire to classify objects into groups of like entities. We classify group elements according to their orbits under actions on the group, orientable 2-surfaces according to their genus, loops in topological spaces according to the elements of the fundamental group. In this paper we seek to find methods of classifying *regular closed curves*.

Definition 1. *Let $C : S^1 \to M$ denote an immersion of the circle S^1 into the manifold M. We call such an immersion a* ***regular closed curve*** *in M.*

By the definition of an immersion, this means that the tangent vector of C is always defined and non-zero.

Definition 2. *Let C_0 and C_1 be curves in a manifold M. C_0 and C_1 are said to be* ***regularly deformable*** *to one another if there exists a function $H(s,t)$ mapping $[0,1] \times [0,1]$ into M such that $H(0,t) = C_0$, $H(1,t) = C_1$, and $H_t(s,t)$ is defined and nonzero for all s,t. (Here H_t denotes the derivative with respect to t.) We also require that $H(s,t)$ be continuous for all s,t. We call such a function a* ***regular deformation****.*

Recall that Whitney's classical theorem classifies regular closed curves in the plane. It depends on a number associated with the curve which, it is easy to check, is invariant under regular deformations; this number is called the *rotation index*.

Definition 3. *Let $C(t)$ be a parametrized regular closed curve in $\mathbb{R}^2$. The* ***rotation index*** *$\gamma(C)$ of this curve is defined to be either the total angle through which the tangent vector $C'(t)$ passes as t traverses the interval $[0,1]$, or this total angle divided by 2π. It can be calculated by finding the degree of the map $t \to \frac{C'(t)}{|C'(t)|}$ (multiplied by 2π or not as desired).*

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Since either definition of the rotation index is acceptable for our methods, we will use the latter, which results in an integer rotation index.

Now that we have the above definitions, we can present a theorem that classifies regular closed curves according to their rotation indices, as originally presented by Whitney in [Wh].

Theorem 4 (Whitney)**.** *Let C_0 and C_1 be regular closed curves in $\mathbb{R}^2$. Then C_0 is regularly deformable to C_1 if and only if $\gamma(C_0) = \gamma(C_1)$.*

Whitney's theorem also implies that the path components of the space of regular closed curves are enumerated by the rotation indices of the curves in each component.

In the torus, we cannot calculate the rotation index quite as easily as we can in the plane. While we could devise a method of calculation intrinsic to the torus by using the natural frame field on T^2, it is equally easy to take advantage of covering spaces and perform the calculations in the plane itself.

The basic properties of lifts and covering spaces are discussed in most introductory algebraic topology texts; see for instance [Ma]. Here we introduce only the following notation and recall a few relevant facts.

Definition 5. *Let $C(t)$ be a regular closed curve in the torus T^2. We denote the (unique) lift of this curve to the $\mathbb{R}^2$ beginning at the origin by $\hat{C}(t)$.*

Three properties about lifts of regular closed curves are worth noting. First, the lift $\hat{C}$ of a curve C in T^2 has endpoint (p, q), where (p, q) denotes the homotopy class of C. This is a product of the construction of the covering space which I have implicitly assumed; we could as easily define the covering map from $\mathbb{R}^2$ to T^2 so that the endpoint would be $(2\pi p, 2\pi q)$, counting radians of rotation rather than integer numbers of rotations. Second, the lift has the property that $\hat{C}'(0) = \hat{C}'(1)$, because of the regularity condition on C and the process of creating the lift. Finally, the rotation index of C can be calculated from the lift by direct integration, just as in the original definition of rotation indices for closed curves. Note, however, that the double point formula presented by Whitney in [Wh] will *not* work calculating the index of a lift, as is shown by Fig. 1.

Note that the two curves in the above diagram have the same rotation index, but certainly do not have the same number of double points. The problem here seems to be that the curves in question are not closed in $\mathbb{R}^2$; considering the following diagram of the two curves projected back onto the torus, it is clear that when they are closed they do in fact have the same number of double points.

Although Whitney's double point formula fails, Burman and Polyak have in [BP] devised a method of calculating the rotation index from the number of double points in some cases. The mathematics involved is related to homology and somewhat beyond the scope of this paper, but it is worth noting that there is in fact a local method for calculating the rotation index of curves on the torus.

Now we have the necessary equipment to proceed with the main thrust of the paper.

2. Prior Work: Curves on the Sphere

For comparison, we briefly recall a few basic properties of regular curves on the sphere S^2. In prior research for the Oregon State REU, Biringer and Barker discovered (see [BB])

Figure 1. Two curves with the same rotation index but different numbers of double points

Figure 2. The curves from Fig. 1 projected onto T^2.

that classification of regular curves in S^2 is remarkably simple, and in fact that the space of regular curves on the sphere only contains two equivalence classes.

Lemma 6. *Every spherical curve is (regularly) homotopic to either the circle or the figure eight.*

This is primarily a product of the topology of S^2. The sphere is the one-point compactification of $\mathbb{R}^2$, which implies that it is closed, compact, and simply connected (i.e. has trivial fundamental group). As a result, we can, in essence, "pull" loops of curves over the north pole of the sphere, which corresponds to allowing homotopies in $\mathbb{R}^2$ to pass through the point at infinity.

Biringer and Barker construct a planar representative of each curve on the sphere by placing the curve on the sphere so that it does not pass through the north pole; then the curve can be placed on the plane as it is guaranteed not to pass through infinity. From these representatives they calculate rotation indices for the spherical curves in the manner described in the introduction.

They demonstrate that each time a loop is pulled over the point at infinity, the index of the planar representative decreases by 2. This means that by successive pulls, every curve eventually has index either 0 or 1, which correspond to the figure eight and the circle.

The sphere, however, is an excessively simple case because of its trivial topology and its compactness. We would prefer to consider surfaces with somewhat more interesting topological properties.

3. Main Theorem: Curves on the Torus

Unlike the plane and the sphere, the torus has nontrivial first homology (fundamental) group, namely the free abelian group on two elements. It is this characteristic that makes the study of curves on the torus more intriguing than the study of curves on the plane.

In surfaces with nontrivial fundamental group, we can begin to see the effect of topology on this problem. Topologically speaking, we classify curves according to their homotopy classes; geometrically speaking, Whitney's theorem shows us that we may classify them according to an associated geometric invariant. To classify curves on surfaces with nontrivial fundamental group – i.e. on surfaces where the rotation index is not sufficient – we must take advantage of both topological and geometric methods, which leads us to the following result.

Theorem 7 (Whitney Analog)**.** *Let C_1 and C_2 be regular closed curves in T^2. Then C_1 is regularly homotopic to C_2 if and only if they are in the same homotopy class and their lifts $\hat{C}_1$ and $\hat{C}_2$ have the same rotation index γ.*

First we define a term which we will use in the proof of this result.

Definition 8. *The **tangent k-frame bundle** of a manifold M of dimension k is the collection of orthonormal frames on M, i.e., the collection of possible orthonormal collections of k vectors of the tangent space to M at each point $p \in M$. We denote this collection by $T_k(M)$. This is also referred to as a type of **bundle**.*

To prove Theorem 7, we will need the following theorem, which is due to Morris Hirsch and noted in [Hi].

Lemma 9 (Hirsch)**.** *Given a manifold N and a positive integer k such that $k < \dim N$, the regular homotopy classes of immersions $S^k \to N$ are in one-to-one correspondence with the elements of $\pi_k(T_k(N))$, where T_k is the tangent k-frame bundle of N and π_k is the k-th homotopy group.*

This exciting lemma speaks directly to the subject at hand, and gives rise naturally to the following proof.

Proof. (Whitney Analog) Set $k = 1$ and $N = T^2$ in Lemma 9 above. Then we know that there is a one-to-one correspondence between the regular homotopy classes of the regular closed curves in the torus and the elements of the fundamental group of the bundle $T_1(T^2)$, which has two-dimensional fibre $\mathbb{R}^2 - (0,0)$ (as we shall show).

First we calculate $\pi_1(T_1(T^2))$.

$T_1(T_2)$ is the bundle of nonzero vectors on T^2, which can be viewed as the plane $\mathbb{R}^2$ with the origin removed. By means of a function that projects every point along a radial ray to the point where the ray intersects the unit circle, this is equivalent to the circle S^1. So at any given point of T^2, the set of nonzero vectors is topologically simply S^1, and we must consider this set for each of the $S^1 \times S^1$ points of the torus. Thus it follows that $T_1(T^2) = S^1 \times S^1 \times S^1$. The underlying reason that we can simply form the direct product here is that T^2 is parallelizable – that is, there is a single frame (namely $(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi})$) that is valid for all points of T^2.

Then since $T_1(T^2) = S^1 \times S^1 \times S^1$, we conclude that $\pi_1(T_1(T^2)) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Intuitively, we can interpret $\pi_1(T_1(T^2)) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ by understanding the first two terms $\mathbb{Z}$ to correspond to the torus and its homotopy classes, and the last term $\mathbb{Z}$ to correspond to the rotation index of a curve, since it effectively measures the rotations of the tangent vector.

So by Lemma 9, there is a one-to-one map that takes a curve $C \to [(C, C')] \in S^1 \times S^1 \times S^1 = T^2 \times S^1$, which from our intuitive interpretation of $\pi_1(T_1(T^2))$ means that two curves are mapped to the same element of the fundamental group precisely when they have the same homotopy class in T^2 and the same rotation index γ as measured by the degree of C' in S^1. □

Although the preceding proof was a remarkable breakthrough for us, it is nevertheless a nonconstructive proof. We had initially hoped for a constructive proof that produced a homotopy between two given curves, in the spirit of Whitney's original proof.

4. Alternate Methods

Intially we had hoped to find a constructive proof of Theorem 7 similar to Whitney's constructive proof of his original theorem. We made a number of attempts, each of which fell short of a complete proof.

4.1. Attempt to Apply Whitney's Theorem.

Originally we had hoped to convert the lifts, which are in general not closed, into closed curves so that Whitney's theorem would be applicable. Therefore we introduce the following lemma.

Lemma 10. *It is always possible to construct a closed curve, call it $\bar{C}$, from a lift $\hat{C}$ by pasting a smooth curve onto both endpoints of $\hat{C}$ in such a way that the new curve $\bar{C}$ is regular and has the same number of double points as $\hat{C}$.*

As a result of the pasting, $\gamma(\bar{C}) = \gamma(\hat{C}) + 1$, a result which is fairly obvious if one draws a few examples.

We also need the following definition for much of the rest of the paper.

Definition 11. *Let C_0 and C_1 be curves in a space M. A function $F : [0,1] \times [0,1] \to M$ is said to be a* ***path homotopy*** *between C_0 and C_1 if it satisfies the following conditions:*

(1) $F(0,t) = C_0(t)$

(2) $F(1,t) = C_1(t)$
(3) $F(s,t)$ *is continuous* $\forall s,t \in [0,1]$
(4) $F(s,0) = C_0(0) = C_1(0)$
(5) $F(s,1) = C_0(1) = C_1(1)$

Additionally, such a function is said to be a ***regular*** *path homotopy if it also fulfills the condition that* $F_t(s,t) \neq 0$ *for any* $s,t \in [0,1]$*, where* F_t *denotes the first derivative with respect to* t*.*

This gave rise to the following first form of Theorem 7.

Conjecture 12. *Let* C_0 *and* C_1 *be curves in* T^2*,* $\hat{C}_0$ *and* $\hat{C}_1$ *be their lifts, and* $\bar{C}_0$ *and* $\bar{C}_1$ *be the closed curves constructed from them. Then the following three statements are equivalent.*

(1) C_0 *is regularly homotopic to* C_1 *in* T^2*.*
(2) $\hat{C}_1$ *is regularly path homotopic to* $\hat{C}_2$ *in* $\mathbb{R}^2$*.*
(3) $\gamma(\bar{C}_1) = \gamma(\bar{C}_2)$ *and* $\hat{C}_1$ *and* $\hat{C}_2$ *share the same endpoint.*

Note that C_0 is only regularly homotopic to C_1 if they share the same homotopy class, and that the specification that the homotopy in statement (2) is a path homotopy ensures that the endpoints of $\hat{C}_0$ and $\hat{C}_1$ are equivalent and fixed by the homotopy. We need not explicitly state as a hypothesis of the conjecture that C_0 and C_1 share the same homotopy class; if they do not, none of the statements can be true, and thus it is moot whether they are equivalent to each other or not.

Below I present the proof attempt in four sections, adding comments after each section. This attempted proof should not be construed as completely valid; I present it in its original form for instructive purposes, with the accompanying commentary illustrating the flaws in the arguments as motivation for later parts of the paper. All apparent assumptions and generalizations should be taken with a grain of salt.

Proof. ($1 \Rightarrow 2$) Let the regular homotopy between C_0 and C_1 be denoted by F. We can also lift homotopies to the covering space and have them remain homotopies between the lifts of their endpoints (since $F(s,t)$ is a path between $F(0,t)$ and $F(1,t)$ for each fixed t). Let the lift of F be denoted $\hat{F}$. So at the least we know that a regular homotopy between C_0 and C_0 implies a homotopy between $\hat{C}_0$ and $\hat{C}_1$. We must then argue that this homotopy is regular. □

The regularity argument essentially amounts to the fact that $\mathbb{R}^2$ and T^2 are locally diffeomorphic (i.e., that the torus is locally Euclidean).

Proof. ($2 \Rightarrow 1$) Let the regular path homotopy between $\hat{C}_0$ and $\hat{C}_1$ be denoted by $\hat{F} : [0,1] \times [0,1] \to \mathbb{R}^2$. Since the projection map $p : \mathbb{R}^2 \to T^2$ is a local diffeomorphism, $p(\hat{F})$ is a continuous deformation between C_0 and C_1 in T^2 since $p(\hat{F}(s,0)) = p(\hat{C}_0) = C_0$ and $p(\hat{F}(s,1)) = p(\hat{C}_1) = C_1$. (We know from topology that if f is a map and $\hat{f}$ its lift, then $p(\hat{f}) = f$.) So we again know that a regular path homotopy between $\hat{C}_0$ and $\hat{C}_1$ implies a regular homotopy between C_0 and C_1. □

At first glance it appears that although we know the beginning and end of the projected homotopy $p(\hat{F})$ are closed curves, the intermediate curves might not be. However, by specifying $\hat{F}$ as a *path* homotopy, we ensure that the endpoints of $\hat{F}(s,t)$ remain fixed at the endpoints of $\hat{C}_0$ and $\hat{C}_1$, and therefore any intermediate curve $p(\hat{F}(s,t))$ will also be closed.

The same comments about regularity discussed after $(1 \Rightarrow 2)$ also apply here, and this section too is essentially sound.

Proof. $(2 \Rightarrow 3)$ Let $\hat{F} : [0,1] \times [0,1] \to \mathbb{R}^2$ be a regular path homotopy between $\hat{C}_0$ and $\hat{C}_1$ and let $\hat{G}$ be a regular homotopy between their extensions, so that together $\hat{F}$ and $\hat{G}$ are a regular homotopy between $\bar{C}_0$ and $\bar{C}_1$. We know that $\hat{G}$ exists by construction of the extension as a smooth curve with no self-intersections – in short, a path in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply connected, we can construct a homotopy between any two paths in the plane, and the additional smoothness specification on the extensions makes it possible to make this homotopy regular.

If $\hat{C}_0$ and $\hat{C}_1$ are regularly path homotopic, then by the definition of a path homotopy, they share the same endpoint. Furthermore, since $\hat{F}$ is a homotopy, it is continuous; and since $\gamma(\hat{F}(s,t) \star \hat{G}(s,t))$ is also continuous, $\gamma(\hat{F}(s,t) \star \hat{G}(s,t))$ (where $\star$ denotes pasting of paths) is continuous for all $t \in [0,1]$. Since the rotation index is always an integer, and the only continuous functions with integer output are constants, the rotation index must be constant, and thus $\bar{C}_0$ and $\bar{C}_1$ have the same rotation index. □

This section has the problem that while $\hat{F}$ and $\hat{G}$ may be regular in themselves, there is no guarantee that together they will form a regular homotopy. The trouble, of course, is at the endpoints of the curves C_0 and C_1. Regularity of $\hat{F}(s,t)$ and of $\hat{G}(s,t)$ does not guarantee that together they will deform the curves at the endpoint in such a way that the tangent vector will remain the same for both curves at the endpoint. The remaining portion of the argument concerning the constancy of the rotation index, however, remains valid; we merely lack an appropriate homotopy on which to invoke this argument.

Proof. $(3 \Rightarrow 2)$ Since $\bar{C}_0$ and $\bar{C}_1$ are closed curves with the same rotation index, we know they can be deformed into one another by the result of [Wh]. Furthermore, since the extensions of $\hat{C}_0$ and $\hat{C}_1$ used to make $\bar{C}_0$ and $\bar{C}_1$ have no double points, we can scale the deformation so that all deformations involving loops and double points take place on one side of the endpoint of $\hat{C}_0$. Thus we can then scale the deformation taking place near the endpoint such that the endpoint stays fixed as well. Then if we restrict the deformation to that of the original curve $\hat{C}_0$, the deformation turns out to be a path homotopy between $\hat{C}_0$ and $\hat{C}_1$. □

While this argument may appear good from an intuitive standpoint, many of its claims turn out to be largely unfounded when a critical eye is turned to them. Primarily, there is nothing to suggest that the deformation in [Wh] can be modified according to the above ideas. If we simply apply Whitney's deformation to $\bar{C}_0$ and $\bar{C}_1$, we must consider that it may move the endpoints of the intermediate curves of the deformation restricted to $\hat{C}_0$, so that these intermediate curves are no longer closed. This certainly would not create a path homotopy between $\hat{C}_0$ and $\hat{C}_1$.

Since the difficulty with the above proof was that it made too many broad assumptions that, when subjected to careful scrutiny, were revealed as invalid, the next logical step was to attempt a direct computation, in the style of Whitney.

Note, however, that since the portions of the proof that $(1 \Rightarrow 2)$ and $(2 \Rightarrow 1)$ were valid, we shall henceforth deal only with the regular homotopies of the lifts in the plane, since we know that once we have these we can project them to the torus and maintain them as regular homotopies.

4.2. **Direct Computation.** Next we attempt to directly mimic Whitney's proof, constructing a homotopy between two lifts without modifying them at all. We take as an assumption that a regular homotopy in $\mathbb{R}^2$ remains a regular homotopy when projected back into T^2. Consider the following:

Proof. Suppose we have two lifts $\hat{C}_0(t)$ and $\hat{C}_1(t)$, which are maps from the interval I into $\mathbb{R}^2$. Because they are lifts of regular curves in T^2, these lifts have the following characteristics:

(1) $\hat{C}_0(0) = \hat{C}_1(0) = (0, 0)$
(2) $\hat{C}_0(1) = \hat{C}_1(1) = (p, q)$
(3) $\hat{C}_0'(0) = \hat{C}_0'(1)$
(4) $\hat{C}_1'(0) = \hat{C}_1'(1)$
(5) $\gamma(\hat{C}_0) = \gamma(\hat{C}_1) = \gamma \in \mathbb{Z}$.

We want to construct a function $F(s, t) : I \times I \to \mathbb{R}^2$ which is a regular path homotopy between $\hat{C}_0$ and $\hat{C}_1$, as given in Definition 11.

We follow Whitney's argument in our attempt to construct such an F. First we reparametrize the curves $\hat{C}_0(t)$ and $\hat{C}_1(t)$ so that $|\hat{C}_0'(t)| = |\hat{C}_1'(t)| = L$, where L is some positive constant. Then we can express the tangent vectors as

$$\hat{C}_0'(t) = Le^{i\theta_0(t)}$$

$$\hat{C}_1'(t) = Le^{i\theta_1(t)}$$

for appropriate functions θ_0 and θ_1 with the properties that $\theta_0(0) = \theta_1(0) = 0$ and $\theta_0(1) = \theta_1(1) = 2\pi\gamma$. These functions $e^{i\theta(t)}$ are simply complex or polar-coordinate representations of the tangent vector in $\mathbb{R}^2$. Note that we can define $\theta_0(0) = 0$ because we can always deform the curve so that the initial tangent vector points along the x-axis, and then by regularity the tangent vector at the endpoint must be an integer multiple of 2π – in this case γ, as illustrated in Fig. 3.

Then, just as Whitney does, we create a linear homotopy θ_s between these two angle functions, of the form $\theta_s(t) = (1 - s)\theta_0(t) + s\theta_1(t)$. This homotopy also has the properties that $\theta_s(0) = 0$ and $\theta_s(1) = 2\pi\gamma$.

Then define a new function

$$\hat{F}_t(s, t) = Le^{i\theta_s(t)} - L\int_0^1 e^{i\theta_s(u)}du,$$

FIGURE 3. Deformation of a curve to one with a horizontal initial tangent vector

where again $\hat{F}_t$ denotes a derivative with respect to t. Note that at $s = 0, 1$ we have

$$\begin{aligned}\hat{F}_t(0,t) &= Le^{i\theta_0(t)} - L\int_0^1 e^{i\theta_0(u)}du \\ &= Le^{i\theta_0(t)} - (p,q)\end{aligned}$$

and

$$\hat{F}_t(1,t) = Le^{i\theta_1(t)} - (p,q),$$

which is equivalent to saying that

$$\hat{F}_t(0,t) = \hat{C}_0'(t) - (p,q)$$

and

$$\hat{F}_t(1,t) = \hat{C}_1'(t) - (p,q)$$

by the definition of $\hat{C}_0$ and $\hat{C}_1$. Finally, it is also easy to check that

$$\int_0^1 \hat{F}_t(s,t)dt = 0.$$

Then we note, still following Whitney's methods, that

$$F_t(s,t) = \hat{F}_t(s,t) + (p,q)$$

(where F is the function we orignally intended to construct) and also, therefore, that

$$\begin{aligned} F(s,t) &= F(s,0) + \int_0^t F_u(s,u)du \\ &= \int_0^t \hat{F}_u(s,u)du + t(p,q). \end{aligned}$$

It is easy to check that $F(0,t) = \hat{C}_0(t)$ and $F(1,t) = \hat{C}_1(t)$ and that $F(s,1) = (p,q)$ and $F(s,0) = (0,0)$. We can also see that since $F_t(s,t) = \hat{F}_t(s,t) + (p,q)$, $F_t(s,1) - F_t(s,0) = \hat{F}_t(s,1) - \hat{F}_t(s,0) = 0$, so that F satisfies most of the conditions we specified for it earlier.

We need now only check for its regularity, and it is here that the adaptation of Whitney's proof breaks down.

Since we know that $F_t(s,t) = \hat{F}_s'(t) + (p,q)$, proving that the homotopy is regular amounts to proving that $\hat{F}_s'(t) + (p,q) \neq 0$. This is equivalent to showing that

$$Le^{i\theta_s(t)} - L\int_0^1 e^{i\theta_s(u)}du + (p,q) \neq (0,0). \tag{1}$$

Note that the first term is always a point on the circle of radius L centered at the origin because it is the tangent vector at a given point (s,t). Similarly, $\int_0^1 e^{i\theta_s(u)}du$ is the average value of all these tangent vectors, and therefore lies inside the circle of radius 1; thus $L\int_0^1 e^{i\theta_s(u)}du$ lies inside the circle of radius L.

Here Whitney takes advantage of these facts and notes that the two vectors can never sum to $(0,0)$ because of the difference in magnitudes. However, since the curves we are dealing with are not closed, we must also consider the addition of (p,q), which renders Whitney's argument invalid here.

By definition, L is the arclength of the lift, and since arclength is at a minimum when the lift of the curve is a straight line between $(0,0)$ and (p,q), we know that $L^2 \geq p^2 + q^2$ and that (p,q) is therefore also somewhere inside or on the boundary of the circle. This is the point that causes the proof to break down; the first point will, as we traverse the curve, traverse the entire circle, and there are inevitably configurations of the three points which we cannot guarantee to have a nonzero sum, but which we cannot conclusively show to sum to zero, either. □

Consider Fig. 4 as an example of such a configuration.

Here (a,b) corresponds to the first term of the sum (1), (c,d) to the second term. In order to assure regularity of the homotopy in all possible cases, we must be able to show that the following configuration never sums to the zero vector; that is, that

$$\begin{aligned} a - c + p &\neq 0 \\ b - d + q &\neq 0 \Leftrightarrow b + |d| - |q| \neq 0 \end{aligned}$$

It is evident that we cannot guarantee this for all values of (a,b), (c,d), and (p,q). We cannot, however, conclusively guarantee that one of the above statements will fail either. Still, the potential for nonregularity is enough for us to at least partially discard this argument.

FIGURE 4. A configuration of points which cannot be guaranteed to be regular

4.3. **Representative Elements.**

Definition 13. *If C is a curve in T^2 with homotopy class (p,q), let a representative element of this homotopy class be denoted $h_{(p,q)}$.*

Since many of the problems with earlier proofs stemmed from the lift not being a closed curve, we made another attempt to turn the lift into a closed curve so that we could apply Whitney's original theorem. This time, we had the idea of pasting a representative element of $h_{(p,q)}$ of the curve C's homotopy class onto the curve, with an opposite orientation. This results in a new curve, either $h_{(p,q)} \star C$ or $C \star h_{(p,q)}$ (where $\star$ denotes the standard "multiplication" of paths by pasting one onto the other). We know that we can make small deformations in the area of the base point of C such that when we paste the representative element onto C the new curve remains regular at the point of pasting.

This new curve is in the $(0,0)$ homotopy class, so when it is lifted to the plane, it is a closed curve. If we lift two such curves, we can then apply Whitney's theorem to them. However, this multiplication by a representative element is in essence a formalized way of constructing the "extensions" of section 4.1, and is subject to many of the same difficulties.

If we are attempting to create a regular homotopy between two curves, we first know that we can, through a series of rotations on the torus and minor deformations in the area of the base point, place both curves on the torus so that their starting points and initial tangent vectors are the same. Then we paste representative elements to each curve.

The idea here is to lift these new curves into the plane and then apply Whitney's deformation to them. Since they are in the $(0,0)$ homotopy class already, we need not be concerned with endpoints moving and resulting in non-closed curves when projected back into the torus.

Then once they are projected back onto the torus we can paste another element of the (p,q) homotopy class onto them, this time with the appropriate orientation to cancel the previous pasting and restore the curve itself to the (p,q) homotopy class.

The problem with this method lies in the fact that $\hat{C}_0 \star h_{(p,q)} \star h^{-1}_{(p,q)}$ and $\hat{C}_1 \star h_{(p,q)} \star h^{-1}_{(p,q)}$ must be deformed back to $\hat{C}_0$ and $\hat{C}_1$, and it is not clear how this affects the intermediate curves.

5. Conclusion

We would still like to come up with a constructive proof for Theorem 7 that mimics Whitney's. At the very least, we would like to investigate the deeper reasons why the proofs we have attempted have all failed. It appears that Whitney was extremely lucky with the qualities of closed curves that helped him to prove his theorem in the first place, and it is not as easy to come up with similar fortunate qualities for closed curves on other surfaces.

One possible method to pursue is using a nonlinear homotopy in the direct computation, rather than the linear addition of $t(p,q)$ we used. We have not investigated this approach yet to any degree.

The most obvious next step is to extend this theorem to two-manifolds other than the torus. Although the torus is more interesting than the plane and the sphere because it has a nontrivial fundamental group, it does have the simplifying property that its fundamental group is abelian. It is thus expected that classification of regular closed curves for other two-surfaces may not follow the same lines of proof as we have outlined above. However, we present the following conjecture.

Conjecture 14. *Let M be a smooth compact 2-manifold, and let C_0 and C_1 be regular closed curves in M. Then C_0 is regularly homotopic to C_1 if and only if their rotation indices are the same and they share the same homotopy class.*

Clearly for the purposes of this conjecture we would also need to construct a notion of the rotation index that is valid for all 2-manifolds. Since all two-manifolds can be represented with an identification diagram, perhaps one could devise some method of calculating rotation index from curves on the identification diagram. For instance, in this context it might be reasonable to use Whitney's double-point formula; since the lack of closure of the curve seems to be what caused it to break down, and since the curves on an identification diagram would remain closed, the formula might once again apply. Alternately one might find a way to modify the formula in Section 2 of [BP] to be applicable to surfaces other than the torus.

Since a proof of Conjecture 14 was the original intent of this project, it might be advantageous to consider any future work in finding a constructive proof of Theorem 7 with an eye to being able to generalize the result later to other surfaces.

Also, it seems likely that Conjecture 14 is a consequence of Hirsch's Theorem (Lemma 9 above) with appropriate definitions and interpretation, so that is another avenue of approach should direct construction remain unsuccessful.

References

[BB] Ian Biringer and Andrew T. Barker. *On Defect of Plane Curves.* Proceedings of the Oregon State University REU Program, 2003.

[BP] Yurii Burman and Michael Polyak. *Geometry of Whitney-Type Formulas.* Moscow Mathematical Journal, vol. 3, no. 3, 823-832.
[Hi] Morris Hirsch.. *Immersions of Manifolds.* Trans. Amer. Math. Soc. 93:242-276, American Math Society, Providence, 1959.
[Ma] William S. Massey. *A Basic Course in Algebraic Topology.* Springer-Verlag, New York, 1991.
[Wh] Hassler Whitney. *On regular closed curves in the plane.* Compositio Mathematica, 4:276-284, 1937.

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