SPECIAL CLASSES OF WHITEHEAD AUTOMORPHISMS ON THE FREE GROUP

SCOTT WEAVER

ADVISOR: DENNIS GARITY
OREGON STATE UNIVERSITY

ABSTRACT. In this paper, we present several results regarding certain specific classes of Whitehead automorphisms. Also, we investigate their consequences for the proliforation of equivalence classes in various free groups.

1. INTRODUCTION

The purpose of this paper is to continue the work of J.H.C. Whitehead on the equivalence classes of the free group induced by equivalence under automorphisms, and build upon the results of past participants in the REU program, namely the work of Cooper and Rowland, Lau and Virnig. Whitehead proved in 1936 that if two words in a free group are equivalent under an automorphism, then they are equivalent under a finite sequence of a certain limited class of automorphisms [WH1, WH2].

He also showed that the lengths of the words obtained after applying each so-called *Whitehead automorphism* in this sequence are strictly decreasing until the the word achieves minimal length, whereupon the subsequent automorphisms fix the length of the word.

As a result, Whitehead's procedure is particularly amenable to implementation by computers as an algorithm to determine equivalence classes. As shown by (for example) Crisp [C], this allows for classification of certain free homotopy classes of loops on a punctured torus.

The results presented this paper will be the examination of certain special categories of automorphisms on the free group (and more specifically on the free group of two generators), and the implications for the classification of words in the free group up to equivalence.

1.1. **Definitions and Concepts.** What follow are some of the terms that will be used throughout this paper. The terms that are followed by an asterisk are defined either exactly as in [CR], or equivalently.

- $F_k = F(x_1, x_2, \ldots, x_k)$* represents the free group on the generators $x_1, x_2, \ldots, x_k$ (that is, the group consisting of these generators, their inverses and for which no relation exists between these generators except the trivial one between an element and its inverse).

  For this paper, we will, almost without exception, write the explicit generators of F_k as $a, b, \ldots$ and their inverses $\overline{a}, \overline{b}, \ldots$
- A *letter** is a single occurrence of a generator, or the inverse of a generator.
- A *word** is an element of F_k. The identity word is represented by 1.

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- A *string** is an element of F_k which may be part of a longer word.
- $w \sim v$ designates equivalence between two words $w, v \in F_k$ under some automorphism S on F_k. That is, $w = S(v)$.
- $|w|$* represents the length of the word w (after cancellation of adjacent inverses).
  We define the length of the identity word to be 0.
- The set $\{a, b, \ldots, x_k, \overline{a}, \overline{b}, \ldots, \overline{x_k}\}$ of generators and their inverses is referred to as L_k*.

Note 1. *For any automorphism S, it must always be the case that $S(\overline{x}) = \overline{S(x)}$ for any x in F_k. Thus, we will restrict ourselves to defining automorphisms by their effect on specific generators, and the reader may easily determine their effect on the inverses of those generators.*

Definition 2. *An automorphism S is* level* *on a word w if $|S(w)| = |w|$. That is, it fixes the length of a word after cancellation of adjacent inverses.*

Example 3. *Let S be the automorphism defined by*

$$S \begin{cases} a & \mapsto & b \\ b & \mapsto & a \end{cases}$$

, then $S(ababb) = babaa$ and S is level on the word $ababb$.

Definition 4. *A* cycle* *is a level inner automorphism on some word w. Equivalently, this is an automorphism that moves the terminal n letters of w to the front of the word.*

Example 5. $bac\underline{\overline{a}bb\overline{c}} \mapsto \underline{\overline{a}bb\overline{c}}bac$

A cyclic word* *w represents the set of the images of all cycles of w. (The initial letter is thus adjacent to the final letter.)*

The explicit form of any cycle, C on a word $w = w_1 w_2 w_3 \ldots w_{n-1} w_n$ which has the effect of fronting the final i letters (alternatively, taking the first $j = n - i$ letters to the back) is that for any x in L_k, either
$C(x) = w_j w_{j+1} \ldots w_{n-1} w_n x \overline{w_n w_{n-1} \ldots w_{j+1} w_j}$
or
$C(x) = \overline{w_j w_{j-1} \ldots w_2 w_1} x w_1 w_2 \ldots w_{j-1} w_j$.

Definition 6. *A cyclic word w in F_k is* minimal* *if no automorphism on the free group maps w to any word v with shorter length than w.*

We will now define the first of the two special classes of automorphisms defined by Whitehead for use in his theorem.

Definition 7. *A* Whitehead Type I automorphism* *S is a permutation acting on L_k so that $S(\overline{x}) = \overline{S(x)}$ for any x in L_k.*

Example 8. $abba\overline{a} \mapsto baab\overline{b}$

Keeping with the convention of past researchers, Whitehead type I automorphisms will commonly be called *permutations*.

Definition 9. *A* cyclic permutation* *is a cycle composed with a permutation. A cyclic permutation of a word w is the image of w under a cyclic permutation. $\widetilde{w}$ denotes a cyclic permutation of w.*
A CP word *w is the set of all cyclic permutations of w.*

The concept of cyclic permutation allows for a more restrictive equivalence relation to be defined:

Definition 10. *A minimal word w is* CP-equivalent* *to a minimal word v (represented by $w \sim_{CP} v$) if w is a cyclic permutation of v (or equivalently, w and v belong to the same CP word).*

Example 11. *In F_2, $abb\underline{a\bar{b}a} \sim_{CP} \underline{\bar{b}a\bar{b}}\bar{b}\bar{a}\bar{a}$.*

Since the set of cyclic permutations is a proper subset of the set of automorphisms on F_k, any words that are CP equivalent are also equivalent in the more general sense, but the converse is not generally true, as can be seen in the following example.

Example 12. *Let S be the automorphism defined by*

$$S \begin{cases} a & \mapsto & aaab \\ b & \mapsto & \bar{a} \end{cases}$$

Then $S(aabb) = aba\bar{b}$. Thus, $aabb \sim aba\bar{b}$, but there is no cyclic permutation that takes $aabb \mapsto aba\bar{b}$.

Definition 13. *A minimal word w is* RCP**, (that is,* <u>R</u>educed under <u>C</u>yclic <u>P</u>ermutation*) if, out of its whole CP-equivalence class, w itself appears first in the lexicographic ordering specified by*
$L_k = \{a, b, \ldots, x_k, \bar{a}, \bar{b}, \ldots, \overline{x_k}\}$.

This allows us to uniquely select a representative of any CP-class.

Example 14. *In F_3, $aabcb\bar{c} = [\overline{ca}\bar{b}\bar{a}b\bar{c}]$, and moreover $aabcb\bar{c}$ is RCP.*

Definition 15. *An x*-string *of a cyclic word w is a substring of w of the form x^n (with n maximal) so that x is in L_k.*

For any x-string y^n, in a cyclic word w, the ordered pair (y, n) is called the x-string label *of y^n. The first element of the x-string label of y^n is called the* generator element *of the x-string label. The second element of the x-string label is called the* length element *of the x-string label.*

Example 16. *Let $w = aaabb\bar{a}\bar{a}b$. Then the x-string labels of w are:*
$(a,3), (b,2), (\bar{a},2), (b,2)$.

Definition 17. *The x*-string sequence *of a cyclic word w is defined to be the cyclically ordered set of w's x-string labels.*

Example 18. *Let $w = aaab\bar{a}\bar{a}cccab\bar{c}$. Then the x-string sequence of w is:*
$(a,3)(b,1)(\bar{a},2)(c,3)(a,1)(b,1)(\bar{c},1)$.
Note that we could also represent this as (for example):
$(c,3)(a,1)(b,1)(\bar{c},1)(a,3)(b,1)(\bar{a},2)$.

Definition 19. *For a cyclic word w, the* skeleton word *for w is the cyclic word determined by $\lambda_1 \ldots \lambda_m$, where the x-string sequence of w is:*
$(\lambda_1, n_1)(\lambda_2, n_2) \ldots (\lambda_m, n_m)$.

Example 20. *Let $w = aba\bar{b}ccddd\bar{e}\bar{e}abad$, then the skeleton word for w is the word $aba\bar{b}cd\bar{e}abad$.*

Definition 21. *For a CP word w, the* CP x-string sequence *is the set of all the possible permutations on the generator labels of the x-string sequence of w.*

By its definition, the CP x-string sequence is unique to a CP word.

Example 22. *Let $w = aababa\bar{b}$. Then both of the following are elements of the CP x-string sequence of w:*
$(a,2)(b,1)(a,1)(b,1)(a,1)(\bar{b},1)$ *and* $(\bar{b},2)(a,1)(\bar{b},1)(a,1)(\bar{b},1)(\bar{a},1)$.

We owe the following remark to Lau [L],

Remark 23 (Lau)**.** *Any RCP word must begin with the string a^n, and there cannot be a longer x-string in the word.*

We now proceed to define the other class of Whitehead automorphisms, the so-called Whitehead type II automorphisms.

Definition 24. *S is a* Whitehead Type II automorphism *if for some fixed x in L_k, for every $y \neq x$ in L_k, one of the following holds:*

- $S(y) = yx$
- $S(y) = \bar{x}y$
- $S(y) = \bar{x}yx$
- $S(y) = y$

and $S(x) = x$.
We refer to x as the characteristic generator *of S.*

We will generally refer to automorphisms that are either Whitehead type I or Whitehead type II as simply *Whitehead* automorphisms.
We will say that a Whitehead automorphism is *trivial* if is the identity automorphism.

Definition 25. *We call an automorphism S a* chain *if there exists a sequence of Whitehead automorphisms $S_nS_{n-1}\cdots S_1$ so that $S = S_nS_{n-1}\cdots S_1$.*

Definition 26. *A chain composed of n Whitehead automorphisms is called* successively shortening *on a word w if for any $m \leq n$,*
$|S_mS_{m-1}\cdots S_1(w)| \leq |S_{m-1}S_{m-2}\cdots S_1(w)|$, *with equality only when $S_{m-1}S_{m-2}\cdots S_1(w)$ is minimal.*

This leads us to Whitehead's theorem.

Theorem 27 (Whitehead)**.** *Let w,v be in F_k, so that $w \sim v$ and v is minimal. Then there exists a successively shortening chain S so that $w = S(v)$.*

2. Descendence

In general, given a minimal word w, there are several intuitive ways to produce more words. An obvious way is to increase the length of one of the x-strings of the word. Any word so produced, we will call a *descendant* of w. More precisely, we present the following definition:

Definition 28. *We say that a minimal word v is a* descendant *of a minimal word w if one of the following is true:*

(1) *There exists a cyclic permutation of v, $\widetilde{v}$ so that $\widetilde{v} = a\widetilde{w}$, where $\widetilde{w}$ represents some cyclic permutation of w which begins with an a.*

(2) *There exists some u in F_k so that u is a descendant of w and there exists some cyclic permutation of v, $\widetilde{v}$ so that $\widetilde{v} = a\widetilde{u}$, where $\widetilde{u}$ is some cyclic permutation of u that begins with an a.*

This definition is purposely recursive to admit the possibility of descendence through the addition to multiple x-strings (c.f. [CR]).

Note that the number of x-strings is fixed by descendence. That is, the descendant of a word will not acquire new x-strings. Also, if w is a descendant of v, then the skeleton words of w and v will be CP-equivalent.

We note the following result from Cooper and Rowland [CR]:

Result 29. *Descendants of minimal words are necessarily minimal.*

Not all words are descendants of other minimal words. Those that are not are called *root* words. From [CR], we know the following theorem:

Theorem 30 (Cooper and Rowland)**.** *If w in F_2 is a root word, then all $v \sim w$ are root words for $|v| = |w|$.*

We also know the following fact (again from [CR]):

Fact 31. *A minimal alternating (that is, one whose longest x-string is one letter long) word is a root word.*

We call w an *ancestor* of v if v is a descendant of w. If v is derived from w by the addition of a single letter, then we call w a *parent* of v. Note that there may be more than one parent of a word (for example, *aaaabbb* is descended from both *aaaabb* and *aaabbb*). If w is an ancestor of v then the number of *generations* between them is defined to be $|v| - |w|$. If w and z are ancestors of v, then w is said to be a *nearer* ancestor to v than z is, if there are fewer generations between w and v than between z and v.

We call two words that share a common ancestor *coancestral*, and we call two words that share a common parent *coparental*.

Theorem 32. *Let u and v be CP-distinct coancestral words. There are uniquely determined sets W and Z such that for any $w \in W$ and any $s \notin W$, w is an ancestor of both u and v and is nearer to them than s is. Likewise, Z contains the closest descendants.*

Proof. Since u and v both descend from some word, say p, they must have the same CP x-string sequence as p, except that some set of x-strings (call it X_u) are all longer in u, and another (call it X_v) are all longer in v. Let u' and v' be the skeleton words of u and v respectively. As noted above, $u' \sim_{CP} v'$. Let $\{C_i\}$ represent the set of cyclic permutations such that for any $C_j \in \{C_i\}$, $u' = C_j(v')$. There is a clear 1-1 correspondence between the set $\{C_i\}$ and the set of bijective mappings from the set of x-string labels of u and the set of x-string labels of v. It is defined as follows:

First, let C be any element of $\{C_i\}$. Then define the function ϕ_C from the set of x-string labels of u onto the set of x-string labels of v by mapping an x-string label l_1 of u with the x-string label of v, l_2 such that C maps the letter of v' which corresponds to l_2 to the same position as that of the letter of u' to which l_1 corresponds.

Take any C_k and by the process just outlined, determine ϕ_{C_k}. Then we can construct the CP x-string sequence of a common ancestor of u and v, w_k as follows:

for any x-string label l_m in the x-string sequence of u, define ω_m to be the ordered pair $(u_m, min(n(l_m), n(\phi(l_m))))$, where u_m is the generator element of l_m and $n(l)$ is the length element of the x-string label l. We now let w_k be the CP word that corresponds to the CP x-string sequence $\omega_1 \ldots \omega_n$. Essentially, this is an 'intersection' of the CP x-string sequences of u and v.

If we repeat this for all of the elements of $\{C_i\}$, and assemble the set $\{w_i\}$. Now, define $W = \{w | w \in \{w_i\}, |w| = \min_{t \in \{w_i\}} |t|\}$. Thus, it is clear that W must contain all of the nearest ancestors of both u and v. Also, any element of W is also a descendent of p (or possibly CP equivalent to p), and therefore minimal.

By nearly the same procedure, we can also construct the set of the nearest common descendants of u and v. The only procedural difference is that when we are assembling the CP x-string sequence of the descendants and the lengths of some x-string of u differs from the length of the corresponding x-string of v, then here take the longer of the two for the descendant. That is to say, basically, replace all occurrances of the word "min" in the preceeding to paragraphs with the word "max", and replace the occurrances of the letter "W/w" with "Z/z".

It is clear by the construction of both W and Z that there cannot be any words that are not in them, but are nearer ancestors (descendants respectively) to u and v than the elements of W (Z respectively).

Thus, we may construct the sets that contain all of the nearest ancestors or descendants with relative ease, given only coancestral words.

□

Example 33. *Let $p = ab\bar{a}b$, $u = aaaabb\overline{aaaa}b$ and $v = aaaaaa\overline{bb}aaabbbb$. We can see that both u and w are descendents of p. We note that the skeleton word for u (which we denote u') is $ab\bar{a}b$, and that (under the same notation) $v' = a\bar{b}ab$. We also see that the x-string sequence of u is $(a,4)(b,2)(\bar{a},3)(b,1)$ and that that of v is $(a,6)(\bar{b},2)(a,3)(b,4)$. Inspection tells us that there is only one cyclic permutation that takes $v' \mapsto u'$ (that which permutes a with b and takes the final letter of v' to the fore).*

Thus, by the process above, we note that up to cyclic permutation there is only one word in $w \in W$, where W is the set of all of the common ancestors of u and v that are minimally distant from them. We apply the above procedure, and note that the CP x-string sequence of w (up to permutation of the generator elements) is $(a,4)(b,2)(\bar{a},2)(b,1)$. Thus, the CP word w can be represented by the RCP word $aaaabb\overline{aa}b$. Inspection confirms that this is, in fact an ancestor to both u and v.

Lemma 34. *Let u and v be distinct non-root CP words, all of whose parents are minimal. Then, u and v are parents of a common descendant, t, if and only if they are coparental with some parent w.*

Proof. First, we show necessity: We note that if we remove one letter from one of the x-strings of t, we will have u . Similarly, if we remove one other letter from one of the x-strings of t we will get v. If we remove one letter from each of these x-strings, we will get some word w. By adding one letter back to one of the x-strings of t, we get u. Therefore, u is a descendant of w. Similarly, if we add one letter back to another x-string of w, then we obtain v (up to cyclic permutation). Therefore, v, too, is a descendant of w. Thus, w is a parent to both u and v.

Now, we show sufficiency: If u and v are both descended from a common parent t, then we note that the size of X_u (as defined in theorem 32) must be cardinality one. Likewise for X_v. Therefore, the size of the 'union' of their x-strings must be to add one to the length of either one of them. Therefore, the uniquely determined nearest common descendant is one generation away. Thus, u and v are parents of the same word, t.

□

Example 35. *Let u and v be the CP words represented by the strings $aaaba\overline{b}$ and $aabaa\overline{b}$. Then we can see that clearly, by the processes defined above, they must have a common descendant $aaabaa\overline{b}$. They are both actually parents of this word. Therefore, from lemma 34 they must be coparental themselves (assuming that their parents are minimal, which it so happens that they are). We see from figure 1 that they, in fact, are both descended from the parent $aaba\overline{b}$.*

All of these lemmas are highly suggestive of the existence of a genealogical structure on the development of descendants. Therefore, we present this in an intuitive manner.

Figure 1 is a "family tree" showing all of the equivalence classes of RCP words of length ≤ 7 in F_2 with ancestors and descendants displayed in a genealogical manner. Root words are those enclosed in a dashed box.

We call an equivalence class a descendant of another equivalence class if all of its RCP elements are the RCP elements of the ancestor class with an additional initial a.

Example 36. *The equivalence class of $aaaaba\overline{b}$ is a descendant of the equivalence class of $aaaba\overline{b}$*

In some cases, words that are equivalent are no longer equivalent if an additional a is added to both. We define this notion more precisely as follows:

Definition 37. *If for some words w,v, $aw \sim av$ but $w \not\sim v$, then the equivalence class of v (w respectively) is said to* ***branch*** *from that of w (v respectively).*

We motivate this use of terminology by figure 2.

If it is possible to fully understand the manner in which equivalence classes grow in RCP elements through the generations, it is of no small interest for us to determine the conditions under which an equivalence class will branch from another.

3. RCP WORDS AND EQUIVALENCE

The following conjecture would establish necessary and sufficient conditions under which the equivalence classes of two equivalent RCP words will branch from each other.

Conjecture 38. *If w,v are RCP words, then $aw = S(av)$ if and only if $S(a) = a$ and $w = S(v)$.*

This conjecture was proposed in a weaker form by Lau [L], and proved for the case where S is composed of a single Whitehead automorphism of each type by Virnig [V].

This conjecture is not true if the condition of RCP-ness is dropped, because of the following counterexample:

Example 39. *Let S be the automorphism defined by the following:*

$$S\begin{cases} a & \mapsto & aaab \\ b & \mapsto & \overline{a} \end{cases}$$

$S(aaaabbbabbbab\overline{a}\overline{bbb}\overline{a}\overline{bbb}) = aaabaaabaaabaaabbb\overline{a}\overline{bb}$

But both of these words are minimal.

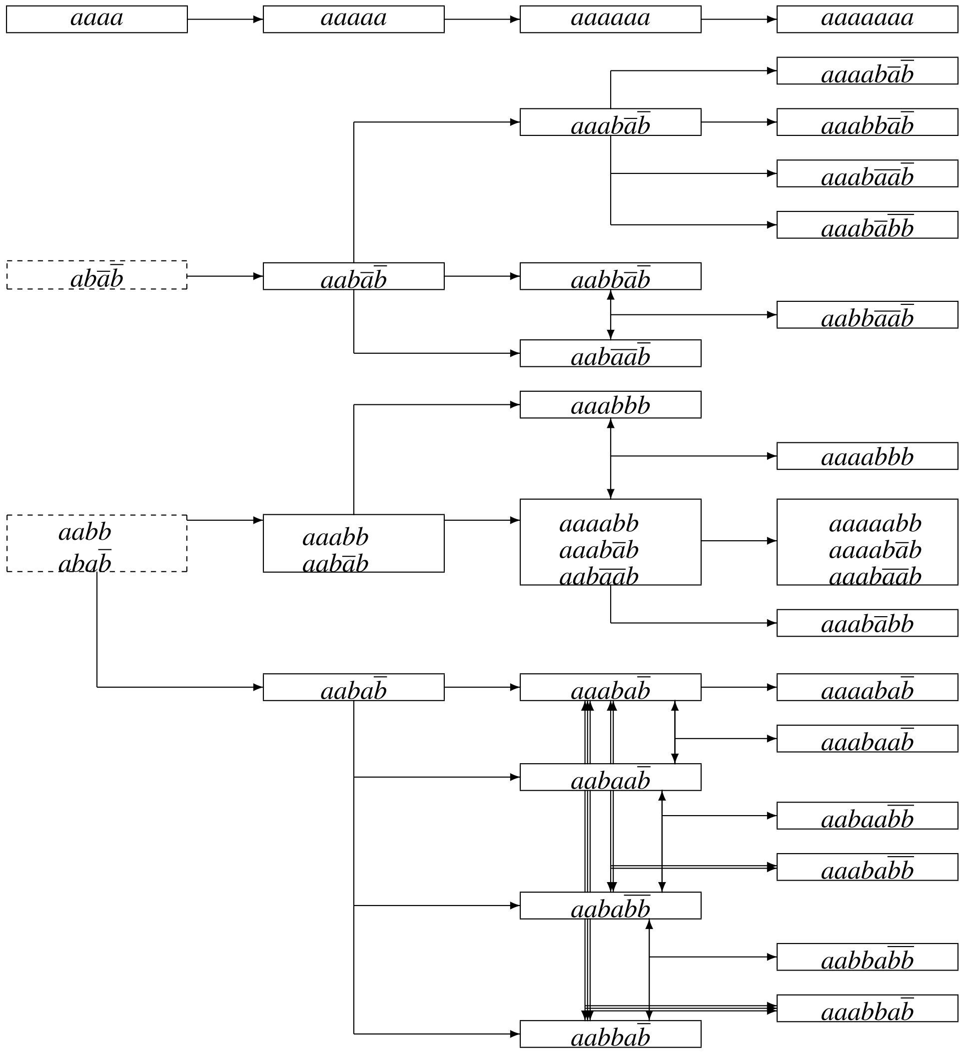

FIGURE 1. Family Tree

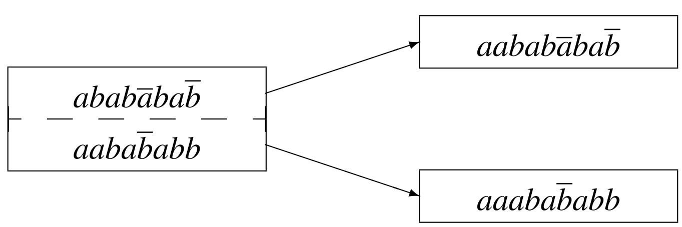

FIGURE 2. Branching Words

Theorem 40. *Given minimal words w,v with $n,m > 0$ initial a's respectively, aw=$S(av)$ for some automorphism S composed of a chain of Whitehead automorphisms only if one of the following holds:*

(1) $S(a) = a$ *and* $S(v) = w$.

(2) *S is composed of of no fewer than* $min(m,n)$ *Whitehead automorphisms.*

Proof. We note that if $S(a) = a$, then $S(av) = aS(v) = aw$ and the theorem clearly holds.

We also note that: $aw = a^{n+1}w'$ $av = a^{n+1}v'$

Since $aw = S(av)$, we know that $w' = \overline{a}^{n+1}S(a)^{m+1}S(v')$. But we also know that w' does not begin with $\overline{a}$. Therefore, the $\overline{a}^{n+1}$ must be canceled by $S(a)^{m+1}$.

If $S(a) \neq a$, then we can break into cases:

Case 1:

$S(a) = a^{n+1}x$ for some $x \in F_k$. This implies that S must consist of at least $n+1$ Whitehead automorphisms.

Case 2:

$S(a) = y$ for some $y \in F_k$, such that $y \neq a$. This implies that one of two possibilities must hold:

Subcase 2A: All of the y^{m+1} are canceled by $S(v')$.

We make the following remark:

Remark 41. *If several of the elements of* v' *combine to produce* $\overline{y}^{n+1}$ *then this implies the cancellation of entire intermediate elements of* v' *with at least some y's. As noted by Virnig,* [V]*, this cancellation could be done prior to the application of S. Therefore, av is not minimal, which is a contradiction. Therefore,* $S({v_1}')$ *alone must have the form* $\overline{y}^{m+1}z$ *for some* $z \in F_k$.

This implies that S is composed of at least $m+1$ Whitehead automorphisms.

Subcase 2B: $y = a^{n'}x$ for some $x \in F_k$, where $n' < n$. In this case, $S({v_1}') = \overline{y}^m\overline{x}z$ for some $z \in F_k$. Thus again, S must be composed of at least m Whitehead automorphisms.

We could now repeat the argument interchanging S with S^{-1}, w with v, and m with n.

□

We note that if $S(a) = a$ in the above theorem, then clearly $aw = S(av)$ if and only if $w = S(v)$. Thus these a-fixing automorphisms guarantee the following:

Proposition 42. *If w,v are RCP words, and there exists an a-fixing automorphism S so that* $w = S(v)$ *then* $aw = S(av)$.

Thus the existence of a-fixing automorphisms between the equivalence classes of two RCP words is already a sufficient condition to ensure that two RCP words will not branch from each other. The conjecture, if proved, will also make that necessary.

We also note that conjecture 38 would have the following consequences:

Consequence 43. *If w,v are RCP words, then the nonexistence of an a-fixing automorphism between w,v implies that* $aw \not\sim av$.

And, it would also imply that:

Consequence 44. *If w,v are RCP then* $aw \sim av$ *implies that* $w \sim v$.

The contrapositive of this statement directly gives rise the following:

Consequence 45. *The number of equivalence classes is strictly nondecreasing with length.*

Although, as yet, a (correct) general proof of the conjecture eludes us, and will have to be the subject of future researches, we believe that its truth will likely be borne out.

3.1. **Partial results on the conjecture in F_2.** If we let w and v be RCP words so that $aw = S(av)$, for some automorphism S, then we may observe the following things about S:

Observation 46. *Whatever S maps a to must begin with at least one a.*

Observation 47. *If v has more than one initial a, then $S(a)$ is either just an a or starts with an a-string and ends with an $\overline{a}$-string.*

We can see the reason for this, because if av has two a's initially, then there will be two repetitions of $S(a)$ to begin aw, but since w is RCP, the second string of a's cannot be as long as the first in aw. Therefore, some of them must cancel with some $\overline{a}$'s at the end of $S(a)$.

Observation 48. $|w| = |v|$*, and therefore, S and S^{-1} are level automorphisms between them.*

Observation 49. $(S(b))^{\pm 1}$ *and* $S^{-1}(b)^{\pm 1}$ *must either start with or end with* $x^{\pm 1}$*, where $x \in L_k$ and $x \notin \{a, \overline{a}\}$.*

This is because neither w nor v may end with $a^{\pm 1}$.

It is our hope that through the use of these observations, or others like them, a researcher in the future will be able to find a contradiction in the case where $S(a) \neq a$.

4. A-FIXING AUTOMORPHISMS AND QCPS

As we have seen, the issue of automorphisms that fix a are of critical importance in the determination whether or not branching will occur between two CP-words. Therefore, we present the following results on the nature of a-fixing automorphisms.

Theorem 50. *If S is an a-fixing chain of finite length on F_2, then $S(b)$ will have exactly one occurrence of $b^{\pm 1}$.*

Proof. First we note that $S(b)$ cannot have fewer than one occurrence of $b^{\pm 1}$. If it did, then $S(b) = a^e$ for some integer e. Therefore, there is no potential word that would ever be mapped by S to a word that contains b. Therefore, no word that contains b would be in the range of S. Thus, S would not be surjective, and hence not an automorphism. This is a contradiction.

Let us consider that $S = S_n S_{n-1} \cdots S_1$, where each S_i is a nontrivial Whitehead automorphism, and $n < \infty$. Let us group together the S_i so that $S = T_m T_{m-1} \cdots T_1$ where each T_j is a chain of Whitehead automorphisms such that $T_j(a) = a$, and T_j cannot be further decomposed into chains that each fix a.

Clearly, if we establish our theorem for an arbitrary T_i then upon composition with other T_j's the theorem will still hold.

Thus, let T be an arbitrary element of $\{T_i\}$.

Case 1:

Let T be composed of a single Whitehead automorphism and T fixes a. Then, clearly, $T(b)$ has exactly one occurrence of $b^{\pm 1}$.

Case 2:

Let T be composed of more than one Whitehead automorphism. Clearly the first Whitehead automorphism applied cannot fix a, otherwise it would constitute its own T_i. Therefore, we know that the first Whitehead automorphism is either a Whitehead type II automorphism with a as its characteristic, or it is a permutation that does not fix a. We will consider those cases separately.

Case 2A:

Let us assume that the first automorphism applied in the T-chain is a Whitehead type II automorphism. Therefore, we see that it maps a and b as follows:

$$\begin{array}{rcl} a & \mapsto & b^{e_1}ab^{e_2} \\ b & \mapsto & b \end{array}$$

Where $e_1, e_2 \in \{0, \pm 1\}$, and $e_1 e_2 \leq 0$.

We can break this up even further into the following three subcases:

- 2A(i):

$$\begin{array}{rcl} a & \mapsto & b^{\pm 1}a \\ b & \mapsto & b \end{array}$$

- 2A(ii):

$$\begin{array}{rcl} a & \mapsto & ab^{\pm 1} \\ b & \mapsto & b \end{array}$$

- 2A(iii):

$$\begin{array}{rcl} a & \mapsto & b^{\pm 1}ab^{\mp 1} \\ b & \mapsto & b \end{array}$$

Case 2A(i:)

We begin by noting that under the first automorphism in T's chain, a and b are affected as follows: either

$$\begin{array}{rcl} a & \mapsto & (b)\{a\} \\ b & \mapsto & b \end{array}$$

or

$$\begin{array}{rcl} a & \mapsto & (\bar{b})\{a\} \\ b & \mapsto & b \end{array}$$

We will hereafter refer the the string lying within the parentheses as (), the string lying within the braces as $\{\}$, top row as the a-row, and the bottom row as the b-row. We will, when representing the effect of successive automorphisms, keep the image of () also inside parentheses, and likewise for $\{\}$. Thus, we will use the same symbols for both () and its subsequent images under succeeding automorphisms.

In either of the above cases, it is clear that under successive automorphisms, the b-row will be a substring of the a-row (up to inversion), as long as cancellations between () and $\{\}$ are suppressed. Specifically, the b-row at any point will be, with possible inversion, ().

After all of the automorphisms are applied, the a-row must reduce to a. Since no automorphism will take either () or $\{\}$ to the trivial word, it must be that () cancels with $\{\}$ to leave only an a remaining. As noted by Virnig [V], we can withhold these cancellations until after all of the automorphisms have been applied. Thus, we will consider that () and $\{\}$ be kept separate until after the final automorphism has been applied.

We now consider the later automorphisms applied. Since it must eventually be the case that () and $\{\}$ must cancel each other leaving an a, it must somewhere in the chain of T occur that one of two things happen: either $() \mapsto [x_1]\overline{\{\}}$ or that $\{\} \mapsto \overline{()}[x_1]$ for some $[x_1] \in F_2$. We will refer to this automorphism as the *inception* of $[x_1]$. Henceforth, we shall treat $[x_1]$ in the same manner as () or $\{\}$ (that is, using the same symbols to refer, also, to the image of $[x_1]$ under subsequent automorphisms).

We assume without loss of generality that it is the first of these (the argument proceeds parallelly in the second case). This may be assumed to occur when $\{\}$ is only one letter long. Otherwise, we could not acquire the circumstance that $\{\}$ becomes a substring of () in one automorphism, unless it was already a substring.

Similarly, at its inception (the point when, for the first time, $() \mapsto [x_1]\overline{\{\}}$) we can consider $[x_1]$ to be one letter long. If it were otherwise, then it must be the case that there were automorphisms applied to () before the one which appended $\overline{\{\}}$ to (). These automorphisms, in order not to ultimately affect the length of $\{\}$ must have either operated with characteristic $\{\}$ (because $\{\}$ is one letter long at this point as noted above), or resulted in a cancelation which leaves $\{\}$ one letter long. Therefore, they must have right or left-affixed $\{\}^{\pm 1}$ to (). If they right-affixed $\{\}$ to (), then the automorphisms which reduce $\{\}$ to one letter must do likewise to the right-affixed copy of $\{\}$. Thus, no automorphism could right-affix $\overline{\{\}}$ to $\{\}$. Therefore, we may assume that they act to the left of (). Since the whole chain of automorphisms from the first one to use $\{\}$ as its characteristic, through the one that inceived $[x_1]$ are all of characteristic $\{\}$, we may reorder them without altering anything. Thus, we may first apply a right-affixing Whitehead II automorphism with characteristic $\{\}$. This is the inception of $[x_1]$. Therefore, $[x_1]$ is only one letter long.

There are two possibilities:

A: $[x_1] \mapsto \cdots \mapsto a$ without ever being lengthened (that is, without there occurring any Whitehead II automorphisms with characteristic other than $[x_1]$), or

B: $[x_1] \mapsto \cdots \mapsto a$ with the appending of new letters.

We will call this dichotomy, D_1. We will say that $D_1 = A$ if the first possibility holds, or $D_1 = B$ if the second holds.

If $D_1 = A$, then there must occur only some combination of Whitehead II automorphisms which only add $[x_1]$'s (up to inversion) into either side of $\{\}$, and (possibly trivial) permutations which eventually send $[x_1] \mapsto a$. Thereupon, we see that $\{\}$ will become some string of the form $a^{\alpha}b^{\pm 1}a^{\beta}$.

If $D_1 = B$, then have two possibilities:

2A(ia): $\{\}$ may be lengthened lengthened, or

2A(ib): $[x_1]$ may be lengthened first,

but not both simultaneously.

Case 2A(ia): $\{\}$ is lengthened first.

Then it will have both generators in it (not regarding inverses), including $[x_1]$. It will thus be of the form $\{\} \mapsto [x_1]^a\{y_1\}[x_1]^b$, where $\{y_1\}$ is in L_k. Notice that we may consider, then, that $[x_1]$

is lengthened before $\{y\}$. Thus, in the following analysis, we may substitute $\{y_1\}$ for $\{\}$ and will therefore arrive at the same conclusion. Thus, this case will be resolved with case 2A(iib).

2A(ib): $\{\}$ is lengthened first, If $[x_1]$ is lengthened first, then it will have both generators in it (not regarding inverses), including the one that represents $\{\}$. Thereafter, $\{\}$ will be a substring (up to inversion) of $[x_1]$. We shall denote this by saying that under some automorphism (without loss of generality) $[x_1] \mapsto \{\}[\zeta_1]$, for some $[\zeta_1]$ in F_2.

By identical reasoning from above, it is clear that at the point of $[\zeta_1]$'s inception, $\{\}$, and $[\zeta_1]$ must each consist of only a single letter. Since it must eventually be the case that $[x_1] \mapsto \cdots \mapsto a$, it must, therefore, also be the case that (without loss of generality) $[\zeta_1] \mapsto \overline{\{\}}[x_2]$, under some automorphism. Also, eventually, $[x_2] \mapsto \cdots \mapsto a$.

Clearly, again, it must be the case that at the inception of $[x_2]$, it is only one letter long. Likewise, at that point, $\{\}$ must also consist of one letter (note that $[x_2]$ and $\{\}$ are necessarily different letters). There are two possibilities, and they are analogous to D_1. Therefore, we define the dichotomy D_2 parallelly with D_1, with $[x_2]$'s in the place of $[x_1]$. In general, we define D_n as the dichotomy D_1 with $[x_1]$'s replaced by $[x_n]$'s.

If $D_2 = A$, then by identical reasoning from above, $\{\}$ must end with the form $a^\alpha b^{\pm 1} a^\beta$. We see that this will generally be the case whenever $D_i = A$.

If $D_2 = B$, then we consider the following. If $\{\}$ is lengthened before $[x_2]$, then we may, by the same argument as before (with the substitution of $\{y_2\}$ instead of $\{y_1\}$, but otherwise the same as before), consider this case to be analogous to the case where $[x_2]$ is first lengthened.

If $[x_2]$ is lengthened first, then by analogous treatment as with $[x_1]$, we see the inception of $[\zeta_2]$ and therefore, too, $[x_3]$. This induces the definition of D_3.

As seen before, if $D_3 = A$, then $\{\}$ becomes of the form $a^\alpha b^{\pm 1} a^\beta$.

Also, as seen before, if $D_3 = B$, then it induces the definition of D_4. We note that since T is a chain of finite length, there are only a finite number of Whitehead II automorphisms in its composition. Therefore, only for a finite set $\{\lambda_1, \ldots, \lambda_l\}$ can $D_{\lambda_i} = B$. Therefore, eventually, we must come to the case where $D_i = A$. At that point, we see that $\{\}$ (or $\{y_i\}$) winds up with the form $a^\alpha b^{\pm 1} a^\beta$.

In the case that $\{\} \mapsto \cdots \mapsto a^\alpha b^{\pm 1} a^\beta$, we observe the following:

Since () cancels with $\{\}$ to leave an a eventually, they must finish with the same number of $b^{\pm 1}$'s. Thus () can have at most one $b^{\pm 1}$. But, up to inversion, the b-row is equal to (). Therefore, the theorem is shown for this case.

If, however, we have the case where $\{y_i\} \mapsto \cdots \mapsto a^\alpha b^{\pm 1} a^\beta$, we note that $\{y_i\}$ may have at most one occurrence of $b^{\pm 1}$. We recall that, by construction, $\{\} \mapsto [x_i]^a \{y_i\} [x_i]^b$. Since $[x_i]$ is eventually shortened into simply a, $\{\}$ must become $a^a \{y_i\} a^b$ where $\{y\}$ has only one $b^{\pm 1}$. Thus, since () cancels with $\{\}$ to leave an a, () can also only have one $b^{\pm 1}$ after all of the automorphisms have been applied. Since the b-row is equal to (), the theorem is again shown in this case.

Clearly, a parallel argument works for Case ii.

Similarly, we may disregard Case iii. This is because we may treat an automorphism of this form as actually being composed of two automorphisms, those represented in Case i and ii. Thus, this case has already been established.

We now proceed to the next of our general cases:

Case 2B:

We now consider the possibility that the first automorphism in the chain T is a permutation that does not fix a. We now note from Lau [L] that we may consider T as a series of Whitehead type II automorphisms followed by a (possibly trivial) permutation.

Thus, this case falls under the treatment of Case 2A.

Therefore, we have shown that there is at least one $b^{\pm 1}$ in $S(b)$, and also that there is at most one. Thus, there is exactly one, and the theorem is proved.

□

As a consequence of this theorem, we see that for any a-fixing automorphism S, $S(b) = a^{\alpha} b^{\pm 1} a^{\beta}$. Thus, we may mimic any automorphism that fixes a by a chain of Whitehead automorphisms which each fix a.

We see this by simply 'piling' a's to either side of b through a chain of Whitehead II automorphisms with $a^{\pm 1}$ as their characteristics, followed by a (possibly trivial) permutation that fixes a.

As another consequence of this theorem, we see that under an a-fixing automorphism, the number of $b^{\pm 1}$'s will be fixed. We can also note that the $b^{\pm 1}$'s will affect the surrounding $a^{\pm 1}$'s in ways that are determined by $S(b)$. For example, if $S(b) = \overline{a}ba$, then the automorphism can be thought of as 'moving' one a rightwards across any $b^{\pm 1}$ (regardless of whether or not there are any to be moved, it could happen that the effect of S on a word is to pile up an $\overline{a}$'s to the left of a b, and an a to the right).

Sometimes, the effect of an automorphism on a word can be the same as that of a cyclic permutation, even though it does not have the explicit form of an inner automorphism composed with a permutation. We call such an automorphism a *QCP* (Quasi Cyclic Permutation).

Example 51. *Let S be the permutation defined as follows:*

$$S \begin{cases} a & \mapsto & a \\ b & \mapsto & \overline{a}b \end{cases}$$

Then, under S, the word $aaab\overline{a}\overline{a}b \mapsto aab\overline{a}\overline{a}\overline{a}b$. But we can easily see that the effect of this automorphism is the same as that of a cyclic permutation which fronted the final three letters and switched a's with $\overline{a}$'s.

We will now proceed to make some observations concerning a-fixing QCPs.

We first note that since a QCP must be a level automorphism on a minimal word w, and an a-fixing automorphism preserves the number of occurrences of $b^{\pm 1}$, any a-fixing QCP must preserve both the number of $a^{\pm 1}$'s and the number of $b^{\pm 1}$'s.

Lemma 52. *The only way that a string of $b^{\pm 1}$'s can be fronted by an a-fixing QCP on a word w is if that string of $b^{\pm 1}$'s immediately follows some initial string of $a^{\pm 1}$'s.*

Proof. If there is anything other than an initial string of $a^{\pm 1}$'s before the string of $b^{\pm 1}$'s, then there is no way that an a-fixing automorphism can 'move' it over the string of $b^{\pm 1}$'s. □

Lemma 53. *Let u be a minimal word that begins with a string a^m, with m, and does not end with a. There is no a-fixing QCP, S, that will have the effect of fronting a string of $b^{\pm n}$ and then changing the $b^{\pm 1} \mapsto a$, except the identity automorphism.*

Proof. Let us assume the existence of such an automorphism.

First, we note that the string of a's in the beginning of u must correspond to a string of $b^{\pm 1}$'s in $S(u)$. Therefore, either S has the effect of 'cycling' a through b (that is, $S(b) = a^l b^{\pm 1} a^{-l}$), and u has a string of the form $b^{\pm m}$ in it, or $S(b) = a^i b^{\pm 1} a^j$ with $i \neq j$ and u has a string of the form $(a^{-i} b^{\pm 1} a^{-j})^{\pm m}$. In the first case, we note the following: The strings between the initial string of a^m and the string of $b^{\pm m}$ and after the b-string must be interchanged upon the application of S, except that all of the a's and b's must be permuted appropriately. This means that they must have the same number of $b^{\pm 1}$'s in them. The only way that two strings can have their a's and b's permuted under such an automorphism as S is if they are of the form $(a^{\pm k} b^{\pm k})^{\pm 1}$. Since these two strings have the same number of $b^{\pm 1}$'s, they must be of the same length. This means that S must be the identity automorphism.

In the second case, the automorphism S will take the string of $b^{\pm n}$ to $(a^i b^{\pm 1} a^j)^{\pm m}$. But this string is not a permutation of a substring of u.

Thus, it too, is impossible. □

Lemma 54. *Let u be a minimal word that begins with a string of n a's and does not end with an $a^{\pm 1}$, and let S be an a-fixing automorphism that takes u to some minimal word v which begins with $n \pm 1$ a's, and also does not end with $a^{\pm 1}$. Then S can only be a QCP that takes some string of $\overline{a}$ to the front and permutes $a \mapsto \overline{a}$ if there are no $a^{\pm 1}$'s in the strings between the initial a's and the $\overline{a}$'s, or after the $\overline{a}$'s.*

Proof. We first note that the two strings between the initial a's and the $\overline{a}$'s and after the $\overline{a}$'s must be the same, except that the a's in one must be $\overline{a}$'s in the other. The only way that an a can be added (or subtracted respectively) to the initial string is if $S(b)$ of $S(\overline{b})$ begins with one a ($\overline{a}$ respectively). But S cannot send b to anything with $a^{\pm 1}$'s on both sides, because then either u or v would end with $a^{\pm 1}$. There must be some $b^{\pm 1}$'s in the strings between the initial a's and the $\overline{a}$'s, or after the $\overline{a}$'s. If there are any $a^{\pm 1}$'s adjacent to those $b^{\pm 1}$'s, then they would only be sent to $\overline{a}$'s if they were surrounded by a $\overline{b}$ and a b. On the other side of the $\overline{a}$ string, however, we must find an a of the opposite sign from the one just discussed, but similarly sandwiched. Therefore, instead of changing the sign on that a, it will add two more copies. This is not the effect of a cyclic permutation. Therefore S can only be a QCP if there are no $a^{\pm 1}$'s found in either of these two strings. □

The reader may rightly wonder about the purpose of these lemmas. The patient reader will find that they do have a role to play, and is advised to postpone frustration with the seemingly arbitrary and useless results.

5. RCP DESCENDENCE

From the table provided, it is clear that new RCP elements arise at each increase in length. As shown by Virnig [V], a descendant is necessarily minimal. Thus, the only way that a minimal non-root aw word can be RCP without w being RCP is if it satisfies the following definition of RCP descendence.

Definition 55. *If as,au and u are all RCP, then as is an* RCP descendant *of u if $s \sim_{CP} u$*

If $s = u$ then as is the trivial *RCP descendant of u.*

Example 56. *$aaabab\overline{aaa}b$ is an RCP descendant of $aaab\overline{aa}b\overline{a}b$.*

We note that an RCP descendant of a CP word is also a descendant in the more general sense. Therefore, no RCP descendant is a root word.

Definition 57. *If u is an RCP word with m initial as then any string of the form $y^{m'}$ where y is in L_k and $m' \geq m-1$ with m' maximal, is called an* egg *of u.*
The number of distinct eggs in a word u is the fertility *of u.*

We note that the only way that an RCP word au can lose its RCP-ness after the truncation of an initial a is if a cyclic permutation of u which takes an egg to the front and permutes that egg to a string of a's alphabetically precedes u. This is the rationale behind the terms 'egg' and 'fertility'.

Theorem 58. *Any RCP word w can have at most F RCP descendants, where F is the fertility of w.*

Proof. Let as and at be two distinct RCP descendants of w. Since as and at are RCP descendants, they are not root words. Therefore, they are not alternating words. Thus, they must have an x-string longer than one. Since they are RCP, this implies that the string of initial a's is, at least, two letters long.

Since as is RCP, but s is not, we know that $s = a^{n'} s_1 x_1^n s_2$, where $n' \in \{n, n-1\}$ and if $w = C_s \circ P_s(s)$, where C_s is a cycle of s, and P_s is a permutation, then x_1 is the element of L_k so that $P_s(x_1) = a$.

Similarly, $t = a^{m'} t_1 x_2^n t_2$, where $m' \in \{n, n-1\}$ and if $w = C_t \circ P_t(t)$, for cycle C_t and permutation P_t, then x_2 is the element of L_k so that $P_t(x_2) = a$.

Thus, w must have the form:

$$\begin{aligned} w &= a^n P_s(s_2) P_s(a)^{n'} P_s(s_1) \\ &= a^n P_t(t_2) P_t(a)^{m'} P_t(t_1) \end{aligned}$$

If we assume that the initial a's of s and the initial a's of t map to the same egg of w, call it x_3^l, then we note the following, $P_s(a) = P_t(a)$, $l = n' = m'$ and

$$a^n P_s(s_2) x_3^l P_s(s_1) = a^n P_t(t_2) x_3^l P_t(t_1)$$

Therefore,

$$P_s(s_2) x_3^l P_s(s_1) = P_t(t_2) x_3^l P_t(t_1)$$

This implies that $P_s(s_2) = P_t(t_2)$ and that $P_s(s_1) = P_t(t_1)$.
Therefore, $s_2 = P_s^{-1} \circ P_t(t_2)$, and $s_1 = P_s^{-1} \circ P_t(t1)$.
We already know that $a = P_s^{-1} \circ P_t(a)$, and $x_1 = P^{-1} \circ P_t(x_2$.
Therefore, we can conclude that $s = P_s^{-1} \circ P_t(t)$. But because $P_s^{-1} \circ P_t$ fixes a, we find that

$$\begin{aligned} as &= aP^{-1} \circ P_t(t) \\ &= P^{-1} \circ P_t(at) \end{aligned}$$

Thus, $as \sim_{CP} at$. But as and at are RCP and distinct. This is a contradiction, so our assumption that s and t map their initial a's to the same egg of w is false. Therefore, s and t must map their initial a's to different eggs of w. Therefore, since w has F eggs, it can have, at most, F RCP descendants. □

Definition 59. *Let as be an RCP descendant of u, then as is said to* hatch *out of some egg e if the initial string of a's in s map to e under the cyclic permutation which takes s to u.*

Thus we have shown that no two distinct RCP descendants may hatch from the same egg. This raises the question of whether two distinct RCP descendants may hatch from different eggs.

We see that if there are, in fact, two distinct nontrivial RCP descendants as and at, of some word, call it u then we know that the form of s, t and u will be as follows:

$$\begin{aligned} s &= a^{n'} s_1 \sigma^{m'} s_2 x_1^n s_3 \\ t &= a^{m'} t_1 x_2^n t_2 \tau^{n'} t_3 \\ u &= a^n u_1 \upsilon_1^{n'} u_2 \upsilon_2^{m'} u_3 \end{aligned}$$

Where $\{\sigma, \tau, \upsilon_1, \upsilon_2\} \subseteq L_k$, and $\{s_i, t_i, u_i | 1 \leq i \leq 3\} \subset F_k$ and $n', m' \in \{n, n-1\}$.

And we know that there must exist permutations P_s and P_t so that:

$$\begin{array}{rcl} & P_s & P_t \\ a \mapsto & \tau & \mapsto \upsilon_1 \\ s_1 \mapsto & t_3 & \mapsto u_2 \\ \sigma \mapsto & a & \mapsto \upsilon_2 \\ s_2 \mapsto & t_1 & \mapsto u_3 \\ x_1 \mapsto & x_2 & \mapsto a \\ s_3 \mapsto & t_2 & \mapsto u_1 \end{array}$$

Therefore, from lemmas 52 through 54, we may show the following theorem:

Theorem 60. *Let as, at and au all be distinct RCP descendants of u. There are no a-fixing automorphisms S_u and S_t so that $u = S_u(t)$ and $t = S_t(s)$.*

Proof. Let us begin by assuming the existence of S_u and S_t from above.

As we know, s, t and u must have the form:

$$\begin{aligned} s &= a^{n'} s_1 \sigma^{m'} s_2 x_1^n s_3 \\ t &= a^{m'} t_1 x_2^n t_2 \tau^{n'} t_3 \\ u &= a^n u_1 \upsilon_1^{n'} u_2 \upsilon_2^{m'} u_3 \end{aligned}$$

From lemmas 52 and 53, we know that $\sigma, \tau, \upsilon_1, \upsilon_2, x_1, x_2 \in a, \bar{a}$. From lemma 54, and from the conservation of b's under a-fixing automorphisms, we know that for some k it is true that, $s_i, t_i, u_i \in b^k, \bar{b}^k$. Therefore, $s_1 = s_2^{\pm 1}$ and $s_2 = s_3^{\pm 1}$ and $s_1 = t_1^{\pm 1}$, etc.

Note that since as, at and au are all RCP, not s, t, nor u may end with $a^{\pm 1}$. Thus, they must all end with $b^{\pm 1}$. Therefore, S_s and S_t may not append $a^{\pm 1}$'s to either side of $b^{\pm 1}$. Also, since the difference between n and n' (m' respectively) is at most 1, $S_s(b)$ and $S_t(b)$ may have at most one occurrence of $a^{\pm 1}$.

Case 1:$n' = m' = n$.

Either (i) we have that S_s or S_t is an a-fixing permutation. In which case, either $as \sim_{CP} at$ or $as \sim_{CP} au$, but either way is a contradiction.

or

(ii) $S_t(s_1) = s_1^{\pm 1} a^{\pm 1}$, which implies that $s_2 = \bar{s}_1$. But then, either t_3 ends with $a^{\pm 1}$ or $n \neq n'$. Either way is a contradiction.

Case 2:$n' = m' = n - 1$.

Thus, $S_u(s_1) = as_1^{\pm 1}$. This implies that $u_2 = \bar{u}_1$. Thus, u_3 ends with $a^{\pm 1}$. This is a contradiction.

Case 3:$n' < m' = n$.

Thus, $S_t(s_1) = as^{\pm 1}$, which implies that $t_2 = \bar{t}_1$. Then, $t_3 = \bar{t}_1$, and therefore must end with an $\bar{a}$. This is a contradiction.

Case 4:$n' > m' = n - 1$.

Thus, $S_u(s_1) = s_1^{\pm 1} a^{\pm 1}$. This implies that $u_2 = u_1$, which means that $u_3 = u_1$, which implies that u_3 ends with $a^{\pm 1}$. This is a contradiction.

Thus, for none of these cases can S_u and S_t both exist. Therefore, if there is more than one distinct RCP descendant of u, then they are not all connected by a-fixing automorphisms. □

As we have seen, the nonexistence of such an a-fixing automorphism between two words makes a word eligible to branch off of the other. Thus, we conjecture that at most one nontrivial RCP descendant of u can be in the equivalence class of au.

6. Open Questions and directions for future researches

There are still many unanswered questions in this area of research. For example, conjecture 38 has now passed through four papers without proof. We believe, however, that the final proof of this conjecture will require a more sophisticated approach than the naive one that has been taken so far. We also hope that the conjecture can be proved in a slightly more general form than that in which it is presented. As we showed with example 39, the condition of RCP-ness cannot be completely dropped, but it is hoped that it can be shown for words that are 'almost' RCP (for example, s and t in theorem 60). That would then strengthen this theorem considerably.

Additionally, we would like to see more progress made in the understanding of the mechanisms governing the increase in the number of RCP elements in any equivalence class as it progresses through generations. Cooper and Rowland put forward the following conjecture [CR]:

Conjecture 61. *After sufficiently many generations, all equivalence classes increase in size by one word every generation except the classes* a^n, $a^{n-m}b\bar{a}^m b$, *and* $a^{n-m}ba^{\pm 1}\bar{b}\bar{a}^m\bar{b}a^{\pm 1}b$.

We believe, however, that it will eventually shown that this is not entirely accurate. Observation of the tables provided in [CR] tells us that, in fact, less than half of the equivalence classes do grow by one RCP element. We believe that it will be possible to establish the necessary form of equivalence classes that are 'pregnant' (that is, capable of spawning new RCP elements).

Additionally, [CR] conjectured that no equivalence class grows by more than one element each generation (in F_2). We believe that that is likely, and its full proof may have to be similar to that of theorem 60. This is only speculation, of course.

We would like to note here that a complete list of all of the RCP words in F_2 whose length is at most 9 are provided in [CR], grouped within their equivalence classes.

REFERENCES

[CR] Cooper, B. and Rowland, E. *On Equivalent Words in the Free Group on Two Generators* Proceedings of the Research Experience for Undergraduates Program in Mathematics, 2002. 52-78. NSF and Oregon State University.

[C] Crisp, D. *The Markoff Spectrum and Geodesics on the Punctured Torus.* PhD Thesis University of Adelaide

[L] Lau, M. *A Computer Implementation of Whitehead's Algorithm.* Proceedings of the Research Experience for Undergraduates Program in Mathematics, 1997. 41-66. NSF and Oregon State University.

[R] Rapaport, E.S. *On Free Groups and Their Automorphisms.* Acta Mathematica, 1958. v. 99: 139-163.

[S] Sanchez, C.M. *Minimal Words in the Free Group of Rank Two.* Journal of Pure and Applied Algebra, 1980. v. 17: 333-337.

[V] Virnig, R. *Whitehead Automorphisms and Equivalent Words* Proceedings of the Resesarch Experience for Undergraduates Program in Mathematics, 1998. 125-167. NSF and Oregon State University.

[WH1] Whitehead, J.H.C. *On Certain Sets of Elements in a Free Group.* Proceedings of the London Mathematical Society, 1936. v. 41: 48-56.

[WH2] Whitehead, J.H.C. *On Equivalent Sets of Elements in a Free Group.* Annals of Mathematics, 1936. v. 36: 782-800.

OREGON STATE UNIVERSITY
E-mail address: sweaver@science.orst.edu