INTERSECTION NUMBERS OF CLOSED CURVES ON THE PUNCTURED TORUS

FRANK CHEMOTTI AND ANDREA RAU

ADVISOR: DENNIS GARITY
OREGON STATE UNIVERSITY

ABSTRACT. The authors consider self-intersections of closed curves on the punctured torus. The *intersection number* of a homotopy class of curves is defined to be the minimum number of transverse self-intersections among all curves in that class. Homotopy classes are denoted by words in the free group on two generators $F(a,b)$. Formulas for intersection number of some particular classes of primitive words are proved, as well as a formula for intersection number of a power of a primitive word given the intersection number of the primitive word. Geodesics of the hyperbolic plane that project to closed curves on the torus are also considered in order to achieve some similar results.

1. INTRODUCTION

Let T be the torus with one puncture. $\pi_1(T)$ is the free group on two generators. Then if w is a word in $F(a,b) \cong \pi_1(T)$, we would like to study the self-intersections of the closed curves on T that represent w. We say that two closed curves f and g on T are equivalent if there is some homeomorphism of T that maps f to a curve that is freely homotopic to g. Then two words w_1 and w_2 in $F(a,b)$ are equivalent if there is some automorphism of $F(a,b)$ that maps w_1 to w_2. Let the *intersection number* of a free homotopy class of a closed curve f on T be the minimum number of transverse self-intersections among all general position curves in that class. Then the intersection number of a word w in $F(a,b)$ is the intersection number of the corresponding class of curves. So given any word w, it would be nice to be able to determine the intersection number by just looking at the form of w. In this paper, we prove formulas for some particular classes of words. In sections 2 through 6 the first author employs some topological and geometric methods: in section 2, a convention for drawing curves is introduced, in section 3, some characterizations of minimal intersection are proved, in section 4, intersection numbers for some primitive classes of words are proved, in section 5, the intersection number for powers of primitive words is proved, and in section 6, some other related results are presented. In sections 7 through 10 the second author employs geometric and algebraic methods, examining geodesics in the hyperbolic plane, the universal cover of T: in section 7, the necessary hyperbolic geometry is introduced, in section 8, algebraic methods are used to investigate particular classes of geodesics, in section 9, geometric methods are used, and in section 10, similar methods are applied to additional classes of words.

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2. A "Canonical Form" for Closed Curves

Throughout the next sections, we assume that every word is reduced (contains no subwords equivalent to the identity word) and that every curve is in general position (passes through no point more than twice). For a word w in $F(a,b)$, $\phi(w)$ denotes the intersection number of w. We will use intersection to mean transverse self-intersection. For a closed curve $f : S^1 \to T$, w_f denotes the word in $F(a,b)$ that f represents. We consider all words to be "cyclic" words (i.e. that the first letter follows the last letter, so that there is no real preferred first or last letter), since they represent free homotopy classes of closed curves. We say that a general position curve has *excess intersection* if it is homotopic to a curve with fewer intersections.

The punctured torus is homeomorphic to a square with opposite edges identified and corners removed. Let this square be in the Euclidean plane, centered at the origin with side length 2. Call this square S. Let loops $a : S^1 \to T$ and $b : S^1 \to T$ (where S^1 is defined as $[0,1]/(0 \sim 1)$) be defined as follows:

$$a(t) = \begin{cases} (2t,0) & \text{if } t \leq \frac{1}{2}, \\ (2t-2,0) & \text{if } t \geq \frac{1}{2}, \end{cases}$$

$$b(t) = \begin{cases} (0,2t) & \text{if } t \leq \frac{1}{2}, \\ (0,2t-2) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

The fundamental group of T is the free group on two generators. Products of a and b (and their inverses A and B) generate loops in all free homotopy classes and there is no relation between a and b, so $\pi_1(T) \cong F(a,b)$.

In our search for minimal self-intersection, we would like to reduce the much too large free homotopy classes to smaller, more manageable classes. We would like to be able to deform every curve to a curve in some sort of "canonical form", and we would like this to decrease self-intersection as much as possible, or at least certainly not increase self-intersection. We will start by examining curves on S.

Let f be any closed curve in T, projected onto S. Define an arc to be a path in S joining two edges of the boundary of S. So f is composed of several arcs. We may continuously deform each arc in f to a straight line segment, while fixing the endpoints (Figure 1.1 to 1.2). Call the resulting curve f'. So f' crosses the edges of S at the same points and in the same order as f, but all paths between consecutive edge crossings are straight line segments. I claim that f' cannot have more intersections than f. Suppose that two arcs of f', denoted by segments AB and CD, intersect each other. Then they must intersect exactly once, and the pairs of points $\{A,B\}$ and $\{C,D\}$ must separate each other on the boundary of S. So any two arcs traveling from point A to point B and from point C to point D, respectively, must intersect. Thus the two paths in f that were deformed into segments AB and CD in f' must intersect at least once. Thus every intersection in f' corresponds to a different intersection in f, so f' has no more intersections than f.

However, f' still may have some unnecessary arcs if f has unnecessary edge crossings. The necessary edge crossings of f are those corresponding to the letters in the reduced word w_f (the correspondence between edges and letters is: right $\leftrightarrow a$, top $\leftrightarrow b$, left $\leftrightarrow A$, bottom $\leftrightarrow B$, which is derived from the definitions of loops a and b above). If there are any unnecessary edge crossings in f, there must be an arc that starts and ends on the same edge of S. This arc, in f', then becomes a segment lying entirely on that edge of S. Let B and C be the points where this segment starts and ends, respectively. Then let A be the starting point of the arc with endpoint B, and let D be the endpoint of the arc with starting point C. Let f'' be the curve obtained from f' by replacing segments AB, BC, and CD with segment AD (Figure 1.2 to 1.3). I claim that f'' does not have more intersections than f'. If some segment intersects segment AD, then it must intersect at least once any path from point A to point D that does not cross the boundary of the square. In particular, it must intersect the path in f' composed of segments AB, BC, and CD at least once. Thus every intersection in f'' corresponds to a different intersection in f', so has no more intersections than f'. This process can be applied repeatedly until we get a curve with no unnecessary edge crossings and no segments lying entirely on an edge. Let g be this final curve.

Then we see that g must cross the edges corresponding to each letter in the same order that they occur in w_f, and it crosses no additional edges. And if necessary, we may also deform g slightly so that no intersections occur on the boundary of the square, without changing the number of intersections. We summarize the previous discussion in the following lemma.

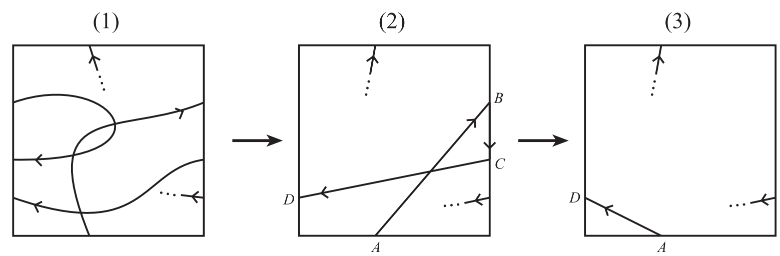

FIGURE 1

Lemma 2.1. *Let f be a closed curve on T. Then f is freely homotopic to a closed curve g which has no more intersections than f and satisfies the following properties:*

(1) *as g is traversed once on S, the letters corresponding to the edges crossed (right $\to a$, top $\to b$, left $\to A$, bottom $\to B$) form w_f,*
(2) *all arcs between consecutive edge crossings are straight line segments,*
(3) *all intersections with edges of S are transverse, and*
(4) *no intersections occur on an edge of S.*

We say that a closed curve satisfying these four properties is in canonical form.

3. METHODS FOR PROVING MINIMALITY

It is easy to prove upper bounds for the intersection number of a word. If we draw any closed curve representing a given word, the number of transverse self-intersections of that curve gives an upper bound for the intersection number. It is more difficult to prove lower bounds or to prove that a curve has no excess intersection. Fortunately, Joel Hass and Peter Scott have proved several helpful theorems. The following definition and two theorems are from [HS], slightly modified for consistency of notation.

Definition 3.1. *If $f : S^1 \to F$ is a closed curve on a surface F, we say that f has a* 1-gon *if there is a subinterval I of S^1 such that f identifies the endpoints of I and $f|_I$ defines a nullhomotopic loop on F. We say that f has a* 2-gon *if there are disjoint subintervals I and J of S^1 such that f identifies* *the endpoints of I and J and $f|_{I\cup J}$ defines a nullhomotopic loop on F. And f has a* weak 2-gon *if there are distinct (but not necessarily disjoint) subintervals I and J of S^1 such that f identifies the endpoints of I and J and $f|_{I\cup J}$ defines a nullhomotopic loop on F.*

Theorem 3.2. [HS] *Let f be a closed curve on an orientable surface F. If f has excess self-intersection, then f has a 1-gon or 2-gon.*

Remark 3.3. *The converse of the above theorem is true, which is easy to see. For if f has a 1-gon, it is a nullhomotopic loop and can be removed to decrease the number of intersections by one. And if f has a 2-gon, its two edges form a nullhomotopic loop, and there is a homotopy that exchanges the two edges, removing two transverse self-intersections (disjointness of I and J is required here).*

Theorem 3.4. [HS] *Let f be a closed curve on a surface F. If f has excess self-intersection, then f has a 1-gon or a weak 2-gon.*

Remark 3.5. *The converse of the above theorem is not true in general: a curve for a^2 must always contain a weak 2-gon, even when drawn without excess self-intersection. I claim that the converse is true if and only if w_f is primitive (it can be seen that any closed curve representing a nonprimitive word must have a weak 2-gon and any closed curve representing a primitive word that has a weak 2-gon must also have a 2-gon).*

Wei will need to introduce some additional concepts and notation to give a useful characterization of 1-gons and 2-gons. We already have a clear correspondence between closed curves on T and words in $F(a,b)$. We will define a similar correspondence between subcurves (a restriction of a closed curve f to a subinterval of S^1) and subwords. Let f be a closed curve in canonical form on S. If g is a subcurve of f, let us form the subword of w_f corresponding to g by traversing g on S and reading off the letters corresponding to the edges crossed (just as in Lemma 2.1). If an endpoint lies on an edge of S, we will consider that to be an edge crossing. And if v is a subword of w_f, we can find the corresponding subcurve of f by restricting its domain from S^1 to an interval such that its image starts at any point whose immediate next edge crossing corresponds to the first letter of v and ends at any point whose immediate previous edge crossing corresponds to the last letter of v (and the edges crossed in between match up with the letters in between, of course). So for a subcurve g of closed curve f, define w_g to be the corresponding subword of w_f. Note that subwords, unlike (cyclic) words, do have well-defined first and last letters, and that subcurves are not necessarily closed. More generally, we can make the same correspondence from any (open) curve to a (noncyclic) word. Then it is clear that if f is a curve traveling from point A to point B (B on the interior of S), g is a curve traveling from point B to point C, and h is the curve traveling from point A to point C along f and g, $w_h = w_f w_g$. An arc is a special case of a subcurve representing a two-letter word whose endpoints lie on the boundary of S, so we will label arcs by their corresponding words.

Lemma 3.6. *Let f be a closed curve in canonical form. Then f does not have any 1-gons. Also, if f has a 2-gon and if $g = f|_I$ and $h = f|_J$ are the subcurves forming the edges of the 2-gon (I and J defined as in Definition 3.1), then $w_g = (w_h)^{\pm 1}$.*

Proof. f cannot have any 1-gons since every arc is a straight line segment and all intersections with edges of S are transverse.

Suppose that f has a 2-gon. Let g and h be the subcurves forming the edges of the 2-gon (as above). The union of the images of g and h must be a null-homotopic loop, so depending on the relative orientation of the two curves, either $w_g w_h$ or $w_g (w_h)^{-1}$ must be equivalent to the identity word. Also, since f is in canonical form, subwords w_g and w_h are reduced. Thus $w_g w_h = id$ implies $w_g = (w_h)^{-1}$ and $w_g (w_h)^{-1} = id$ implies $w_g = w_h$. Moreover, since the words are reduced, this equality is not simply that the two words represent the same element of $F(a,b)$, but that the two words are identical strings of letters. □

In the case where the two subcurves have the same orientation ($w_g = w_h$), the first 2-gon vertex must be an intersection of two arcs with the same second letter (which is the first letter of w_g), the other 2-gon vertex must be an intersection of two arcs with the same first letter (which is the last letter of w_g). And in the case where the two subcurves have the opposite orientation, the same holds if we invert all the labels of arcs that are part of h.

4. Intersection Numbers for Some Primitive Classes of Words

Theorem 4.1. $\phi\left(a^i b^j\right) = (i-1)(j-1)$.

Proof. First, we would like to show that $\phi\left(a^i b^j\right) \leq (i-1)(j-1)$. Figure 2 shows a method of drawing a curve f representing $a^i b^j$ that has $(i-1)(j-1)$ intersections. First, i copies of a are drawn with no intersections, moving upwards in S. Then j copies of b are drawn, moving rightwards in S. This makes $j-1$ complete vertical arcs, each of which intersects each of the $i-1$ complete horizontal arcs. Then the curve can be closed. Thus there are $(i-1)(j-1)$ total intersections for f, which represents $a^i b^j$ (by Lemma 2.1, since it crosses the correct edges in the correct order). So $(i-1)(j-1)$ is an upper bound for the minimum number of transverse self-intersections over the entire free homotopy class of $a^i b^j$.

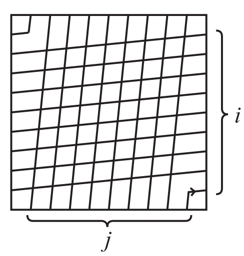

FIGURE 2

Now, if we show that f has no 1-gons or 2-gons, then by Theorem 3.2 f has no excess intersection, that is, f realizes the minimum number of intersections for its free homotopy class. f has no 1-gons by Lemma 3.6 (as drawn in the figure, f is not officially in canonical form, since some arcs were not drawn as straight segments in favor of visual clarity; however it should be clear that f is homeomorphic to a curve in canonical form). Suppose that f has a 2-gon, with edges formed by subcurves g and h. By Lemma 3.6, $w_g = w_h$ (it cannot be that $w_g = (w_h)^{-1}$ since w_f contains no A or B). g and h must also start at a mutual intersection. However, the only intersections in f are between an aa arc and a bb arc, so no two subcurves starting at an intersection of f can represent the same subword. Therefore, f has no 2-gons. So finally, $\phi\left(a^i b^j\right) = (i-1)(j-1)$.

(We will see an alternate method of proving $\phi\left(a^i b^j\right) \geq (i-1)(j-1)$ in section 6.) □

The next three proofs parallel the proof of Theorem 4.1.

Theorem 4.2. $\phi\left(a^i b^j a^k b^l\right) = (i+k-2)(j+l-2)+(i-k)+(j-l)-1$*, for* $i > k$ *and* $j \geq l$.

Proof. Figure 3 shows that $\phi\left(a^i b^j a^k b^l\right) \leq (i+k-2)(j+l-2)+(i-k)+(j-l)-1$. In (1), i copies of a are drawn for the first part of f. In (2), j copies of b are drawn. Each of the resulting $j-1$ complete vertical arcs intersects all of the $i-1$ complete horizontal arcs. So there are $(i-1)(j-1)$ intersections here. In (3), the next k copies of a are drawn, interlaced with the first set of as. This makes $k-1$ complete horizontal arcs, each of which intersects all of the $j-1$ complete vertical arcs. So there are $(j-1)(k-1)$ additional intersections here. In (4), the first of the next set of bs is drawn. This intersects each of the complete horizontal segments above the last a, of which there are $i-k-1$. In (5), the remaining bs are drawn, making $l-1$ complete vertical segments, each of which intersects all of the $i-1+k-1$ complete horizontal segments. So there are $(i+k-2)(l-1)$ additional intersections here. In (6), the final segment to close the curve is drawn. If $l < j$, this intersects each of the $j-l-1$ complete vertical segments to the right of the last b, and it also intersects the one diagonal segment. Otherwise (if $j = l$), the final closing segment intersects nothing. So in either case, there are $j-l$ additonal intersections here.

Therefore, in total there are

$$
\begin{aligned}
& (i-1)(j-1)+(j-1)(k-1)+(i-k-1)+(i+k-2)(l-1)+(j-l) \\
= \; & (i-1)(j-1)+(k-1)(j-1)+(i+k-2)(l-1)+(i-k)+(j-l)-1 \\
= \; & (i+k-2)(j-1)+(i+k-2)(l-1)+(i-k)+(j-l)-1 \\
= \; & (i+k-2)(j+l-2)+(i-k)+(j-l)-1
\end{aligned}
$$

intersections for f.

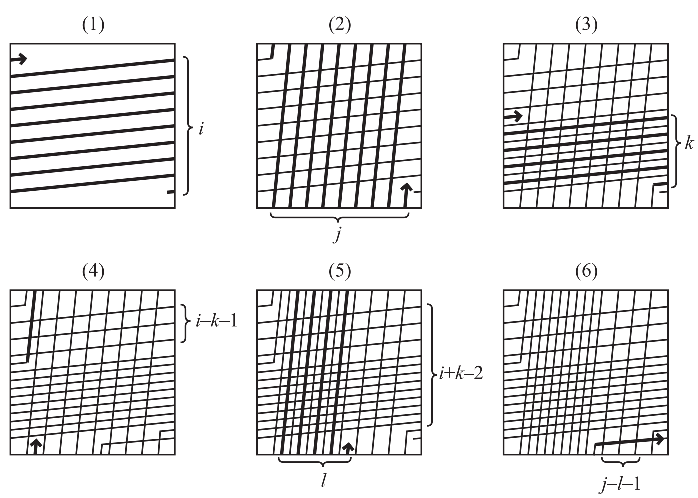

FIGURE 3

Now, we will show that f has no 1-gons or 2-gons. f has no 1-gons by Lemma 3.6. Suppose that f has a 2-gon, with edges formed by subcurves g and h. By Lemma 3.6, $w_g = w_h$ (it cannot be that $w_g = (w_h)^{-1}$ since w_f contains no A or B). Suppose that w_g and w_h each start with an a. There is only one pair of arcs ending in a (both ba) that could intersect (if and only if $l < j$). Supposing that they do intersect, if we follow the two subcurves leading away from the intersection, we find that after $k-1$ aa arcs, the two subcurves diverge: g onto an ab arc, h onto another aa arc (or the other way around), since $i > k$ (see Figure 4. So it cannot be that w_g and w_h each start with an a. But w_g and w_h cannot each start with a b since there is no intersecting pair of arcs ending in b. Therefore, f has no 2-gons. So finally, $\phi\left(a^i b^j a^k b^l\right) = (i+k-2)(j+l-2) + (i-k) + (j-l) - 1$. □

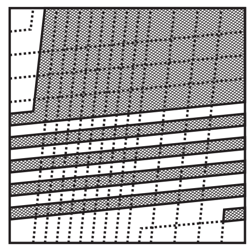

FIGURE 4

Theorem 4.3. $\phi\left(a^i b^j a^k B^l\right) = (i+k-1)(j+l-1)$.

Proof. Figure 5 shows that $\phi\left(a^i b^j a^k B^l\right) \leq (i+k-1)(j+l-1)$. In (1), i copies of a are drawn for the first part of f. In (2), j copies of b are drawn. Each of the resulting $j-1$ complete vertical segments intersects all of the $i-1$ complete horizontal segments. So there are $(i-1)(j-1)$ intersections here. In (3), k more copies of a are drawn. Each of the resulting $k-1$ complete horizontal segments intersects all of the $j-1$ complete vertical segments, and one incomplete horizontal segment intersects all of the $j-1$ complete vertical segments. So there are $(k)(j-1)$ additional intersections here. In (4), the remaining l copies of b are drawn. One incomplete vertical segment intersects $k-1$ complete horizontal segments and one incomplete horizontal segment; each of the $l-1$ complete vertical segments intersects $i-1+k-1$ complete horizontal segments and one incomplete horizontal segment; and one last incomplete vertical segment intersects $i-1$ complete horizontal segments. So there are $(l)(i+k-1)$ additional intersections here.

Therefore, in total there are

$$\begin{aligned} & (i-1)(j-1) + (k)(j-1) + (l)(i+k-1) \\ = \; & (i+k-1)(j-1) + (i+k-1)(l) \\ = \; & (i+k-1)(j+l-1) \end{aligned}$$

intersections for f.

Now, we will show that f has no 1-gons or 2-gons. f has no 1-gons by Lemma 3.6. Suppose that f has a 2-gon, with edges formed by subcurves g and h. Suppose that $w_g = w_h$. Suppose that w_g and w_h each start with an a. There is only one pair of arcs ending in a that intersect (one aa and the Ba). If we follow the two subcurves leading away from the intersection, we find that they overlap (see Figure 6) which means that the corresponding subintervals I and J are not disjoint, so they cannot form the edges of a 2-gon. But it cannot be that w_g and w_h start with a b or a B, either,

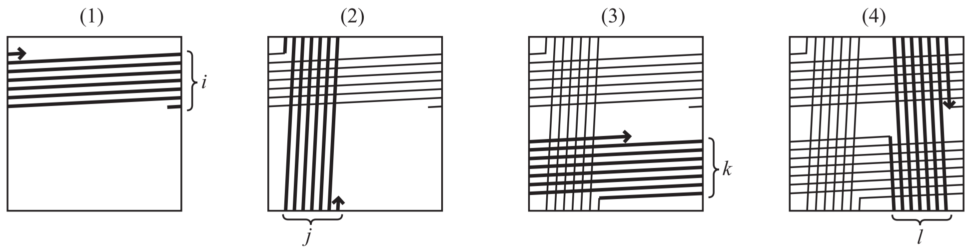

FIGURE 5

since no two arcs ending in b or B intersect. Then we must assume that $w_g = (w_h)^{-1}$. Then w_g must start with a b or B, since starting with an A is impossible here and staring with an a would imply that w_h ends with an A. Without loss of generality, we may assume that w_g starts with a b, so w_h ends with a B. But no arcs that end with b intersect arcs that begin with B, so such g and h cannot form a 2-gon. Therefore, f has no 2-gons. So finally, $\phi\left(a^i b^j a^k B^l\right) = (i+k-1)(j+l-1)$. □

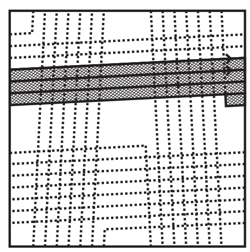

FIGURE 6

Theorem 4.4. $\phi\left(a^i b^j A^k B^l\right) = (i+k-1)(j+l-1)-1$.

Proof. Figure 7 shows that $\phi\left(a^i b^j A^k B^l\right) \leq (i+k-1)(j+l-1)-1$. In (1), i copies of a are drawn for the first part of f. In (2), j copies of b are drawn. Each of the resulting $j-1$ complete vertical segments intersects all of the $i-1$ complete horizontal segments. So there are $(i-1)(j-1)$ intersections here. In (3), k copies of A are added. Now, one incomplete horizontal segment and each of the $k-1$ complete horizontal segments intersect all of the $j-1$ complete vertical segments. So there are $(k)(j-1)$ additional intersections here. In (4), l copies of B are drawn, closing f. One incomplete vertical segment intersects $k-1$ complete horizontal segments; each of the $l-1$ complete vertical segments intersects $i-1+k-1$ complete horizontal segments and one incomplete horizontal segment; and one last incomplete vertical segment intersects $i-1$ complete horizontal segments. So there are $(k-1)+(l-1)(i+k-1)+(i-1) = (l)(i+k-1)-1$ additional intersections here.

Therefore, in total there are

$$\begin{aligned} & (i-1)(j-1)+(k)(j-1)+(l)(i+k-1)-1 \\ = \; & (i+k-1)(j-1)+(i+k-1)(l)-1 \\ = \; & (i+k-1)(j+l-1)-1 \end{aligned}$$

intersections for f.

Now we will show that f has no 1-gons or 2-gons. f has no 1-gons by Lemma 3.6. Suppose

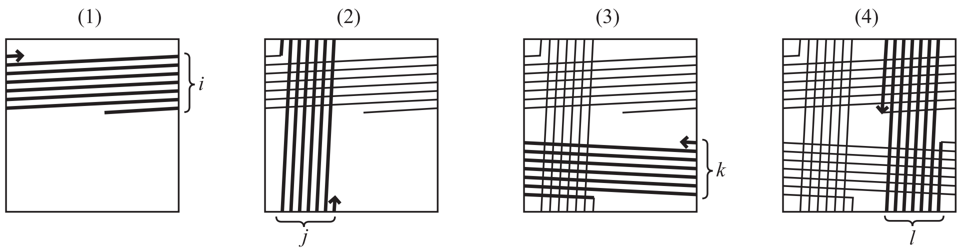

FIGURE 7

that f has a 2-gon, with edges formed by subcurves g and h. Suppose that $w_g = w_h$. Suppose that w_g and w_h each start with an a. There is only one pair of arcs ending in a that intersect (one aa and the Ba). If we follow the two subcurves leading away from the intersection, we find that they overlap (see Figure 8) which means that the corresponding subintervals I and J are not disjoint. But it cannot be that w_g and w_h start with A, b, or B, either, since no two arcs ending in A, b, or B intersect. Then we must assume that $w_g = (w_h)^{-1}$. However, no arcs that end with a intersect arcs that begin with A, no arcs that end with b intersect arcs that begin with B, no arcs that end with A intersect arcs that begin with a, and no arcs that end with B intersect arcs that begin with b. Thus no choice of initial letter for w_g (and resulting final letter for w_h) is consistent. So such g and h cannot form a 2-gon. Therefore, f has no 2-gons. So finally, $\phi\left(a^i b^j A^k B^l\right) = (i+k-1)(j+l-1)-1$. $\square$

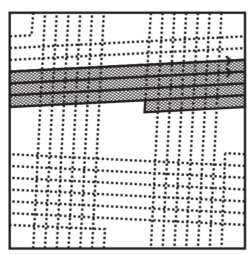

FIGURE 8

Remark 4.5. *The theorems proved in this section are enough to find the intersection number of any any primitive word with four or fewer blocks of letters (a block is a subword x^n where x is a single letter such that no other subword x^m with $m > n$ contains this subword x^n).*

Proof. A word with zero blocks is the identity word, which has intersection number 0. The only primitive words with one block are those with only one letter, all of which have intersection number 0. It is clear that a reduced word with two blocks must be equivalent to a word of the form $a^i b^j$ $(i, j > 0)$ under a Whitehead Type I automorphism (a permutation S of $\{a, b, A, B\}$ such that $S\left(x^{-1}\right) = S(x)^{-1}$; see [CR] for more on automorphisms of $F(a, b)$), whose intersection number is given by 4.1. Words with more than one block must have an even number of blocks since blocks must alternate between as or As and bs or Bs (no letter can be adjacent to its inverse) and the first and last block are adjacent. Thus there are no words with three blocks. It also follows that a word with four blocks must be in one of the following forms: $x^i y^j x^k y^l$, $x^i y^j x^{-k} y^l$, $x^i y^j x^k y^{-l}$, $x^i y^j x^{-k} y^{-l}$ (where $x \neq y^{\pm 1}$, $i, j, k, l > 0$). $x^i y^j x^k y^l$ is equivalent to $a^i b^j a^k b^l$ under some Whitehead Type I automorphism, which, if primitive, is equivalent to some word of the same form with $i > k$ and $j \geq l$ by the following Lemma (4.6), whose intersection number is given by 4.2. $x^i y^j x^{-k} y^l$ is equivalent to $b^i a^j B^k a^l$ under some Whitehead Type I automorphism, which is in turn equivalent to $a^l b^i a^j B^k$ under a cycle (i.e. conjugation by A^l), whose intersection number is given by Theorem 4.3. $x^i y^j x^k y^{-l}$ is equivalent to $a^i b^j a^k B^l$ under some Whitehead Type I automorphism, whose intersection number is given by Theorem 4.3. And finally, $x^i y^j x^{-k} y^{-l}$ is equivalent to $a^i b^j A^i B^j$ under some Whitehead Type I automorphism, whose intersection number is given by Theorem 4.4. □

Lemma 4.6. *Every primitive word w of the form $a^i b^j a^k b^l$ is equivalent to a word of the same form with $i > k$ and $j \geq l$.*

Proof. Since w is primitive, it cannot be that $i = k$ and $j = l$. Then if it is not the case that $i > k$ and $j \geq l$ already, we have the following cases ($\sim$ indicates equivalence under either a Whitehead Type I automorphism or a cycle):

For $i < k$ and $j \leq l$: $a^i b^j a^k b^l \sim a^k b^l a^i b^j$, so let $i' = k$, $j' = l$, $k' = i$, and $l' = j$.

For $i \leq k$ and $j > l$: $a^i b^j a^k b^l \sim b^i a^j b^k a^l \sim a^j b^k a^l b^i$, so let $i' = j$, $j' = k$, $k' = l$, and $l' = i$.

For $i \geq k$ and $j < l$: $a^i b^j a^k b^l \sim b^i a^j b^k a^l \sim a^l b^i a^j b^k$, so let $i' = l$, $j' = i$, $k' = j$, and $l' = k$.

Then $a^i b^j a^k b^l \sim a^{i'} b^{j'} a^{k'} b^{l'}$ and $i' > k'$ and $j' \geq l'$. □

5. Intersection Numbers for Powers of Words

Theorem 5.1. *For a primitive word w and a positive integer p, $\phi(w^p) = p^2\phi(w) + p - 1$.*

Proof. First we will prove by induction on p that $\phi(w^p) \leq p^2\phi(w) + p - 1$. We would like to find a curve f for word w^p with $p^2\phi(w) + p - 1$ intersections. We would also like f to be formed of p parallel subcurves f_1 to f_p, where $w_{f_i} = w$, such that each successive f_i is drawn shifted by a small distance to the right (from the direction of the curve) of f_{i-1}, an addition to a segment connecting the end of f_p to the beginning of f_1 (see Figure 9). When $p = 1$, we can draw a curve for w without any excess intersection, and all desired conditions will be met $(\left(p^2\phi(w) + p - 1\right)|_{p=1} = \phi(w))$. So we assume that f, with $w_f = w^p$, drawn as described above, has $p^2\phi(w) + p - 1$ intersections for some $p \geq 1$. We would like to insert one more subcurve, f_{p+1}, to get a drawing of w^{p+1}. First, remove the final segment from f that connects f_p to f_1, crossing the $p-1$ curves between (Figure 9.2). Then add the subcurve f_{p+1} (representing subword w) after f_p, traveling parallel, just to the right of f_p (Figure 9.3). As we draw f_{p+1}, the first time it crosses through a group of intersections (all corresponding to one intersection in the original drawing of w), it intersects the p transverse sections of f_1 through f_p (Figure 10.2). The second time through such a group of intersections, it intersects all p curves f_1 through f_p, plus itself, f_{p+1} (Figure 10.3). Then to connect the end of f_{p+1} to the beginning of f_1, all p previous copies of must be intersected (Figure 9.4). Call this new closed curve f'. So we started with f, with $p^2\phi(w) + p - 1$ intersections, subtracted $p - 1$ intersections, added $(2p+1)\phi(w)$ intersections, and then added p more intersections to get f'. $w_{f'} = w^p$, so we have a drawing of w^p with $p^2\phi(w) + (2p+1)\phi(w) + p = (p+1)^2\phi(w) + (p+1) - 1$ intersections, drawn as we had desired. Therefore, $\phi(w^p) \leq p^2\phi(w) + p - 1$ for all $p \geq 1$.

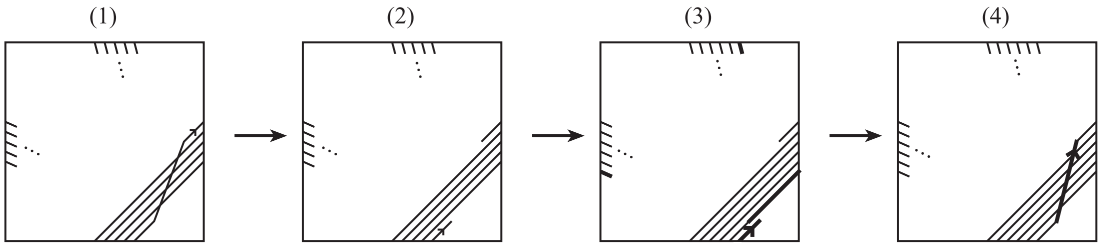

FIGURE 9

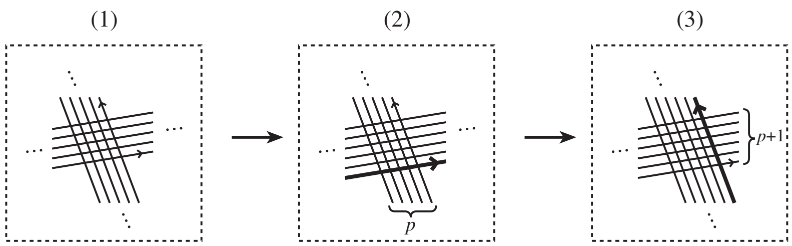

FIGURE 10

Basically, in drawing a curve f for w^p, we have replaced a curve e representing w (without excess intersection) with a ribbon of p parallel curves, reconnected properly at its ends so that it is one closed curve. Now we can identify two types of intersections in f. Define Type I intersections to be the $p^2\phi(w)$ intersections within transverse self-intersections of the entire ribbon. Define Type II intersections to be the $p-1$ intersections within the ribbon, where the rightmost subcurve crosses over to become the leftmost subcurve. There is a natural bijective correspondence between blocks of p^2 Type I intersections in f and intersections of e, which can be extended to a p-fold covering map $\gamma: f \to e$ (if the correct topology is chosen). Intuitively, γ contracts the ribbon of p parallel subcurves of f down to the single curve e.

Now with this map γ, we will prove that a closed curve f for which $w_f = w^p$, drawn as above, has no 1-gons or 2-gons. It is easy to see that f can be drawn as above to be in canonical form. Thus it will not contain any 1-gons. Now suppose that f has a 2-gon, with edges formed by subcurves g and h. Suppose that both 2-gon vertices are Type I intersections. Then $\gamma(g)$ and $\gamma(h)$ are edges of a weak 2-gon in e. But e is a primitive curve without excess intersection, which contains no weak 2-gons by Remark 3.5. So at least one vertex of the 2-gon must be a Type II intersection. But if we follow the two subcurves leaving any Type II intersection, we find that they overlap before intersecting again (at a Type II intersection), which contradicts the disjointness assumption of the intervals I and J in the definition of a 2-gon. Therefore, f has no 1-gons or 2-gons. Then by Theorem 3.2, f has no excess intersection.

Therefore $\phi(w^p) = p^2\phi(w) + p - 1$. □

6. Other Results

Theorem 6.1. *Let w be a word in $F(a,b)$. Let n_{aa} be the number of aa or AA subwords in w and n_{bb} be the number of bb or BB subwords in w. Then $\phi(w) \geq n_{aa}n_{bb}$.*

Proof. Subwords *aa* and *AA* correspond to arcs joining the left and right sides of S. Subwords *bb* and *BB* correspond to arcs joining the top and bottom sides of S. Therefore, the endpoints of each *aa* and *AA* arc separate the endpoints of each *bb* and *BB* arc on the boundary of S. Therefore each of the n_{aa} horizontal arcs must intersect each of the n_{bb} vertical arcs, so any closed curve representing w must have at least $n_{aa}n_{bb}$ transverse self-intersections. □

Remark 6.2. *This lower bound gives an alternate proof that $\phi(a^ib^j) = (i-1)(j-1)$. It is easy to see that $\phi(a^ib^j) \leq (i-1)(j-1)$, and since for $w = a^ib^j$, $n_{aa} = i-1$ and $n_{bb} = j-1$, $\phi(a^ib^j) \geq (i-1)(j-1)$.*

The following theorem is presented to give a possible alternate approach to finding intersection numbers. Unfortunately, this technique became much more complex when applied to any larger cases.

Theorem 6.3. $\phi(a^i) = i-1$.

Proof. Suppose that f is a closed curve represengting a^i in canonical form on S with k intersections. So f intersects the vertical boundary of S at i distinct points. Let Q be the uppermost of these points. Then let P and R be the other points at which the segments leading to and leaving from Q intersect the boundary, respectively. Segments PQ and QR must intersect since PQ starts below Q on the left and ends at Q on the right while QR starts at Q on the left and ends below Q on the right. Let S be this point of intersection. So if we form a curve f' from f by removing segments PQ and QR and replacing them with the path along segments PS and SR (see Figure 11), one a loop is removed from the f, along with at least one intersection. Moreover, no intersections are added since the set of points in f' is a subset of the set of points in f. Then f' represents a^{i-1} with with no more than $k-1$ self-intersections. In particular, if f has no excess intersection, $k = \phi(a^i)$, so $\phi(a^{i-1}) \leq \phi(a^i) - 1$.

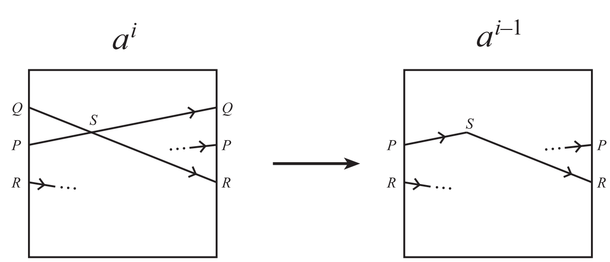

FIGURE 11

Now suppose that g is a closed curve representing a^{i-1} in canonical form on S with l intersections. So g intersects the vertical boundary of S at $i-1$ distinct points. Let R be the uppermost of these points, and let P be the other point at which the segment leading to R intersects the boundary.

Then let Q be any point on the boundary above R and let S be any point on segment PR which is not separated from the top of the square by any other segments (there is a neighborhood of R containing such a point). So if we form a curve g' from g by removing segment PR and replacing it with the path composed of segments PS, SQ, QS, and SR (always traveling left to right) (see Figure 12), one a loop is added, and exactly one intersection is added. Then g' represents a^i with $l+1$ intersections. In particular, if g has no excess intersection, $l = \phi\left(a^{i-1}\right)$, so $\phi\left(a^i\right) \leq \phi\left(a^{i-1}\right)+1$.

Therefore, $\phi\left(a^i\right) = \phi\left(a^{i-1}\right)+1$. Then since $\phi(a)=0$, $\phi\left(a^i\right) = i-1$.

□

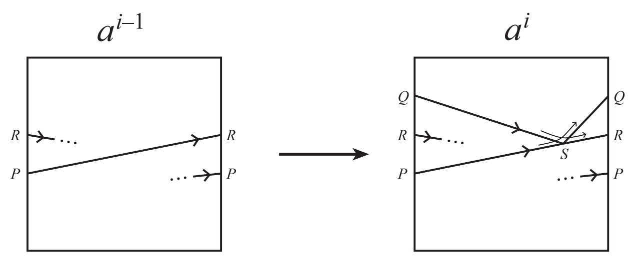

FIGURE 12

Here we describe an algorithm for determining intersection number and minimal configurations. In the canonical form, two arcs intersect if and only if the two pairs of endpoints separate on the boundary of S. Moreover, if two arcs intersect, they intersect exactly once. So all intersections of a curve in canonical form are determined only by the ordering of edge crossings. Since there is only a finite number of permutations of the edge crossings, a computer program can check them all and tell us the minimum number of intersections in addition to the permutations that give us that minimum. This algorithm is extremely inefficient (factorial time), but a Mathematica implementation still proved to be invaluable in the development of much of the results of this paper. What follows is a more detailed explanation of how the algorithm is implemented. Given a word w, with a or A in positions i_1 through i_{n_a} and b or B in positions j_1 through j_{n_b}, and a list of integers $\{p_1, p_2, \ldots, p_{n_a+n_b}\}$, where $\left\{p_{i_1}, p_{i_2}, \ldots, p_{i_{n_a}}\right\} = \{1,2,\ldots,n_a\}$ and $\left\{p_{j_1}, p_{j_2}, \ldots, p_{j_{n_a}}\right\} = \{1,2,\ldots,n_b\}$ (setwise), we label n_a points on the vertical boundary (1 through n_a) and n_b points on the horizontal boundary (1 through n_b) and draw a curve f (where $w_f = w$) in the following way. We may assume that the first letter of w is a, so we start at the point labeled 1 on the vertical boundary of S. Then for each successive letter, we draw a straight segment to the yet unvisited point with the smallest label on the corresponding edge of S. At the end, connect back to the point labeled 1 on the right edge. Complete code for the following relevant Mathematica functions can be found in the appendix. `IntersectByDrawing[word_String, order_List, max_]`, if `max` $=-1$, returns the number of intersections of f, drawn as just described for $w =$ `word` and $(p_1, \ldots, p_{n_a+n_b}) =$ `order`. Otherwise, it returns the minimum of `max` and the number of intersecitons. `DrawingPermutations[word_String]` returns a list of all $n_a!n_b!$ possible lists $\{p_1, \ldots, p_{n_a+n_b}\}$ for word `word`. And finally, `MinimalPermutations[word_String]` returns the intersection number of `word` with all lists $(p_1, \ldots, p_{n_a+n_b})$ that give the minimum.

7. INTRODUCTION TO HYPERBOLIC GEOMETRY

In order to examine the intersections of geodesics representing closed curves on the torus, we need to review some hyperbolic geometry. This background is taken directly from [DIW] and [C]. The upper half plane model of the hyperbolic plane, $\mathbb{H}$, is defined on the set $\{x+iy\colon y>0\}$. Note that in this plane, geodesics are represented by semicircles centered on the real axis or infinite vertical lines. All geodesics in $\mathbb{H}$ that project to closed geodesics on $\mathbb{T}$ will have irrational endpoints, and the geodesics on the torus are a projection of these geodesics. The pair of points on the real axis that determines the position of the geodesic are called the *feet* of the geodesic. We use the group

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}$$

to act upon $\mathbb{H}$ through the homomorphism defined by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto Tz = \tfrac{az+b}{cz+d}$$

This group of fractional linear transformations is $\Gamma = PSL(2,\mathbb{Z})$, and we will denote the matrix in $SL(2,\mathbb{Z})$ and the transformation in $\Gamma = PSL(2,\mathbb{Z})$ by the same symbol. Let Γ' be the commutator subgroup of Γ. Γ' is a free group on the two generators a and b, where

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Note that Γ' is isomorphic to the commutator subgroup of $SL(2,\mathbb{Z})$. We will denote the inverses of a and b as A and B respectively. We can now consider the fundamental region $\mathcal{D}$ of $\mathbb{H}$. This

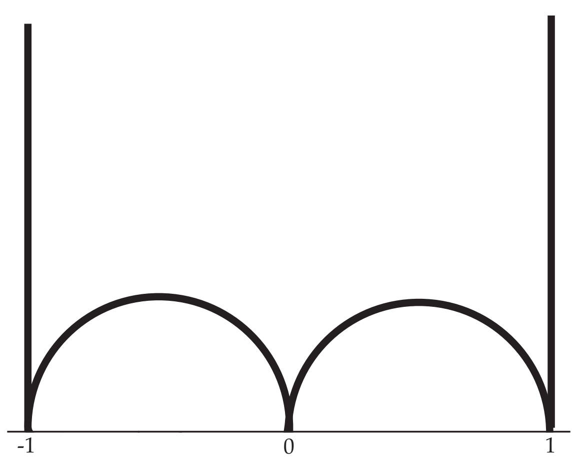

FIGURE 13. The fundamental region $\mathcal{D}$

region is enclosed between two infinite vertical lines at -1 and 1 and above two semicircles with feet at -1 and 0 and at 0 and 1 (see Figure 13). The operations a and b act on $\mathcal{D}$ by identifying opposite edges of the region. In this fashion, we can construct a torus. Note that the point which is missing (namely, that at infinity) represents the "puncture" of the torus.

8. Algebraic approaches to Determining the Feet of Geodesics

8.1. **Monotonicity of roots.** From the following tables, it appears that both roots of a word of the form $a^j b^k$ increase when adding a's to the word, and decrease when adding b's to the word. These tables were created with the help of a program created by [B]. We will prove this monotonicity in two lemmas, using a nested induction.

Word	Root	
$aabb$	1.76759	.565741
a^1aabb	1.781	.610308
a^2aabb	1.7831	.616905
a^3aabb	1.7834	.617869
a^4aabb	1.78345	.61801
a^5aabb	1.78346	.61803
a^6aabb	1.78346	.618033

Word	Roots	
a^4bb	1.7831	0.616905
a^4bbb^1	1.640685	0.6168
a^4bbb^2	1.62131	0.616785

Word	Roots	
a^5bb	1.7834	0.617869
a^5bbb^1	1.640725	0.617855
a^5bbb^2	1.621315	0.61785
a^6bbb^3	1.61851	0.61785

Word	Roots	
a^6bb	1.78345	.61801
a^6bbb^1	1.64073	.618008
a^6bbb^2	1.62132	.618007
a^6bbb^3	1.61851	.618007
a^6bbb^4	1.6181	.618007

Word	Roots	
a^7bb	1.78346	0.61803
a^7bbb^1	1.640735	0.61803
a^7bbb^2	1.621315	0.61803
a^7bbb^3	1.61851	0.61803
a^7bbb^4	1.618105	0.61803
a^7bbb^5	1.618045	0.61803

Word	Roots	
a^8bb	1.78346	0.618035
a^8bbb^1	1.640735	0.618035
a^8bbb^2	1.621315	0.618035
a^8bbb^3	1.61851	0.618035
a^8bbb^4	1.618105	0.618035
a^8bbb^5	1.618045	0.618035
a^8bbb^6	1.618035	0.618035

To begin, we introduce some information essential to the following proofs.

Definition 8.1. *Let* $\Phi = \frac{\sqrt{5}+1}{2}$ *represent the golden ratio.*

Remark 8.2. *The Fibonacci numbers satisfy the recurrence equation* $F_n = F_{n-2} + F_{n-1}$ *for* $n \geq 3$*, with the convention that* $F_1 = 1$ *and* $F_2 = 1$*.*

Remark 8.3. *The* $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$ *and the* $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\Phi} = \Phi - 1$*.*

Lemma 8.4. *The matrix of* a^n *is* $\begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix}$ *and that of* b^n *is* $\begin{pmatrix} F_{2n-1} & -F_{2n} \\ -F_{2n} & F_{2n+1} \end{pmatrix}$*, where* F_n *represents the nth Fibonacci number.*

Proof. We will prove this using an inductive argument. It is given that the matrix of a is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Now assume $a^{n-1} = \begin{pmatrix} F_{2n-3} & F_{2n-2} \\ F_{2n-2} & F_{2n-1} \end{pmatrix}$. Then $a^n = aa^{n-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} F_{2n-3} & F_{2n-2} \\ F_{2n-2} & F_{2n-1} \end{pmatrix} = \begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix}$. It is also given that the matrix of b is $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Assume $b^{n-1} = \begin{pmatrix} F_{2n-3} & -F_{2n-2} \\ -F_{2n-2} & F_{2n-1} \end{pmatrix}$. Then $b^n = bb^{n-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} F_{2n-3} & -F_{2n-2} \\ -F_{2n-2} & F_{2n-1} \end{pmatrix} = \begin{pmatrix} F_{2n-1} & -F_{2n} \\ -F_{2n} & F_{2n+1} \end{pmatrix}$. □

Remark 8.5. *We know that* ${F_{2n}}^2 - {F_{2n-1}}^2 - F_{2n}F_{2n-1} = -1$*. To see this, note that for* $n = 1$*, that is* F_2*,* $1^2 - 1 * 1 - 1^2 = -1$*. Assume this is true for* F_{2n}*, that is* ${F_{2n}}^2 - {F_{2n-1}}^2 - F_{2n}^2 = -1$*. Then for* $F_{2n+2} = F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1}$ *we have* $(2F_{2n} + F_{2n-1})^2 - (2F_{2n} + F_{2n-1}) - (F_{2n-1} + F_{2n})^2 = F_{2n}^2 - F_{2n-1}^2 - F_{2n}F_{2n-1} = -1$*.*

Using this information, we can now begin the essential steps of the proof of monotonicity. As a warning to the reader, in these proofs we exclude some complicated algebraic steps in order to clarify the arguments presented.

Lemma 8.6. *The roots of a word increase monotonically when adding a's to a word of the form $a^j b^k$, where k is fixed.*

Proof. Suppose we have a word of the form *aabb* (the smallest case we will consider). The roots of this word are approximately (within .00001) 1.767590 and .565741. Now looking at the case where we add one *a* to this word (*aaabb*), we find that the roots occur at approximately 1.781 and .610308, both of which are greater than the corresponding roots of *aabb*. Assume this monotonicity holds when adding *n* *a*'s to the word *aabb*, that is, the roots of $a^n bb$ are greater than those of $a^{n+1}bb$ and so on. To show the monotonicity holds when adding $n+1$ *a*'s to *aabb*, we find the matrices of the words $a^n bb$ and $a^{n+1}bb$. These are equal to

$$\begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -3F_{2n}+2F_{2n-1} & 5F_{2n}-3F_{2n-1} \\ -F_{2n}-3F_{2n-1} & 2F_{2n}+5F_{2n-1} \end{pmatrix}$$

and

$$\begin{pmatrix} F_{2n} & F_{2n-1}+F_{2n} \\ F_{2n-1}+F_{2n} & F_{2n-1}+2F_{2n} \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -F_{2n}-3F_{2n-1} & 2F_{2n}+5F_{2n-1} \\ -4F_{2n}-F_{2n-1} & 7F_{2n}+2F_{2n-1} \end{pmatrix}$$

respectively. Then the respective roots are

$$\frac{1}{2}\left(\frac{5F_{2n}+3F_{2n-1}}{F_{2n}+3F_{2n-1}} \pm \sqrt{\left(\frac{5F_{2n}+3F_{2n-1}}{-F_{2n}-3F_{2n-1}}\right)^2 + 4\frac{5F_{2n}-3F_{2n-1}}{-F_{2n}-3F_{2n-1}}}\right)$$

and

$$\frac{1}{2}\left(\frac{8F_{2n}+5F_{2n-1}}{4F_{2n}+F_{2n-1}} \pm \sqrt{\left(\frac{8F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}\right)^2 + 4\frac{2F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}}\right).$$

To prove monotonicity, we must show for our roots of the form $e \pm \sqrt{f}$ and $g \pm \sqrt{h}$ that $e+\sqrt{f} < g+\sqrt{h}$ and $e-\sqrt{f} < g-\sqrt{h}$. To simplify these inequalities algebraically, we will then consider the inequalities $e-g < \sqrt{h}-\sqrt{f}$ and $e-g < \sqrt{f}-\sqrt{h}$. So, continuing to use this notation, we find that $e-g$ is equal to

$$\frac{(5F_{2n}+3F_{2n-1})(4F_{2n}+F_{2n-1})-(8F_{2n}+5F_{2n-1})(F_{2n}+3F_{2n-1})}{(-4F_{2n}-F_{2n-1})(-F_{2n}-3F_{2n-1})}$$
$$= 12\frac{-F_{2n-1}{}^2+F_{2n}{}^2-F_{2n}F_{2n-1}}{(-F_{2n}-3F_{2n-1})(-4F_{2n}-F_{2n-1})} = \frac{-12}{(-F_{2n}-3F_{2n-1})(-4F_{2n}-F_{2n-1})}$$

By expanding and simplifying the expression $\sqrt{h}-\sqrt{f}$, we find that

$$\sqrt{\left(\frac{8F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}\right)^2 + 4\frac{2F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}} - \sqrt{\left(\frac{5F_{2n}+3F_{2n-1}}{-F_{2n}-3F_{2n-1}}\right)^2 + 4\frac{5F_{2n}-3F_{2n-1}}{-F_{2n}-3F_{2n-1}}}$$
$$= \frac{(-F_{2n}-3F_{2n-1})\sqrt{32F_{2n}{}^2+5F_{2n-1}{}^2-8F_{2n}F_{2n-1}}+(4F_{2n}+F_{2n-1})\sqrt{5F_{2n}{}^2+45F_{2n-1}{}^2-18F_{2n}F_{2n-1}}}{(-4F_{2n}-F_{2n-1})(-F_{2n}-3F_{2n-1})}$$

Then our inequality of $e-g<\sqrt{h}-\sqrt{f}$ is equivalent to

$$-12<(-F_{2n}-3F_{2n-1})\sqrt{32F_{2n}{}^2+5F_{2n-1}{}^2-8F_{2n}F_{2n-1}}+(4F_{2n}+F_{2n-1})\sqrt{5F_{2n}{}^2+45F_{2n-1}{}^2-18F_{2n}F_{2n-1}}.$$

As we know F_{2n} is at the least $1.618026F_{2n-1}$ and at the most ΦF_{2n-1} when $n\geq 7$, we substitute these into the above inequality. Note that for cases where $n<7$, the monotonicity of the roots can easily be verified by performing the computations by hand. In doing so, to ensure that the inequality holds true we will substitute the larger value, ΦF_{2n-1}, whenever F_{2n} has a negative coefficient, and the smaller value, $1.618026F_{2n-1}$ whenever F_{2n} has a positive coefficient. After this substitution, the inequality reduces to $-33302<F^2_{2n-1}$. We know this is true as F^2_{2n-1} is always positive.

Now we consider the inequality $e-g<\sqrt{f}-\sqrt{h}$. After expanding and simplifying the expression $\sqrt{f}-\sqrt{h}$, we find that it is equivalent to

$$\sqrt{\left(\frac{5F_{2n}+3F_{2n-1}}{-F_{2n}-3F_{2n-1}}\right)^2+4\frac{5F_{2n}-3F_{2n-1}}{-F_{2n}-3F_{2n-1}}}-\sqrt{\left(\frac{8F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}\right)^2+4\frac{2F_{2n}+5F_{2n-1}}{-4F_{2n}-F_{2n-1}}}$$
$$=\frac{(-4F_{2n}-F_{2n-1})\sqrt{5F_{2n}{}^2+45F_{2n-1}{}^2-18F_{2n}F_{2n-1}}+(F_{2n}+3F_{2n-1})\sqrt{32F_{2n}{}^2+5F_{2n-1}{}^2-8F_{2n}F_{2n-1}}}{(-4F_{2n}-F_{2n-1})(-F_{2n}-3F_{2n-1})}$$

Then the inequality of $e-g<\sqrt{f}-\sqrt{h}$ is equivalent to

$$-12<(-4F_{2n}-F_{2n-1})\sqrt{5F_{2n}{}^2+45F_{2n-1}{}^2-18F_{2n}F_{2n-1}}+(F_{2n}+3F_{2n-1})\sqrt{32F_{2n}{}^2+5F_{2n-1}{}^2-8F_{2n}F_{2n-1}}$$

Again we substitute the maximum and minimum values of F_{2n} as discussed above. When $n\geq 7$ this inequality reduces to $-.000017470<F^2_{2n-1}$. Again as F^2_{2n-1} is positive, this inequality is true. Thus the proof is complete. □

Lemma 8.7. *The roots of a word decrease monotonically when adding b's to a word where the number of a's is fixed.*

Proof. This proof is similar to that of Lemma 8.6. We will consider a word, say a^4b^2. Then we know the roots of this word occur at approximately 1.783095 and .616905. Then, adding one b to this word, a^4b^3, we find that the roots occur at approximately 1.640685 and .6168, each of which is smaller than the corresponding roots of a^4b^2. Assume this monotonicity holds when adding n b's to a word of the form a^jb^k. The matrices of a^jb^n and a^jb^{n+1} are respectively

$$\begin{pmatrix}1&1\\1&2\end{pmatrix}^4\begin{pmatrix}F_{2n-1}&-F_{2n}\\-F_{2n}&F_{2n}+F_{2n-1}\end{pmatrix}=\begin{pmatrix}-21F_{2n}+13F_{2n-1}&8F_{2n}+21F_{2n-1}\\-34F_{2n}+21F_{2n-1}&13F_{2n}+34F_{2n-1}\end{pmatrix}$$

and

$$\begin{pmatrix}1&1\\1&2\end{pmatrix}^4\begin{pmatrix}F_{2n}&-F_{2n}-F_{2n-1}\\-F_{2n}-F_{2n-1}&F_{2n-1}+2F_{2n}\end{pmatrix}=\begin{pmatrix}-8F_{2n}-21F_{2n-1}&29F_{2n}+8F_{2n-1}\\-13F_{2n}-34F_{2n-1}&47F_{2n}+13F_{2n-1}\end{pmatrix}$$

Then the respective roots are

$$\frac{1}{2}\left(\frac{34F_{2n}+21F_{2n-1}}{34F_{2n}-21F_{2n-1}}\pm\sqrt{\left(\frac{34F_{2n}+21F_{2n-1}}{-34F_{2n}+21F_{2n-1}}\right)^2+4\frac{8F_{2n}+21F_{2n-1}}{-34F_{2n}+21F_{2n-1}}}\right)$$

and

$$\frac{1}{2}\left(\frac{55F_{2n}+34F_{2n-1}}{13F_{2n}+34F_{2n-1}} \pm \sqrt{\left(\frac{55F_{2n}+34F_{2n-1}}{-13F_{2n}-34F_{2n-1}}\right)^2 + 4\frac{29F_{2n}+8F_{2n-1}}{-13F_{2n}-34F_{2n-1}}}\right)$$

Now, we wish to show that for our roots of the form $e \pm \sqrt{f}$ and $g \pm \sqrt{h}$, that $e+\sqrt{f} > g+\sqrt{h}$ and $e-\sqrt{f} > g-\sqrt{h}$. Again for simplicity, we will then consider the equivalent inequalities $e-g > \sqrt{h}-\sqrt{f}$ and $e-g > \sqrt{f}-\sqrt{h}$, using a method similar to that above. Using this same notation, we find $e-g$ equal to

$$\left(\frac{34F_{2n}+21F_{2n-1}}{34F_{2n}-21F_{2n-1}}\right) - \left(\frac{55F_{2n}+34F_{2n-1}}{13F_{2n}+34F_{2n-1}}\right)$$
$$= -1428\left(\frac{-F_{2n-1}{}^2+F_{2n}{}^2-F_{2n}F_{2n-1}}{(13F_{2n}+34F_{2n-1})(34F_{2n}-21F_{2n-1})}\right) = \frac{1428}{(-13F_{2n}-34F_{2n-1})(-34F_{2n}+21F_{2n-1})}$$

Then after factoring and simplifying the expression $\sqrt{h}-\sqrt{f}$, we have

$$\sqrt{\left(\frac{34F_{2n}+21F_{2n-1}}{-34F_{2n}+21F_{2n-1}}\right)^2 + 4\frac{8F_{2n}+21F_{2n-1}}{-34F_{2n}+21F_{2n-1}}} - \sqrt{\left(\frac{55F_{2n}+34F_{2n-1}}{-13F_{2n}-34F_{2n-1}}\right)^2 + 4\frac{29F_{2n}+8F_{2n-1}}{-13F_{2n}-34F_{2n-1}}} =$$

$$\frac{(-34F_{2n}+21F_{2n-1})\sqrt{1517F_{2n}{}^2+68F_{2n-1}{}^2-620F_{2n}F_{2n-1}} - (-13F_{2n}-34F_{2n-1})\sqrt{68F_{2n}{}^2+2205F_{2n-1}{}^2-756F_{2n}F_{2n-1}}}{(-13F_{2n}-34F_{2n-1})(-34F_{2n}+21F_{2n-1})}$$

Then the inequality $e-g > \sqrt{h}-\sqrt{f}$ simplifies to

$$1428 > (-34F_{2n}+21F_{2n-1})\sqrt{(1517F_{2n}^2+68F_{2n-1}^2-620F_{2n}F_{2n-1})} + (13F_{2n}+34F_{2n-1})\sqrt{(68F_{2n}^2+2205F_{2n-1}^2-756F_{2n}F_{2n-1})}$$

As above, we substitute the bounds of F_{2n} into this inequality, namely at the least $1.618026F_{2n-1}$ and at the most ΦF_{2n-1} (where $n \geq 7$). In this case, to ensure that the inequality is true we must substitute the larger value, ΦF_{2n-1}, when F_{2n} has a positive coefficient and the smaller value, $1.618026F_{2n-1}$, when F_{2n} has a negative coefficient. Doing so, we find $-.000018396 < F_{2n-1}^2$. This inequality is true as F_{2n-1}^2 is always positive. Similarly, the inequality for $e-g > \sqrt{f}-\sqrt{h}$ simplifies to

$$-1428 < (13F_{2n}+34F_{2n-1})\sqrt{(68F_{2n}^2+2205F_{2n-1}^2-756F_{2n}F_{2n-1})} + (-34F_{2n}+21F_{2n-1})\sqrt{(1517F_{2n}^2+68F_{2n-1}^2-620F_{2n}F_{2n-1})}$$

Once again, we will substitute the values discussed above for F_{2n} (note that because of the sign change from the first inequality, ΦF_{2n-1} will replace F_{2n} associated with negative coefficients and $1.618026F_{2n-1}$ will replace those with positive coefficients). The inequality then reduces to $-61818 < F_{2n-1}^2$. As $F_{2n-1}^2 > 0$, the inequality is true. Thus we know by induction that both roots of a word a^4b^n are greater than the corresponding roots of the word a^4b^{n+1}. □

Theorem 8.8. *Through this double induction in Lemma 8.6 and Lemma 8.7, we know that for any word of the form a^jb^k, both roots increase monotonically when adding a's and decrease monotonically when adding b's.*

8.2. **Calculating roots of words.** We will calculate the roots of the permutations of a^jb^k using the matrices for the respective symbols and the resulting quadratic equation to demonstrate that the word a^jb^k has $(j-1)(k-1)$ intersections. In order to do so, we will introduce several lemmas. Note that these results depend on the previous double induction proof of monotonicity. Moreover, the intervals found below are approximate within 0.000001. We enclose below tables that initially motivated results that follow. All tables presenting approximate numerical roots of permutations were created with the help of the program by [B].

Permutation	Roots	
aabb	1.76759	.565741
aaabb	1.781	.610308
aaaabb	1.7831	.616905
aaabbb	1.64039	.609612
aaaaabb	1.7834	.617869
aaaabbb	1.64068	.616801

Lemma 8.9. *The roots of the permutation $a^j b^k$ occur in the intervals (.381966, .618034) and (1.618034, 2.618035).*

Proof. Using the matrices defined for a and b, where $a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ then the matrix for W is

$$\begin{pmatrix} F_{2j-1}F_{2k-1} - F_{2j}F_{2k} & -F_{2j-1}F_{2k} + F_{2j}F_{2k} + F_{2j}F_{2k-1} \\ F_{2j}F_{2k-1} - F_{2j}F_{2k} - F_{2j-1}F_{2k} & F_{2j}F_{2k-1} + F_{2j-1}F_{2k} + F_{2j-1}F_{2k-1} \end{pmatrix}$$

where F_n represents the nth Fibonacci number. Now we apply the transformation $T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}$ and solve the resulting quadratic equation $T(z) = z$ for z. So, using this notation we would have $z^2 + \left(\frac{d-a}{c}\right) z - \frac{b}{c} = 0$. Thus, for the above matrix for W, we find that the roots of its quadratic equation are

$$\frac{1}{2}\left(-\frac{F_{2j}F_{2k-1} + F_{2j-1}F_{2k} + F_{2j}F_{2k}}{F_{2j}F_{2k-1} - F_{2j}F_{2k} - F_{2j-1}F_{2k}} \pm \sqrt{\left(\frac{F_{2j}F_{2k-1} + F_{2j-1}F_{2k} + F_{2j}F_{2k}}{F_{2j}F_{2k-1} - F_{2j}F_{2k} - F_{2j-1}F_{2k}}\right)^2 + 4\frac{-F_{2j-1}F_{2k} + F_{2j}F_{2k} + F_{2j}F_{2k-1}}{F_{2j}F_{2k-1} - F_{2j}F_{2k} - F_{2j-1}F_{2k}}}\right)$$

We take the limits of the roots as j and k approach ∞. Dividing each term of the numerator and denominator of these roots by $F_{2j}F_{2k}$ we find that as j and k approach ∞, the roots of this permutation approach $\frac{2\Phi - 1 \pm \sqrt{(1-2\Phi)^2 - 4}}{2} = \Phi, \Phi - 1 \approx$1.618034, .618034, where Φ represents the golden ratio. Using the smallest word of the form $a^j b^k$, ab, to establish the bounds of the intervals of the feet, we find that the roots of ab occur at .381966 and 2.618035. Thus the lower root is in (.381966, .618034) and the upper root is in (1.618034, 2.618035). □

Definition 8.10. [M] *A* fixed point *of a function $g(x)$ is a point p such that $g(p) = p$.*

Definition 8.11. [M] *The iteration $p_n = g(p_{n-1})$ for $n = 0, 1, \ldots$ is called* fixed point iteration.

Lemma 8.12. *Words of the form $a^{j-m} b^k a^m$ follow the fixed point iteration of $G = \frac{2z-1}{-z+1}$.*

Proof. The method of this proof is taken from Lemma 3.9 found in [B]. Let W represent the matrix of $a^j b^k$, and without loss of generality suppose $W = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$. Then $W(z) = \frac{tz+u}{uz+w}$. Suppose the fixed points of W are some p and q. For the word $a^{j-1}b^k a^1$ we must cyclically permute W by bringing one a from the front to the back. To do so, we consider $AWa(x)$, and we find its fixed points p_1, q_1. Now if we let $x = A(p)$, we have $AWa(A(p)) = AW(aA(p)) = AW(p) = A(p) = p_1$. Thus when moving m a's to the back of W, we have $p_m = A^m(p)$. Then for each successive image of the fixed points of W, we need only consider the iterations of the A function. As $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, we know $A(x) = \frac{2x-1}{-x+1}$ (see Figure 14). □

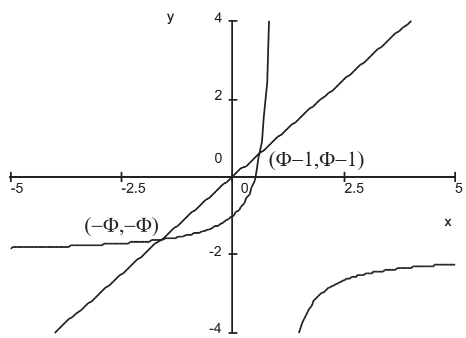

FIGURE 14. Graph of $A(x) = \frac{2x-1}{-x+1}$

The table below is an analysis of the behavior of the fixed point iteration of $A(x) = \frac{2x-1}{-x+1}$ on different intervals.

Interval	Behavior
$(-\infty, -\Phi)$	Increase monotonically to $-\Phi$
$(-\Phi, \Phi-1)$	Decrease monotonically to $-\Phi$
$(\Phi-1, 1)$	Flip once to (1, ∞), flip to (-∞,-Φ), increase monotonically to $-\Phi$
$(1, \infty)$	Flip to (-∞,-Φ), increase monotonically to $-\Phi$

Lemma 8.13. *Words of the form $b^{k-n}a^j b^n$ follow the fixed point iteration of $G = \frac{2z+1}{z+1}$.*

Proof. Using the same process as Lemma 8.12, we see that for $b^{k-n}a^j b^n$ we consider cyclic permutations of $b^k a^j$ sending b's from the front to the back. Then we need only consider iterations of the B function. As $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, we know $B(x) = \frac{2x+1}{x+1}$ (see Figure 15). □

The table below is an analysis of the behavior of the fixed point iteration of $A(x) = \frac{2x-1}{-x+1}$ on different intervals.

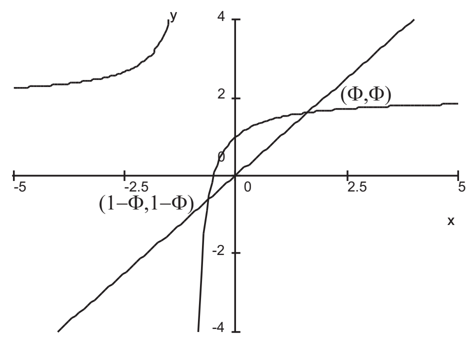

FIGURE 15. Graph of $B(x) = \frac{2x+1}{x+1}$

Interval	Behavior
$(-\infty,-1)$	Flip to (Φ, ∞), decrease monotonically to Φ
$(-1,1-\Phi)$	Flip once to (-∞, -1), flip to (Φ, ∞), decrease monotonically to Φ
$(1-\Phi,\Phi)$	Increase monotonically to Φ
(Φ,∞)	Decrease monotonically to Φ

Permutation	Roots	
abba	-3.30278	0.302776
aabba	-3.28042	0.56613
abbaa	-1.76638	0.304839
aaabba	-3.27698	0.610317
aabbaa	-1.76619	0.56619
abbaaa	-1.63849	0.305159

Lemma 8.14. *The permutation $a^{j-m}b^ka^m$ (where $0 < m \leq j$) creates a series of j concentric semicircles with feet occuring in ($-\infty$, -1.618034) and (-.618034, .618034).*

Proof. This follows by applying Lemma 8.12 on the intervals found for the roots of the original word W (see Lemma 8.9). These iterations then create a series of j concentric semicircles, none of which intersect. □

Permutation	Roots	
bbaa	-1.76759	-.565741
bbaaa	-1.63852	-.561484
bbbaaa	-1.64039	-.609612
bbbaaaa	-1.62127	-0.609502
bbbbaaaa	-1.62131	-0.616785

Lemma 8.15. *The roots of the permutation b^ka^j occur in (-2.618035, -1.618034) and (-.618034, -.381966).*

Proof. Using once again the matrices defined for a and b, we find that the matrix is

$$\begin{pmatrix} F_{2k-1}F_{2j-1}-F_{2k}F_{2j} & F_{2k-1}F_{2j}-F_{2k}F_{2j+1} \\ -F_{2k}F_{2j-1}+F_{2k+1}F_{2j} & -F_{2k}F_{2j}+F_{2k+1}F_{2j+1} \end{pmatrix}$$

Taking the limit of the general equation for the roots as k and j approach ∞ we have

$$\frac{1}{2}\left(-\frac{\Phi^2-(\Phi-1)^2}{-(\Phi-1)+\Phi} \pm \sqrt{\left(\frac{\Phi^2-(\Phi-1)^2}{-(\Phi-1)+\Phi}\right)^2+4\frac{(\Phi-1)-\Phi}{-(\Phi-1)+\Phi}}\right) \approx -1.618034, -.618034$$

We will use the roots of the smallest word of this form, *ba*, to establish the bounds of the intervals. We find that the roots occur at -2.618035 and -.381966. So the intervals for the feet of the geodesic representing $b^k a^j$ are (-2.618035, -1.618034) and (-.618034, -.381966). Moreover, this geodesic is the innermost of the nested geodesics for words of the form $a^{j-m}b^k a^m$. □

Permutation	Roots	
baab	3.30278	-.302776
baaab	3.56613	-.280416
bbaaab	3.56155	-.561553
baaabb	1.78078	-.280776
bbbaaaab	3.6095	-0.609502
bbaaaabb	1.78306	-0.560835
baaaabbb	1.64068	-0.277047

Lemma 8.16. *The permutation $b^{k-n}a^j b^n$ (where $0 < n < k$) creates a series of $k-1$ concentric semicircles with feet occuring in (-.618034, .618034) and (1.618034, ∞).*

Proof. This follows by applying Lemma 8.13 to the intervals from Lemma 8.15. These form a series of $k-1$ nested geodesics that do not intersect with each other. □

Lemma 8.17. *The geodesics of $b^{k-n}a^j b^n$ intersect the geodesics of $a^m b^k a^{j-m}$ (with the exception of $b^k a^j$).*

Proof. We already know that these feet of $b^{k-n}a^j b^n$ are to the right of the the relevant foot of $b^k a^j$, as we applied the iteration Lemma 8.16 to $b^k a^j$ and found that the roots of $b^{k-n}a^j b^n$ are monotonically increasing from this point. Now we must find an interval for the rightmost foot of $ab^k a^{j-1}$. We already know from Lemma 8.14 that this foot must be in the interval (-.618034, .618034) as it follows the form $a^{j-m}b^k a^m$. Now, we find the matrix associated with the word $ab^k a^x$ and find the limit of the upper root as x approaches ∞ in the same fashion as above. We find this value to be .2763932, and as the values of the roots of $a^{j-m}b^k a^m$ were monotonically decreasing from at most .618034, we know that the right foot of $ab^k a^{j-1}$ thus lies in the interval (.2763932, .618034).

Now we must only find an interval for the left foot of $ba^j b^{k-1}$. Similar to the method above, we know this foot to be in the interval (-.618034, .618034) as it is of the form $b^{k-n}a^j b^n$. We find the matrix associated with the word $ba^j b^y$ and find the limit of the upper root as y approaches ∞.

We compute this value to be -.276392, and as the values of the roots of $b^{k-n}a^jb^n$ were monotonically increasing from -.618034, we know the left foot of ba^jb^{k-1} lies in the interval (-.618034, -.2763932). As this indicates that the leftmost feet of $b^{k-n}a^jb^n$ are between the rightmost feet of b^ka^j and that of ab^ka^{j-1}, we know they intersect. □

Lemma 8.18. *The intersections of the geodesics of the word a^jb^k occur within the fundamental region.*

Proof. Because the leftmost feet of the geodesics of $b^{k-n}a^jb^n$ are greater than -.618034 and the rightmost feet of the geodesics of ab^ka^{j-1} are less than .618034, we know the right and left limiting cases of intersections between these two groups of geodesics occur between -1 and 1. Futhermore, because the geodesics of ab^ka^{j-1} and ba^jb^{k-1} have radii greater than .5, we know that the lower limiting case of their intersection occurs within the fundamental region. □

The table below organizes the intervals discussed above in a table to facilitate clarity.

Permutation	Lower Root	Upper Root
a^jb^k	(.381966, .618034)	(1.618034, 2.618035)
$a^{j-m}b^ka^m$	(-∞, -1.618034)	(-.618034, .618034)
b^ka^j	(-2.618035, -1.618034)	(-.618034, -.381966)
$b^{k-n}a^jb^n$	(-.618034, .618034)	(1.618034, ∞)
ab^ka^{j-1}	(-∞, -1.618034)	(.2763932, .618034)
ba^jb^{k-1}	(-.618034, -.2763932)	(1.618034, ∞)

Theorem 8.19. *A word W of the form a^jb^k (where $j \geq k \geq 2$) has $(j-1)(k-1)$ intersections.*

Proof. By Lemma 8.17, we know that the $k-1$ geodesics of $b^{k-n}a^jb^n$ intersect the $j-1$ geodesics of $a^{j-m}b^ka^m$. Moreover, by Lemma 8.18, we know all these intersections occur within the fundamental region. Hence, the number of intersections within the fundamental region is $(j-1)(k-1)$. □

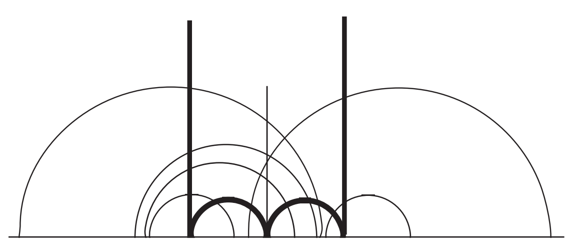

FIGURE 16. Intersections in $\mathcal{D}$ of *aaaabb*

Example 8.20. *The word aaaabb has 3 intersections in the fundamental region (see Figure 16).*

9. Geometric Approaches to Determining the Feet of Geodesics

9.1. **Background Information.** In this section we will use cutting sequences to examine once again the monotonicity of roots proved above, as well as other simple patterns. This background information is taken directly from [C]. A *cutting sequence* $S(\gamma)$ of a geodesic γ is a doubly infinite sequence composed of the symbols a, b, A, and B. When listing these sequences we usually omit the commas, e.g.

$$S(\gamma) = \ldots\ldots X_{-1}X_0X_1\ldots\ldots$$

where each $X_i \in \{a,b,A,B\}$. We also use the notation

$$W^n = \overbrace{W\ldots W}^{n}$$

where n is a positive integer and W is a word composed of the generators a, b, A, and B. We will let W^{-1} denote the word obtained by reversing W and replacing each symbol by its inverse. We

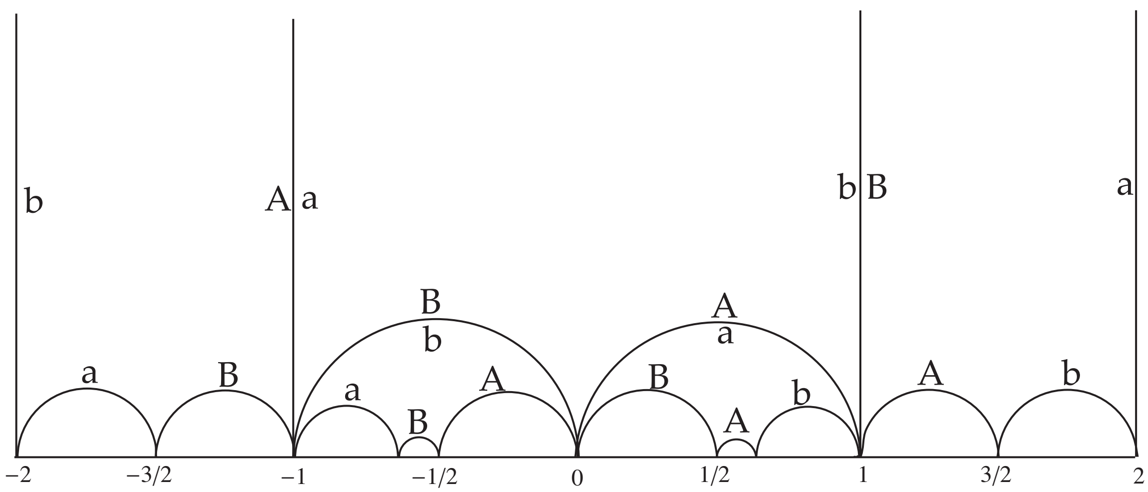

FIGURE 17. The labelled grid Λ

will now describe a labelled grid of geodesics in $\mathbb{H}$. This grid is obtained by taking images under Γ' of the sides of $\mathcal{D}$ (see Figure 17). We can now label this grid using the cyclic ordering

$$\ldots\ldots a < B < A < a\ldots\ldots$$

We begin by labelling the vertical line between -1 and ∞ within $\mathcal{D}$ as a, and then moving counter-clockwise around $\mathcal{D}$ we label the semicircle with feet at -1 and 0 as B, that with feet at 0 and 1 as A, and finally the vertical line between 1 and ∞ as b. The labels of the other side of each gridline correspond to the inverses of those discussed above. All labels can thus be obtained using this labelling in $\mathcal{D}$. This grid is referred to as the *labelled grid induced by* Γ and is denoted by Λ. Recall that as all geodesics in $\mathbb{H}$ that project to closed geodesics on $\mathbb{T}$ have irrational endpoints, each foot will cut the lines of Λ infinitely often. Note that reading $S(\gamma)$ from left to right corresponds to the geodesic γ traversing a grid line, with the convention that only the label immediately after the line is listed. [C] also notes that the sequence of grid lines crossed by γ is completely determined by the initial one, and that initial one belongs to some fan. That is to say, at some point γ will entire a fan of tiles never to leave again. This is important to the work we include in this section, particularly in Case 1 below. From this idea arise the following definition and theorem.

Definition 9.1. [C] *The* boundary expansion $S(\xi)$ *of an irrational point* ξ *is the cutting sequence of any geodesic which begins within* $\mathcal{D}$ *and ends at* ξ. *We write*

$$S(\xi) = X_0X_1X_2\ldots\ldots$$

where each $X_i \in \{a,b,A,B\}$.

Theorem 9.2. [C] *Let* ξ *and* ξ' *be distinct irrationals with boundary expansions*

$$S(\xi) = X_0X_1X_2\ldots\ldots \text{ and } S(\xi') = X_0'X_1'X_2'\ldots\ldots,$$

respectively. Then $\xi < \xi'$ *if and only if* $X_0X_1X_2\ldots < X_0'X_1'X_2'\ldots$.

From this theorem, we see that when comparing two words, we need only look at the first place the words differ to apply the cyclic ordering of the symbols to determine which foot is larger than the other. This result leads to the method applied in Case 2 below.

9.2. **Monotonicity of roots.** Using the cutting pattern discussed in the previous section, it is not difficult to prove the monotonicity of roots when adding a's or b's to a word of the form a^jb^k. To do so then, we will make use of the following diagrams to prove the monotonicity once again by double induction.

Case 1: We will first demonstrate that both roots of a^jb^k increase monotonically when adding a's. We will consider the legs of the geodesic one at a time. First we see that for a word of the form a^jb^k, one of the legs begins by cutting the right semicircle of $\mathcal{D}$. For simplicity, we will denote the leftmost, centermost, and rightmost semicircles representing a particular reflection of the fundamental region as left, center, and right respectively, unless noted otherwise.

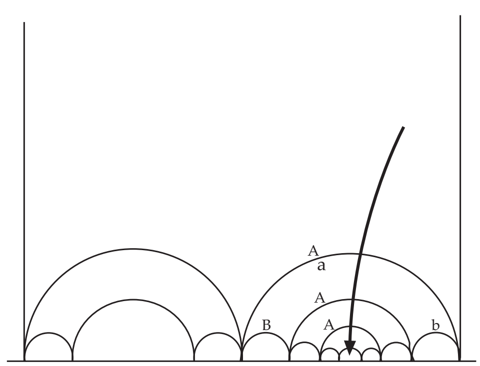

FIGURE 18

After this, the foot will cut the center a total of $j-1$ times (see Figure 18). After cutting through the $j-1$th center semicircle, (and thus accounting for all the a's), the foot then will cut the left.

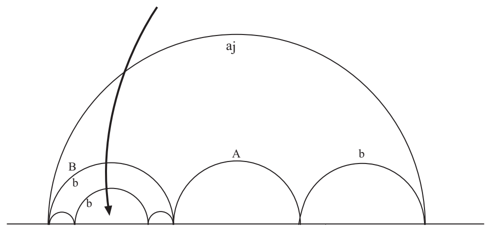

FIGURE 19. Cutting sequence of $a^j b^k$

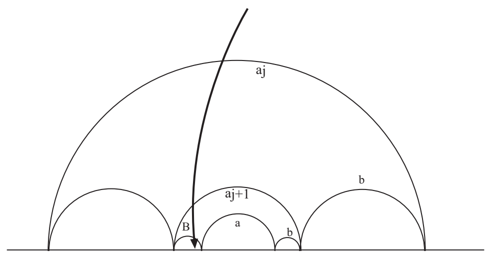

FIGURE 20. Cutting sequence of $a^{j+1} b^k$

From this point on, the foot will cut the center a total of k times (see Figure 19). This process will continue, as it is cyclic and the roots of the geodesic are irrational. However, considering a word of the form $a^{j+1} b^k$, we know this foot will follow the same path up to the point where it cuts the kth center (see Figure 20). After this, the foot will cross once more the center before crossing the left and eventually centers representing the b's. By these figures then, we see that this foot is monotonically larger for $a^{j+1} b^k$ than the respective foot of $a^j b^k$.

Now, for the other leg, we consider the word $B^k A^j$. We see that it crosses to the horizontal

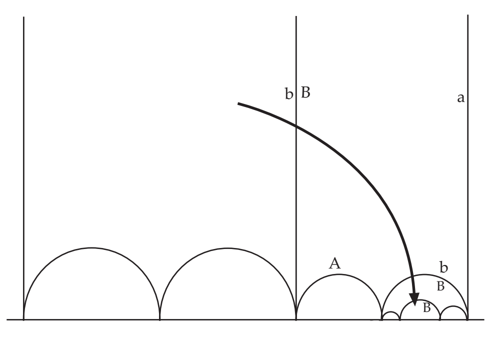

FIGURE 21

translation of $\mathcal{D}$, of which it cuts the rightmost semicircle (see Figure 21. Once the B's are accounted for, the foot cuts the right and then proceeds to cut the center $j-1$ times, until the A's are all represented. Because of the cyclic nature of the cutting, the foot then cuts the left and repeats

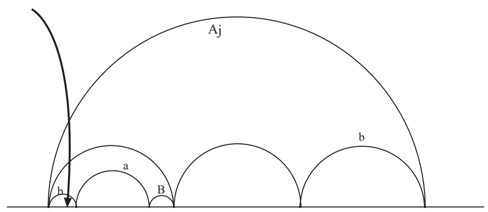

FIGURE 22. Cutting sequence of B^kA^j

the pattern of above. (see Figure 22). For the word B^kA^{j+1}, the foot follows the same pattern up

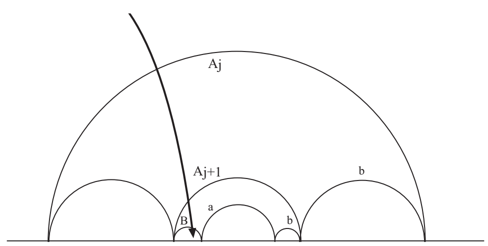

FIGURE 23. Cutting sequence of B^kA^{j+1}

to the point where it crosses the kth center. From here, it cuts once more the center before cutting the left and repeating the previous pattern. Then, as seen in the figures, this foot is monotonically larger (see Figure 23).

Case 2: We will now show that both roots of a^jb^k decrease monotonically when adding b's, again considering one foot as a time. We may do so using the same method as above, with a different approach. Comparing the words a^jb^k and a^jb^{k+1}, we see that the words are identical to a certain point, namely where a^jb^k cycles back to a and a^jb^{k+1} has one additional b before cycling through the word again (see Figure 24). As such, we know a^jb^k cuts through an A at the same time as a^jb^{k+1} cuts a B. Using the cyclic ordering used in the cutting pattern of the fundamental region, $\ldots a < B < A < b < \ldots$, we see that $B < A$. Hence, this foot of a^jb^{k+1} is smaller than that of a^jb^k. Similarly, to determine the monotonicity of the other foot we see that B^kA^j and $B^{k+1}A^j$ are identical until B^kA^j begins to cut the series of a's, while $B^{k+1}A^j$ cuts one additional b. By the same cyclic ordering, as $b < a$, we know this foot of $B^{k+1}A^j$ is smaller than that of B^kA^j. Thus by simply looking at a word and examining the first symbol at which W and W_1 are different, we know which word has larger or smaller feet. This completes the proof of monotonicity.

$$aa\ldots a_j bb\ldots b_k (a\ldots)$$
$$aa\ldots a_j bb\ldots b_k b_{k+1} (a\ldots)$$

$$BB\ldots B_k A\ldots A_j\ldots$$
$$BB\ldots B_k B_{k+1} A\ldots A_j\ldots$$

FIGURE 24

10. PATTERNS OF OTHER WORDS

In this section, we will briefly discuss the patterns of the feet of geodesics of two other simple word forms, $a^j b^k a^l b^m$ and $a^j b^k a^l B^m$. We begin with $a^j b^k a^l b^m$, where without loss of generality we can assume that $j \geq k$. To do so, we use a method similar to that of Case 2 in the section above. That is, we examine the first point of divergence between words, and using the cyclic ordering of symbols mentioned above, we can say which word has larger roots. For a word of the form $a^j b^k a^l b^m$, we need only examine 16 possibilities of how roots can change when varying any of j, k, l, or m. Although there are many more possibilities for how j, k, l, and m vary, note that we are only concerned with the first location of divergence between words in our analysis, due to Theorem 9.2. To see this, notice that the roots of words $a^{j+1} b^k a^l b^{m+1}$ would change in the same general fashion from $a^j b^k a^l b^m$ as would those of $a^{j+1} b^k a^{l+1} b^{m+1}$. In the table below, each entry indicates a comparison of the position of feet between a particular W_1 and W_1^{-1} with the original W and W^{-1}. An entry of “greater” is thus read as indicating that the foot of W_1 or $W_1^{-1}$is greater than that of W or W^{-1}. Note that when only one variable is changed, as in the first four lines, we consider only the case where it increases (the other case is easily obtained).

Change to word $a^j b^k a^l b^m$	Change in foot of W	Change in foot of W^{-1}
Fix k, l, m (j increases)	Greater	Greater
Fix j, l, m (k increases)	Less	Less
Fix j, k, m (l increases)	Greater	Greater
Fix j, k, l (m inceases)	Less	Less
Fix l, m		
(j, k, increase)	Greater	Less
(j increases, k decreases)	Greater	Greater
Fix k, m		
(j, l, increase)	Greater	Greater
(j increases, l decreases)	Greater	Less
Fix k, l		
(j, m, increase)	Greater	Less
(j increases, m decreases)	Greater	Greater
Fix j, l		
(k, m, increase)	Less	Less
(k increases, m decreases)	Less	Greater
Fix j, m		
(k, l, increase)	Less	Greater
(k increases, l decreases)	Less	Less
Fix j, k		
(l, m, increase)	Greater	Less
(l increases, m decreases)	Greater	Greater

From this table then, we can understand the behavior of the geodesics of words of the form $a^j b^k a^l b^m$ when changing the value of j, k, l, and m, in relation to the original word. Using the same method, we also present a table of the same type as above compiling the behavior of words of the form $a^j b^k a^l B^m$ when changing j, k, l, and m. Although this is just a brief analysis of the behavior of words of these forms, it could perhaps serve as a starting point for a further study of this type. Certainly, as these tables only analyze one permutation of each of these forms, more work could be done to consider the behavior of all cyclic permutations.

Change to word $a^j b^k a^l B^m$	Change in foot of W	Change in foot of W^{-1}
Fix k**,** l**,** m (j increases)	Greater	Less
Fix j**,** l**,** m (k increases)	Less	Less
Fix j**,** k**,** m (l increases)	Less	Greater
Fix j**,** k**,** l (m inceases)	Greater	Greater
Fix l**,** m		
(j, k, increase)	Greater	Less
(j increases, k decreases)	Greater	Greater
Fix k**,** m		
(j, l, increase)	Greater	Greater
(j increases, l decreases)	Greater	Less
Fix k**,** l		
(j, m, increase)	Greater	Greater
(j increases, m decreases)	Greater	Less
Fix j**,** l		
(k, m, increase)	Less	Greater
(k increases, m decreases)	Less	Less
Fix j**,** m		
(k, l, increase)	Less	Greater
(k increases, l decreases)	Less	Less
Fix j**,** k		
(l, m, increase)	Less	Greater
(l increases, m decreases)	Less	Less

11. CONCLUSION

We have found formulas for the intersection number of all words composed of no more than four blocks of letters and of words that are a power of a primitive word (given the intersection number of the primitive word). We have proved the monotonicity of roots of words of the pattern $a^j b^k$ when adding b's and a's. We have also used both algebraic and geometric methods as a means for better understanding the patterns of geodesics of words of simple patterns, namely $a^j b^k$, $a^j b^k a^l b^m$, and $a^j b^k a^l B^m$.

Although much work has been done on the intersection numbers of curves on T, there are still open questions that remain. A study of the behavior of geodesics of words of the form $a^j b^k a^l b^m$ and $a^j b^k a^l B^m$ could be continued to better understand which changes lead to intersections. More research could be done on words composed of simple patterns (as one example, $a^j b^k A^l b^m$) to better understand the intersections in $\mathcal{D}$. Moreover, it would clearly be advantageous to extend the formulas found above to a general formula in order to calculate the number of intersections, given simply a specific word. Some of the results of this paper may also be able to be generalized to other surfaces.

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Appendix A

```
IntersectByDrawing[word_String, order_List, max_] :=
  Module[{na, nb, i, j, count, l1, r1, l2, r2},
    For[na = 0; nb = 0, na + nb < StringLength[word],
      If[ToLowerCase[StringTake[word, {na + nb + 1}]] == "a", na++, nb++]];
    word2 = word <> StringTake[word, {1}];
    order2 = Join[order, {order[[1]]}];
    For[i = 1; count = 0, i <= StringLength[word] - 1, i++,
      For[j = i + 1, j <= StringLength[word], j++,
        If[StringTake[word2, {i}] == "a", l1 = na - order2[[i]] + 1,
          If[StringTake[word2, {i}] == "b", l1 = na + order2[[i]],
            If[StringTake[word2, {i}] == "A", l1 = na + nb + order2[[i]],
              l1 = 2*na + 2*nb - order2[[i]] + 1]]];
        If[StringTake[word2, {i + 1}] == "A", r1 = na - order2[[i + 1]] + 1,
          If[StringTake[word2, {i + 1}] == "B", r1 = na + order2[[i + 1]],
            If[StringTake[word2, {i + 1}] == "a",
              r1 = na + nb + order2[[i + 1]],
              r1 = 2*na + 2*nb - order2[[i + 1]] + 1]]];
        If[StringTake[word2, {j}] == "a", l2 = na - order2[[j]] + 1,
          If[StringTake[word2, {j}] == "b", l2 = na + order2[[j]],
            If[StringTake[word2, {j}] == "A", l2 = na + nb + order2[[j]],
              l2 = 2*na + 2*nb - order2[[j]] + 1]]];
        If[StringTake[word2, {j + 1}] == "A", r2 = na - order2[[j + 1]] + 1,
          If[StringTake[word2, {j + 1}] == "B", r2 = na + order2[[j + 1]],
            If[StringTake[word2, {j + 1}] == "a",
              r2 = na + nb + order2[[j + 1]],
              r2 = 2*na + 2*nb - order2[[j + 1]] + 1]]];
        {l1, r1} = Sort[{l1, r1}];
        {l2, r2} = Sort[{l2, r2}];
        If[(l1 < l2 && l2 < r1 && r1 < r2) || (l2 < l1 && l1 < r2 && r2 < r1),
           count++];
```

```
        If[max != -1 && count >= max, Return[max]]]];
    count]

DrawingPermutations[word_String] :=
  Module[{i, j, k, ik, jk, list, acount, bcount, lista, listb, listboth,
      finallist, pa, pb},
    list = Characters[word];
    finallist = {};
    acount = Count[list, "a"] + Count[list, "A"];
    bcount = Count[list, "b"] + Count[list, "B"];
    pa = Permutations[Table[i, {i, 1, acount}]];
    pb = Permutations[Table[i, {i, 1, bcount}]];
    For[i = 1, i <= Length[pa], i++,
      For[j = 1, j <= Length[pb], j++,
        lista = pa[[i]];
        listb = pb[[j]];
        listboth = {};
        For[k = 1; ik = 1; jk = 1, k <= Length[list], k++,
          If[list[[k]] == "a" || list[[k]] == "A",
            listboth = Join[listboth, {lista[[ik++]]}],
            listboth = Join[listboth, {listb[[jk++]]}]]];
        finallist = Join[finallist, {listboth}]]];
    finallist]

MinimalPermutations[word_String] := Module[{i, all, minint, minlist, int},
    all = DrawingPermutations[word];
    minint = IntersectByDrawing[word, all[[1]], -1];
    minlist = {all[[1]]};
    For[i = 2, i <= Length[all], i++,
      int = IntersectByDrawing[word, all[[i]], minint + 1];
      If[int < minint,
        minint = int;
        minlist = {all[[i]]},
        If[int == minint,
          minlist = Join[minlist, {all[[i]]}]]]];
    {minint, minlist}]
```

DAVIDSON COLLEGE
E-mail address: frchemotti@davidson.edu

ST. OLAF COLLEGE
E-mail address: rau@stolaf.edu