A GRAPHICAL EXPLORATION OF STABLE CHARACTERISTICS OF SIMPLE POPULATION MODELS

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ABSTRACT. Seven simple population models are widely used in biological literature. These models display global stability when they display local stability. We examine various graphical representations of these seven models. (How each demonstrates the stable or unstable behavior of each model due to varying parameters such as reproductive rates.) This graphical exploration is done through the use of basic curves, time plots, time-plus-2 curves, bifurcation maps and complex convergence plots. We study the effect of various parameters like reproductive rate on the models' behaviors. We conclude that while these models differ in details, they can generally be used interchangeably.

1. INTRODUCTION

Typical population growth and decay can be modeled by discrete one-dimensional difference equations. The models of interest share the characteristic that they increase to a certain carrying capacity and decrease thereafter. When these models are globally stable, they reach equilibrium where the birth and death rates are equal, regardless of initial population. For our purposes, these models have been normalized so that the equilibrium is at x=1. When they are locally stable, they converge to this equilibrium only for initial populations that are already near equilibrium. These models display global stability if they display local stability. Previous work has found a condition that demonstrates this characteristic is that these seven models have been shown to demonstrate local stability and therefore global stability if they are enveloped by linear fractional functions. These simple models can demonstrate complex behavior for high reproductive rates. Both the stable and the complex behavior will be demonstrated graphically through the use of basic curves, time plots, time-plus-2 curves, bifurcation maps and black and white complex convergence plots.

1.1. **Background and Definitions.**

Definition 1.1. *The following definitions and theorems are from* [1, 2, 3]. *A one-dimensional population model is a function of the form*

$$x_{t+1}=f(x_t)$$

where f is a continuous function from the nonnegative reals to the nonnegative reals and there is a positive number $\bar{x}$, the equilibrium point, such that:

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f(0)=0
$f(x) > x$ *for* $0 < x < \bar{x}$
$f(x) = x$ *for* $x = \bar{x}$
$f(x) < x$ *for* $x > \bar{x}$
and if $f'(x_m) = 0$*, where* x_m *is a critical point and* $x_m \leq \bar{x}$ *then*
$f'(x) > 0$ *for* $0 \leq x < x_m$
$f'(x) < 0$ *for* $x > x_m$ *such that* $f(x) > 0$.

Definition 1.2. *A model is globally stable if and only if for all* x_o *such that* $f(x_o) > 0$ *we have*

$$\lim_{x\to\infty} x_t = \bar{x}.$$

where $\bar{x}$ *is the unique equilibrium point of* $x_{t+1}=f(x_t)$*.*

Definition 1.3. *A model is locally stable if and only if for every small enough neighborhood of* $\bar{x}$ *if* x_o *is in this neighborhood, then* x_t *is in this neighborhood for all t, and*

$$\lim_{x\to\infty} x_t = \bar{x}.$$

While difficult to test, the following theorems determine when a model is locally or globally stable.

Theorem 1.4. *If* $f(x)$ *is differentiable then, a model is locally stable if* $|f'(\bar{x})| < 1$*, and if the model is locally stable then* $|f'(\bar{x})| \leq 1$*.*

Theorem 1.5. *A continuous model is globally stable if and only if it has no cycle of period 2. (That is, there is no point except* $\bar{x}$ *such that* $f(f(x)) = x$*.)*

If we examine the following example, we can ascertain a somewhat simpler method of determining local stability.

Example 1.6. Theorem 1.4 gives that for x slightly less than 1, $f(x)$ is below a straight line with slope -1, and if for x slightly greater than 1, $f(x)$ is above the same straight line, then the model is locally stable. If we examine model 1 with $r = 2$: $x_{t+1} = x_t e^{2(1-x_t)}$, it can be seen that the local stability bounding line is $2 - x$. It can also be seen that this line is an upper bound on $f(x)$ for all x in [0,1) and a lower bound for all $x > 1$. From Theorem 1.5 we note that since the bounding line has a cycle of period 2, $2 - (2 - x) = x$, then our model cannot have a cycle of period 2 and hence is globally stable. The next definition follows from this idea.

Definition 1.7. *A function* $\phi(x)$ *envelops a function* $f(x)$ *if and only if*

- $\phi(x) > f(x)$ *for* $x \in (0,1)$
- $\phi(x) < f(x)$ *for* $x > 1$ *such that* $\phi(x) > 0$ *and* $f(x) > 0$

Definition 1.8. *A linear fractional function is a function of the form*

$$\phi(x) = \frac{1-\alpha x}{\alpha-(2\alpha-1)x} \quad \textit{where } \alpha \in [0,1).$$

These functions have the following properties:

- $\phi(1)=1$
- $\phi'(1) = -1$
- $\phi(\phi(x)) = x$
- $\phi'(x)$¡0.

As the models are meant to represent real populations, for practicality reasons these functions are only of interest when $x > 0$ and $\phi(x) > 0$.

The main argument in [1],[2],[3] is that if $f(x)$ is enveloped by a linear fractional function, then $f(x)$ is locally stable and therefore globally stable. As such, additional results and proofs of the following theorem appear in [1],[2] and [3].

The following theorem assumes that the model of interest is $x_{t+1} = f(x_t)$, and that the model is normalized so that the equilibrium point is 1, that is f(1) = 1.

Theorem 1.9. *Let ϕ be a monotone decreasing function which is positive on $(0,x_-)$ and $\phi(\phi(x)) = x$. If $f(x)$ is a continuous function such that:*

- $\phi(x) > f(x)$ *on (0,1)*
- $\phi(x) < f(x)$ *on* $(1,x_-)$
- $f(x) > x$ *on (0,1)*
- $f(x) < x$ *on* $(1,\infty)$
- $f(x) > 0$ *on* $(1,x_\infty)$

then for all $x \in (0,x_\infty)$,

$$\lim_{k\to\infty} f^{(k)}(x) = 1.$$

Corollary 1.10. *If $f_1(x)$ is enveloped by $f_2(x)$, and $f_2(x)$ is globally stable, then $f_1(x)$ is globally stable.*

Corollary 1.11. *If $f(x)$ is enveloped by a linear fractional function then $f(x)$ is globally stable.*

Additionally, from [1, 2, 3] we know that population models with one choice of parameters will envelop the same model with a different choice of parameters. In these papers, the enveloping technique was applied to the seven models from literature, however it was noted that enveloping was not necessary for global stability.

2. Characteristics and Methods of Plots

For the following models basic curves, time plots, time-plus-2 curves, bifurcation maps and black and white complex convergence plots will be used to examine stable and unstable behavior. The basic curves, time plots and time-plus-2 curves were created using Maple while the bifurcation maps and black and white convergence plots were created using programs written in Java.

2.1. **Basic Curves.** The purpose of the basic curve is to show the $y = f(x)$ curve for a specific parameter or rate. That is, we can find the maximum size of the population for a particular rate after one time step by solving $f'(x) = 0$ and also where the population dies out by solving $f(x) = 0$, excluding the obvious solution of f(0)=0.

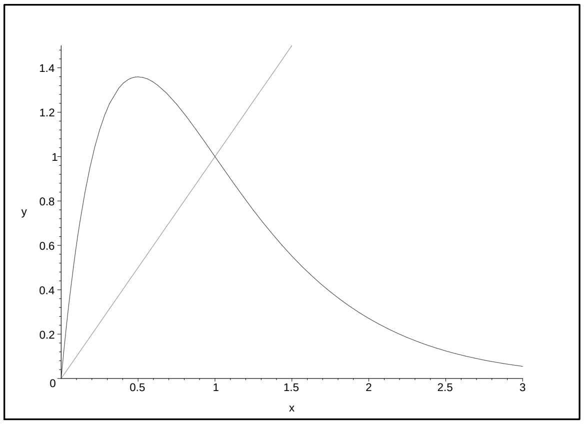

FIGURE 1. A basic curve

2.2. **Time Plots.** The purpose of the time plot is to show what happens to the population over several time steps for a particular reproduction rate and initial population. In these plots, the size of the population is plotted against time. For different reproduction rates or initial populations, the behavior of the model could demonstrate stable behavior where the population either approaches equilibrium (one population size or oscillates between two or more population sizes or unstable behavior with the model degenerating into chaos. Unless otherwise specified, all time plots in this paper will begin with initial population $x = .1$.

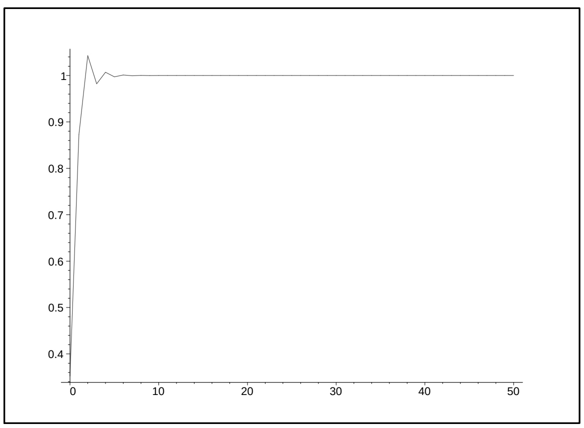

FIGURE 2. Time plot with population approaching equilibrium

2.3. **Time-Plus-2 Curves.** Unlike the basic curve, the time-plus-2 curve shows the behavior of the population after two time steps. That is, it plots $y = f(f(x))$ and the $y = x$ line. A series of time-plus-2 curves can demonstrate the rate where the population will oscillate between two different sizes. If the population still approaches the equilibrium for a given rate, the $f(f(x))$ curve will only intersect the $y = x$ line at one place, the equilibrium, in our case $x = 1$. When a rate is where the population oscillates, the $f(f(x))$ curve will be tangent to the $y = x$ line at the equilibrium point. The time-plus-2 curves make it easy to distinguish when a rate is beyond the

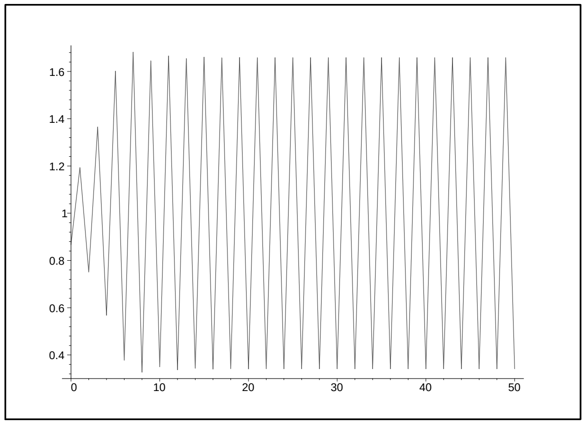

FIGURE 3. Time plot with period 2 oscillation

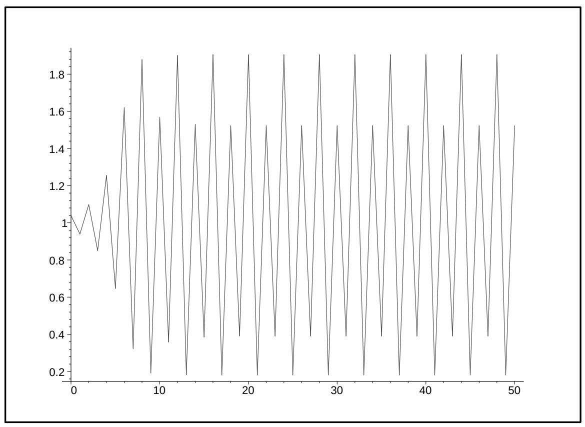

FIGURE 4. Time plot with period 4 oscillation

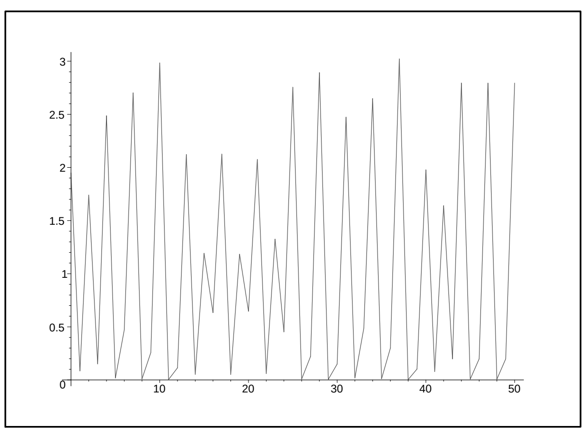

FIGURE 5. Time plot with chaotic behavior

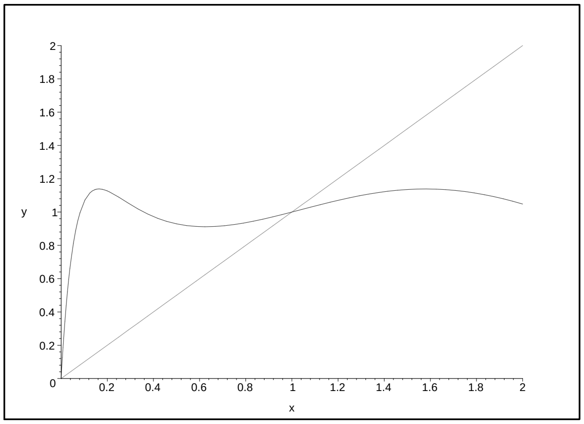

FIGURE 6. Time-Plus-2 Curve before bifurcation point

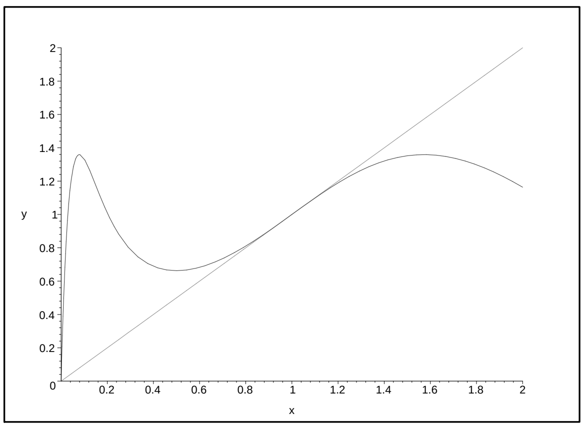

FIGURE 7. Time-Plus-2 Curve at bifurcation point

bifurcation point because the $f(f(x))$ curve will intersect the $y = x$ line at exactly three places, the unstable equilibrium, and the population values that have oscillation of period 2. Thus if one solved $f(f(x)) = x$, they would find the population values for which the equilibrium is no longer stable because an equilibrium point is stable if and only if there is no oscillation of period 2 (see theorem 1.5 above). This result can be found as a result of Sarkovskii's Theorem given in [6] and a modification of Sarkovskii's Theorem given in [5]. As demonstrated in [15], the rates that generate oscillations of period 4 can be found be plotting time-plus-4 curves, which would yield five intersection points with the $y = x$ line: the unstable equilibrium and the four population values that have oscillation of period 4. Thus one can easily find oscillations of period k by looking at time-plus-k curves.

2.4. **Bifurcation Maps.**

Definition 2.1. *A bifurcation is a split in two, typically due to a parameter change in a system. The parameter at which the bifurcation occurs is typically known as a bifurcation value. Bifurcation values occur where a system is structurally unstable.* [6, 12]

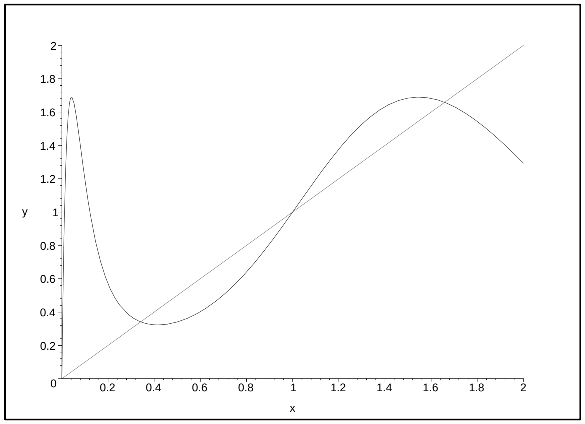

FIGURE 8. Time-Plus-2 Curve after bifurcation point

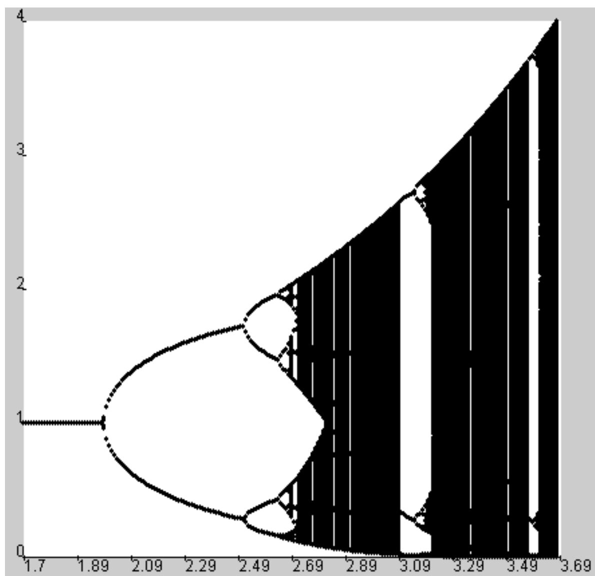

FIGURE 9. Bifurcation Map

Bifurcation maps are useful because they demonstrate the long term behavior of a population affected by many reproductive rates and initial population values. The program that creates the bifurcation maps iterates the function an infinite number of times (in our case 500 represents infinity) for many initial population values and rates but does not plot the population. This is done in order to eliminate bifurcation transients. Bifurcation transients are unusual behavior after a bifurcation due to too few iterations [12]. It then plots the population value after an additional 500 iterations as the vertical value and the rate it corresponds to as the horizontal value. By examining the plot, one will see a single line where the population is at equilibrium and then see it split in two, or bifurcate, representing an oscillation of period two. It can then bifurcate again into oscillations of period four and so on until the plot becomes chaotic and one can no longer distinguish where the model bifurcates except in the white stripes. The white stripes represent areas of stability until they bifurcate into chaos again. One can even predict where the next bifurcation value B_k occurs, from a period 2^k orbit to a period 2^{k+1} orbit as it is

$$\lim_{k\to\infty} \frac{B_k - B_{k-1}}{B_{k+1} - B_k}$$

This is known as Feigenbaum's number and it is approximately 4.669 [7]. In particular, we are interested in the period-doubling bifurcations mentioned in [6] as a change from an attraction to a fixed point to the creation of a period two orbit. When used in conjunction with the time plots and time-plus-2 curves, one can confirm the rates and population values that the population will bifurcate at.

2.5. **Black and White Complex Convergence Plots.** Complex convergence plots give indications of a model's stability at particular reproduction rates along the real axis. The program to create the complex plots begins with a complex initial population then iterates for an infinite number (again 500) of times. If the final population value is within a certain range or box, then the population value is considered to be converging to the equilibrium and the point corresponding to the starting value is colored white. However, if the final value is not within the range, then the population is considered to be shooting toward infinity and the starting value will be colored black. The range of the plot (or bounds of the box) are the horizontal range from 0 to 2, which is meant to be the real part of the complex number, and the vertical range from -1 to 1, which is the imaginary part of the complex number. It is necessary to make plots for various rates for each model to gain an accurate idea of when the equilibrium is stable and when it is not. If the equilibrium is stable, the real axis should be white, and the population values are considered to be converging to one. However if there are breaks on the real axis, the equilibrium is considered to be unstable, and thus these plots might suggest rates where the population again bifurcates.

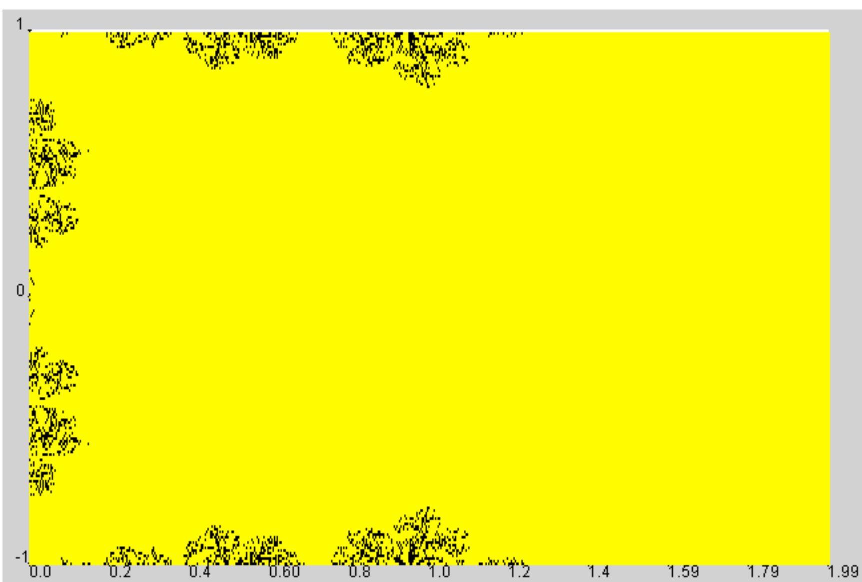

FIGURE 10. Complex Convergence Plot

3. THE SEVEN MODELS

3.1. Model I. $x_{t+1} = x_t e^{r(1-x_t)}$

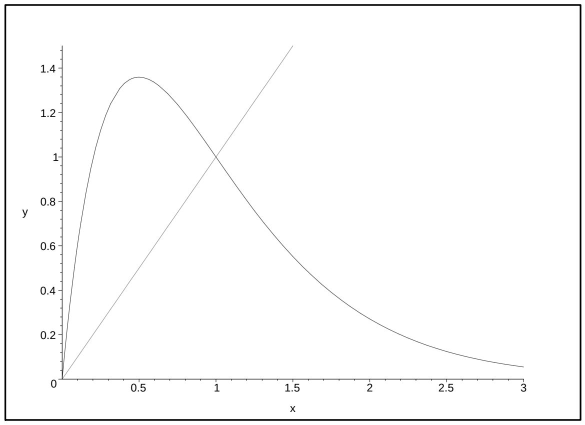

FIGURE 11. Model I basic curve

Model I from [9],[10] and [14] is one of the most commonly used population models. From [1, 2, 3] we know that the model is is globally stable when $0 < r \leq 2$. We observe this behavior of the model by examining the following plots.
When examining the time plot of model 1 with $r = 1.4$ we observe that the model does behave as predicted and the population approaches the equilibrium.

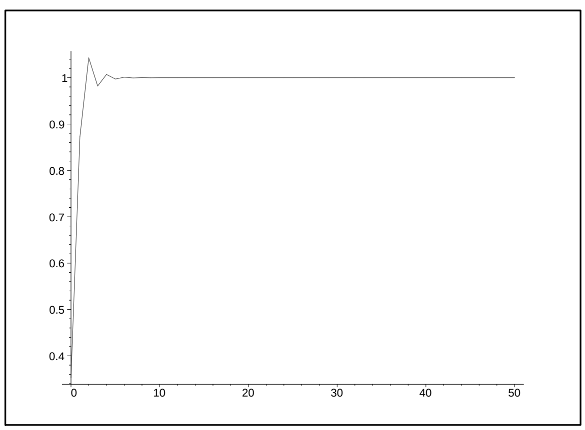

FIGURE 12. Model I Time Plot with $r = 1.4$

However for $r = 2$, it appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at this rate as Cull suggests. The explanation for this discrepancy is actually due to computer approximation and the actual time plot for $r = 2$ should look similar to that of $r = 1.4$.
For $r > 2$ we expect the population to not converge to the equilibrium, and actually for values of $2 < r \leq 2.5$ the population oscillates between 2 population values as demonstrated by the time plot with $r = 2.4$.

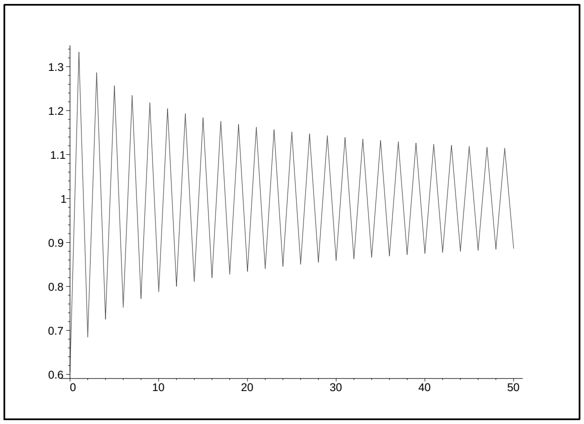

FIGURE 13. Model I Time Plot with $r = 2$

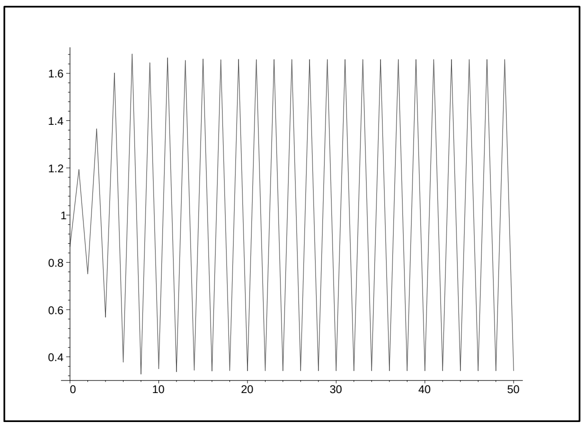

FIGURE 14. Model I Time Plot with $r = 2.4$ with a period 2 oscillation

For $r \geq 2.5$ the population bifurcates again and again from a period 4 oscillation to period 8 and so forth until it descends into chaos. This is demonstrated by the time plot for $r = 2.6$ which demonstrates the period 4 oscillation and the time plot $r = 3.4$ which demonstrates the chaotic behavior of the population exhibited at this reproduction rate.

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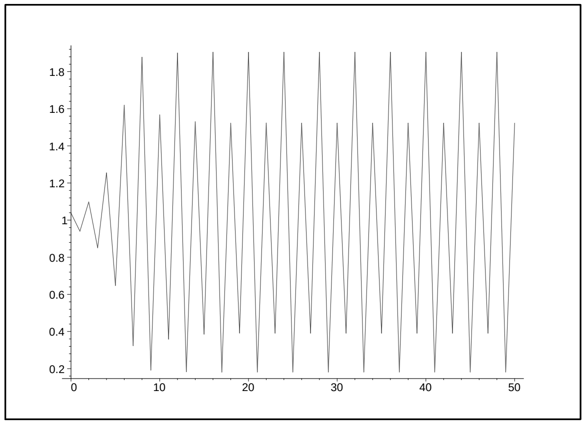

FIGURE 15. Model I Time Plot with $r = 2.6$ with a period 4 oscillation

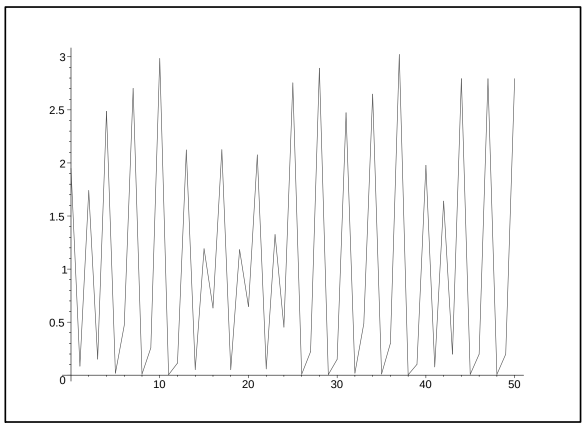

FIGURE 16. Model I Time Plot with $r = 3.4$ exhibiting chaotic behavior

Example 3.1. *We find by solving $f(f(x)) = x$ when $r = 2.5$ results in the values $x = .2895$ and $x = 1.71$. It can be shown for any initial population when $r = 2.5$ that the population still oscillates between these exact values. We note this behavior demonstrated by the time plots for $r = 2.5$ with initial populations $x = .3, x = 1.1$ and $x = 2.6$ respectively.*

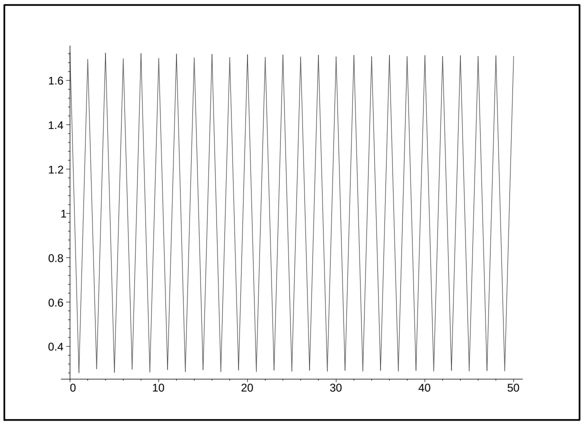

FIGURE 17. Model I Time Plot with $r = 2.5$ and initial population $x = .3$

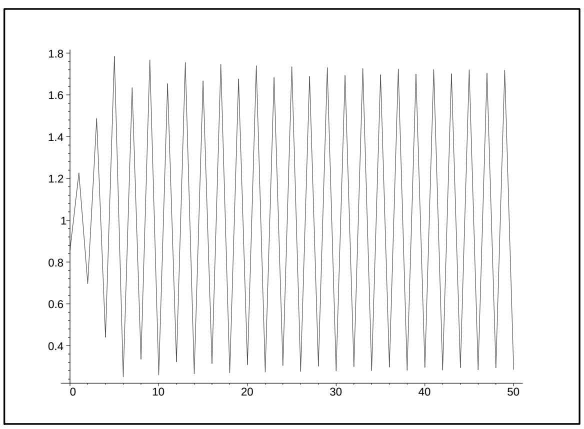

FIGURE 18. Model I Time Plot with $r = 2.5$ and initial population $x = 1.1$

Since there are many indications that the behavior of the model is independent of the initial population for all seven models, this part of the discussion will be neglected for the following six models.

We can also verify that the population for Model I is globally stable at $r \leq 2$ by examining the time-plus-two curves. For the globally stable values of $r \leq 2$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $r = 1.6$.
When $r \leq 2$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $r = 2$.

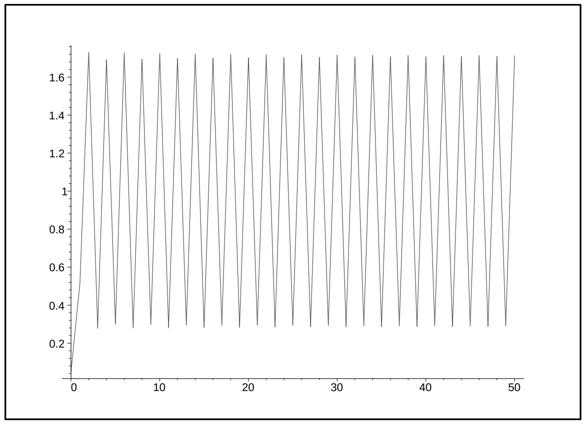

FIGURE 19. Model I Time Plot with $r = 2.5$ and initial population $x = 2.6$

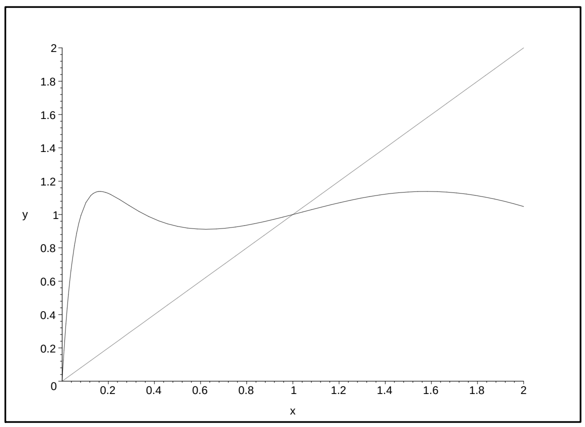

FIGURE 20. Model I Time-Plus-2 Curve with $r = 1.6$

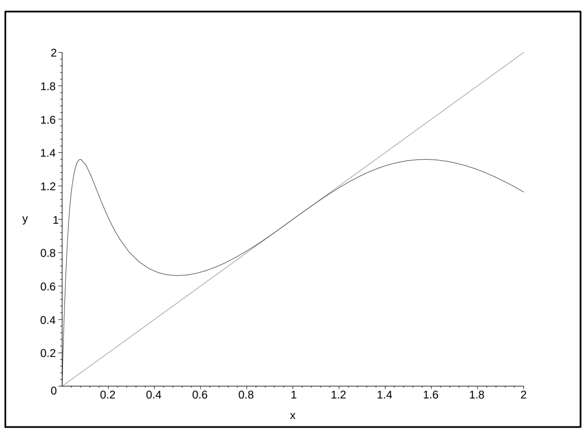

FIGURE 21. Model I Time-Plus-2 Curve with $r = 2$

When $r > 2$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the $y = x$ line as demonstrated by the time-plus-two curve for $r = 2.4$.

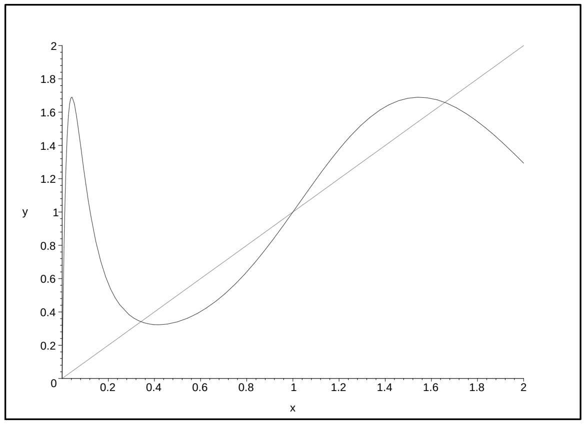

FIGURE 22. Model I Time-Plus-2 Curve with $r = 2.4$

The stability of Model I can also be demonstrated by looking at the bifurcation map for Model I. It can be seen that the model remains at the equilibrium until it bifurcates at $r = 2$, the period 4 oscillation is also visible near $r = 2.6$.

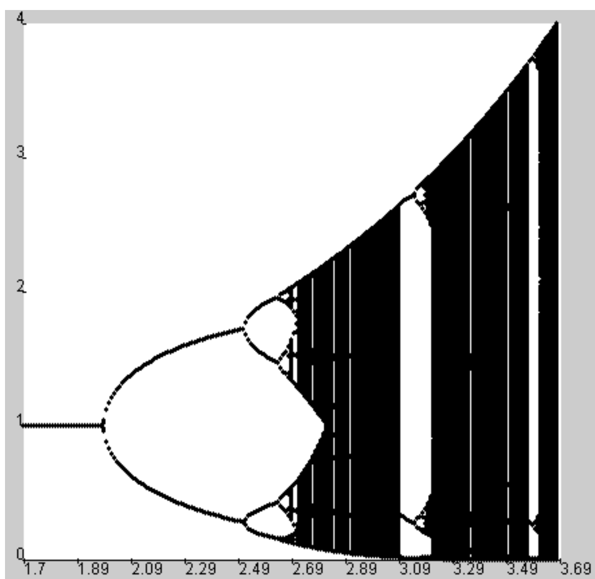

FIGURE 23. Bifurcation Map Model I

One final exploration of the behavior of Model I can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. The amount of white surrounding the real axis in the plots is an indication of how stable the model is at that reproductive rate. At $r = 1.7$ the real axis is entirely

FIGURE 24. Model I Complex Convergence Plot for $r = 1.7$

surrounded in white, thus providing further evidence for the stability of the model at this rate.
At the bifurcation value of $r = 2$, the real axis is still surrounded in white, however there is evidence that the convergent area is beginning to ”collapse” around the real axis at the equilibrium.

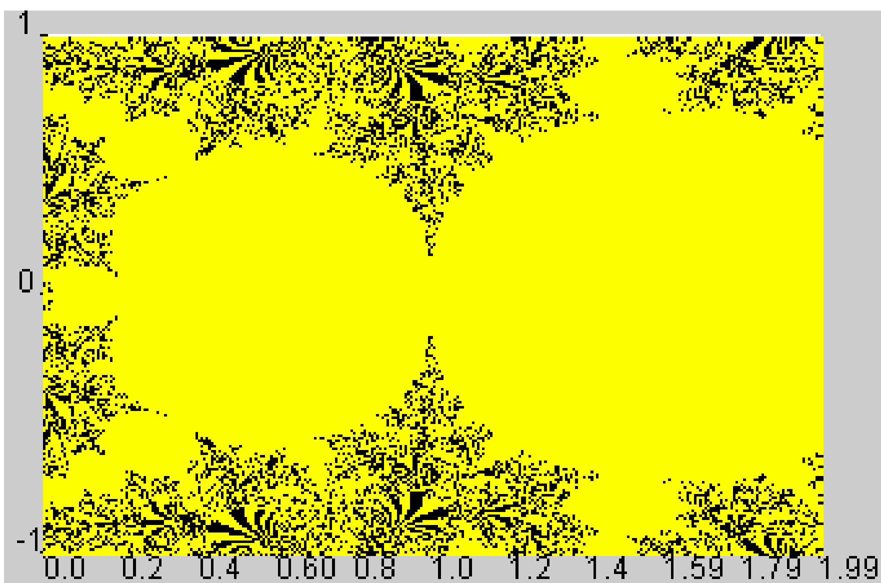

FIGURE 25. Model I Complex Convergence Plot for $r = 2$

Shortly after the bifurcation value of $r = 2$, in this case $r = 2.3$, the convergent area has in fact collapsed around the real axis at the equilibrium point.
Long after the first bifurcation value of $r = 2$, for example the possible second bifurcation value

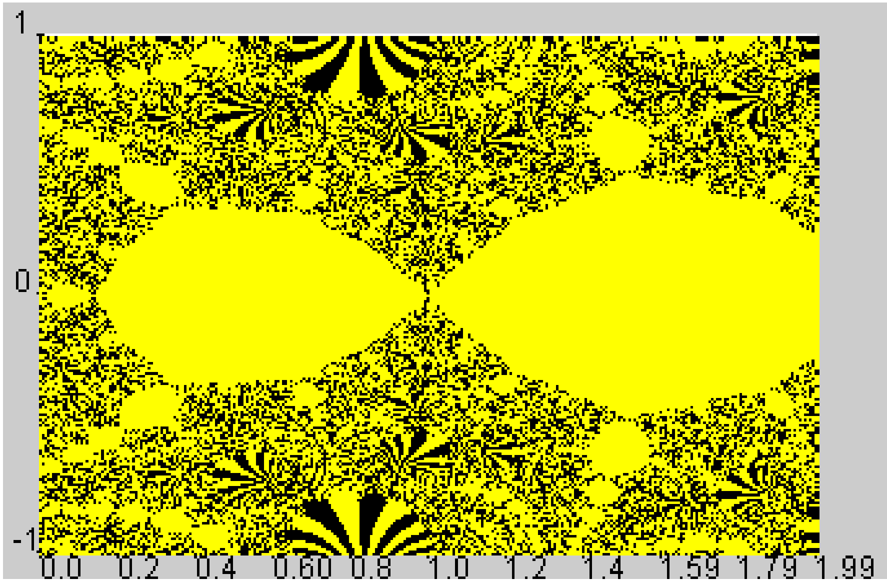

FIGURE 26. Model I Complex Convergence Plot for $r = 2.3$

$r = 2.6$, the convergent area has collapsed in many areas around the real axis, indicating the instability of the equilibrium.

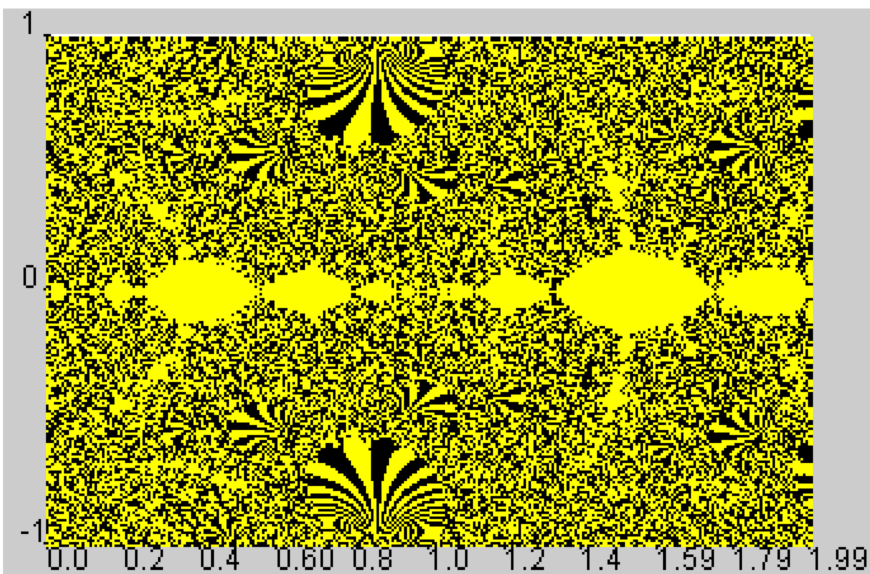

FIGURE 27. Model I Complex Convergence Plot for $r = 2.6$

3.2. **Model II.** $x_{t+1} = x_t[1 + r(1 - x_t)]$

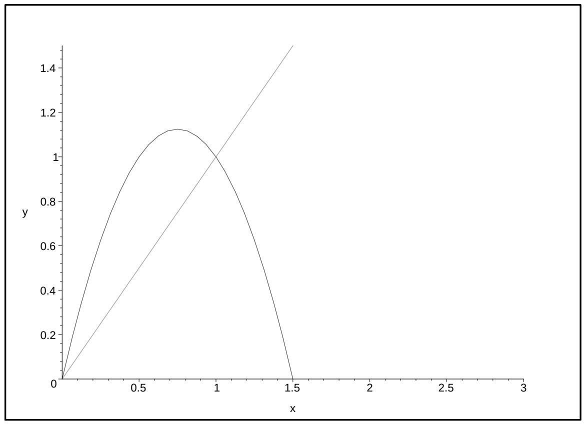

FIGURE 28. Model 2 basic curve

Model II from [16] is also commonly used and is considered to be a variation on Model I [1, 2, 3]. From [1, 2, 3] we know that this model, like Model I is also globally stable when $0 < r \leq 2$. We observe this behavior of the model by examining the following plots.

When examining the time plot of model 2 with $r = 1.8$ we observe that the model does behave as predicted and the model approaches the equilibrium. However for $r = 2$, it again appears as

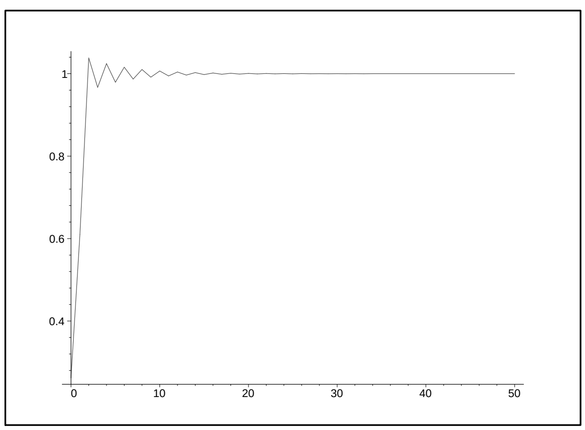

FIGURE 29. Model II Time Plot with $r = 1.8$

if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at this rate as Cull suggests. Again, this is due to computer approximation and the actual time plot for $r = 2$ should look similar to that of $r = 1.8$.

For $r > 2$ we expect the model to not converge to the equilibrium, and actually for values of $2 < r \leq 2.4$ the population oscillates between 2 population values. It then bifurcates into a period 4 oscillation as demonstrated by the time plot with $r = 2.5$.

For $r \geq 2.5$ the model bifurcates again and again from a period 4 oscillation to period 8 and so forth until it descends into chaos. This is demonstrated by time plot $r = 2.7$ which demonstrates the chaotic behavior of the population exhibited at this reproduction rate.

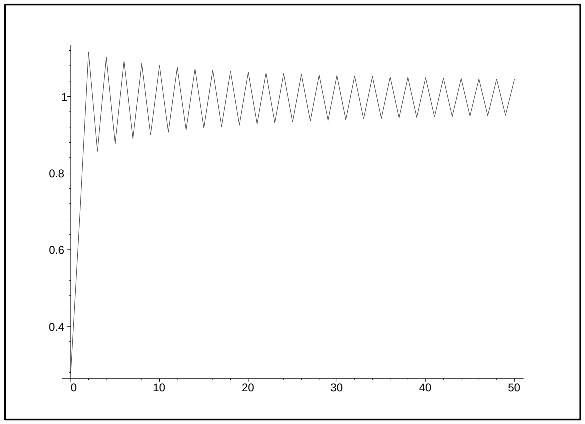

FIGURE 30. Model II Time Plot with $r = 2$

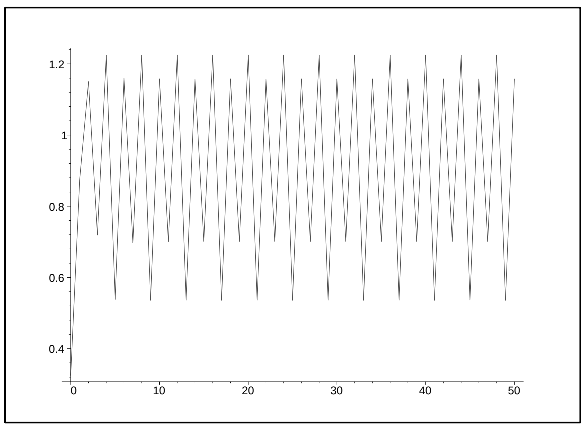

FIGURE 31. Model II Time Plot with $r = 2.5$ with a period 4 oscillation

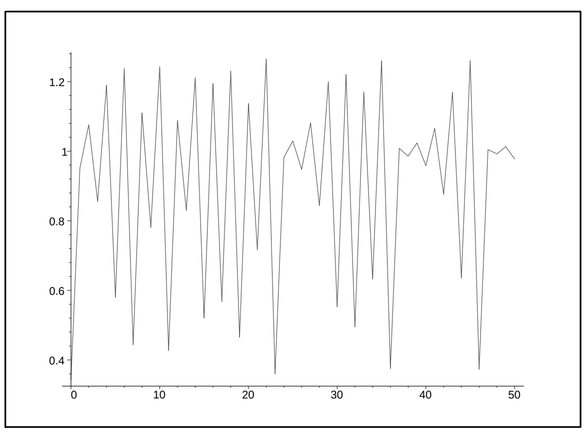

FIGURE 32. Model II Time Plot with $r = 2.7$ displaying chaotic behavior

We can also verify that the population for Model II is globally stable at $r \leq 2$ by examining the time-plus-two curves. For the globally stable values of $r \leq 2$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $r = 1.5$.

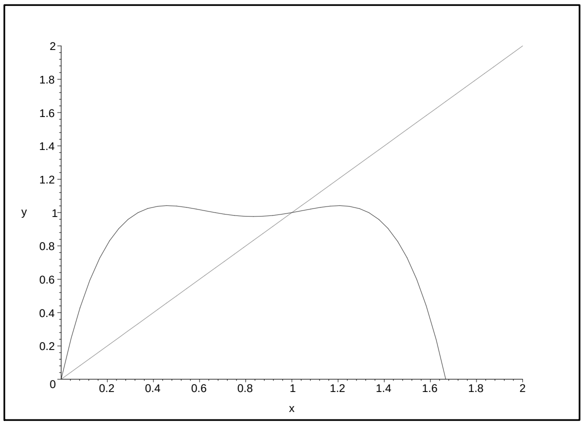

FIGURE 33. Model II Time-Plus-2 Curve with $r = 1.5$

When $r = 2$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $r = 2$.

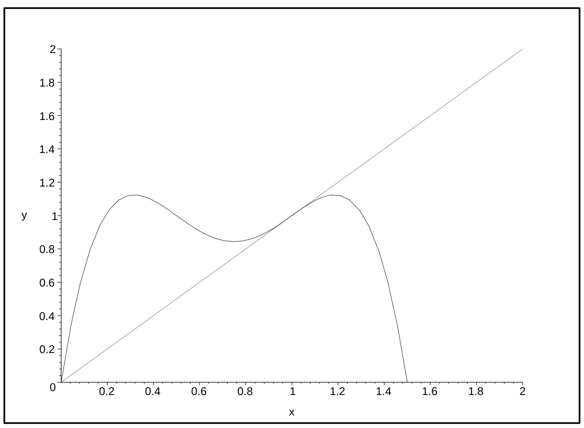

FIGURE 34. Model II Time-Plus-2 Curve with $r = 2$

When $r > 2$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the $y = x$ line as demonstrated by the time-plus-two curve for $r = 2.4$.

The stability of Model II can also be demonstrated by looking at the bifurcation map for Model II. It can be seen that the model remains at the equilibrium until it bifurcates at $r = 2$, the period 4 oscillation is also visible near $r = 2.5$.

One final exploration of the behavior of Model II can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. At $r = 1.9$ the real axis is entirely surrounded in white, thus providing further evidence for the stability of the model at this rate.

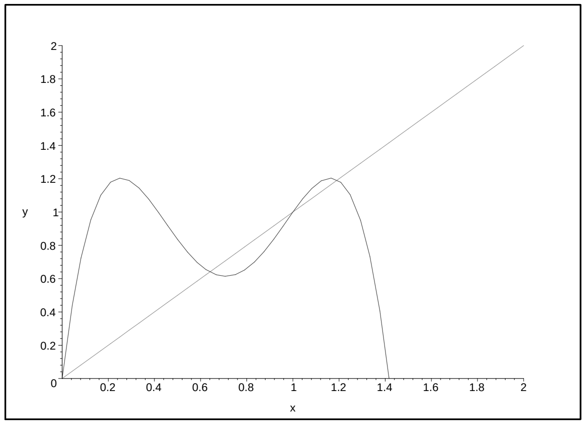

FIGURE 35. Model II Time-Plus-2 Curve with $r = 2.4$

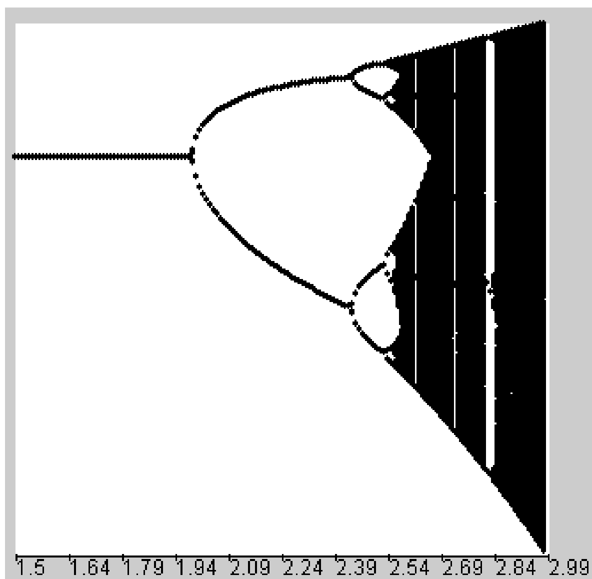

FIGURE 36. Bifurcation Map Model II

Shortly after the bifurcation value of $r = 2$, in this case $r = 2.1$, the convergent area has in fact collapsed around the real axis.

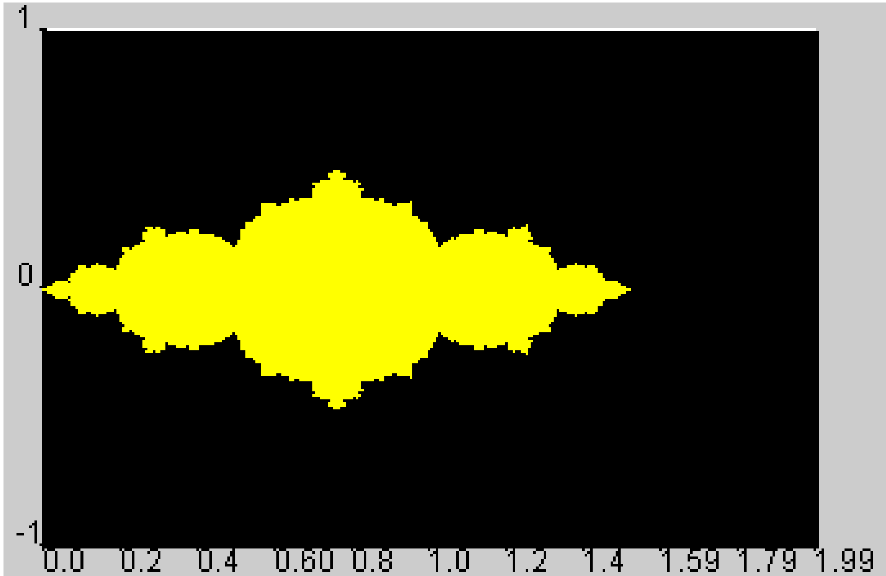

FIGURE 37. Model II Complex Convergence Plot for $r = 1.9$

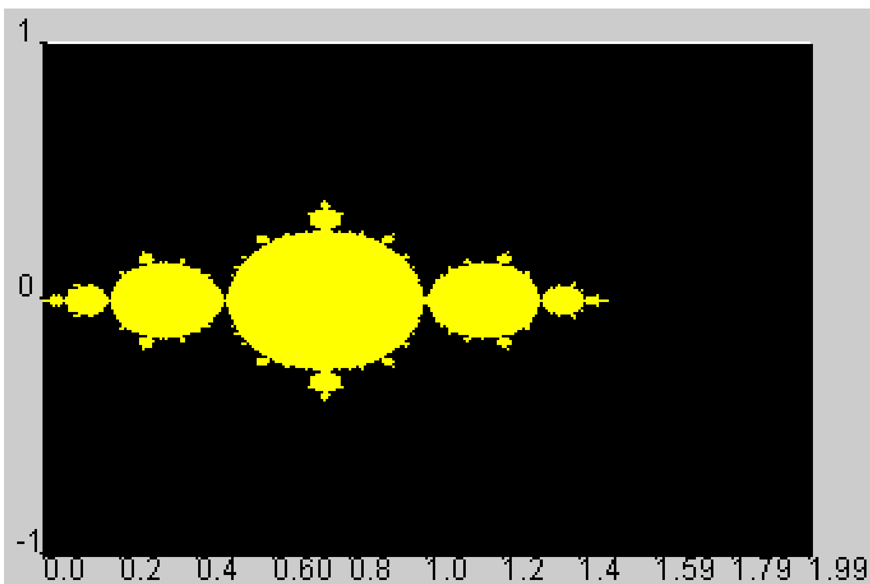

FIGURE 38. Model II Complex Convergence Plot for $r = 2.1$

At $r = 2.4$, the collapse of the convergent area along the real axis is even more pronounced than at $r = 2.1$, indicating the instability of the equilibrium.

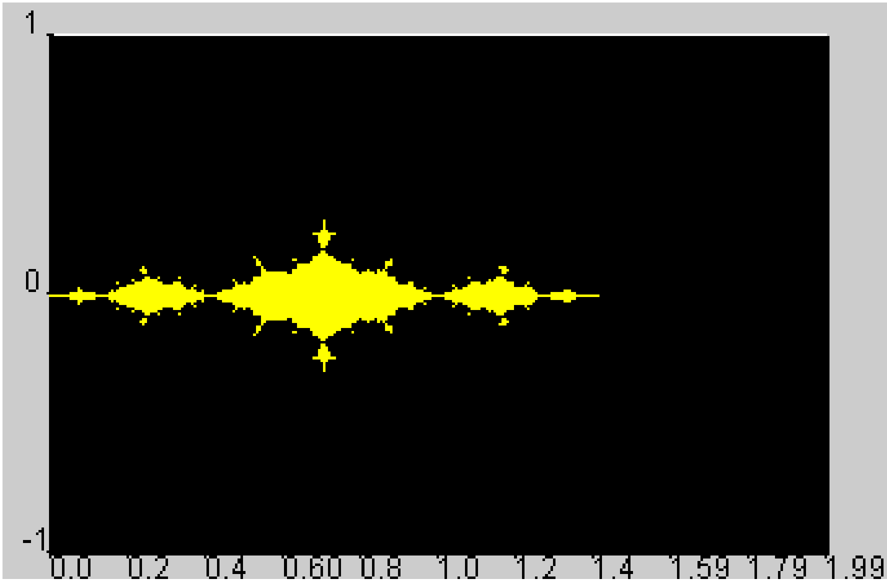

FIGURE 39. Model II Complex Convergence Plot for $r = 2.4$

3.3. **Model III.** $x_{t+1} = x_t[1 - rlnx_t]$

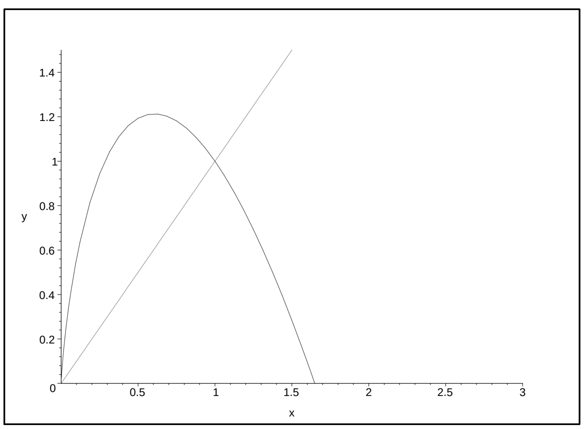

FIGURE 40. Model 3 basic curve

Model III is from [11]. From [1, 2, 3] we know that this model, like Model I and Model II is also globally stable when $0 < r \leq 2$. We observe this behavior of the model by examining the following plots.

When examining the time plot of Model III with $r = 1.8$ we observe that the model does behave as predicted and the population approaches the equilibrium. However for $r = 2$, it again appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium

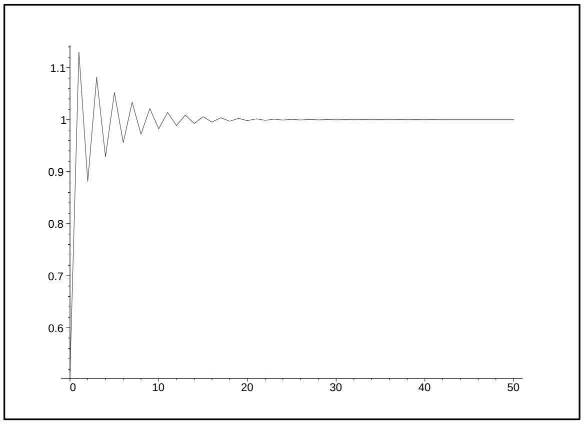

FIGURE 41. Model III Time Plot with $r = 1.8$

if the model were globally stable at this rate as Cull suggests. Again, this is due to computer approximation and the actual time plot for $r = 2$ should look similar to that of $r = 1.8$.

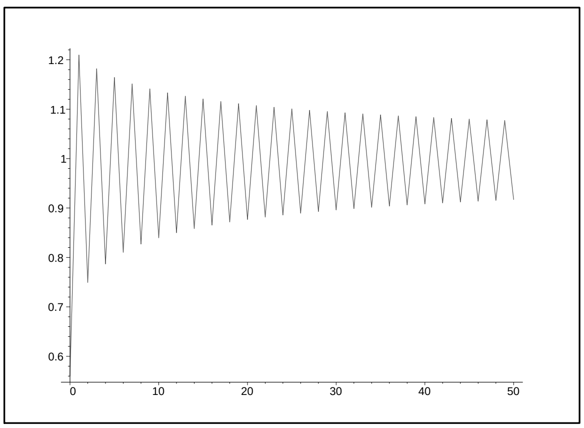

FIGURE 42. Model III Time Plot with $r = 2$

For $r > 2$ we expect the population to not converge to the equilibrium, and actually for values of $2 < r \leq 2.3$ the population oscillates between 2 population values as demonstrated by the time plot with $r = 2.2$.
At $r = 2.8$ there is an example where the population actually dies out after only eight iterations of the function as demonstrated by the time plot with $r = 2.8$

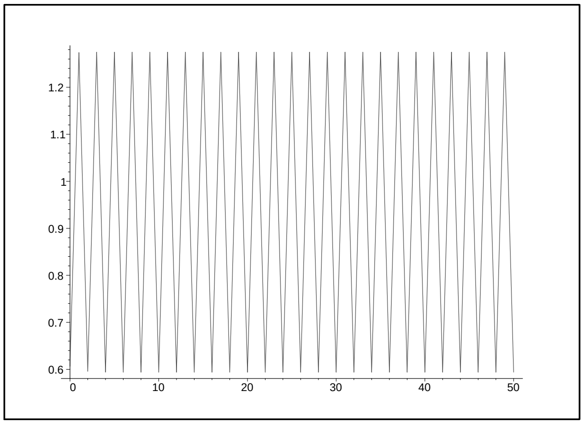

FIGURE 43. Model III Time Plot with $r = 2.2$

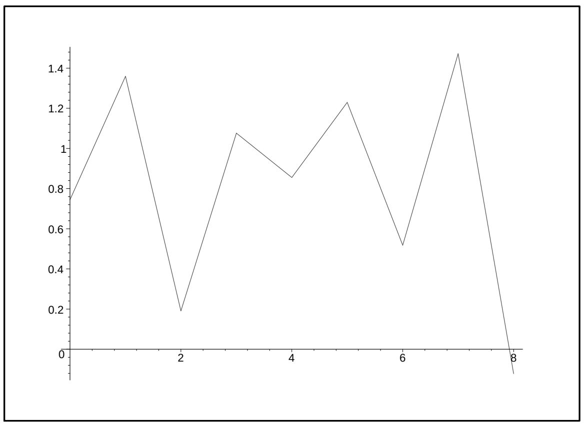

FIGURE 44. Model III Time Plot with $r = 2.8$

We can also verify that the population for Model III is globally stable at $r = 2$ by examining the time-plus-two curves. For the globally stable values of $r \leq 2$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $r = 1.8$.

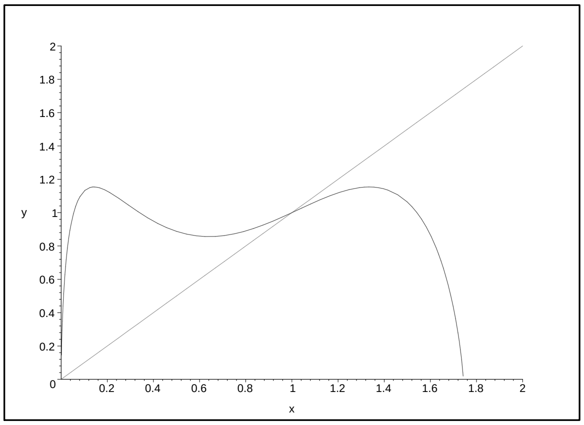

FIGURE 45. Model III Time-Plus-2 Curve with $r = 1.8$

When $r = 2$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $r = 2$.

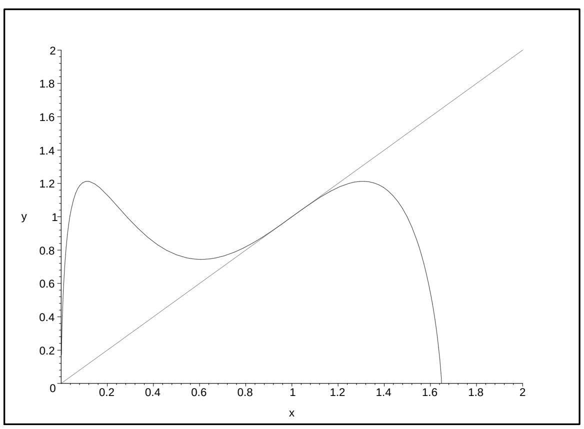

FIGURE 46. Model III Time-Plus-2 Curve with $r = 2$

When $r > 2$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the $y = x$ line as demonstrated by the time-plus-two curve for $r = 2.4$. Also, for $r = 2.8$ we can detect chaotic behavior (in this case the population dying out) by the time-plus-two curve.

The stability of Model III can also be demonstrated by looking at the bifurcation map for Model III. It can be seen that the model remains at the equilibrium until it bifurcates at $r = 2$.

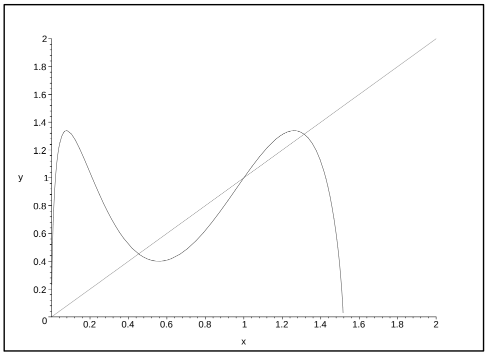

FIGURE 47. Model III Time-Plus-2 Curve with $r = 2.4$

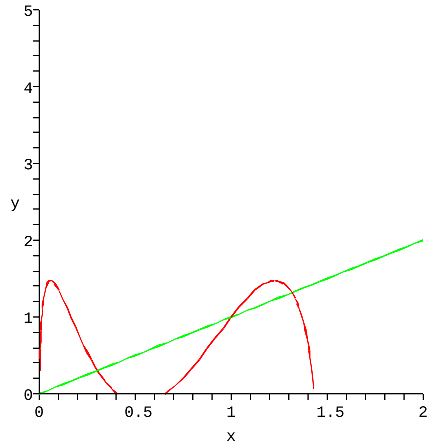

FIGURE 48. Model III Time-Plus-2 Curve with $r = 2.8$

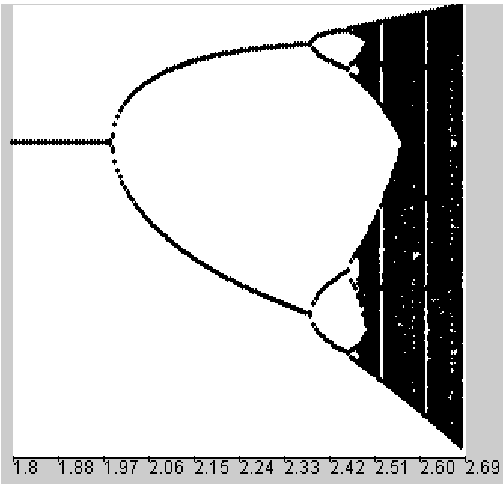

FIGURE 49. Bifurcation Map Model III

One final exploration of the behavior of Model III can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. At $r = 1.9$ the real axis is entirely surrounded in white, thus providing further evidence for the stability of the model at this rate.

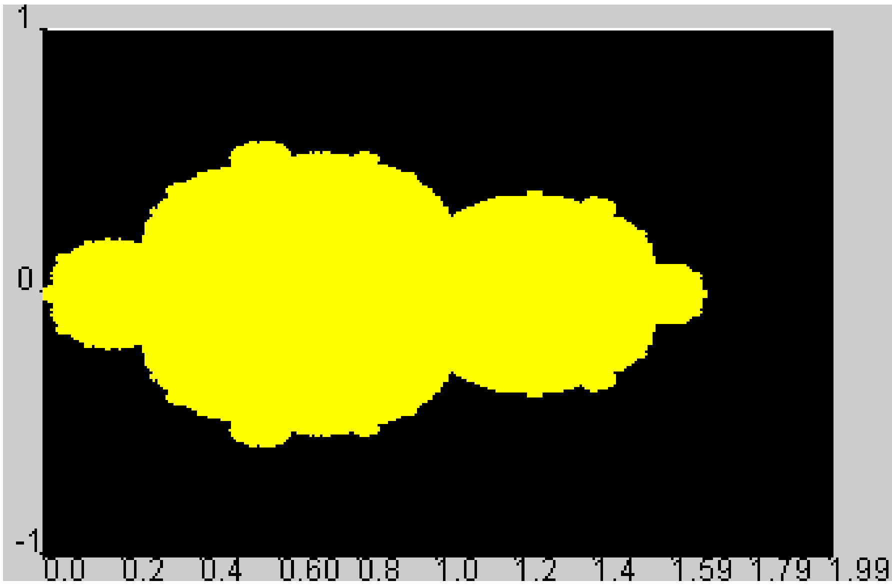

FIGURE 50. Model III Complex Convergence Plot for $r = 1.9$

Shortly after the bifurcation value of $r = 2$, in this case $r = 2.1$, the convergent area has in fact collapsed around the real axis.

At $r = 2.4$, the collapse of the convergent area along the real axis is even more pronounced than at $r = 2.1$, indicating the instability of the equilibrium.

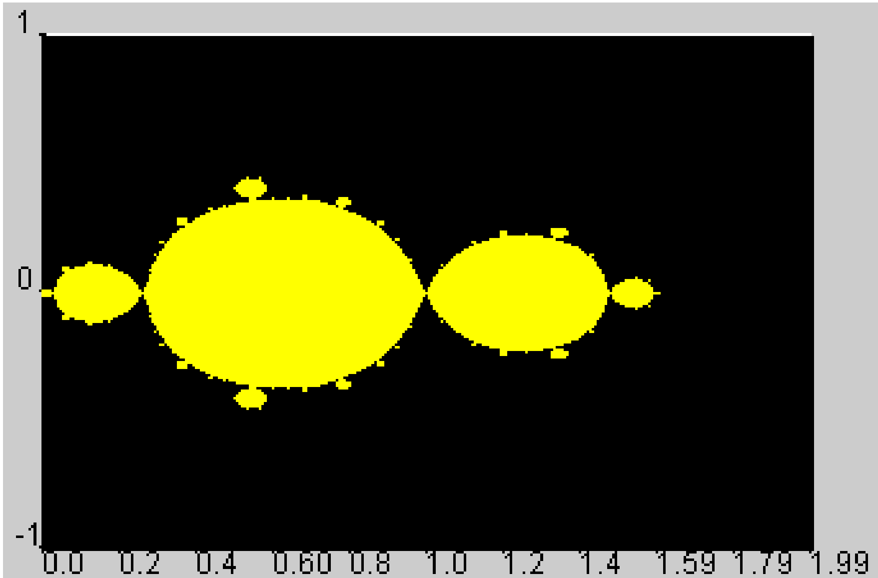

FIGURE 51. Model III Complex Convergence Plot for $r = 2.1$

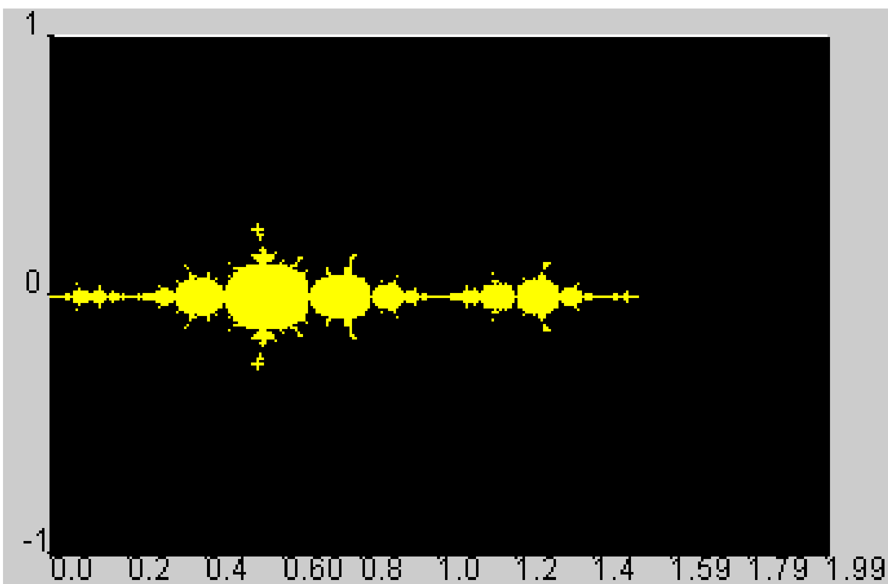

FIGURE 52. Model III Complex Convergence Plot for $r = 2.4$

3.4. **Model IV.** $x_{t+1} = x_t(\frac{1}{b+cx_t} - a)$ where $c = \frac{1}{a+1} - b$

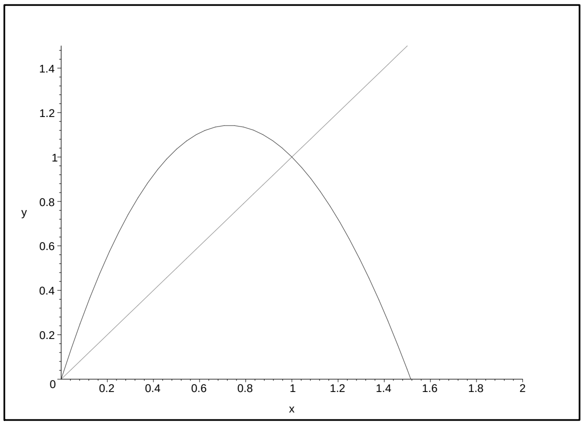

FIGURE 53. Model IV basic curve

Model IV from [18] differs from the first three models in that it has two parameters or reproduction rates. From [1, 2, 3] we know that the model is is globally stable when

$$\frac{a-1}{(a+1)^2} \leq b < \frac{1}{a+1}.$$

To avoid assymptotes for $x > 0$, we must have $a > 1$. We observe this behavior of the model by examining the following plots. In order to investigate the plots however, one parameter must be fixed, in our case a, and we vary the other.

When examining the time plot of model IV with $a = 20, b = .0435$ we observe that the model does behave as predicted and the population approaches the equilibrium. However for $a = 20, b =$

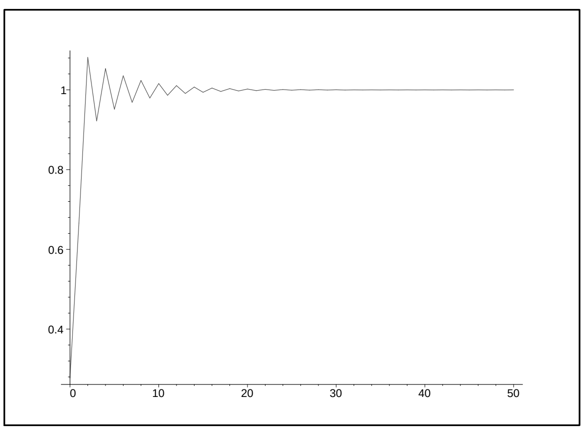

FIGURE 54. Model IV Time Plot with $a = 20, b = .0435$

.0430, it again appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at these rates as Cull suggests. Again, this is due to computer approximation and the actual time plot for $b = .0430$ should look similar to that of $b = .0435$

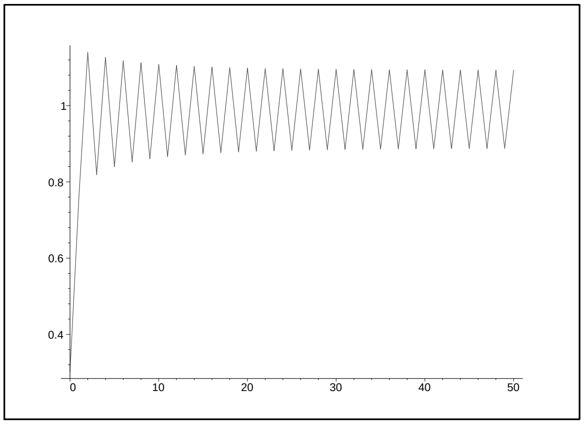

FIGURE 55. Model IV Time Plot with $a = 20, b = .0430$

For $b < .0430$ or $b > .0476$ we expect the population to not converge to the equilibrium, and actually for values of $.0421 < b < .0430$ the population oscillates between 2 population values as demonstrated by the time plot with $b = .0427$.

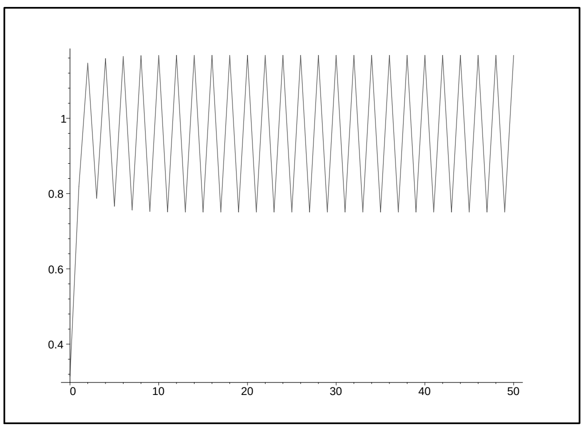

FIGURE 56. Model IV Time Plot with $a = 20, b = .0427$

For $b < .0430$ or $b > .0476$ the population bifurcates again and again from a period 4 oscillation to period 8 and so forth until it descends into chaos. This is demonstrated by the time plot for $b = .0417$ which demonstrates the chaotic behavior of the model and the time plot $b = .0477$ which demonstrates the population immediately crashing and approaching 0.

We can also verify that the population for Model IV is globally stable at $b = .0430$ by examining the time-plus-two curves. For the globally stable values of $.0430 < b \leq .0476$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $b = .0435$.
When $b = .0430$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $b = .0430$.
When $.0421 < b < .0430$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the $y = x$ line as demonstrated by the time-plus-two curve for $b = .0427$.

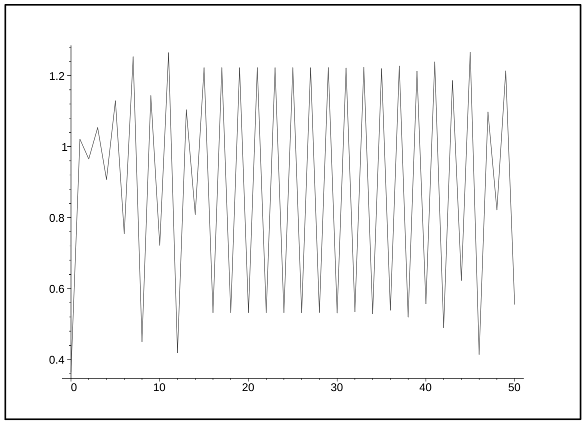

FIGURE 57. Model IV Time Plot with $a = 20, b = .0417$

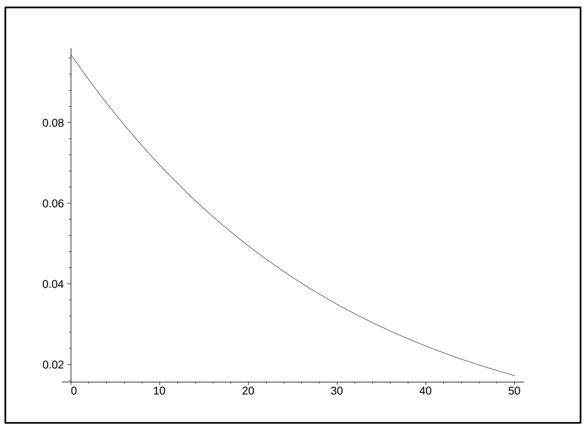

FIGURE 58. Model IV Time Plot with $a = 20, b = .0477$

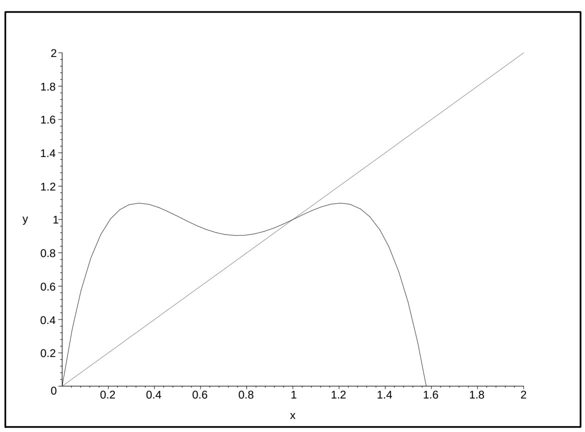

FIGURE 59. Model IV Time-Plus-2 Curve with $a = 20, b = .0435$

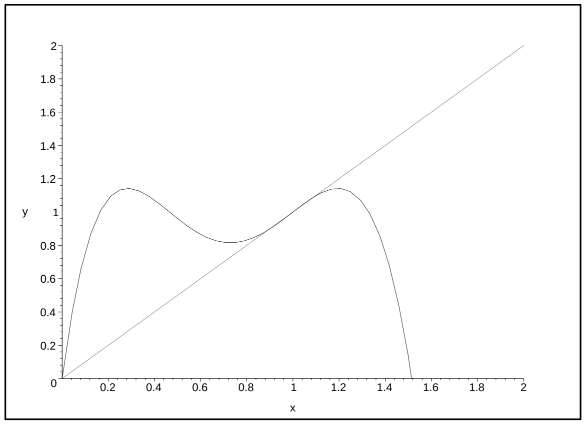

FIGURE 60. Model IV Time-Plus-2 Curve with $a = 20, b = .0430$

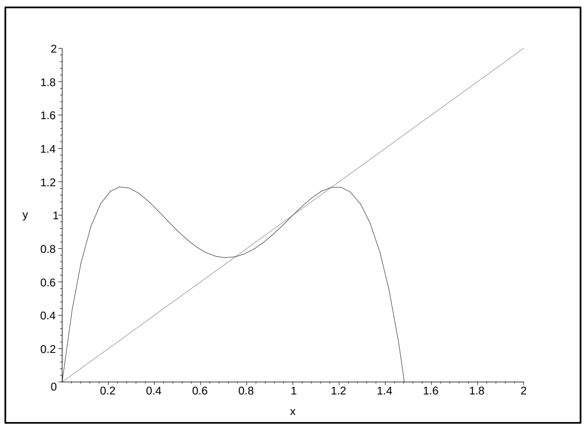

FIGURE 61. Model IV Time-Plus-2 Curve with $a = 20, b = .0427$

The stability of Model IV can also be demonstrated by looking at the bifurcation map for Model IV. This bifurcation map appears to be a mirror image of the others in that it can be seen that the model does not reach the equilibrium until $b = .0430$.

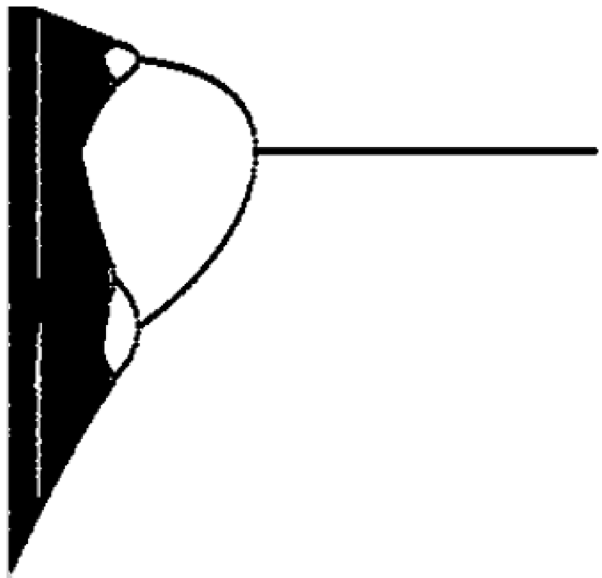

FIGURE 62. Bifurcation Map Model IV

One final exploration of the behavior of Model IV can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. At $b = .0435$ the real axis is entirely surrounded in white, thus providing further evidence for the stability of the model at this rate.

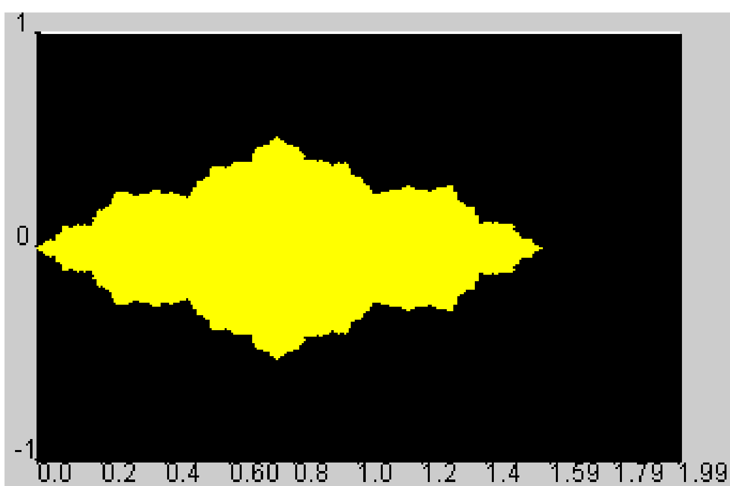

FIGURE 63. Model IV Complex Convergence Plot for $a = 20, b = .0435$

Shortly before the bifurcation value of $b = .0430$, in this case $b = .0427$, the convergent area has in fact collapsed around the real axis.

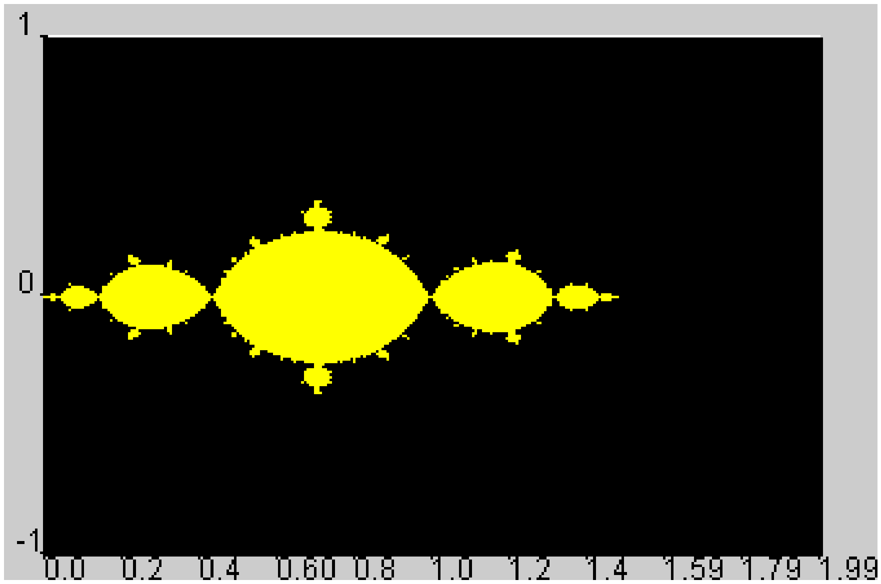

FIGURE 64. Model IV Complex Convergence Plot for $a = 20, b = .0427$

At $b = .0420$, the collapse of the convergent area along the real axis is even more pronounced than at $b = .0427$, indicating the instability of the equilibrium.

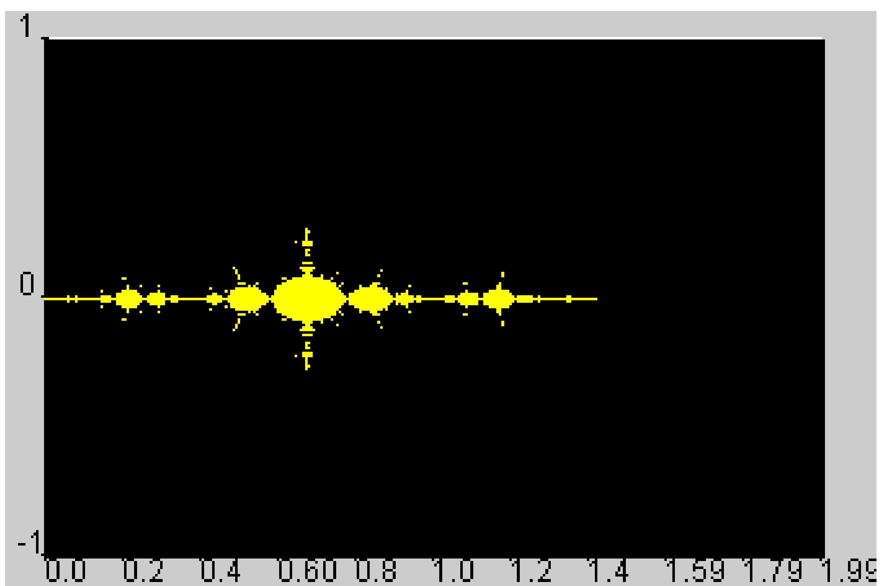

FIGURE 65. Model IV Complex Convergence Plot for $a = 20, b = .0420$

3.5. **Model V.** $x_{t+1} = \frac{(1+ae^b)x_t}{1+ae^{bx_t}}$

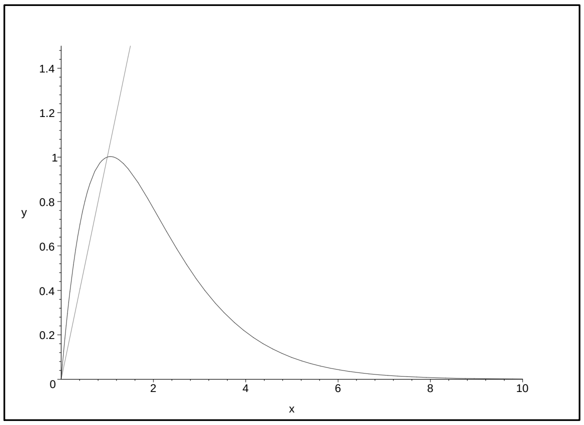

FIGURE 66. Model V basic curve

Model V is from [13] and also has two parameters or reproduction rates. From [1, 2, 3] we know that the model is is globally stable when

$$a(b-2)e^b \leq 2$$

It is also assumed for this model that $a > 0$ and $b > 0$. We observe this behavior of the model by examining the following plots. In order to investigate the plots however, one parameter must be fixed, in our case a, and we vary the other.

When examining the time plot of model V with $a = 5, b = 1.8$ we observe that the model does behave as predicted and the population approaches the equilibrium.

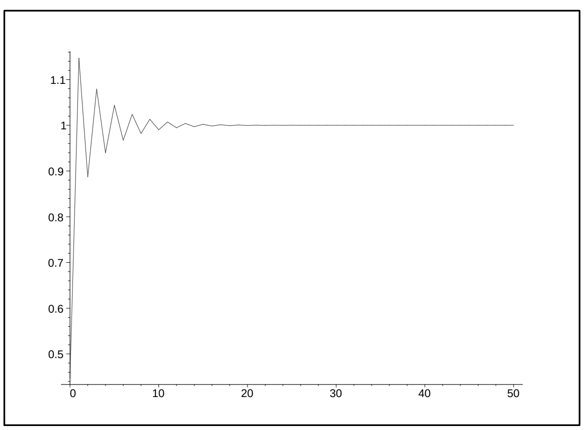

FIGURE 67. Model V Time Plot with a=5, b=1.8

However for $a = 5, b = 2$, it again appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at these rates as Cull suggests. Again, this is due to computer approximation and the actual time plot for $b = 2$ should look similar to that of $b = 1.8$

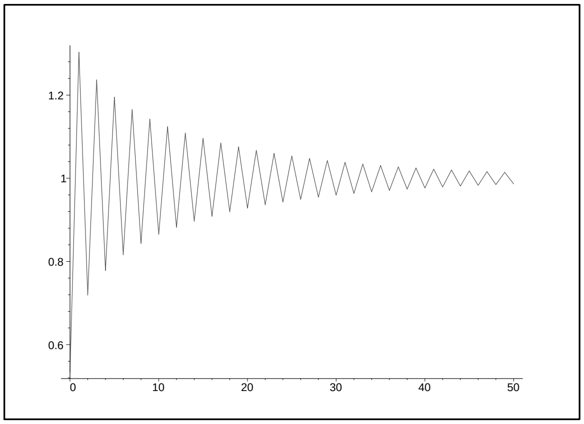

FIGURE 68. Model V Time Plot with $a = 5, b = 2$

For $b > 2$ we expect the population to not converge to the equilibrium, and actually for values of $2 < b < 2.6$ the population oscillates between 2 population values as demonstrated by the time plot with $b = 2.3$.

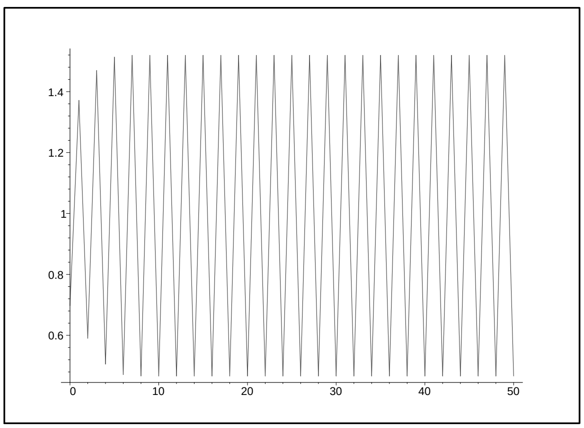

FIGURE 69. Model V Time Plot with $a = 5, b = 2.3$

We can also verify that the population for Model V is globally stable at $b \leq 2$ by examining the time-plus-two curves. For the globally stable values of $b \leq 2$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $b = 1$.
When $b = 2$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $b = 2$.
When $b > 2$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the y=x line as demonstrated by the time-plus-two curve for $b = 3$.

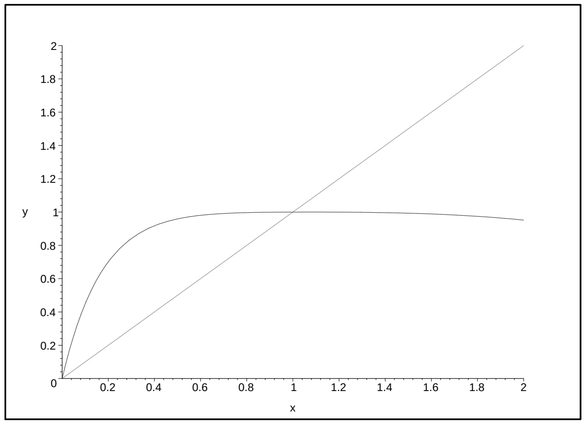

FIGURE 70. Model V Time-Plus-2 Curve with $a = 5, b = 1$

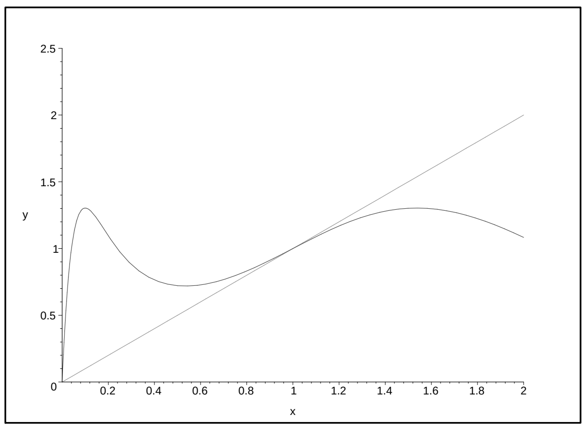

FIGURE 71. Model V Time-Plus-2 Curve with $a = 5, b = 2$

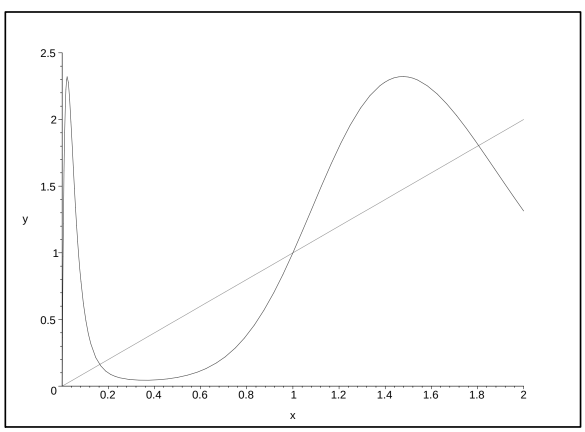

FIGURE 72. Model V Time-Plus-2 Curve with $a = 5, b = 3$

The stability of Model V can also be demonstrated by looking at the bifurcation map for Model V. It can be seen that the model remains at the equilibrium until it bifurcates at $b = 2$.

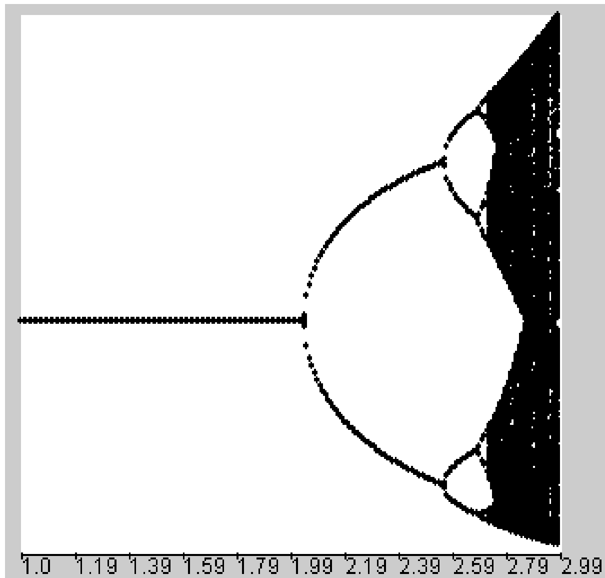

FIGURE 73. Bifurcation Map Model V

One final exploration of the behavior of Model V can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. At $b = 2$ the real axis is entirely surrounded in white, thus providing further evidence for the stability of the model at this rate.

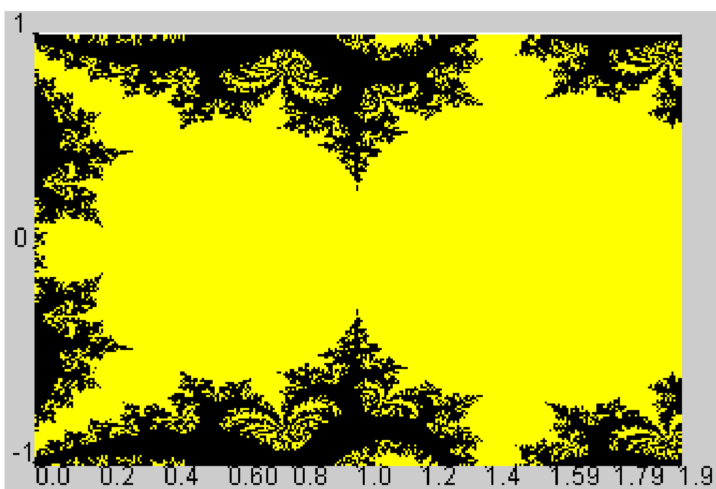

FIGURE 74. Model V Complex Convergence Plot for $a = 5, b = 2$

Shortly after the bifurcation value of $b = 2$, in this case $b = 2.3$, the convergent area has in fact collapsed around the real axis.

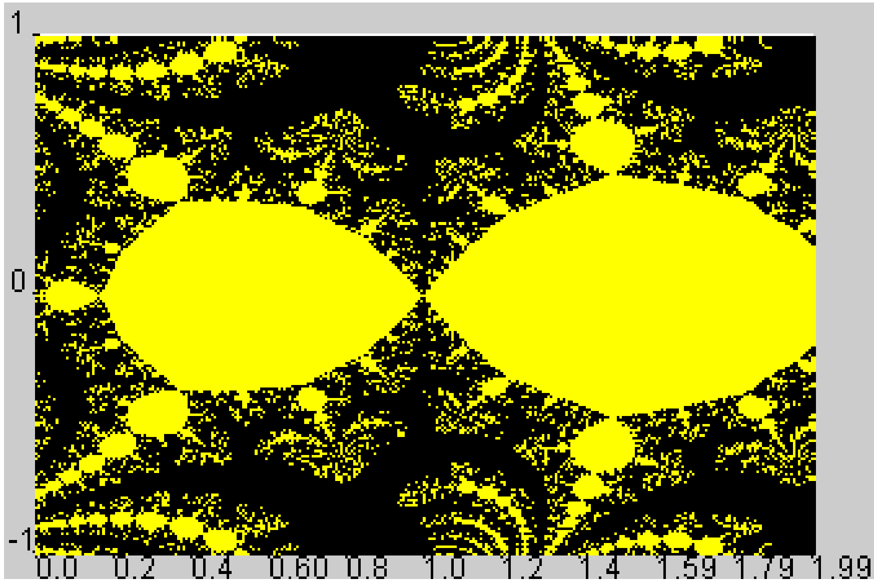

FIGURE 75. Model V Complex Convergence Plot for $a = 5, b = 2.3$

At $b = 2.5$, the collapse of the convergent area along the real axis is even more pronounced than at $b = 2.3$, indicating the instability of the equilibrium.

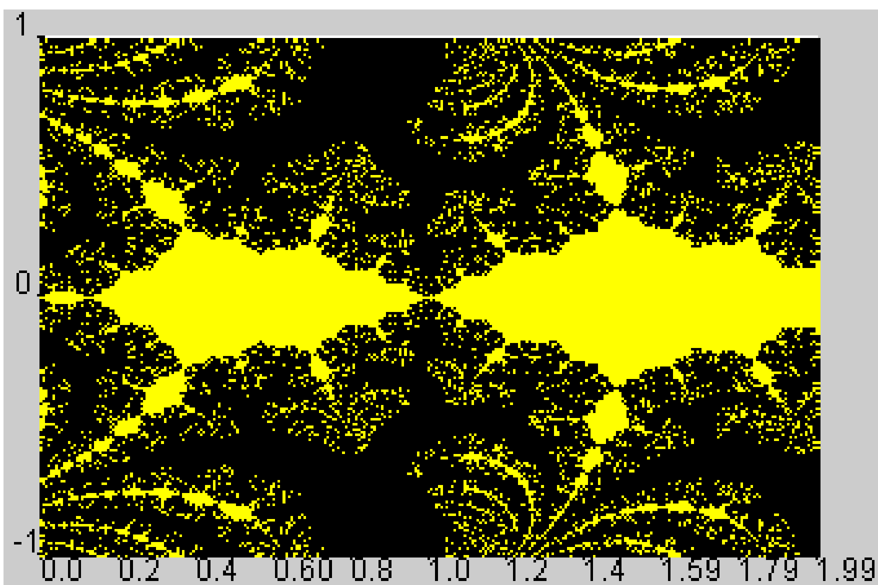

FIGURE 76. Model V Complex Convergence Plot for $a = 5, b = 2.5$

3.6. **Model VI.** $x_{t+1} = \frac{(1+a)^b x_t}{(1+ax_t)^b}$

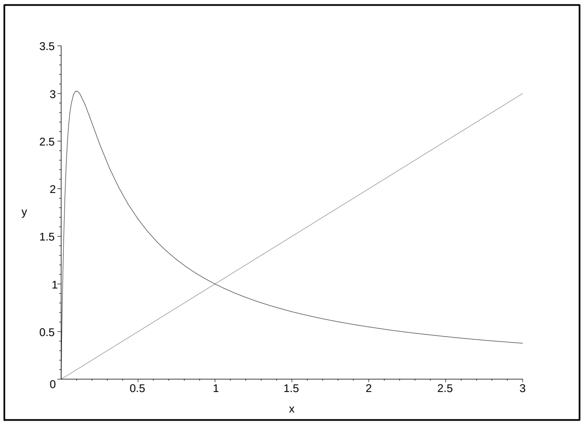

FIGURE 77. Model VI basic curve

Model VI is from [8] and also has two parameters or reproduction rates. It also has two cases to consider with respect to stability, when $0 < b \leq 2$ and $b > 2$. For our purposes we will look at the case when $b > 2$. From [1, 2, 3] we know that the model is is globally stable when

$$a(b-2) \leq 2$$

It is also assumed for this model that $a > 0$ and $b > 0$. In our case, when $a = 10$, this gives $b = 2.2$. We observe this behavior of the model by examining the following plots. To investigate the plots however, one parameter must be fixed, in our case a, and we vary the other.
When examining the time plot of model VI with $a = 10, b = 2$ we observe that the model does behave as predicted and the population approaches the equilibrium. However for $a = 10, b = 2.2$, it

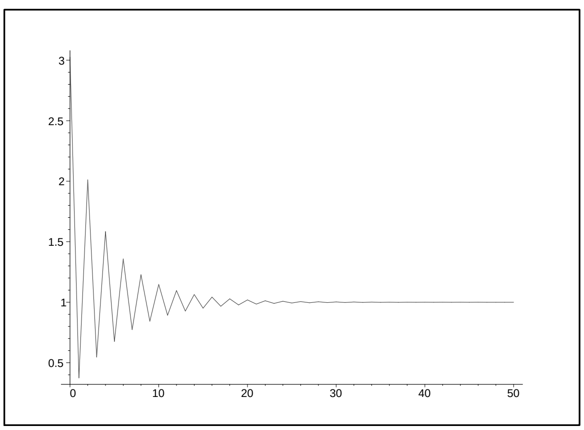

FIGURE 78. Model VI Time Plot with $a = 10, b = 2$

again appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at these rates as Cull suggests. Again, this is due to computer approximation and the actual time plot for $b = 2.2$ should look similar to that of $b = 2$.

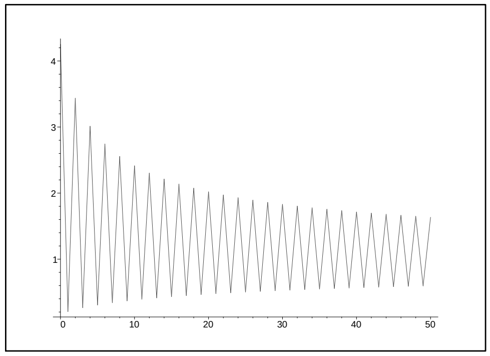

FIGURE 79. Model VI Time Plot with $a = 10, b = 2.2$

For $b > 2.2$ we expect the population to not converge to the equilibrium, and actually for values of $2.2 < b < 2.8$ the population oscillates between 2 population values as demonstrated by the time plot with $b = 2.4$.

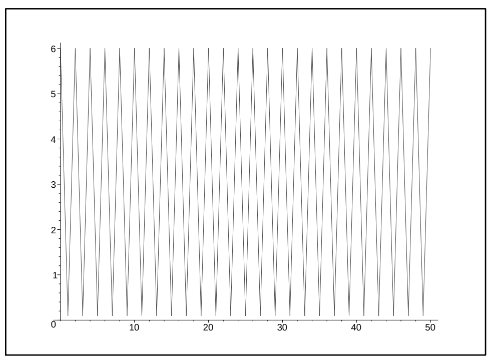

FIGURE 80. Model VI Time Plot with $a = 10, b = 2.4$

At $b = 3$ we find a nice example of the population demonstrating a period 4 oscillation with the time plot with $b = 3$.

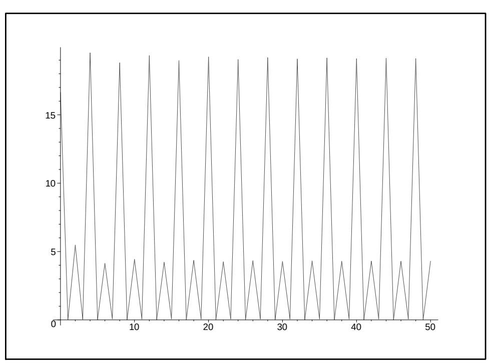

FIGURE 81. Model VI Time Plot with $a = 10, b = 3$

We can also verify that the population for Model VI is globally stable at $b = 2.2$ by examining the time-plus-two curves. For the globally stable values of $b \leq 2.2$ the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $b = 1.8$.

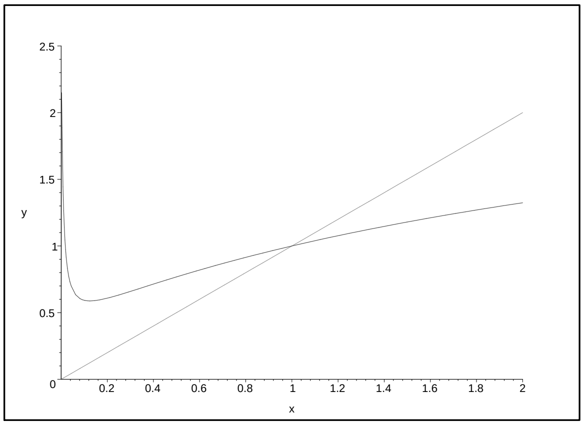

FIGURE 82. Model VI Time-Plus-2 Curve with $a = 10, b = 1.8$

When $b = 2.2$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $b = 2.2$.

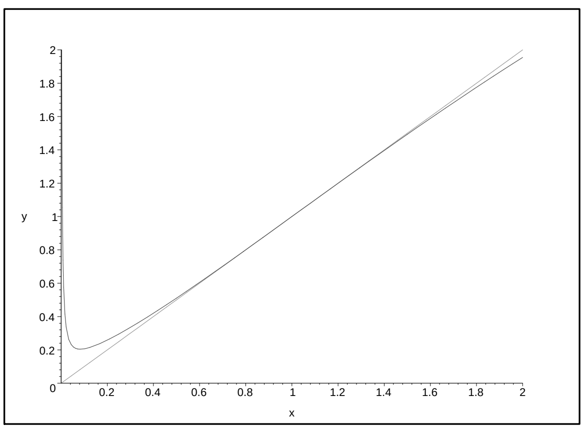

FIGURE 83. Model VI Time-Plus-2 Curve with $a = 10, b = 2.2$

In this particular case when $b > 2.2$, and the model is no longer stable, the cycle of period two cannot be seen by the three intersections of the curve with the $y = x$ line, there are only two intersections as demonstrated by the time-plus-two curve for $b = 2.3$.

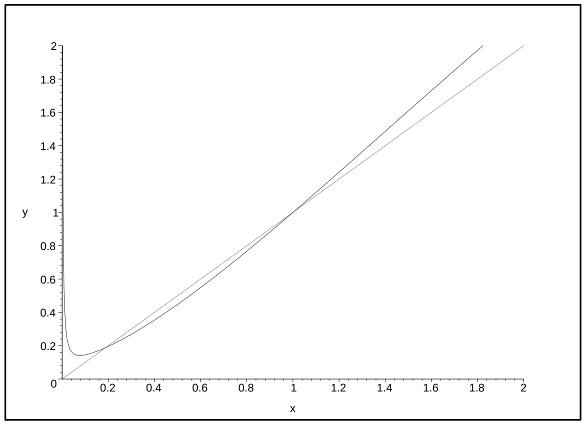

FIGURE 84. Model VI Time-Plus-2 Curve with $a = 10, b = 2.3$

The stability of Model VI can also be demonstrated by looking at the bifurcation map for Model VI. It can be seen that the model remains at the equilibrium until it bifurcates at $b = 2.2$. However it bifurcates in such a way that helps explain the behavior of the previous time-plus-two curve because one can note that one of the bifurcation values is at or very near 0.

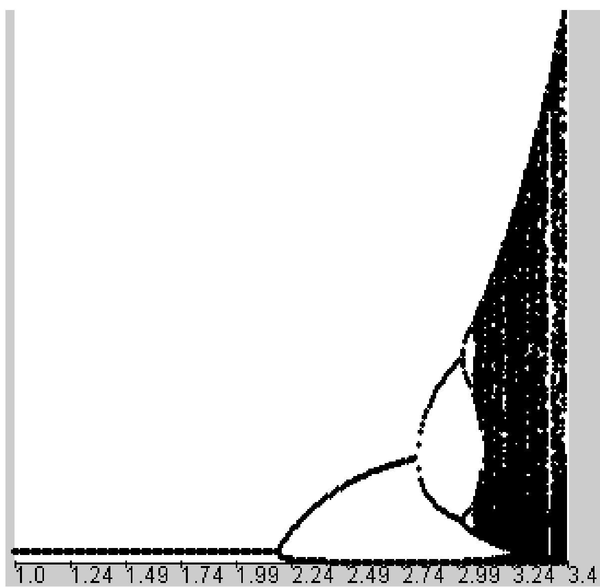

FIGURE 85. Bifurcation Map Model VI

One final exploration of the behavior of Model VI can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. At $b = 2$ the entire plot surrounded in white, thus providing

further evidence for the stability of the model at this rate.(As such, a plot of this type will not be shown).
Shortly after the bifurcation value of $b = 2.2$, in this case $b = 2.25$, the convergent area has collapsed around the real axis in a rather unusual way.

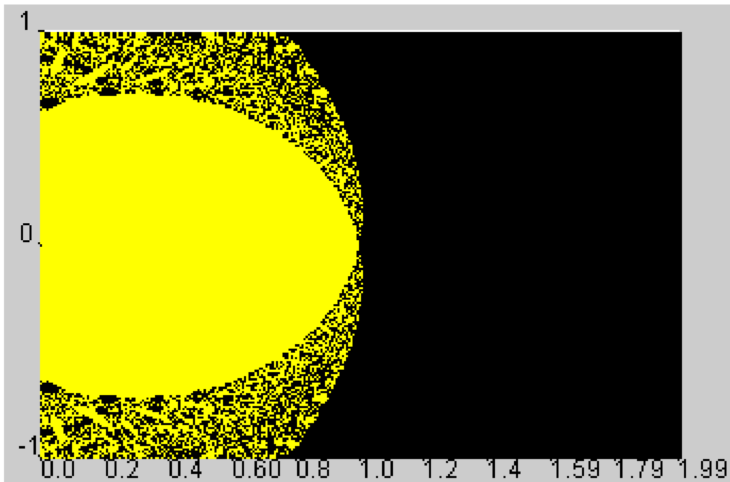

FIGURE 86. Model VI Complex Convergence Plot for $a = 10, b = 2.25$

At $b = 2.3$, the collapse of the convergent area along the real axis is even more pronounced than at $b = 2.25$, indicating the instability of the equilibrium.

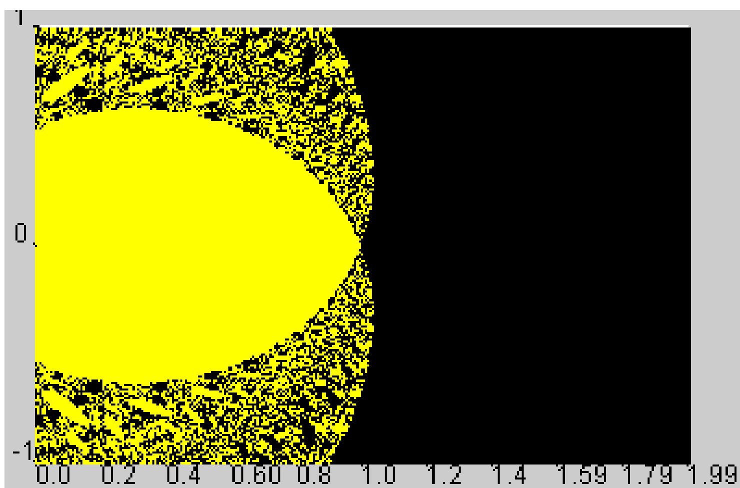

FIGURE 87. Model VI Complex Convergence Plot for $a = 10, b = 2.3$

3.7. Model VII. $x_{t+1} = \frac{(ax_t)}{1+(a-1)x_t^b}$

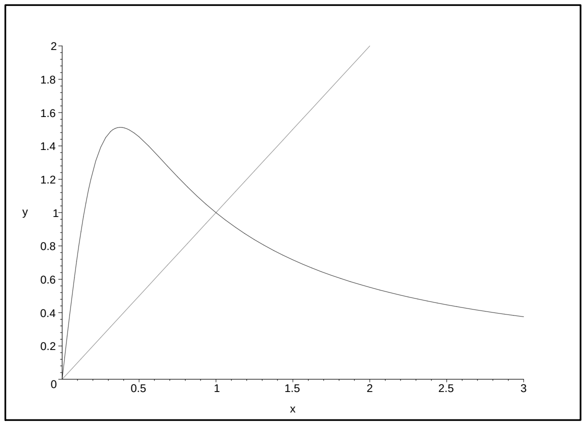

FIGURE 88. Model VII basic curve

Model VII is from [17] and also has two parameters or reproduction rates. It also has three cases to consider with respect to stability, when $0 < b \leq 2$, $b > 2$ and $b \geq 3$. For our purposes we will look at the case when $b > 2$. From [1, 2, 3] we know that the model is is globally stable when

$$a(b-2) \leq b$$

We observe this behavior of the model by examining the following plots. In order to investigate the plots however, one parameter must be fixed, in our case a, and we vary the other.
When examining the time plot of Model VII with $a = 8, b = 1.8$ we observe that the model does behave as predicted and the population approaches the equilibrium. However for $a = 10, b = 2.3$, it

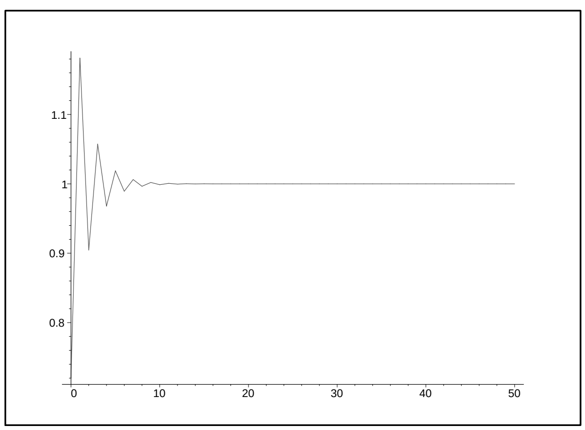

FIGURE 89. Model VII Time Plot with $a = 8, b = 1.8$

again appears as if the population is cycling between 2 values when in fact it should be approaching equilibrium if the model were globally stable at these rates as Cull suggests. Again, this is due to computer approximation and the actual time plot for $b = 2.3$ should look similar to that of $b = 1.8$.

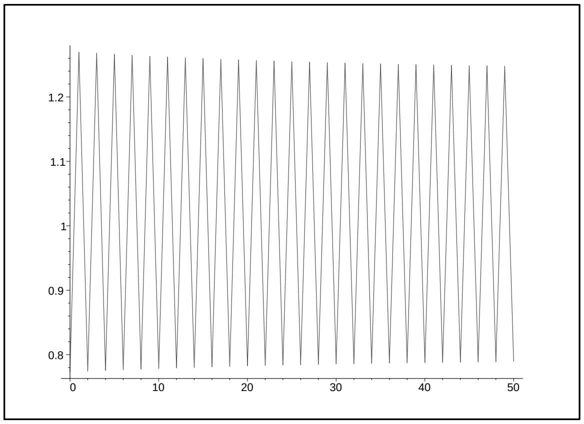

FIGURE 90. Model VII Time Plot with $a = 8, b = 2.3$

For $b > 2.3$ we expect the population to not converge to the equilibrium, and actually for values of $2.3 < b < 2.9$ the population oscillates between two population values as demonstrated by the time plot with $b = 2.7$.

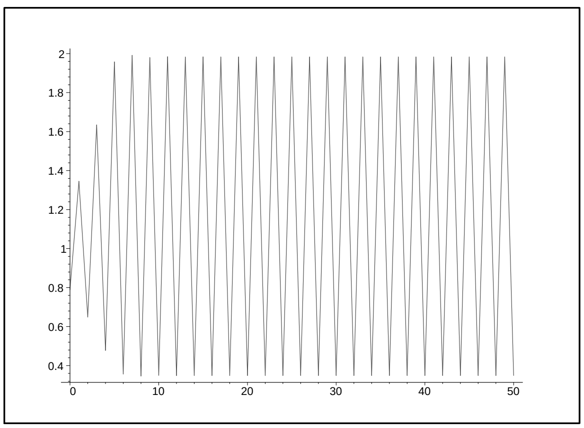

FIGURE 91. Model VII Time Plot with $a = 8, b = 2.7$

We can also verify that the population for Model VII is globally stable at $b = 2.3$ by examining the time-plus-two curves. For the globally stable values of $b \leq 2.3$, the curve intersects the $y = x$ line only at $x = 1$ as demonstrated by the time-plus-two curve for $b = 2$.
When $b = 2.3$, and the model is globally stable, the time-plus-2 curve lies tangent to the $y = x$ line at the equilibrium point as demonstrated by the time-plus-two curve for $b = 2.3$.
When $b > 2.3$, and the model is no longer stable, the cycle of period two can be seen by the three intersections of the curve with the $y = x$ line as demonstrated by the time-plus-two curve for $b = 2.7$.

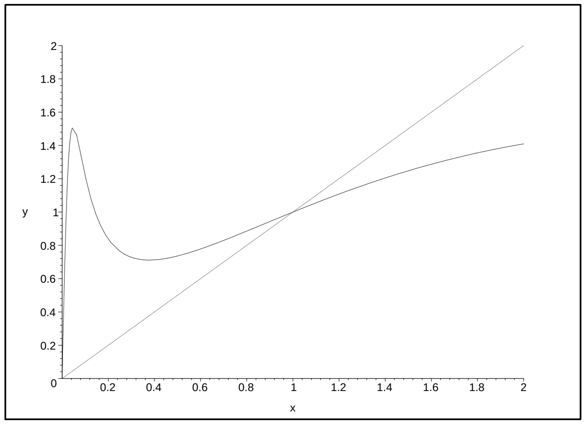

FIGURE 92. Model VII Time-Plus-2 Curve with $a = 8, b = 2$

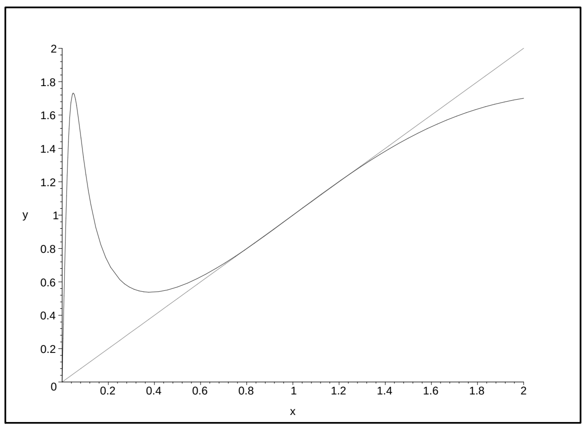

FIGURE 93. Model VII Time-Plus-2 Curve with $a = 8, b = 2.3$

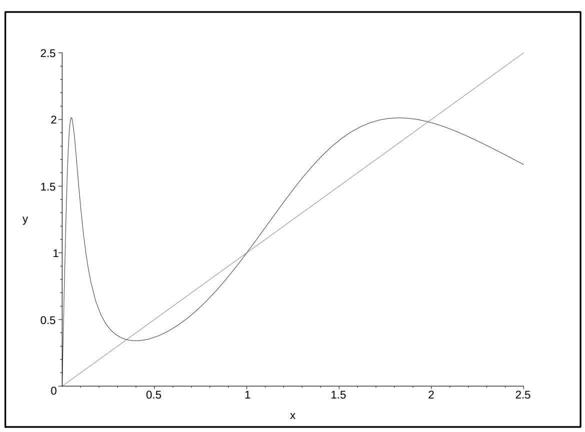

FIGURE 94. Model VII Time-Plus-2 Curve with $a = 8, b = 2.7$

The stability of Model VII can also be demonstrated by looking at the bifurcation map for Model VII. It can be seen that the model remains at the equilibrium until it bifurcates at $b = 2.3$.

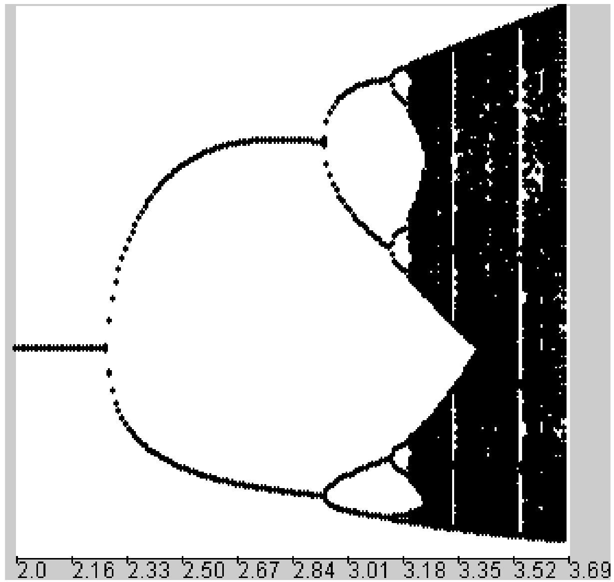

FIGURE 95. Bifurcation Map Model 7

One final exploration of the behavior of Model VII can be performed by looking at a series of complex convergence plots. These plots can suggest at what reproductive rates the bifurcations of the population may be taking place. However, very interesting plots were obtained by this time holding b fixed and varying a. At $a = 1.5$ the real axis is surrounded in white, thus providing evidence for the stability of the model at this rate.

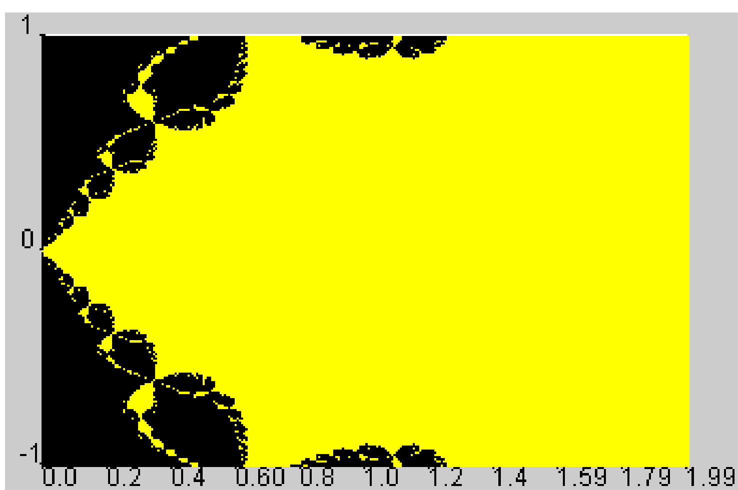

FIGURE 96. Model VII Complex Convergence Plot for $a = 1.5, b = 3$

Shortly after the bifurcation value, in this case $a = 5$, the convergent area has collapsed around the real axis.

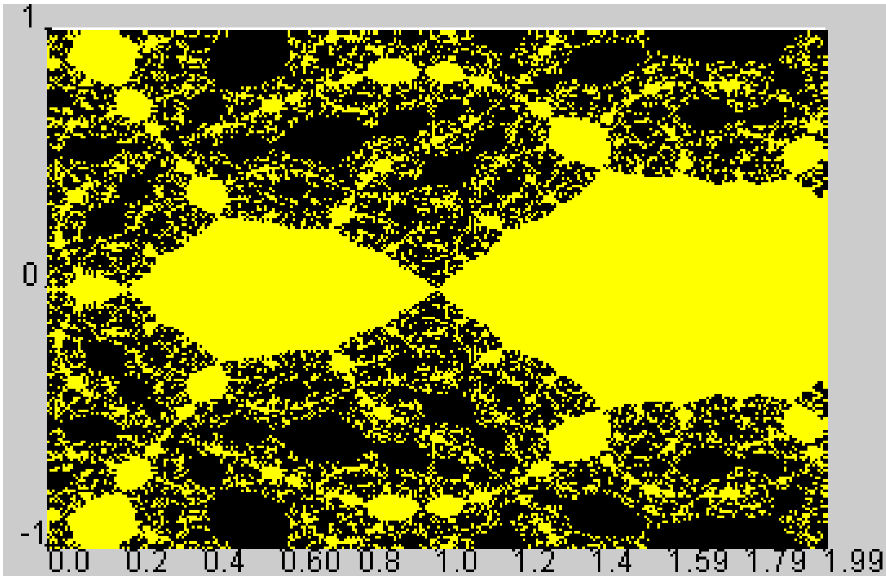

FIGURE 97. Model VII Complex Convergence Plot for $a = 5, b = 3$

At $a = 7$, the collapse of the convergent area along the real axis is even more pronounced than at $a = 5$, indicating the instability of the equilibrium.

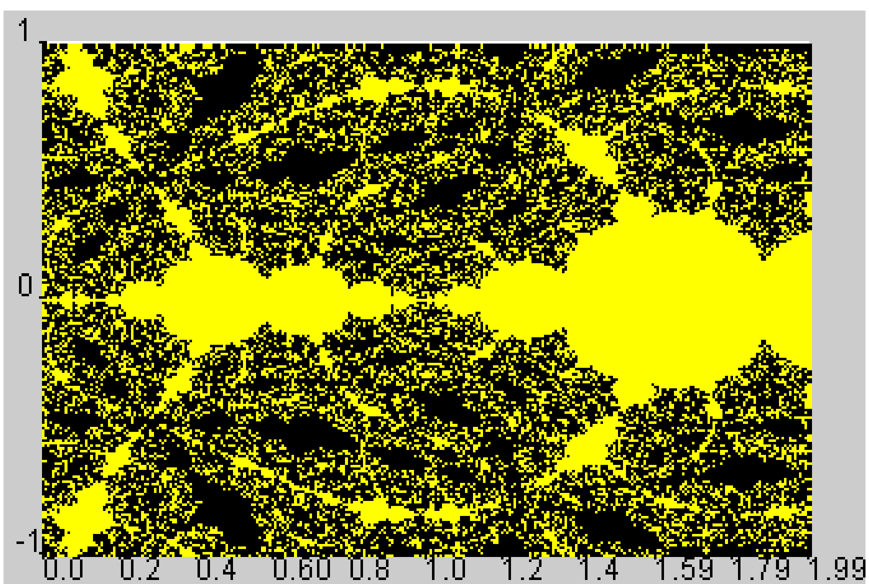

FIGURE 98. Model VII Complex Convergence Plot for $a = 7, b = 3$

4. What's Next

In order to further understand the complex behavior of these models, we intend to look at what happens when a higher order polynomial is added to the existing parameters. After preliminary investigations for two such models, it is our suspicion that these new models behave in a similar fashion.

5. Conclusion

Through graphical analyis, we have shown that the required stability conditions found by [1, 2, 3] are correct. We have also confirmed that for all but Model IV, reproduction rates slightly larger than those for which the model is globally stable will result in period-two doubling bifurcations and an eventual descent into chaos for even larger rates. Additionally, we have demonstrated that the models can appear very similar with respect to certain graphical representations such as basic curves, time-plus-2 curves and bifurcation diagrams, however their striking differences are obvious when one examines the time plots and complex convergence plots for these same models with the same rates. Therefore, we find it necessary to examine each type of plot to fully understand the stable characteristics of these models.

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