NONNEGATIVE DEFECT

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ABSTRACT. V.I. Arnold introduced axiomatic definitions for invariants St and $J^{\pm}$ of plane curves. In addition, Arnold showed that under the ordinary connected sum of two planar curves these invariants are additive. Moreover, this result implies that defect is additive under the connected sum. We use a notion of interjected sum to extend the idea of the ordinary connected sum, and show that defect is additive under this operation as well. This enables us to show that loops, the connected or interjected sum of a planar curve with a standard curve of defect zero (*i.e.*, K_0 or K_2), can be removed from a planar curve without changing defect. By examining the signed Gauss word of a planar curve, we give the construction of a Crossing graph. In turn, we are then able to show that negative $J^{\pm}$ perestroikas, in conjunction with removing loops, are enough to reduce any planar curve to the trivial loop. Inductively, this then implies that the defect of any planar curve is nonnegative.

1. INTRODUCTION

A *planar* curve is a smooth mapping $\varphi\colon S^1 \to \mathbb{R}^2$ of a circle into the plane whose derivative vanishes nowhere [Arn94]. A generic immersion is one having only finitely many transverse self-intersections, or double points (*i.e.*, crossings). Generic planar curves, or *normal* curves, have no points of self-tangency nor self-intersection points of multiplicity greater than 2.

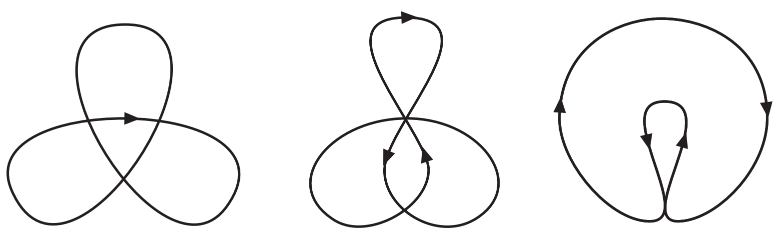

FIGURE 1. A normal curve (left), a non-generic curve with a self-intersection of multiplicity 3, *i.e.* a triple point (center), and a non-generic curve with a point of self-tangency (right).

The goal of this paper is the study of the defect of normal curves. Foundational to this study is the work of many authors which will be introduced here and outlined in more detail throughout this paper. In 1937, H. Whitney [Whi37] gave a classification of normal curves by showing that two curves with the same rotation number (*i.e.*, Whitney index) may be deformed into each other.

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More recently in [Arn94], V.I. Arnold axiomatically introduced the invariants St and $J^{\pm}$ for normal curves by normalizing their values on standard curves and showing how they change under different deformations of a curve. Several authors, such as A. Shumakovitch [Shu96], F. Aicardi [Aic94], and C. Luo [Luo97], then presented and refined explicit expressions for these invariants. In particular, Aicardi showed that the defect $(2St + J^{+})$ of any tree-like curve was zero. Then in 1998, M. Polyak gave a method for determining these invariants via Gauss diagrams [Pol98].

This paper proposes the result that the defect of normal curves is always nonnegative. Integral to this result were the explicit expressions for normal curves, principally those presented by Aicardi and Polyak. Moreover, the work of G. Cairns and D. Elton [CE93] in Gauss words motivated the construction of the Crossing graph, a major tool in the proof of our result. Additionally, we would like to thank Scott Weaver, a fellow REU participant, for his insight into the proof of Proposition 10.3.

2. Invariants

The Whitney index is an invariant that helps classify normal curves up to regular homotopies. The Arnold invariants are defined via classes of elementary standard curves and are affected by rules describing their change under deformations.

2.1. **Whitney Index.** Given a curve γ with a prescribed orientation, the Whitney index of γ is the total rotation number of the tangent vector as γ is traversed once in the direction determined by the orientation. The Whitney index of γ is denoted by $ind(\gamma)$. Note that reversing the orientation of γ would change the sign of $ind(\gamma)$. Hence, for a nonoriented curve γ only $|\,ind(\gamma)\,|$ can be defined.

Theorem 2.1 ([Whi37])**.** *Two curves γ_1 and γ_2 may be deformed into each other if and only if $ind(\gamma_1) = ind(\gamma_2)$.*

The standard representatives of curves with indices $0, \pm 1, \pm 2, \ldots$ are shown in Figure 2.

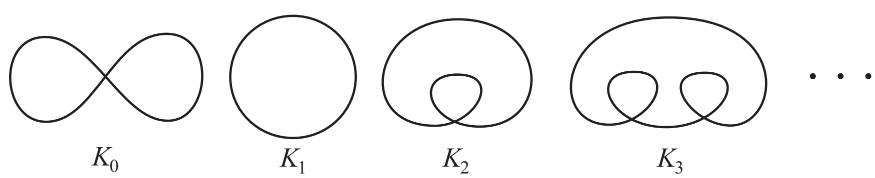

FIGURE 2. The standard curves K_0 and $K_1, K_2, \ldots$ with different orientations have indices $0, \pm 1, \pm 2, \ldots$.

Note that for $i = 0, 1, \ldots$, K_{i+1} is the simplest curve with i double points. Theorem 2.1 gives rise to equivalence classes based on Whitney index. That is, any normal curve γ where $ind(\gamma) = i$ can be deformed into K_i.

2.2. **The Arnold Invariants.** The Arnold invariants, normalized on the standard curves $K_0, K_1, K_2, \ldots$, are intimately tied to deformations of planar curves. These deformations, or perestroikas, take planar curves through intermediate points of self-tangency or through triple points.

Definition 2.2. *A point of self-tangency of a curve is called* direct *if the velocity vectors of the curve point in the same direction at the point of self-tangency and it is called* inverse *if the velocity vectors point in opposite directions.*

Definition 2.3. *A* perestroika *is a deformation of a curve in which an intermediate curve possesses a point of self-tangency or a triple point. There are three kinds of perestroikas: direct self-tangency, inverse self-tangency, and triple-point.*

Figure 3 shows two types of perestroikas. Note here that the inverse self-tangency perestroika is similar to the direct self-tangency perestroika.

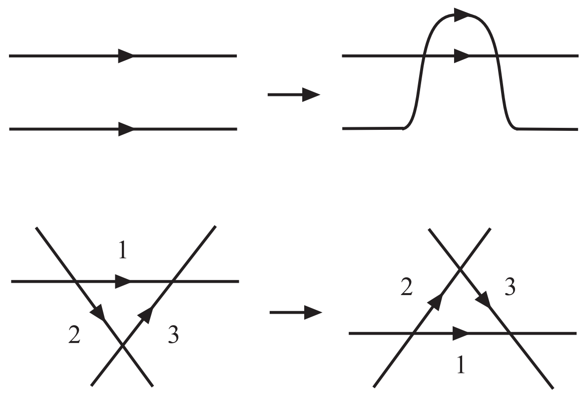

FIGURE 3. A direct self-tangency perestroika (above) and a triple-point perestroika (below).

Definition 2.4. *A* positive *(resp.* negative*) crossing of a self-tangency perestroika, or simply positive perestroika, increases (resp. decreases) the number of double points (by 2).*

The definition of a positive transversal crossing requires a more involved explanation. Here, as presented in [Arn94], a *vanishing triangle* is the triangle formed by three branches of a curve corresponding to three double points interconnected by arcs containing no other double points. The *sign of a vanishing triangle* is defined by the following construction. The orientation of the normal curve defines a cyclic ordering of the sides of the vanishing triangle (*i.e.*, the order of the visits of the triple point by the three branches). In this way, the sides of the triangle acquire orientations via this ordering. However, each side also has its own direction (from the original orientation of the curve) that may coincide, or not, with the orientation given by the ordering. For each vanishing triangle define a quantity q which takes the value of 0, 1, 2, or 3 according to the number of sides equally oriented by the ordering and by their respective directions. Then, the sign of a vanishing triangle is $(-1)^q$. Note that q does not change under a change of orientation on the curve, as both the cyclic order and the direction of the sides are reversed.

Definition 2.5 ([Arn94])**.** *A transversal crossing of a triple point is* positive *if the new-born vanishing triangle is positive.*

Arnold defined the invariants Strangeness St and $J^{\pm}$ of plane curves which correspond to these perestroikas.

Theorem 2.6. *(*[Arn94]*) There exist unique (up to additive constants) invariants St, J^+, and J^- of generic immersions of fixed index where,*

(1) *St remains constant under self-tangency perestroikas but increases by 1 after a positive crossing of a triple-point perestroika.*

(2) *J^+ remains constant under inverse self-tangency perestroikas and triple point perestroikas, but increases by a constant number a_+ under a positive crossing of a direct self-tangency perestroika.*
(3) *J^- remains constant under direct self-tangency perestroikas and triple point perestroikas, but increases by a constant number a_- under a positive crossing of the inverse self-tangency perestroika.*

The three invariants are normalized by the following conditions:

$$\left.\begin{array}{rclrcl} St(K_0) &=& 0, & St(K_{i+1}) &=& i, \\ J^+(K_0) &=& 0, & J^+(K_{i+1}) &=& -2i, \\ J^-(K_0) &=& -1, & J^-(K_{i+1}) &=& -3i, \end{array}\right\} \quad \text{for } i = 1, 2, \ldots.$$

In addition, $a_+ = 2$ and $a_- = -2$.

The two invariants St and J^+ give an expression for the defect of a normal curve.

Definition 2.7. *The* defect *of a planar curve γ, denoted $\delta(\gamma)$, is given by the formula,*

$$\delta(\gamma) = 2St + J^+.$$

In this way, the defect is also an invariant of normal curves. In section §5, the idea of defect is explored further.

3. The Seifert Decomposition

A Seifert decomposition is a primary tool for the study of planar curves. For example, as any planar curve is the projection of an alternating knot, a Seifert surface of a knot is an orientable surface with the knot as its boundary. Although Seifert surfaces will not be discussed in this paper, the Seifert decomposition encodes useful information for the calculation of certain invariants of planar curves.

Definition 3.1. *Let γ be a normal curve. Given a double point d of γ, a* Seifert splitting *changes the direction of γ locally at d as shown in Figure 4.*

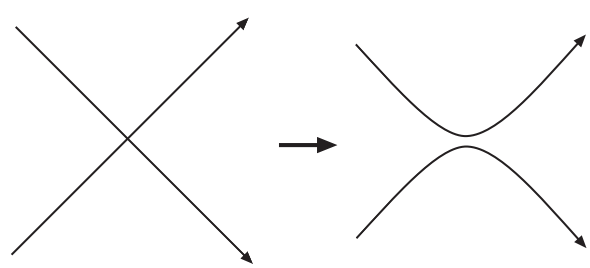

FIGURE 4. A Seifert splitting.

Remark. A Seifert splitting locally changes a double point from a transversal self-intersection to a tangential self-intersection. In other words, the crossing is eliminated by connecting each of the incoming branches to the adjacent branch leaving the crossing. For visual clarity, the two branches are typically pulled apart.

When the Seifert splitting process is applied to every crossing, a set of closed oriented cycles is produced, called *Seifert cycles*. Since only tangential self-intersections remain, one cycle must lie completely inside or outside every other cycle.

Definition 3.2. *Let c and s be Seifert cycles of a plane curve. If c and s share an exterior connection (*i.e.*, c and s are tangent to each other and neither is contained in the other), then c is said to be* exteriorly adjacent *to s. If c and s share an interior connection and c is contained in s, then c is said to be* interiorly adjacent *to s.*

In addition, the Seifert cycles of a normal curve have a natural orientation arising from the prescribed orientation of the curve.

Definition 3.3. *Given a Seifert cycle c of a normal curve γ, the index of c, denoted by $ind(c)$, is defined to be $+1$ (resp. -1) if c is oriented counterclockwise (resp. clockwise).*

4. EXPLICIT FORLUMAS FOR INVARIANTS

Here, explicit expressions for normal curves are given for the Whitney index as well as the Arnold invariants. However, necessary definitions for terms used in these expressions are presented first.

4.1. **Definitions.** As described in [Shu96], any normal curve γ gives rise to the partition of $\mathbb{R}^2$ into the connected components of $\mathbb{R}^2 - \gamma$, the arcs of γ between double points, and the double points themselves. This is a stratification of $\mathbb{R}^2$. Denote by Σ_k the set of all k-dimensional strata. Then, 0-strata are called *vertices*, 1-strata are called *edges*, and 2-strata are called *regions*. Moreover, all regions except the *exterior* region (which is homeomorphic to an annulus) are homeomorphic to a disk.

For any stratum, we can define its index with respect to an oriented normal curve γ.

Definition 4.1. *The index of a* region $\sigma \in \Sigma_2$ *is the total rotation number of the radius-vector connecting a point x in the interior of σ to a point moving along γ in the direction of the orientation. The index of an* edge $\sigma \in \Sigma_1$ *is the average of the indices of two regions adjacent to σ. The index of a* vertex $\sigma \in \Sigma_0$ *is the average of the indices of four regions adjacent to σ.*

Remark. The index of an arbitrary stratum σ is denoted by $ind_\gamma(\sigma)$.

The Seifert decomposition gives an alternative way to calculate the index of a region.

Proposition 4.2 ([Luo97])**.** *The index of a region is equal to the sum of the indices of the Seifert cycles which contain the region.*

Seifert cylces make the calculation of region index more intuitive than the already simple Shumakovitch method.

4.2. **Whitney Index.** The Seifert decomposition of a normal curve γ provides a simple formula for the Whitney index [BB03]:

$$ind(\gamma) = \sum_c ind(c) \tag{1}$$

over all Seifert cycles c of γ.

4.3. **Self-Tangency Invariants.** The Seifert decomposition is also instumental in the calculation of the self-tangency invariants $J^{\pm}$.

Definition 4.3. *For any cycle c of a normal curve γ, let*

$$t(c) = ind(c)ind_{\gamma}(R_c),$$

where R_c is the region exteriorly adjacent to c.

An alternative calculation of the t-function is given by the following proposition.

Proposition 4.4 ([Luo97]). *For any plane curve γ, there exists one and only one function t on the set of Seifert cycles, such that for cycles $c,s \in \gamma$,*

$$t(c) = \begin{cases} 0 & \textit{if } c \textit{ is a cycle with exterior boundary}, \\ t(s)+1 & \textit{if } c \textit{ is interiorly adjacent to } s, \\ -t(s) & \textit{if } c \textit{ is exteriorly adjacent to } s. \end{cases}$$

The following theorem gives rise to explicit expressions for $J^{\pm}$.

Theorem 4.5 ([Luo97]). *For a curve of n crossings and s Seifert cycles,*

$$\begin{aligned} J^{+} &= 1+n-s-2\sum_{c} t(c), \\ J^{-} &= 1-s-2\sum_{c} t(c), \end{aligned}$$

where c is over all Seifert cycles and t is the function defined in Proposition 4.4.

Note that the expressions for J^{+} and J^{-} differ only by the number of double points in the curve.

4.4. **Strangeness.** Again as presented in [Shu96], consider an oriented normal curve γ with a fixed initial point $x \in \gamma$, such that for all vertices $v \in \gamma$, $x \neq v$. Enumerate all the edges of γ from 1 to $2n$, where n is the number of vertices of γ, advancing in the direction imposed by the prescribed orientation. Note that the label 1 is assigned to the edge containing x.

Then, consider a vertex $v \in \gamma$. There are two edges adjacent to v and directed towards v according to the given orientation. This couple of edges are ordered (with (i, j)) by the condition that the tangent vectors to the edges i and j determine the positive orientation of the plane. Then the *weight* of v, denoted by $w(v)$, is defined to be $sgn(i-j)$. This structure gives rise to the following expression for Strangeness.

Theorem 4.6 ([Shu96]). *Let γ be a normal curve with a fixed initial point x. Then,*

$$St(\gamma) = \sum_{\sigma\in\Sigma_0} w(\sigma)ind_{\gamma}(\sigma) + \delta^2 - \frac{1}{4},$$

where δ is the index of the edge containing x.

Remark. The strangeness of a normal curve γ is always an integer.

Here, we will point out that combining this expression for St with that of J^{+} presented in §4.3 do not give rise to an expression for defect that is especially simple. Alternative expressions for defect are given in §5 and §6.

5. Tree-like Curves

A normal curve is *tree-like* if the removal of any double point is the union of two branches with no common points. More formally,

Definition 5.1 ([Luo97]). *A double point d of a normal curve γ is called* reducible *if the removal of this point, or $\gamma - \{d\}$, is the union of two disjoint branches. A normal curve is called* tree-like *if each of its crossings is reducible (See Figure 5).*

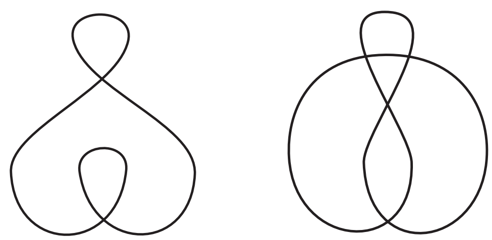

FIGURE 5. A tree-like curve (left) and a nontree-like curve (right).

Aicardi proved the following theorem concerning tree-like curves:

Theorem 5.2 ([Aic94]). *Arnold's invariants of a tree-like curve with n ordinary double points and with Seifert cycles c are given by*

$$St = \sum_c t(c), \quad J^+ = -2St, \quad J^- = -2St - n$$

Theorem 5.2 admits the following Corollary.

Corollary 5.3 ([Aic94]). *Every tree-like curve has defect zero.*

Here, notice that defect can be thought of as a measure of how close a curve is to being tree-like. An *almost* tree-like curve has defect zero, but is not tree-like according to Definition 5.1. The nontree-like curve in Figure 5 is an example of an almost tree-like curve. Notice that one J^- perestroika (a defect preserving operation) would deform this almost tree-like curve into one that was tree-like. In general, it seems that any almost tree-like curve could be continuously deformed into one that was tree-like via a sequence of J^- perestroikas or defect preserving combinations of J^+ and St.

6. Gauss Diagrams

Gauss diagrams provide a way to classify normal curves since many planar curves share a common Gauss diagram. Similarly, however, there are more Gauss diagrams than there are curves. A chord diagram is a generalized form of a Gauss diagram.

Definition 6.1 ([Aic94],[Pol98]). *A* chord diagram *is a finite set of chords of the standard circle having distinct endpoints. A* based *chord diagram includes a base point and an orientation on the circle.*

Definition 6.2 ([Pol98]). *The* Gauss diagram of a curve *is the standard circle with the preimages of each double point connected with a chord.*

The *Gauss diagram* of a normal curve γ with n double points is formed in the following manner. Let x be an arbitrary point on γ distinct from the double points of γ. Given an orientation of γ, traverse γ in the direction of the chosen orientation, starting at x. Enumerate each instance of a double point of γ with the integers in sequence, from 1 to $2n$, in the order in which each crossing occurs. Next, order the integers $1,\ldots,2n$ around the edge of a standard circle. For all n pairs of numbers that are identified with a single double point of γ, connect each pair with a chord (See Figure 6).

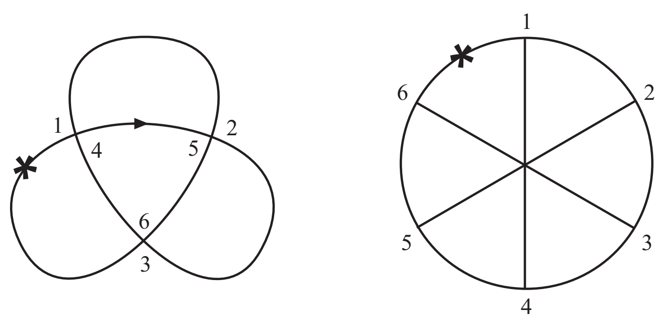

FIGURE 6. A planar curve and the corresponding Gauss diagram.

Definition 6.3 ([Aic94])**.** *A Gauss diagram is called* planar *if the chords do not intersect.*

This property of Gauss diagrams was shown by Aicardi to be closely related to tree-like curves.

Theorem 6.4 ([Aic94])**.** *The Gauss diagram of a normal curve is planar if and only if the curve is tree-like.*

It is sometimes convenient to consider *based* Gauss diagrams which assign to each of the chords of the Gauss diagram either a positive or negative sign. Given an orientation of γ and a base point x distinct from the double points of γ, follow the convention described in §4.4. As each chord c represents a double point d of γ, each chord is labeled with a ± 1, denoted by $sign(c)$, according to whether $w(d) = \pm 1$.

Remark. Notice that the signs prescribed to the chords depend upon the orientation of γ and the choice of x. If the orientation is reversed, the signs of the chords will change. Similarly, if x is moved through a double point, the sign of that chord changes.

As introduced in [Pol98], a *representation* $\phi\colon A \to G$ of a chord diagram A in a Gauss diagram G, is an embedding of A to G mapping the circle of A to the circle of G (preserving orientation), each of the chords of A to a chord of G and a basepoint to a basepoint. For such a representation $\phi\colon A \to G$ define

$$sign(\phi) = \prod sign(\phi(c))^{m(c)}$$

by taking the product over all chords c of A of signs of the chords $\phi(c)$ in G with the multiplicity $m(c)$ of c. Denote by $\langle A, G\rangle$ the sum

$$\langle A, G\rangle = \sum_{\phi: A \to G} sign(\phi)$$

over all representations $\phi : A \to G$.

Consider the chord diagram B_4, which is shown in Figure 7.

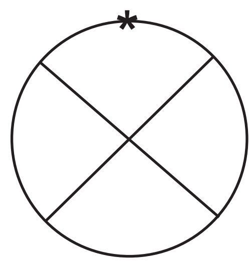

FIGURE 7. The chord diagram B_4

This chord diagram is instrumental in calculating the defect of a curve via its Gauss diagram.

Theorem 6.5 ([Pol98])**.** *Let γ be a normal curve with Gauss diagram G_γ. The defect of γ is given by,*

$$\delta(\gamma) = -2\langle B_4, G_\gamma \rangle.$$

Compared to the expression for defect given in §4, the method of calculating defect in Theorem 6.5 is less computationally intensive. Also, note from this expression that defect depends only on how signed chords cross in a Gauss diagram.

7. SUMMATIONS

This section presents two forms of summations: ordinary connected sums and interjected sums. In [Arn94], Arnold showed that defect is additive under the ordinary connected sum. We will show that defect is additive under the interjected sum.

7.1. **Ordinary Connected Sums.** Arnold introduced both an ordinary and a strange connected sum in [Arn94]. However, for the purposes of this paper, only ordinary connected sums need be considered.

Definition 7.1 ([Arn94])**.** *Let φ_1 and φ_2 be two immersions of a circle into the right and left half-planes $\mathbb{H}_r$ and $\mathbb{H}_l$ respectively. The* (ordinary) connected sum *of φ_1 and φ_2 is the new immersion shown in Figure 8.*

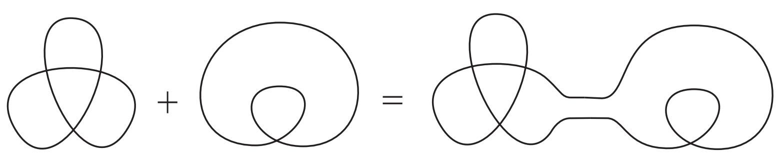

FIGURE 8. The ordinary connected sum of the trefoil and K_2.

Based on Arnold's work, the relationship between ordinary connected sums and defect is clear.

Proposition 7.2 ([Arn94]). *Given normal curves γ_1 and γ_2, the invariants $St, J^{\pm}$ are additive under the ordinary connected sum of γ_1 and γ_2:*

$$St(\gamma_1+\gamma_2) = St(\gamma_1)+St(\gamma_2),$$

$$J^{\pm}(\gamma_1+\gamma_2) = J^{\pm}(\gamma_1)+J^{\pm}(\gamma_2).$$

Corollary 7.3. *Defect is additive under the ordinary connected sum.*

Proof. The Corollary follows from Theorem 7.2. □

7.2. **Interjected Sums.** Unlike the ordinary connected sum defined by Arnold, in the case of the interjected sum the immersions φ_1 and φ_2 do not reside in the right and left halfplanes. Rather, φ_2 is immersed into a bounded region of φ_1.

Definition 7.4. *Let φ_1 and φ_2 be two immersions of a circle such that for $\sigma \in \Sigma_2$ of φ_1, $\varphi_2 \in \sigma$. The* interjected sum *of φ_2 into φ_1 is the new immersion shown in Figure 9.*

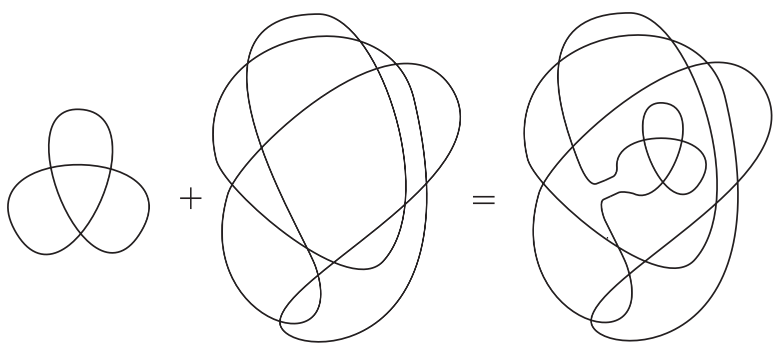

FIGURE 9. The interjected sum of the trefoil into a nonstandard normal curve.

Similar to the ordinary connected sum, defect is also additive under the interjected sum.

Theorem 7.5. *Defect is additive under the interjected sum.*

Proof. Let η and ν be normal curves, and let γ be the interjected sum of ν into η. Further, let G_γ, G_η, and G_ν be the Gauss diagrams of these respective curves. Let a be the arc of η where the bridge from ν is connected, and let a' be the section of the perimeter of G_η which corresponds to a in η. Then G_γ is formed by inserting the chords of G_ν into a'.

Since the defect of a curve depends only on the crossings of the chords of the Gauss diagram of the curve and since G_γ contains exactly the crossings of G_η and G_ν, then $\delta(\gamma) = \delta(\eta)+\delta(\nu)$.

Note that it might be necessary to reverse the orientation of ν before performing the interjected sum. However, since reversing the orientation of ν changes the sign of every chord in G_ν, the defect of ν remains unchanged. □

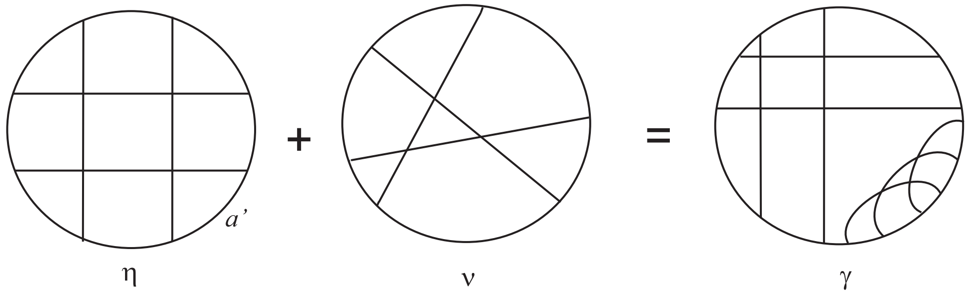

FIGURE 10. The effect of interjected sum on Gauss diagrams.

8. GAUSS WORDS

An unsigned Gauss word is a word in some finite alphabet $\{a_1, a_2, ..., a_k\}$ where each letter occurs exactly twice. A signed Gauss word is an unsigned Gauss word where each letter is given a superscript $+1$ or -1, such that the two occurrences of a_i are given indices of opposite sign.

We can describe the construction of a Gauss word as follows. Given a normal curve γ, label the intersection points with the letters $a_1, \ldots, a_k$. Let x be an arbitrary point on γ. Choosing an orientation for γ, traverse γ beginning at x.

Remark. Recall that a crossing is positive (resp. negative) if while traversing through the double point the arc that one crosses is oriented left to right (resp. right to left). Alternatively, the first occurence of a_i is assigned the superscript $w(a_i)$, while the second occurence is defaulted to the index of opposite sign.

At each crossing a_i, record the symbol a_i^{+1} if the crossing is of positive nature; else record the symbol a_i^{-1}. The list that is constructed in this manner is called the *Gauss word* of γ. In practice, the positive indices are omitted.

Let w be a signed Gauss word. Note that w can be written in the form

$$w = a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_{2k}}^{j_{2k}} = \prod_{n=1}^{2k} a_{i_n}^{j_n}$$

where $j_n = \pm 1$. Define a function $\phi \colon w \to w$ by the mapping

$$\prod_{n=1}^{2k} a_{i_n}^{j_n} \longmapsto \prod_{n=1}^{2k} a_{i_n}$$

In this way, $\phi(w)$ admits the unsigned equivalent of w.

Definition 8.1 ([CE93])**.** *A signed Gauss word is called* planar *if it is the intersection sequence of a normal closed curve. (An unsigned Gauss word is planar if it can be given a signing for which it is planar.)*

Given a set of the form $S = \{a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}\}$, one can consider an *abstract* Gauss word.

Definition 8.2 ([CE93])**.** *An* abstract *Gauss word is a permutation of the set $S = \{a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}\}$ (*i.e.*, a word in the elements of S where each element occurs exactly once).*

Definition 8.3. *A Gauss word w is called* minimal *if all possible cancellations (*i.e. $a_i a_i^{-1}$ *or* $a_i^{-1} a_i$*) are made. If a minimal Gauss word w is the identity, then w is said to be trivial; else, it is called non-trivial.*

Abstract Gauss words motivate the following question: given an abstract Gauss word w, when is w planar? Although this question will not be explored in depth in this paper, Gauss gives a necessary (but not sufficient) condition that will be of use. As found in [CE93], Gauss showed that for an abstract Gauss word w to be the Gauss word of a normal curve, then for every letter $a_i \in w$, an even number of letters must occur in w between a_i and a_i^{-1}. According to the notation in [CE93], consider an abstract word w with letters $a_1, \ldots, a_k$ and let $i \in \{1, \ldots, k\}$. By cyclically permuting w, it can be assumed that a_i occurs in w before a_i^{-1}. Under this assumption,

(1) let S_i denote substring of letters between, but excluding, a_i and a_i^{-1}, and
(2) let $\alpha_i(w)$ denote the sum of the indices of the elements of S_i.

Theorem 8.4 ([CE93])**.** *If an abstract Gauss word w with k crossings is the Gauss word of a closed normal planar curve, then* $\alpha_i(w) \equiv 0$ *(mod 2), for all* $i \in \{1, \ldots, k\}$.

9. Crossing Graph

Given a planar Gauss word w, a crossing graph is a way to reconstruct the normal curve that w represents. The construction is as follows. Without loss of generality, cyclically permute w so that a_1 is the first letter. Then, for the first occurance of each letter $a_1, \ldots, a_k$ in w, associate an arrow oriented up or down according to the sign of each letter. Let a downward (resp. upward) arrow represent a positive (resp. negative) superscript. Align the arrows in a row from left to right, preserving the order that the associated letters first appear in w (See Figure 11).

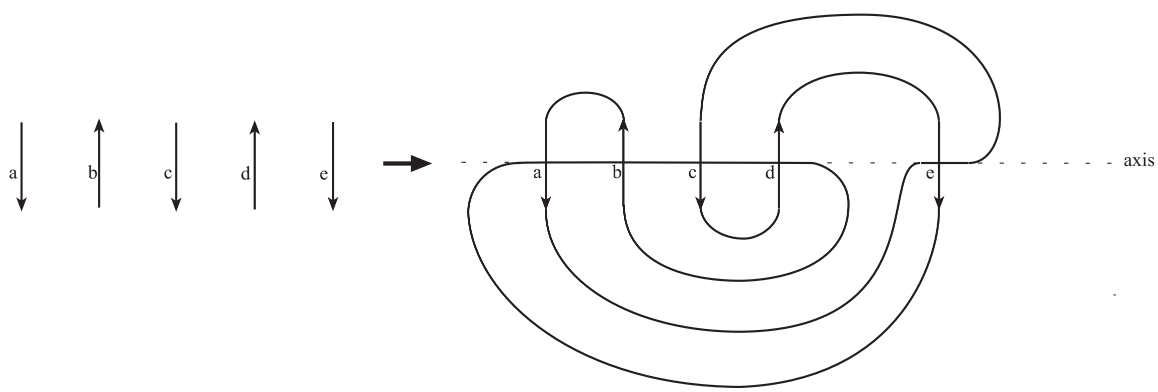

Figure 11. The arrow placements (left) and the crossing graph (right) for $ab^{-1}cd^{-1}ba^{-1}ec^{-1}de^{-1}$. Notice that crossing e begins a second partition of the axis of C.

Next, horizontally cross each arrow from left to right as the first occurance of each letter appears in w. This horizontal line is called the *axis*. In this way, each crossing has a horizontal and a vertical component. It is often the case that for two letters a_i and a_j, both a_i and a_i^{-1} occur in w before a_j. In this case, draw an arc (either clockwise or counterclockwise) back to the arrow corresponding to a_i such that the arrow will be traversed in the appropriate direction. Note that when this occurs, the axis is partitioned into sections, as the crossing a_j will be in a different section of the axis than a_i. According to w continue the process of either crossing arrows along the axis (for the first occurance of a letter) or drawing an arc back to an arrow (for the second occurance of a letter). Throughout this process, no additional crossings other than the original k are permitted.

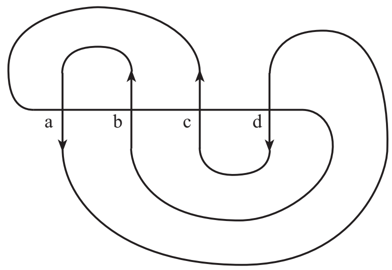

FIGURE 12. The crossing graph for $ab^{-1}c^{-1}dba^{-1}d^{-1}c$

Remark. A natural extension of the axis of a crossing graph C determines two halfplanes $\mathbb{H}_1$ and $\mathbb{H}_2$. In general, a halfplane of C is denoted by $\mathbb{H}_C$.

Definition 9.1. *Let C be a crossing graph of a planar Gauss word w, let $a,b \in w$, and let $A_1,\dots,A_m$ be partitions of the axis of C. First, any crossings $c_1,\dots,c_k$ that occur between a and b in A_i are said to be* intermediate *crossings with respect to a and b. Second, if*

(1) *$a,b \in A_i$, and*
(2) *the vertical component of a is joined to the vertical component of b with an arc α such that $\alpha \in \mathbb{H}_C$,*

then a and b are said to be arc-joined.

In a crossing graph C, a *loop* is formed when an arc connects the horizontal component of a crossing directly to the vertical component of the same crossing.

10. J-ADJACENCY

The property of J-adjacency relates a possible $J^{\pm}$ perestroika of a normal curve to the corresponding Gauss word or Crossing graph.

Definition 10.1. *Let w be a Gauss word. For some letters $a,b \in w$, consider the sets $S_1 = \{a, b^{-1}\}$ and $S_2 = \{a^{-1}, b\}$. If for S_1 and S_2 the respective elements appear adjacent to each other in w, then a and b are said to be* J-adjacent*. Alternatively, a and b are called* J-adjacent *if they are arc-joined with no intermediate crossings.*

Lemma 10.2. *Given a non-trivial Gauss word w of a planar curve γ, let C be a crossing graph of w. Then, at least two crossings in C must be arc-joined.*

Proof. Let w be a non-trivial Gauss word of a planar curve γ and let C be a crossing graph of w. Suppose that $A_1,\dots,A_m$ are partitions of the axis of C, labelled from left to right. Consider the partition A_1. First, we show that A_1 must contain three or more crossings.

If A_1 contains only one crossing a, the crossing must represent a substring aa in $\phi(w)$. This contradicts the hypothesis that w is minimal. Suppose that A_1 contains two crossings a and b, where a preceeds b. There are two cases:

(1) A substring of the form *abba* occurs in $\phi(w)$, or
(2) $\alpha_a(w) = 1$.

In the first case, two trivial loops are formed which again contradicts the fact that w is minimal. In the second case, the non-zero α-function contradicts the hypothesis that w is a planar Gauss word. Hence, A_1 must have three or more crossings.

Now, again consider A_1 and suppose it has crossings $a_1, \dots, a_k$. Assume without loss of generality that a_r preceeds a_s in w, for $1 \leq r < s \leq k$ (else relabel). Note that $a_1 a_2 \dots a_k$ must be a substring of $\phi(w)$. Furthermore, the horizontal component of a_k must connect to the vertical component of a_j, where $(k-j) mod\ 2 = 0$, else $\alpha_{a_j}(w) = 1$. Given this restriction on j, it follows that $a_j, \dots, a_k$ present an even number of crossings. Notice that the arc η joining a_k to a_j together with the axis of A_1 bound a region $\sigma \in \Sigma_2$ of C.

Suppose a_l is intermediate to a_j and a_k and without loss of generality assume that the vertical component of a_l is directed into the region σ. Then, an arc must connect a_l to another crossing a_m that is also intermediate to a_j and a_k, else an additional double point would be necessary somewhere along η to ensure the closure of γ. Note that the vertical component of a_m is directed out of σ. Since the arc connecting a_l to a_m necessarily lies in the interior of $\sigma \in \mathbb{H}_C$, it follows that a_l and a_m are arc-joined. □

Proposition 10.3. *Given a non-trivial Gauss word w of a planar curve γ, let C be a crossing graph of w. Then for some $a, b \in w$, a and b are J-adjacent in C.*

Proof. Let w be a non-trivial Gauss word of a planar curve γ and let C be a crossing graph of w. Suppose that no two letters of w are J-adjacent. Consider two letters $a, b \in w$ of a segment A_i, where a and b are arc-joined in such a way that no other pair of letters in w is arc-joined with fewer intermediate crossings. By Lemma 10.2 this arc-joined pair exists in C. Since a and b are not J-adjacent, there must be a crossing c intermediate to a and b. Furthermore, without loss of generality it can be assumed that the vertical component of c is directed into the region $\sigma \in \Sigma_2$ bounded by the arc connecting a and b and the axis of A_i. In order for γ to be closed, the part of γ that entered σ via c must also exit. This necessitates a second crossing $d \in A_i$ intermediate to a and b such that the vertical component is directed out of σ. Since crossings $c, d \in A_i$ are arc-joined, a and b cannot be arc-joined with the fewest number of intermediate crossings and we have arrived at a contradiction. □

11. The effect of removing J-adjacent double points

Just as J-adjacent double points represent possible $J^{\pm}$ perestroikas, the removal of a pair of J-adjacent double points corresponds to the deformation of a curve via a negative $J^{\pm}$ perestroika.

Lemma 11.1. *Given a non-trivial Gauss word w of a normal curve γ, let C be the crossing graph of w. Then, for a J-adjacent pair of double points a and b, removing a and b from C is equivalent to a negative $J^{\pm}$ perestroika.*

Proof. Let w be a non-trivial Gauss word of a normal curve γ, and let a and b be a pair of J-adjacent double points in the crossing graph C of γ. Then, the elements of $\{a, b^{-1}\}$ and $\{a^{-1}, b\}$ are pairwise adjacent in w and a and b are arc-joined in C. If in $\phi(w)$ it happens that a and b have the structure *...ab...ba...*, then the arc joining a and b has opposite direction from that of the axis of C. In this case, the arc connecting a and b can be pulled across the axis in a negative J^{-}

perestroika. Similarly, if a and b have the structure $...ab...ab...$ in $\phi(w)$, then the arc connecting a and b shares the same direction as the axis. In this case, the arc can be pulled across the axis in a negative J^+ perestroika.

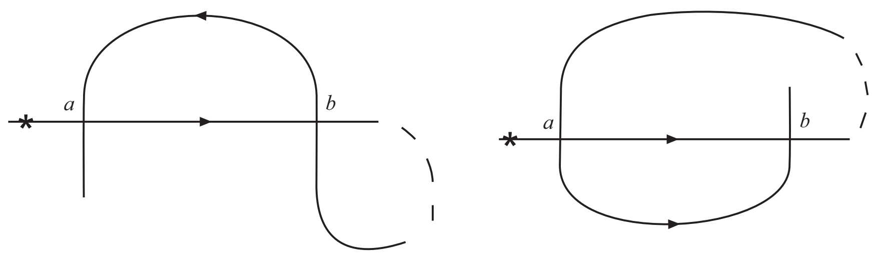

FIGURE 13. Right: $...ab^{-1}...a^{-1}b...$ Left: $...ab^{-1}...ba^{-1}...$

Hence, the proof is complete. □

According to Lemma 11.1, the relationship between the removal of J-adjacent double points and $J^{\pm}$ perestroikas is clear. The correllation between negative $J^{\pm}$ perestroikas and the curve itself is also obvious.

Definition 11.2. *Let γ be a normal curve and let a and b be a J-adjacent pair of double points of γ. Then, γ is* reduced *to γ' if γ' is the result of performing a negative $J^{\pm}$ perestroika on γ via removing a and b.*

Remark. Note that if γ' is the reduction of γ, then γ' is also a normal curve, as performing a $J^{\pm}$ perestroika does not map a normal curve to one that is non-generic.

A natural question that then arises is, how many $J^{\pm}$ perestroikas can be performed on a given normal curve?

Definition 11.3. *The* reduction number *of a normal curve γ is the minimum number of negative $J^{\pm}$ perestroikas needed to reduce γ to the trivial curve, where the corresponding Gauss word is minimized before and after each deformation.*

Though there may be multiple sequences of $J^{\pm}$ perestroikas that reduce a curve to the identity, there clearly exists a sequence with a minimum number of deformations.

12. NONNEGATIVE DEFECT

The following Lemma utilizes the fact that defect is additive under connected and interjected sums.

Lemma 12.1. *Let w be the Gauss word of a normal curve γ. Minimizing w preserves the defect of γ.*

Proof. Let w be the Gauss word of a normal curve γ. In order to minimize w, obvious cancellations of the form aa^{-1} or $a^{-1}a$ are made, where $a \in w$. In γ, these types of substrings correspond to loops. Any loop can be thought of as either the connected or interjected sum of the curve K_0 with another curve $\bar{\gamma}$. Since K_0 has defect zero, the it follows that $\delta(\bar{\gamma}) = \delta(\gamma)$ under the sum, since defect is additive in both cases (Corollary 7.3 and Theorem 7.5). In the same way, reversing the sum will preserve the defect of γ. □

The ability to remove loops without altering defect is essential in arriving at a reduction number. This is true because if loops could not be removed, there would not necessarily exist a sequence of $J^{\pm}$ perestroikas that would reduce a given curve to the identity. In addition, if removing loops altered defect, the invariant *St* would have to be considered in order to reduce curves to the trivial loop.

Theorem 12.2. *Every normal curve has nonnegative defect.*

Proof. Let γ be a normal curve. Suppose γ has reduction number 0. Then γ can be reduced to a circle simply by removing some number of loops. Since γ is tree-like, by Corollary 5.3, $\delta(\gamma) = 0$.

Assume that every curve with reduction number k has nonnegative defect. Suppose that η is a curve with reduction number $k+1$. By Proposition 10.3, after removing any loops from η, there is at least one pair of double points of η which are J-adjacent. By Lemma 11.1, η can be reduced to η' by a negative $J^{\pm}$ perestroika, where η' has reduction number k by definition. Note that removing loops from η preserves defect (Lemma 12.1) and negative J^{-} perestroikas preserve defect (evident from the defect formula). Thus, since a negative J^{+} perestroika reduces defect of η by 2, it follows that $\delta(\eta) \geq \delta(\eta')$. By assumption, $\delta(\eta') \geq 0$. Hence by induction, the proof is complete. □

In addition, Theorem 12.2 yields an algorithm for the calculation of defect of a normal curve γ. Given the reduction number of γ, record the number of J^{+} perestroikas, p, that occur. The defect of γ is then given by $\delta(\gamma) = 2p$. However, though this calculation is straight forward, determining the reduction number of a curve becomes more difficult as the number of double points increases. Given a Gauss word of such a curve, there may be several initial $J^{\pm}$ perestroikas that could be performed. However, removing different pairs of double points gives rise to different loops that may be cancelled and different $J^{\pm}$ perestroikas that may be performed following the reduction. Since each reduction often depends on the previous reduction, this iterative process becomes increasingly complex with each additional double point, and hence determining the reduction number may prove to be quite tedious.

13. Conclusion

In this paper, we have shown that defect is never negative for normal curves. To do this we built upon the idea of a connected sum, introducing the interjected sum. By showing that under both operations defect is additive, we were able to remove loops from curves without changing defect. Then, via Crossing graphs and the structure of corresponding Gauss words, we were able to show that along with removing loops, $J^{\pm}$ perestroikas could be sequentially performed in order to reduce a normal curve to the trivial loop. This allowed us to prove the main result that defect is nonnnegative.

Through this study, two ideas for future investigation become evident. One persuit is the classification of almost tree-like curves, or more generally, curves with defect $2n$. Based on Polyak's formula for the defect of curves (derived from their representative Gauss diagrams), curves can be classified according to how many crossings occur between chords of the same sign and chords of opposite sign in the Gauss diagram.

A second investigation stems from the work of Cairns and Elton [CE93]. Along with the α-funtion that was defined in §8, in [CE93] an additional β-funtion was defined. Work could be done relating this β-function to Gauss diagrams (recall that there are more Gauss diagrams than there are curves). The hope would be to define a complete set of criteria that would determine whether a Gauss diagram actually corresponds to a curve.

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