

KNOT INVARIANTS AND THEIR IMPLICATIONS FOR CLOSED PLANE CURVES

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ABSTRACT. A dominating feature of knot theory is the problem of knot classification. In this paper, we hope to simplify the task of classification through strong connections between Arnold's work with knot invariants and that of Xiao-Song Lin and Zhenghan Wang. We show that the defect of a plane curve associated with a flattened alternating knot may be considered an invariant for alternating knots. We also give simple method of computing a certain Vassiliev knot invariant from Arnold's plane curve invariants and signed crossings.

1. INTRODUCTION

The goal of this paper is to further pursue the connections between knots and plane curves. By a knot, we mean a smooth mapping $\phi : S^1 \rightarrow \mathbb{R}^3$ and by a plane curve we mean a smooth mapping $\psi : S^1 \rightarrow \mathbb{R}^2$. Shortly, we will define these terms more rigorously, but the provided definitions will suit our introductory purposes.

Integral to this study is the previous work in plane curve theory by Whitney [W], Arnold [Ar], and Aicardi [Ai]. Furthermore, the research of Luo [Lu] and Lin and Wang [L-W] provided an impetus for our own methods and subsequent results.

Lin and Wang [L-W] use integral geometry to create an integral form for a known knot invariant. We suggest possible extensions of their work that might produce new invariants or at least new forms for known invariants.

A key idea native to the work of [Lu] and [L-W] involves the lifting of curves to knots and the flattening of knots onto the plane. We introduce a new notation to characterize both knots and plane curves, which is invariant under liftings from or flattening onto the plane. Using this notation, we are able to construct a new combinatorial form for a preexisting Vassiliev knot invariant.

Following the work of Aicardi [Ai] we consider plane curves whose defect is 0. These curves are studied because their Arnold invariants may be easily computed from formulas in [Lu]. We suggest a possible form for the Gauss diagrams of defect 0 curves.

From results in [L-W], we provide a simple proof of the fact that if two plane curves have the same Gauss diagram, then they have the same defect. More importantly, we show that the nonintersecting chords in a Gauss diagram have no effect on defect.

We also consider the effect of knot moves on defect of a plane curve that has been lifted to a knot. We are able to show that the creation of simple loops in a knot corresponds to nonintersecting chords in a Gauss diagrams and thus does not affect the defect of the plane curve created by flattening the knot. Also, we show that the flipping of a tangle in a knot, known as a flype, has

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no effect on the defect of the associated plane curve. These conditions show that any plane curves created by flattening a projection of the same alternating knot will have the same defect. Thus, defect may be considered an invariant for alternating knots.

Finally, we begin consider possible restrictions for Gauss diagram to correspond to a valid plane curve. We show that each intersecting chord in the diagram must intersect other chords an even number of times.

2. A BRIEF INTRODUCTION TO PLANE CURVES

2.1. **Basics of plane curves.** We will begin with a few basic definitions.

Definition 2.1. A plane curve is an immersion of S^1 into \mathbb{R}^2 with only transversal double points.

Definition 2.2. The index of a plane curve is the number of complete turns made by the velocity vector [W].

To compute the index of a curve, choose a starting point on the curve and imagine a tangent line to it. As the curve loops about, count the number of times the tangent line makes a complete rotation.

Example 2.3. A simple circle has index 1.

Whitney showed that two plane curves may be regularly homotoped into each other if they have the same index [W].

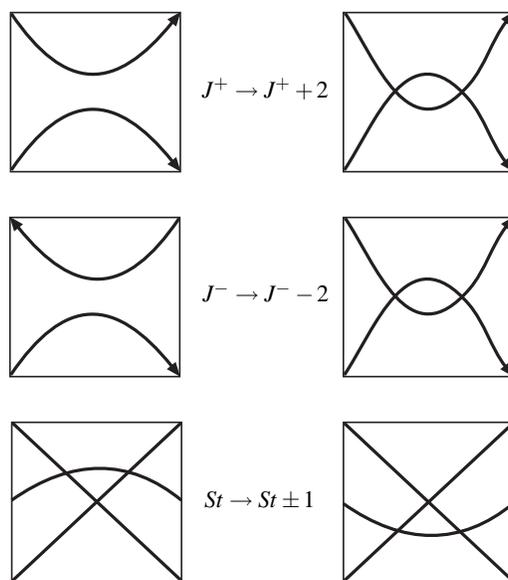


FIGURE 1. Arnold's plane curve moves or *perestroikas*.

Arnold defined three invariants of homotopy classes of plane curves: Strangeness, abbreviated St , J^+ , and J^- [Ar]. Arnold computed these invariants for a plane curve C by assigning values for the invariants to a standard set of plane curves (see Figure 2) with indices $0, 1, 2, \dots$. The curve from the standard set with the same index as C is selected and then homotoped by a series

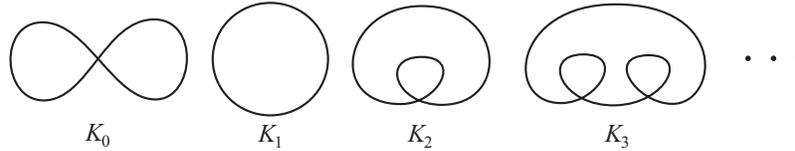


FIGURE 2. The standard set of plane curves.

of moves into C . We will refer to these moves by the name of the invariant they affect, as shown in Figure 2.1. Each move is associated with one of the Arnold invariants and so as the curve is transformed into C , the invariants of C are being computed. For a more complete description of the calculation, the reader is referred to Arnold’s paper [Ar].

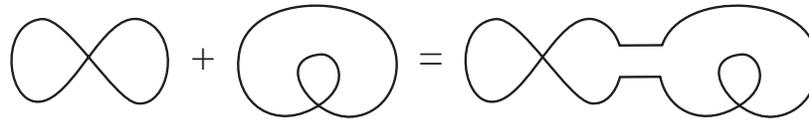


FIGURE 3. The connected sum of two plane curves.

Definition 2.4. [Ar] *The connected sum of two plane curves is the plane curve formed by placing the curves beside each other in the plane, taking a small section from the outside edges of each curve and joining the curves with two line segments as shown in Figure 3*

One property of the Arnold invariants is that they are additive under the connected sum. One must be careful, however, because the connected sum is not defined for all plane curves. For example, two simple circles with opposite orientations cannot be joined using a connected sum.

2.2. Gauss Diagrams. We may associate a simple diagram with each plane curve which gives another way of studying them.

Definition 2.5. *Let C be a plane curve. The Gauss diagram or chord diagram of C is the set of chords on a round circle formed by connecting the preimages of each double point of C .*

See Figure 3 for an example of a plane curve and its associated Gauss diagram.

3. TREE-LIKE PLANE CURVES

Aicardi [Ai] studied a subset of plane curves, called tree-like, for which the Arnold invariants are easily calculated. Later, Luo [Lu] simplified the calculations for these curves and generalized Aicardi’s results for J^+ and J^- .

3.1. When is a Curve Tree-like? The first definition of tree-like is due to Aicardi:

Definition 3.1. [Ai] *A tree-like curve is one in which every double point separates the curve into two closed curves that do not intersect.*

See Figure 4. If any double point on the tree-like curve is cut, the curve is separated into two disjoint curves. However, if the non-tree-like curve is cut at a double point, the remaining curves may still be connected at another double point.

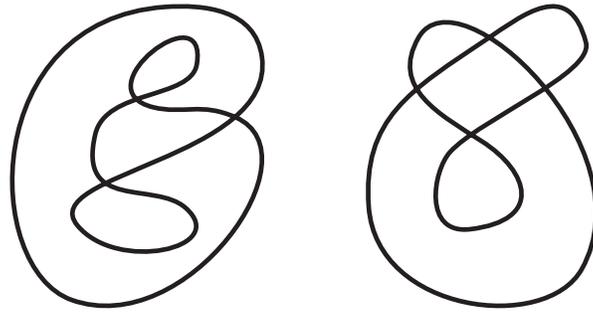


FIGURE 4. Tree-like (left) and non-tree-like (right) curves.

3.2. Properties of Tree-like Curves. Aicardi noted several interesting properties of tree-like curves.

Definition 3.2. [Ai] A planar *Gauss diagram* consists of only non-intersecting chords.

Theorem 3.3. [Ai] A curve is tree-like if and only if its Gauss diagram is planar.

The above theorem provides an easy way of telling whether or not a curve is tree-like.

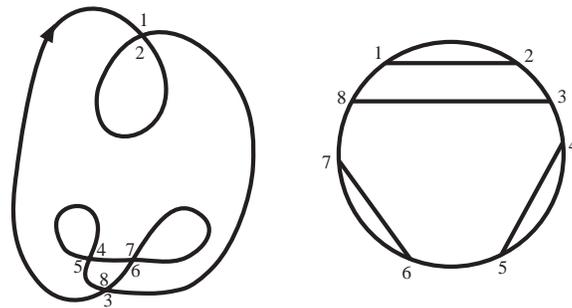


FIGURE 5. A tree-like curve and its associated Gauss diagram.

Example 3.4. In Figure 5 the Gauss diagram associated with the tree-like curve has only non-intersecting chords

Definition 3.5. The defect of a plane curve, denoted here by D , is the quantity $2St + J^+$.

Theorem 3.6. [Ai] The defect of a tree-like plane curve is 0.

The converse of this theorem is not true: if the defect of a curve is 0, it is not necessarily a tree-like curve, as seen in the following example. Aicardi's theorem was later extended by Lin and Wang [L-W] to describe all curves with the property $D = 0$.

Example 3.7. See Figure 6. The plane curve is non-tree-like, as seen by its associated chord diagram, but has defect 0.

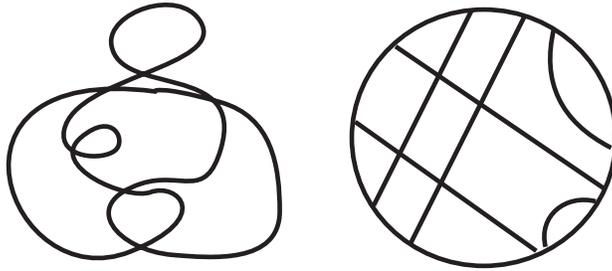


FIGURE 6. A non-treelike curve with defect 0.

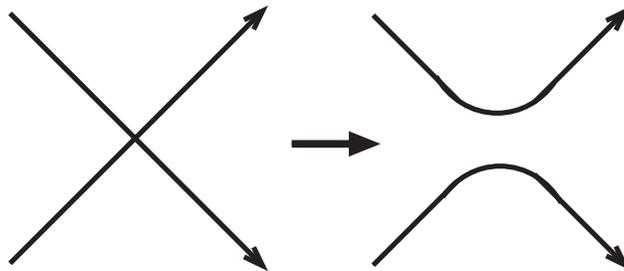


FIGURE 7. Seifert splitting.

3.3. Associating Seifert Cycles with a Plane Curve.

Definition 3.8. [Lu] A Seifert splitting separates a plane curve at its double points as shown in Figure 7.

Definition 3.9. [Lu] Seifert Cycles are the closed, oriented curves that remain after all the double points in a plane curve have been split (see Figure 3.3).

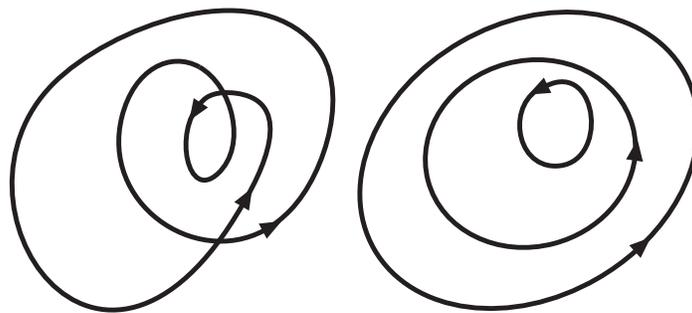


FIGURE 8. A plane curve and its associated Seifert cycles.

Luo simplified Aicardi’s methods of computing Arnold invariants by using Seifert cycles instead of graphs. The following function was first defined by Aicardi, but was redefined in terms of Seifert cycles by Luo.

Definition 3.10. [Lu] The t function is defined as follows on a set of Seifert cycles:

- (1) $t_i = 0$ for any cycle i whose boundary is exterior;
- (2) $t_j = t_i + 1$ if cycles i and j are adjacent and j is inside i ;
- (3) $t_j = -t_i$ if cycles i and j are adjacent and outside each other.

Theorem 3.11. [Lu] *There exists one and only one t function on a set of Seifert cycles.*

Using the t function, we can compute all the Arnold invariants easily for tree-like curves.

Theorem 3.12. [Lu] *Let C be a tree-like plane curve with n double points. Then*

- (1) $St(C) = \sum t_i(C)$,
- (2) $J^+(C) = -2St(C)$,
- (3) $J^-(C) = -2St(C) - n$,

where i is over all Seifert cycles associated with C .

For any plane curve, J^+ and J^- are also easily computed using the t function and the associated set of Seifert cycles.

Theorem 3.13. [Lu] *Let C be a plane curve with n double points. Let s be the number of Seifert cycles associated with C . Then*

- (1) $J^+(C) = 1 + n - s - 2\sum t_i$,
- (2) $J^-(C) = 1 - s - 2\sum t_i$.

Here i is over all Seifert cycles.

Luo's paper lacks a formula in terms of the t function for the strangeness of any plane curve. Using defect, however, we can easily compute the value of St . Later in this paper, we will discuss these alternate methods for this computation.

4. A BRIEF INTRODUCTION TO KNOT THEORY

4.1. Basics of knot theory.

Definition 4.1. *A knot, K , is an embedding of the unit circle into three-space by the mapping $\varphi: S^1 \rightarrow \mathbb{R}^3$.*

We can describe a knot as a closed, non-self-intersecting, continuous curve, which the term embedding implies. This mapping would define a knot of one link whereas multiple mappings of the unit circle into the same space would define knots of multiple links. In this paper, we will examine knots of one link.

$$\begin{array}{ccccc}
 S^1 & \xrightarrow{f} & \mathbb{R}^3 & & \\
 \varphi \downarrow & & & \downarrow & \Psi \\
 S^1 & \xrightarrow{g} & \mathbb{R}^3 & &
 \end{array}$$

FIGURE 9. Diagram for knot equivalence.

Definition 4.2. [D] *Two knots, f and g , are equivalent if there is a commutative diagram, as shown in Figure 4.1, where φ and ψ are orientation preserving diffeomorphisms of the circle S^1 and \mathbb{R}^3 , respectively.*

By this definition, the composition of functions, $\psi \circ f$ and $g \circ \phi$ lead to an equivalent embedding in \mathbb{R}^3 . With small changes to this definition, we can reach one that satisfies embeddings from one space into any Euclidean space. In 1926, however, Reidemeister [R] refined the more generic definition of equivalent embeddings to knots. His definition allows us to visually consider a diffeomorphism of a knot. But we must first understand knots visually before we can consider Reidemeister's theorem. We can do so with knot projections.

Definition 4.3. *A knot projection is a projection of a knot onto the plane that indicates its over-under crossings.*

For instance, we have such a projection in Figure 10.

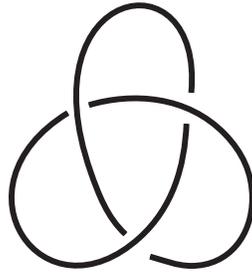


FIGURE 10. A projection of the trefoil knot.

In describing these projections or the knots themselves, we will use the terms overstrand and understrand to designate portions of a knot at each crossing, and we will understand these terms in their most literal sense. In this way, the overstrand is the portion of the knot that passes over the understrand at a particular crossing. One should note, however, that these terms only address those overlapping strands at a crossing.

For our work, it is necessary to understand the manner in which we may describe these crossings.

Definition 4.4. *Let i correspond to a crossing of a knot. Then $\text{sign}(i)$ will be either $+1$ or -1 , as defined in Figure 11.*

To compute the sign of a crossing, first choose an orientation for the knot (i.e. give the knot a direction by which one might trace the figure). Then, a crossing has $\text{sign}(i) = 1$ if one can rotate the understrand of the knot clockwise to align it with the orientation of the overstrand. Alternatively, one must rotate the understrand counterclockwise if a crossing has $\text{sign}(i) = -1$.

4.2. Characterizing and identifying knots. Suppose that we have two different projections of the same knot. With the knowledge that they are the same knot, we can deform one projection into the other. Generically, an isotopy is a homotopy of one embedding where its properties are preserved throughout the mapping. Specifically, to knots, we have the following definition.

Definition 4.5. [Ad] Ambient isotopy is the deformation of the knot in \mathbb{R}^3 .

Consider the knot as composed of an infinitely flexible rubber. Then we can bend and stretch the knot in any fashion, but it is not possible to shrink portions of the knot to singular points (see Figure 12). We can discuss two types of ambient isotopies for knots: planar isotopy and Reidemeister moves.

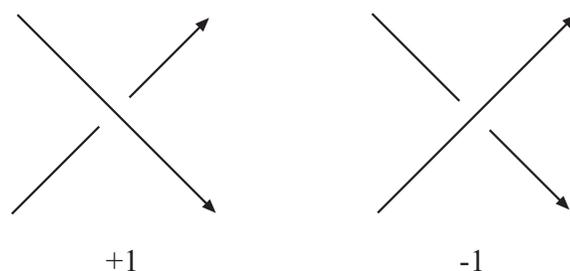


FIGURE 11. Types of crossings.

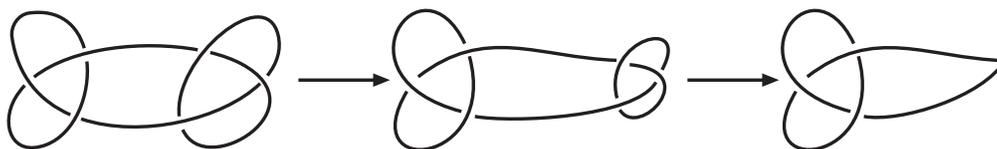


FIGURE 12. Portions of a knot may not be shrunk to single points.

Definition 4.6. Planar isotopy is a deformation of a knot projection that leaves every crossing intact.

Definition 4.7. A Reidemeister move is one of three ways in which to change a knot projection as defined in Figure 13.

We will refer to these deformations as type I, II, and III moves.

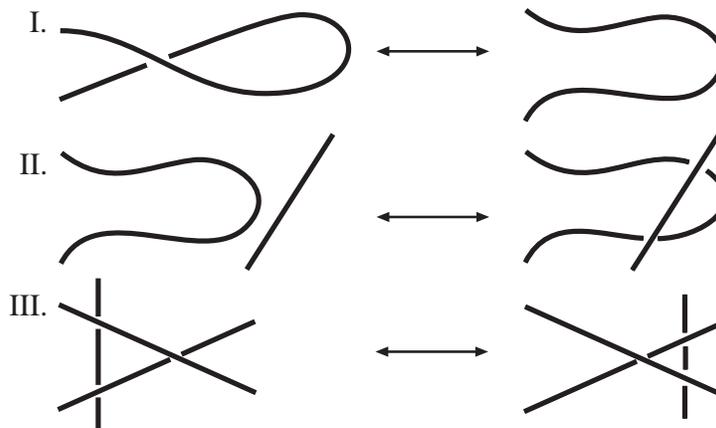


FIGURE 13. The Reidemeister moves.

When Reidemeister first studied these moves, he proved the following theorem:

Theorem 4.8. [R] Two knots are equivalent (as in Definition 4.2) if and only if a projection of the first knot can be deformed into a projection of the second by a finite number of Reidemeister moves.

This theorem is somewhat analogous to Whitney's result for plane curves with the distinction that a knot's index (or winding number) is not invariant under Reidemeister moves. The type I move will alter a knot projection's index by ± 1 . Thus the classification of knots is more difficult than for plane curves and continues to be studied. These pursuits have taken the form of several different invariants for knots.

Definition 4.9. *A knot invariant is a property of a knot that does not change under ambient isotopy.*

Alternatively, a knot invariant is independent of the projections of the same knot. Hence, if we compute an invariant, a , for two different knots, K_1 and K_2 . If $a_1 \neq a_2$, then $K_1 \neq K_2$. The inverse, however, is not generally true, and two knots with equal invariants need not necessarily be equivalent. In other words, proving that a given invariant uniquely classifies all knots is still an open question. Currently, the most effective known invariants are knot polynomials and Vassiliev invariants.

Definition 4.10. *A knot polynomial is an invariant for a knot in the form of a polynomial whose coefficients encode properties of the given knot and are the same for equivalent knots.*

Some polynomial invariants are the Jones, Alexander, HOMFLY, Conway, and Kauffman polynomials. Each has its strengths and weaknesses in its ability to characterize knots. For instance, every prime knot of nine or fewer crossings has a unique Jones polynomial. Finding a knot invariant, however, that uniquely classifies all knots of n -crossings is still an open problem. We will not detail the derivation of these polynomials; should a fuller treatment be desired, one may consult [Ad], chapter 6.

5. INTEGRAL GEOMETRY OF PLANE CURVES AND KNOT INVARIANTS

Knot invariants may be formed from integrals. Lin and Wang [L-W] give two integrals such that the difference between them is a Vassiliev knot invariant of second order. This quantity is also related to the second coefficient of the Conway polynomial mentioned in the preceding section. One of the integrals produced by Lin and Wang may be considered a generalized form of Crofton's Formula [S] for convex plane curves. This integral, denoted I_Y , may be used to create plane curve invariants that are related to Arnold's invariants for plane curves.

5.1. Integral knot invariants. To begin, we need some basic definitions from differential geometry.

Definition 5.1. [NE] *A differential form, ω , on \mathbb{R}^n , denoted $\omega \in \Lambda(\mathbb{R}^n)$, is a combination obtained by adding and multiplying real-valued functions and the differentials dx_1, dx_2, \dots, dx_n , where dx_1, dx_2, \dots, dx_n are the differentials of the natural coordinate functions of \mathbb{R}^n and multiplication is defined so that $dx_i dx_j = -dx_j dx_i$ for any $i, j \in \{1, 2, \dots, n\}$. If each summand of ω has p differentials, then ω is called a p -form. If ω is an n -form, then ω is called a top-dimensional form and denoted $\omega \in \Lambda^n(\mathbb{R}^n)$.*

A differential form can be thought of as a combination formed by adding and multiplying $x \in \mathbb{R}^n$ and differentials dx_1, dx_2, \dots, dx_n . Two forms may be multiplied together using the *wedge product*, which is the usual associative or distributive multiplication with the added condition of antisymmetry. Notice the antisymmetry restriction in Definition 5.1 implies that if $i = j$, then $dx_i dx_i = 0$. Thus if ω is a q -form in \mathbb{R}^n and $q > n$ then $\omega = 0$.

Example 5.2. Let $\omega = dx dy + dx dz + dy dz$, where $(x, y, z) \in \mathbb{R}^3$. Then ω is a 2-form on \mathbb{R}^3 .

We can compute the wedge product, denoted by \wedge or understood to be the operation between two differentials, of two forms by carrying out the usual operation of multiplication with the condition of antisymmetry.

Example 5.3. Suppose $\omega = dx dy + dx dz$ and $\tau = dx + dy$ are 2 and 1-forms, respectively, in \mathbb{R}^3 . Then $\omega \wedge \tau$ is $(dx dy + dx dz) \wedge (dx + dy)$ which after usual multiplication becomes $dx dy dx + dx dz dx + dy dz dx + dx dy dy$. Each term with a repeated differential in it is 0 so the product becomes $dy dz dx$ which under the antisymmetry condition is $dx dy dz$.

Notice that when a q -form is wedged with a p -form, a $(q + p)$ -form results.

We can compute derivatives of forms using the exterior derivative.

Definition 5.4. [NE] If $\omega_i = f_i dx_1 dx_2 \dots dx_p$ is a summand of the p -form $\omega \in \Lambda(\mathbb{R}^n)$ and f_i is a mapping from \mathbb{R}^n then the exterior derivative of ω , $d\omega$, is $\sum_i d\omega_i$ where $d\omega_i = df_i \wedge dx_1 dx_2 \dots dx_p$. Here df_i denotes the differential of f_i , given by $df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$.

Example 5.5. Let $\omega = x^2 z^2 dx dy + z dy dz + x z^3 dx dz$. Then $d\omega = d(x^2 z^2) \wedge dx dy + d(z) \wedge dy dz + d(x z^3) \wedge dx dz$. Performing the differentiation, we have $(2xz^2 dx + 2x^2 z dz) \wedge dx dy + dz \wedge dy dz + (z^3 dx + 3xz^2 dz) \wedge dx dz$. Next we compute the wedge products and find that all terms except $2x^2 z dz dx dy$ cancel, which under antisymmetry is $2x^2 z dx dy dz$.

Our definition for a differential form holds for Euclidean spaces; however, forms may also be defined for surfaces and, more generally, for manifolds. We will need the following definition for forms on manifolds.

Definition 5.6. [NE] A form ω is exact if $\omega = d\tau$ for some form τ .

Differential forms are useful for describing geometry because they may be moved from one manifold to another by a mapping known as a *pullback*. If F is a mapping between manifolds, then we denote the pullback of F as F^* . Pullbacks may be used to change variables in integrals. A simple example from calculus is the change of variables from cartesian coordinates to polar coordinates used to compute an integral. In general, if M and N are surfaces, F is a mapping from M to N , and $\omega \in \Lambda(N)$ then $\int_{F(M)} \omega = \int_M F^*(\omega)$.

The following is a special case of Stokes Theorem.

Theorem 5.7. If M is a compact manifold and $\tau \in \Lambda(M)$, then $\int_M d\tau = 0$.

Suppose that M and N are compact, orientable manifolds, $\dim(N) = n$, and that F_t is a 1-parameter family of mappings from M to N . Let $\omega \in \Lambda^n(N)$. Using a theorem from De Rham cohomology, we see that $F_1^*(\omega) - F_0^*(\omega)$ is an exact form. Using Theorem 5.7 we have $\int_M F_1^*(\omega) = \int_M F_0^*(\omega)$. Suppose for a plane curve C we associate a mapping F_C with its image in N and choose $\omega \in \Lambda^n(N)$. Define $I_C = \int_M F_C^*(\omega)$. I_C will be an invariant under isotopy of C if we can construct F_C so that if C_0 and C_1 are two homotopic curves, then F_{C_0} and F_{C_1} are also homotopic. It would then follow that $I_{C_0} = I_{C_1}$ and I_C , if nontrivial, would be an invariant of curves.

Lin and Wang use the method described above to construct a knot invariant and relate it to Arnold's plane curve invariants.

We will use the permutation symbol summed over all permutations of its indices to denote forms in \mathbb{R}^n .

Definition 5.8. The permutation symbol, denoted by $\varepsilon_{ijk\dots l}$, is defined by

$$\varepsilon_{ijk\dots l} = \begin{cases} -1 & \text{if } ijk\dots l \text{ is a odd permutation of } \{1, 2, \dots, n\}; \\ 1 & \text{if } ijk\dots l \text{ is an even permutation of } \{1, 2, \dots, n\}; \\ 0 & \text{if any two indices of } \{1, 2, \dots, n\} \text{ are repeated.} \end{cases}$$

Example 5.9. We can denote the cross product between two vectors v and $w \in \mathbb{R}^3$ by $v \times w = \varepsilon_{ijk} v^j w^k$ where $v = (v^1, v^2, v^3)$ and $w = (w^1, w^2, w^3)$ because $v \times w = (v^2 w^3 - w^2 v^3, w^1 v^3 - v^1 w^3, v^1 w^2 - w^1 v^2)$.

Definition 5.10. (Lin and Wang) Let $x \in \mathbb{R}^3 \setminus \{0\}$. Let

$$\omega(x) = \frac{1}{4\pi} \frac{\varepsilon_{ijk} x^i dx^j dx^k}{|x|^3}$$

denote the unit area form of S^2 . Suppose that $\gamma: S^1 \rightarrow \mathbb{R}^3$ is a smooth imbedding, where S^1 is identified with \mathbb{R}/\mathbb{Z} . Denote

$$\Delta_4 = \{(t_1, t_2, t_3, t_4); 0 < t_1 < t_2 < t_3 < t_4 < 1\}$$

and

$$\Delta_3(\gamma) = \{(t_1, t_2, t_3, z) : 0 < t_1 < t_2 < t_3 < 1, z \in \mathbb{R}^3 \setminus \{\gamma(t_1), \gamma(t_2), \gamma(t_3)\}\}$$

Define

$$I_X(\gamma) = \int_{\Delta_4} \omega(\gamma(t_3) - \gamma(t_1)) \wedge \omega(\gamma(t_4) - \gamma(t_2))$$

and

$$I_Y(\gamma) = \int_{\Delta_3(\gamma)} \omega(z - \gamma(t_1)) \wedge \omega(z - \gamma(t_2)) \wedge \omega(z - \gamma(t_3))$$

Throughout the paper we will assume the above definition when we refer to I_X or I_Y .

Proposition 5.11. [L-W] We have

$$I_X(\gamma) = -\frac{1}{2\pi^2} \int_{\Delta_4} \frac{[\gamma(t_3) - \gamma(t_1), \dot{\gamma}(t_3), \dot{\gamma}(t_1)]}{|\gamma(t_3) - \gamma(t_1)|^3} \cdot \frac{[\gamma(t_4) - \gamma(t_2), \dot{\gamma}(t_4), \dot{\gamma}(t_2)]}{|\gamma(t_4) - \gamma(t_2)|^3} dt_1 dt_2 dt_3 dt_4.$$

Let

$$E(z, t) = \frac{(z - \gamma(t)) \times \dot{\gamma}(t)}{|z - \gamma(t)|^3}.$$

Then

$$I_Y(\gamma) = -\frac{1}{2\pi^3} \int_{\Delta_3(\gamma)} [E(z, t_1), E(z, t_2), E(z, t_3)] d^3 z dt_1 dt_2 dt_3.$$

Proof. Both formulas follow from computing the wedge products in I_X and I_Y from Definition 5.10 and noting that the triple scalar product, denoted $[a, b, c]$, is equal to $\varepsilon_{ijk} a^i b^j c^k$ for $a, b, c \in \mathbb{R}^3$. QED

Theorem 5.12. (Lin and Wang) Let $\gamma: S^1 \rightarrow \mathbb{R}^3$ be a smooth imbedding and let $v_2(\gamma) = I_X(\gamma) - I_Y(\gamma)$. Then v_2 is invariant under isotopy of γ .

Lin and Wang [L-W] computed that if γ is a simple closed plane curve then $I_X(\gamma)$ is 0. By Theorem 5.12, $I_Y(\gamma)$ is invariant under deformation of γ because v_2 is invariant under deformation of γ . The following theorem gives the value of a multiple of $I_Y(\gamma)$ evaluated on the round circle.

Theorem 5.13. (Lin and Wang) (Generalized Crofton's Formula for convex plane curves) Let $\gamma(t) : S^1 \rightarrow \mathbb{R}^2$ be a simple closed plane curve in \mathbb{R}^2 . Then

$$\int_{\Delta_3} \int_{z \in \mathbb{R}^3} \frac{V}{\prod_{i=1}^3 |z - \gamma(t_i)|^3} d^3z dt_1 dt_2 dt_3 = \frac{\pi^3}{6}$$

where V is the oriented volume of the parallelepiped spanned by $(z - \gamma(t_i)) \times \dot{\gamma}(t_i)$, $i = 1, 2, 3$.

Notice that if a singularity, such as a double point, occurs as the image of γ is deformed in \mathbb{R}^3 , then I_Y may change in value. If we limit γ to a plane curve, we may thus relate I_Y to Arnold's invariants, which also describe how curves will change upon addition of double points.

Proposition 5.14. [L-W] $v_2(\text{unknot}) = -\frac{1}{24}$.

5.2. New Invariants. We may create new invariants by defining integrals as in Definition 5.10. Lin and Wang [L-W] choose γ to be an embedding of S^1 into \mathbb{R}^3 ; we will consider cases where γ is an embedding of S^2 into \mathbb{R}^q , $q \in \mathbb{N}$. Alternatively one might consider embedding T^2 , or any compact 2-manifold, into \mathbb{R}^q .

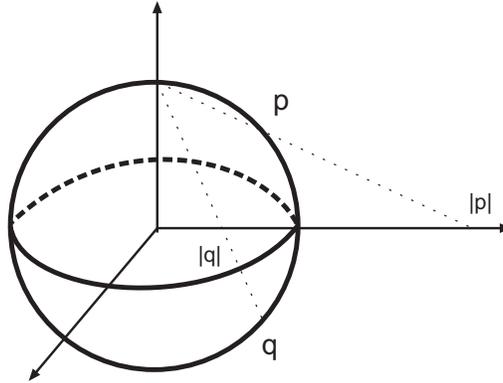


FIGURE 14. $|q| < |p|$

Note. Suppose $p, q \in S^2$. We will use a stereographic projection of S^2 . Thus $|q| < |p|$ if when p and q are projected onto the plane by a chord from the north pole of S^2 centered at the origin of \mathbb{R}^3 , as shown in Figure 14, then the distance from the origin to q is less than the distance from the origin to p . Notice that points near the top of the sphere are thus considered larger than points on the bottom of the sphere.

We will first consider the case where S^2 is embedded in \mathbb{R}^3 .

Definition 5.15. Let $\gamma : S^2 \rightarrow \mathbb{R}^3$ be a smooth embedding. Let $x \in \mathbb{R}^3 / \{0\}$. Let

$$\omega_2(x) = \frac{1}{4\pi} \frac{\varepsilon_{ijk} x^i dx^j dx^k}{|x|^3}$$

be the unit area form of S^2 . Let $\Delta_6 = \{(t_1, t_2, t_3) : t_1, t_2, t_3 \in S^2, 0 \leq |t_1| < |t_2| < |t_3|\}$. Define

$$I_A = \int_{\Delta_6} \omega_2(\gamma(t_2) - \gamma(t_1)) \wedge \omega_2(\gamma(t_3) - \gamma(t_2)) \wedge \omega_2(\gamma(t_1) - \gamma(t_3))$$

Notice that ω_2 is a 2-form so $\omega_2 \wedge \omega_2 \wedge \omega_2$ is a 6-form, which is a top-dimensional form on $S^2 \times S^2 \times S^2$.

Next we will consider the case when S^2 is embedded in \mathbb{R}^4 . We found two possible integral invariants.

Definition 5.16. Let $\gamma: S^2 \rightarrow \mathbb{R}^4$ be a smooth embedding. Let $x \in \mathbb{R}^4 / \{0\}$. Let

$$\omega_3(x) = \frac{1}{2\pi^2} \frac{\varepsilon_{ijkl} x^i dx^j dx^k dx^l}{|x|^4}$$

be the unit area form of S^3 . Let Δ_6 be as in Definition 5.15 and $\Delta_{4(\gamma)} = \{(t_1, t_2, t_3, t_4, z) : t_1, t_2, t_3, t_4 \in S^2, 0 \leq |t_1| < |t_2| < |t_3| < |t_4|, z \in \mathbb{R}^4 \setminus \{\gamma(t_1), \gamma(t_2), \gamma(t_3), \gamma(t_4)\}\}$ Define

$$I_{B_1} = \int_{\Delta_6} \omega_3(\gamma(t_3) - \gamma(t_1)) \wedge \omega_3(\gamma(t_3) - \gamma(t_2))$$

and

$$I_{B_2} = \int_{\Delta_{4(\gamma)}} \omega_3(z - \gamma(t_1)) \wedge \omega_3(z - \gamma(t_2)) \wedge \omega_3(z - \gamma(t_3)) \wedge \omega_3(z - \gamma(t_4))$$

Proposition 5.17. Let $\gamma: S^2 \rightarrow \mathbb{R}^4$ be a smooth embedding. Then

$$I_{B_1} = \frac{9}{\pi^4} \int_{\Delta_6} \frac{V_2}{|(\gamma(\theta_3, \phi_3) - \gamma(\theta_1, \phi_1))|^4 \cdot |(\gamma(\theta_3, \phi_3) - \gamma(\theta_2, \phi_2))|^4},$$

where

$$\begin{aligned} V_2 = & \varepsilon_{i_1 j_1 k_1 l_1} \varepsilon_{i_2 j_2 k_2 l_2} (\gamma^{i_1}(\theta_3, \phi_3) - \gamma^{i_1}(\theta_1, \phi_1)) (\gamma^{j_2}(\theta_3, \phi_3) - \gamma^{j_2}(\theta_2, \phi_2)) \\ & (\gamma^{k_1} \gamma_{\phi_1}^{l_1} \gamma_{\theta_2}^{k_2} \gamma_{\phi_2}^{l_2} \gamma_{\theta_3}^{j_1} \gamma_{\phi_3}^{j_2} d^{k_1} \theta_1 d^{l_1} \phi_1 d^{k_2} \theta_2 d^{l_2} \phi_2 d^{j_1} \theta_3 d^{j_2} \phi_3 \\ & - \gamma_{\theta_1}^{k_1} \gamma_{\phi_1}^{l_1} \gamma_{\theta_2}^{k_2} \gamma_{\phi_2}^{l_2} \gamma_{\theta_3}^{j_2} \gamma_{\phi_3}^{j_1} d^{k_1} \theta_1 d^{k_1} \phi_1 d^{l_1} \theta_2 d^{k_2} \phi_2 d^{l_2} \theta_3 d^{j_1} \phi_3), \end{aligned}$$

Proof. Suppose that $t_1, t_2, t_3 \in S^2$ are denoted by the coordinates (θ_1, ϕ_1) , (θ_2, ϕ_2) and (θ_3, ϕ_3) respectively. Note that for $a, b \in \{1, 2, 3\}$, $d(\gamma(t_a) - \gamma(t_b)) = d(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$, which becomes

$$\gamma_{\theta_a} d\theta_a + \gamma_{\phi_a} d\phi_a - \gamma_{\theta_b} d\theta_b - \gamma_{\phi_b} d\phi_b$$

where γ_{θ} and γ_{ϕ} denote the partial derivatives of γ with respect to θ and ϕ respectively.

Now

$$\omega_4(\gamma(t_a) - \gamma(t_b)) = \frac{3}{\pi^2} \frac{\varepsilon_{ijkl} (\gamma^i(\theta_a, \phi_a) - \gamma^i(\theta_b, \phi_b)) (N)}{|\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b)|^4},$$

where

$$N = -\gamma_{\theta_a}^j \gamma_{\phi_a}^k \gamma_{\theta_b}^l d\theta_a d\phi_a d\theta_b - \gamma_{\theta_a}^j \gamma_{\phi_a}^k \gamma_{\phi_b}^l d\theta_a d\phi_a d\phi_b + \gamma_{\theta_a}^j \gamma_{\theta_b}^k \gamma_{\phi_b}^l d\theta_a d\theta_b d\phi_b + \gamma_{\phi_a}^j \gamma_{\theta_b}^k \gamma_{\phi_b}^l d\phi_a d\theta_b d\phi_b$$

Consider wedging together $\omega_4(\gamma(t_3) - \gamma(t_1))$ and $\omega_4(\gamma(t_3) - \gamma(t_1))$. Both of these forms will have $a = t_3$ in the above form. Thus the summands in N with two a 's will cancel out when the two forms are wedged. What remains from multiplying N_1 and N_2 is thus

$$\begin{aligned} & 9\gamma_{\theta_1}^{k_1}\gamma_{\phi_1}^{l_1}\gamma_{\theta_2}^{k_2}\gamma_{\phi_2}^{l_2}\gamma_{\theta_3}^{j_1}\gamma_{\phi_3}^{j_2}d^{k_1}\theta_1d^{l_1}\phi_1d^{k_2}\theta_2d^{l_2}\phi_2d^{j_1}\theta_3d^{j_2}\phi_3 \\ & -9\gamma_{\theta_1}^{k_1}\gamma_{\phi_1}^{l_1}\gamma_{\theta_2}^{k_2}\gamma_{\phi_2}^{l_2}\gamma_{\theta_3}^{j_1}\gamma_{\phi_3}^{j_2}d^{k_1}\theta_1d^{l_1}\phi_1d^{k_2}\theta_2d^{l_2}\phi_2d^{j_1}\theta_3d^{j_2}\phi_3, \end{aligned}$$

and our result follows. QED

Finally we consider S^2 embedded in \mathbb{R}^5 .

Definition 5.18. Let $\gamma: S^2 \rightarrow \mathbb{R}^5$ be a smooth embedding. Let $x \in \mathbb{R}^5 \setminus \{0\}$. Let

$$\omega_4(x) = \frac{\varepsilon_{ijklm}x^i dx^j dx^k dx^l dx^m}{|x|^5}$$

be the area form of S^4 . Let

$$\Delta_8 = \{(t_1, t_2, t_3, t_4) : t_1, t_2, t_3, t_4 \in S^2, 0 \leq |t_1| < |t_2| < |t_3| < |t_4|\}$$

and

$$\Delta_{16(\gamma)} = \{(t_1, t_2, t_3, z_1, z_2) : t_1, t_2, t_3 \in S^2, 0 \leq |t_1| < |t_2| < |t_3|, z_1, z_2 \in \mathbb{R}^5 \setminus \{\gamma(t_1), \gamma(t_2), \gamma(t_3)\}\}$$

Define

$$I_{C_1} = \int_{\Delta_8} \omega_5(\gamma(t_2) - \gamma(t_1)) \wedge \omega_4(\gamma(t_4) - \gamma(t_3))$$

and

$$I_{C_2} = \int_{\Delta_{16(\gamma)}} \omega_4(z_1 - \gamma(t_1)) \wedge \omega_4(z_1 - \gamma(t_2)) \wedge \omega_4(z_2 - \gamma(t_2)) \wedge \omega_4(z_2 - \gamma(t_3))$$

Proposition 5.19. Let $\gamma: S^2 \rightarrow \mathbb{R}^5$ be a smooth embedding. Then

$$I_{C_1} = (24)^2 \int_{\Delta_8} \frac{V_4}{|\gamma(\theta_2, \phi_2) - \gamma(\theta_1, \phi_1)|^5 \cdot |\gamma(\theta_4, \phi_4) - \gamma(\theta_3, \phi_3)|^5}$$

where

$$\begin{aligned} V_4 &= \varepsilon_{i_1 j_1 k_1 l_1 m_1} \varepsilon_{i_2 j_2 k_2 l_2 m_2} (\gamma^{i_1}(\theta_2, \phi_2) - \gamma^{i_1}(\theta_1, \phi_1)) (\gamma^{j_2}(\theta_4, \phi_4) - \gamma^{j_2}(\theta_3, \phi_3)) \\ & \quad (\gamma^{l_1}(\theta_1, \phi_1) - \gamma^{l_1}(\theta_2, \phi_2)) (\gamma^{m_2}(\theta_3, \phi_3) - \gamma^{m_2}(\theta_4, \phi_4)). \end{aligned}$$

Proof. Note that in the computation of $\omega_4(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$ the only terms remaining after computing the wedge products between $d^j(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$, $d^k(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$, $d^l(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$ and $d^m(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b))$ are those that have one of each of $d\theta_a$, $d\phi_a$, $d\theta_b$ and $d\phi_b$. By exchanging indices and using the antisymmetry property of the wedge product we obtain

$$\omega_4(\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b)) = \frac{24\varepsilon_{ijklm}(\gamma^i(\theta_a, \phi_a) - \gamma^i(\theta_b, \phi_b))\gamma_{\theta_b}^j \gamma_{\phi_b}^m \gamma_{\theta_a}^k \gamma_{\phi_a}^l d\theta_b d\phi_b d\theta_a d\phi_a}{|\gamma(\theta_a, \phi_a) - \gamma(\theta_b, \phi_b)|^5}.$$

Noting that the two wedge products in I_{C_1} share no common variables, our result follows. QED

It remains to be seen if the integrals defined in this section yield an invariant.

5.3. Knot Invariants in the Plane. From the computation that $v_2(\text{unknot}) = -1/24$, Lin and Wang develop some interesting limiting forms of the integral knot invariants, I_Y and I_X . For I_Y , we will consider the computation on a projection of K onto the plane. We cannot calculate I_X in the plane because the integral blows up when double points arise in the projection. For this reason, we will employ the following notion of a limiting plane curve.

Definition 5.20. Let K_ϵ be the knot whose crossings are over-passes of radius ϵ and whose remaining parts lie completely within the plane.

Alternatively, we can first consider flattening a knot, K , onto the plane except at crossings. The crossings are projected by placing the understrand of each crossing in the plane and the overstrand on a half-circle of radius ϵ that lies perpendicular to the plane, as shown in Figure 15. If we let ϵ approach zero, we refer to the resulting diagram by K_0 , the limiting plane curve of K_ϵ . To avoid confusion, this definition for K_0 is not related to the generic plane curve, K_0 of index 0. Should we need to reference that plane curve, we will make a special note. Otherwise, we will use K_0 to refer to the limiting plane curve of a knot.

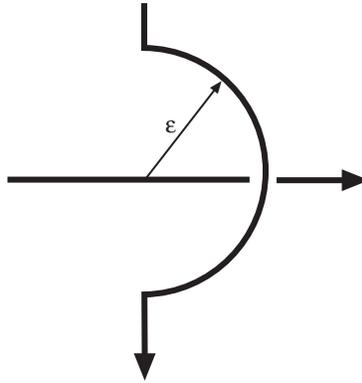


FIGURE 15. A local picture of K_ϵ at a crossing

Given a knot K , we can create a signed chord diagram. To each chord in the chord diagram of K , we associate the sign of the corresponding crossing given by the knot projection. For each pair of intersecting chords in the chord diagram of K , we will give the appropriate sign, which is the product of the respective signs of the two chords. We will employ this notion of signing when considering K_0 .

Definition 5.21. Let c_+ (respectively c_-) be sum of intersecting chords in the chord diagram of K_0 whose products are positive (respectively, negative).

Example 5.22. In Figure 16, $c_+ = 0$ and $c_- = 4$.

Theorem 5.23. [L-W] (4.3) Let K be a knot diagram with n crossings with K_ϵ and K_0 defined as above. Then the limit

$$\lim_{\epsilon \rightarrow 0} K_\epsilon = K_0$$

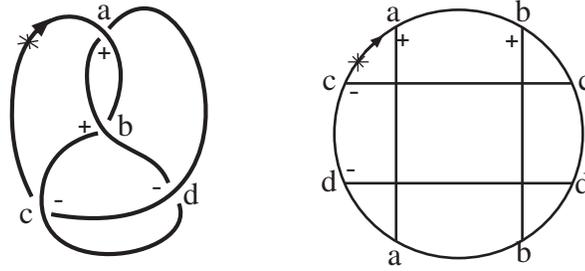


FIGURE 16. A figure-eight knot and its associated Gauss diagram

exists (see Figure 15), and

$$I_X(K_0) = \frac{n}{16} + \frac{(c_+ - c_-)}{4},$$

where c_+ and c_- are as in Definition 5.21.

Using merely chord diagrams, however, the computation of c_{\pm} can become cumbersome. Thus, to simplify the process, we will define a new value that can replace c_{\pm} . But first, we will have to address some intermediate ideas and new notation.

First, we will define the mathematical meaning of an alphabet and words. Let an alphabet be a set of *letters* from which we can draw a subset. Then, a sequence of letters is called a word. In each knot (or plane curve), all n -crossings (or double points) are letters in our alphabet.

Definition 5.24. Given an orientation and base point for a signed plane curve, K_0 , let P define a word onto which the preimages of all double points are mapped.

By this definition, we can form $2n$ -ary words of length $2n$.

Remark. This definition is identical for knots, and we will use words for both knots and plane curves interchangeably. Keeping a base point and orientation intact, this permutation is an invariant under projection onto the plane and lifting from the plane into \mathbb{R}^3 .

Example 5.25. Given K in Figure 16, we have this alphabet: $(a b c d)$. And from our alphabet, we can form the following word:

$$P(K) = \begin{pmatrix} a & b & c & d & b & a & d & c \\ + & + & - & - & + & + & - & - \end{pmatrix}.$$

Note that we have incorporated the sign of each crossing in $P(K)$. Although this step is optional, it can speed up computations that employ this notation. We will use this notation to define a new function on our knots or plane curves. Given a double point $i \in K_0$, there are exactly two instances of $i \in P(K_0)$. Then, the *betweenness* function on i gives all preimages of double points in K_0 that lie between both instances of i . If we consider a in Figure 16, the following preimages of crossings are between both instances of $a \in P(K)$: $\{b c d b\}$.

Definition 5.26. Define $b(i)$ as the betweenness function which maps i to the set of all single instances of double point preimages between $i \in P$. For two generic double points, $p, q \in K_0$, $q \in b(p)$ if and only if $P = (\cdots p \cdots q \cdots p \cdots q \cdots)$ or $(\cdots q \cdots p \cdots q \cdots p \cdots)$. And $q \notin b(p)$ if and only if $P = (\cdots p \cdots q \cdots q \cdots p \cdots)$ or $(\cdots q \cdots p \cdots p \cdots q \cdots)$.

Example 5.27. For K in Figure 16, $b(a) = \{c d\}$.

Definition 5.28. Define d_i by

$$d_i = \text{sign}(i) \sum \text{sign}(b(i))$$

where the summation is computed over all the signed crossings of the double points in the output of $b(i)$.

Note that the computation of d_i is invariant under both orientation of the curve and placement of a base point. For a given double point, i , there are two entries for i in P . And between each i in P , there exists the same number of singularly appearing entries as do outside each i in P because of the manner in which the permutation is defined. For this reason, the base point will not affect $b(i)$. Similarly, the orientation of the curve will not affect the computation as the betweenness for each i in P is equal to that of i in P^{-1} .

Lemma 5.29. Given K_0 ,

$$c_+ - c_- = \sum \frac{d_i}{2}.$$

Proof. Consider two double points of K_0 , m and n with associated chords M and N in the chord diagram D of K_0 . Without loss of generality, let the base point on K_0 be placed just before m with orientation moving towards this intersection. If M and N do not intersect, the curve must pass through m twice before reaching n . In the permutation form of K_0 , this path is: $(m \cdots m \cdots n \cdots n \cdots)$. From our previous definition of $b(i)$, $b(m) \neq n$. Thus in the computation of both $\sum d_i/2$ and $c_+ - c_-$ are invariant under non-intersection of any two chords in D of K_0 . To complete this proof, assume that

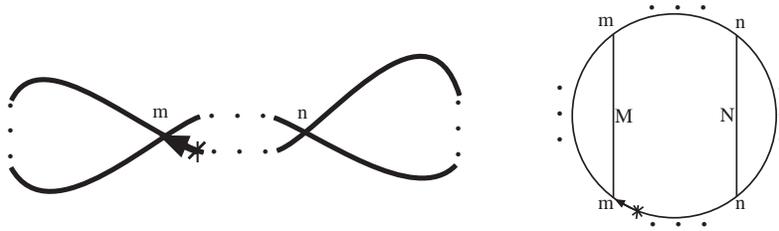


FIGURE 17. K_0 and its associated chord diagram illustrate the case of non-intersecting chords.

we have counted every non-intersecting chord in D of K_0 . Then,

$$c_+ - c_- = \sum \frac{d_i}{2} = 0.$$

By way of contradiction, assume that there exists at least one pair of double points $p, q \in K_0$ with associated chords $P, Q \in D$ such that P intersect Q implies $c_+ - c_- \neq \sum d_i/2$. Before the inclusion of p, q in our computation, let $c_+ - c_- = \sum d_i/2 = k$ for some $k \in \mathbb{Z}$. Then P intersect Q implies $\text{sign}(p)\text{sign}(q) = \pm 1$. First consider the positive case. Then $c_+ - c_- = k + 1$. By our earlier

argument, the associated permutation of K_0 must be $(\cdots p \cdots q \cdots p \cdots q \cdots)$, which implies the following equality:

$$\sum \frac{d_i}{2} = \frac{1}{2}(2k + \text{sign}(p)\text{sign}(q) + \text{sign}(q)\text{sign}(p)),$$

where $\text{sign}(p)\text{sign}(q) = +1$ by our designation. Then, $\sum d_i/2 = k + 1$ which is a contradiction. The case for which $\text{sign}(p)\text{sign}(q) = -1$ follows by an identical argument.

Therefore, given any two pairs of double points in K_0 , $c_+ - c_- = \sum d_i/2$. QED

Note. From this result, we can write I_X in an alternative form:

$$I_X(K_0) = \frac{n}{16} + \frac{(c_+ - c_-)}{4} = \frac{n}{16} + \frac{\sum d_i}{8}.$$

Just as we can flatten knots into the plane by the limiting notion of K_ϵ , we can also raise plane curves by arbitrating crossings with the usual notion of signing with a ± 1 . In this manner, we can move at will from the plane into three-space, and if we maintain the original crossing indices of a knot, we can consider a knot as a plane curve with signed crossings. Furthermore, we can lift that plane curve back to \mathbb{R}^3 with additional information as to its formation. The following corollary of [L-W] follows from this understanding and Definition 5.21.

Corollary 5.30. [L-W](4.4) *Let C be an immersed circle in the plane with only transverse double points.*

- (1) *If C is resolved to two knots K^1 and K^2 of the same knot type, then $I_X(K_\epsilon^1)$ and $I_X(K_\epsilon^2)$ have the same limit.*
- (2) *$I_Y(C)$ is invariant when C is deformed in the plane without changing its chord diagram.*
- (3) *If C is resolved to an unknot K^u , then*

$$I_Y(C) = \frac{1}{24} + \frac{n}{16} + \frac{\sum d_i}{8},$$

where c_\pm are computed using the signed chord diagram of K^u .

Corollary 5.30(3) provides a method of computing I_Y without having to compute an integral. However, the plane curve must be raised to an unknot to do, which Lin and Wang [L-W] note can always be done by the following method: chose a base point on the curve and proceed around it. The first time a double point is encountered, chose the branch currently being travelled to be the overstrand. For a plane curve with many double points, however, this method is somewhat laborious.

Definition 5.31. [L-W] (5.4) *Let K_0 be a plane curve. Then we define*

$$\alpha(K_0) = I_Y(K_0) + \frac{n}{16} - \frac{1}{24},$$

where n is the number of double points of K_0 .

Drawing from 5.30 (3), we can rewrite this formula in terms of n and $\sum d_i$:

$$\alpha(K_0) = \frac{n}{8} + \frac{\sum d_i}{8}.$$

Once again, we must calculate $\sum d_i(K_0)$ on an unknot projection. This approach may seem at first cumbersome, but [L-W] define this invariant to note the changes on $\alpha(K_0)$ by each Reidemeister move in the plane. In the following theorem, there is a distinction placed upon type II moves. The distinction involves a superscript of \pm indicating its similarity to a J^\pm move based upon the orientation of the curve.

Theorem 5.32. [L-W] (5.5) *The invariant α of plane curves has the following properties:*

- (1) α equals 0 for every simple closed plane curve;
- (2) α decreases by $1/8$ when a type I move is performed;
- (3) α is unchanged when a type II^+ move is performed;
- (4) α increases by $1/4$ when a type II^- move is performed;
- (5) α decreases by $1/4$ if a type III move is performed, which corresponds to the perestroika for which $St \rightarrow St - 1$;
- (6) $|\alpha(C)| \leq n^2/8$, where n is the number of double points on K_0 ;
- (7) $\alpha(C)$ is independent of the orientation of K_0 .

The next corollary allows us to construct our new form for v_2 .

Corollary 5.33. [L-W] (5.6) *We have*

$$\alpha = -\frac{2St + J^-}{8}.$$

By rearranging 5.31, we can define I_Y in terms of α :

$$I_Y(K_0) = \alpha(K_0) - \frac{n}{16} + \frac{1}{24},$$

and by further substitution from 5.33, we arrive at the expression

$$I_Y(K_0) = -\frac{2St + J^-}{8} - \frac{n}{16} + \frac{1}{24},$$

which we can write as

$$I_Y(K_0) = -\frac{2St + J^+}{8} + \frac{n}{16} + \frac{1}{24}$$

by the equality that $J^+ - J^- = n$.

Theorem 5.34. *Given a knot K , the invariant, v_2 , can be expressed in terms of Arnold's plane curve invariants and combinatorics of its signed crossings.*

Proof. We will first derive v_2 , and secondly, we will show that it is invariant under planar isotopy and all three Reidemeister moves.

By substitution into $v_2 = I_X - I_Y$, we have

$$v_2 = \frac{2St + J^+}{8} + \frac{\sum d_i}{8} - \frac{1}{24}.$$

Next, we will show that this form for v_2 is invariant under each Reidemeister move and planar isotopy. But planar isotopy only changes K trivially. Therefore, it will not affect our form of v_2 , and we may move immediately to examining the variations of I_X and I_Y after each Reidemeister transformation. We will designate the variables and invariants of K after a given Reidemeister transformation with a prime. For instance, we can express I_X and I_Y after a given move by $I'_X =$

$I_X + a$ and $I'_Y = I_Y + b$ for some $a, b \in \mathbb{Q}$. Then, if $a = b$, $I'_X - I'_Y = I_X - I_Y$ and $v'_2 = v_2$.

(1) Type I: We know from (5.32) that $\alpha' = \alpha - 1/8$. And with this move, we also lose one double point in the projection onto the plane of K . Thus

$$I'_Y = \alpha - \frac{1}{8} - \frac{n-1}{16} - \frac{1}{24}.$$

It follows that

$$I'_Y = I_Y - \frac{1}{16}.$$

For I_X , our immediate concern involves the computation of $\sum d_i$. Under a type I move, however, $\sum d_i$ does not change because the representation for that double point in K_0 is $(\cdots p p \cdots)$. Thus, this point does not contribute to $\sum d_i$, and therefore, its removal will leave $\sum d_i$ unchanged. Then,

$$I'_X = \frac{\sum d_i}{8} + \frac{n-1}{16},$$

and by simplifying, we have

$$I'_X = I_X - \frac{1}{16}.$$

Therefore, v_2 is invariant under a type I move.

(2) Type II⁺: we know that $n' = n + 2$ and $\alpha' = \alpha$. Thus,

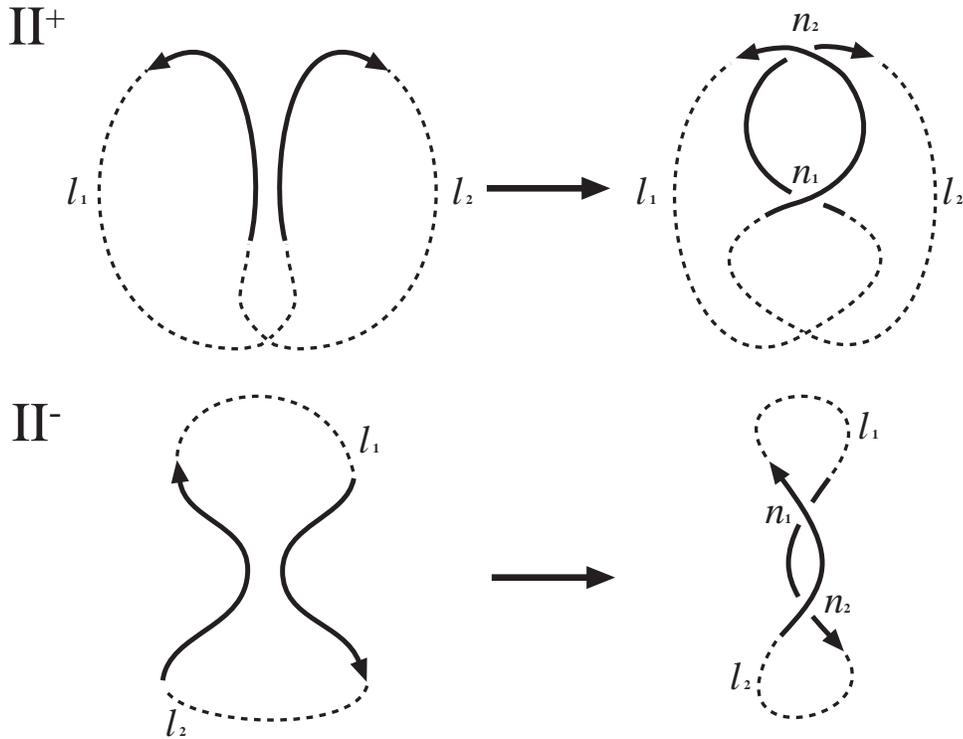


FIGURE 18. Type II moves upon a generic knot K .

$$I'_Y = \alpha - \frac{n+2}{16} + \frac{1}{24}, \quad \text{so} \quad I'_Y = I_Y - \frac{1}{8}.$$

For $\sum d_i$, we can examine the isolated portion of K at which the move occurred. In performing the type II^+ move, we create two new crossings: n_1 and n_2 (see Figure 18). Let us consider the remainder of the knot as two loops: l_1 and l_2 . After the type II^+ move, we have this permutation word, P , for K : $(n_1 n_2 l_1 n_1 n_2 l_2)$. For any crossing exclusive to either l_1 or l_2 , we have no change in $\sum d_i$ by definition of our betweenness function. So let us consider the nontrivial case. Let q be a generic crossing that is shared between l_1 and l_2 . Thus, $b(n_1) = b(n_2) = q$. But

$$\text{sign}(n_1)\text{sign}(q) + \text{sign}(n_2)\text{sign}(q) = 0,$$

because n_1 and n_2 are oppositely signed crossings in K . But we must still account for the $b(n_1) = n_2$ and $b(n_2) = n_1$. Thus, we have

$$\text{sign}(n_1)\text{sign}(n_2) + \text{sign}(n_2)\text{sign}(n_1) = -2,$$

and therefore,

$$I'_X = \frac{\sum d_i - 2}{8} + \frac{n+2}{16}, \quad \text{so} \quad I'_X = I_X - \frac{1}{8}.$$

Therefore, v_2 is invariant under type II^+ transformations.

(3) Type II^- : For a type II^- move, $\alpha' = \alpha + 1/4$, and $n' = n + 2$. Thus,

$$I'_Y = \alpha + \frac{1}{4} - \frac{n+2}{16} + \frac{1}{24}, \quad \text{so} \quad I'_Y = I_Y + \frac{1}{8}.$$

By inspection (see Figure 18), we know that the word, P , of K following the transformation is $(n_1 n_2 l_2 n_2 n_1 l_1)$. As with our proof of invariance for type II^+ moves, no crossings shared between l_1 and l_2 may affect $\sum d_i$ because of the opposite signs of n_1 and n_2 . So we have

$$I'_X = \frac{\sum d_i}{8} + \frac{n+2}{16}, \quad \text{so} \quad I'_X = I_X + \frac{1}{8}.$$

Therefore, v_2 is invariant under type II^- transformations.

(4) Type III: As we examine the type III transformations, we will consider those moves that correspond to a deformation in the plane for which $St' = St - 1$. Thus, by Arnold's conventions, we will move from a positively signed triangle to one that is signed negatively. Thus, we can refer to Theorem 5.32 and note that $\alpha' = \alpha + 1/4$ for every move of this type. This move does not change n , leaving

$$I'_Y = \alpha + \frac{1}{4} - \frac{n}{16} + \frac{1}{24}, \quad \text{so} \quad I'_Y = I_Y + \frac{1}{4}.$$

For I'_X , we must consider two cases (see Figure 19) because there are two unique orderings on the three crossings associated with the transformation of K . Let n_1, n_2 , and n_3 be the crossings before this transformation. After the transformation, we have n'_1, n'_2, n_3 . We will note that n_3 remains unchanged under the type III move and therefore is not n'_3 .

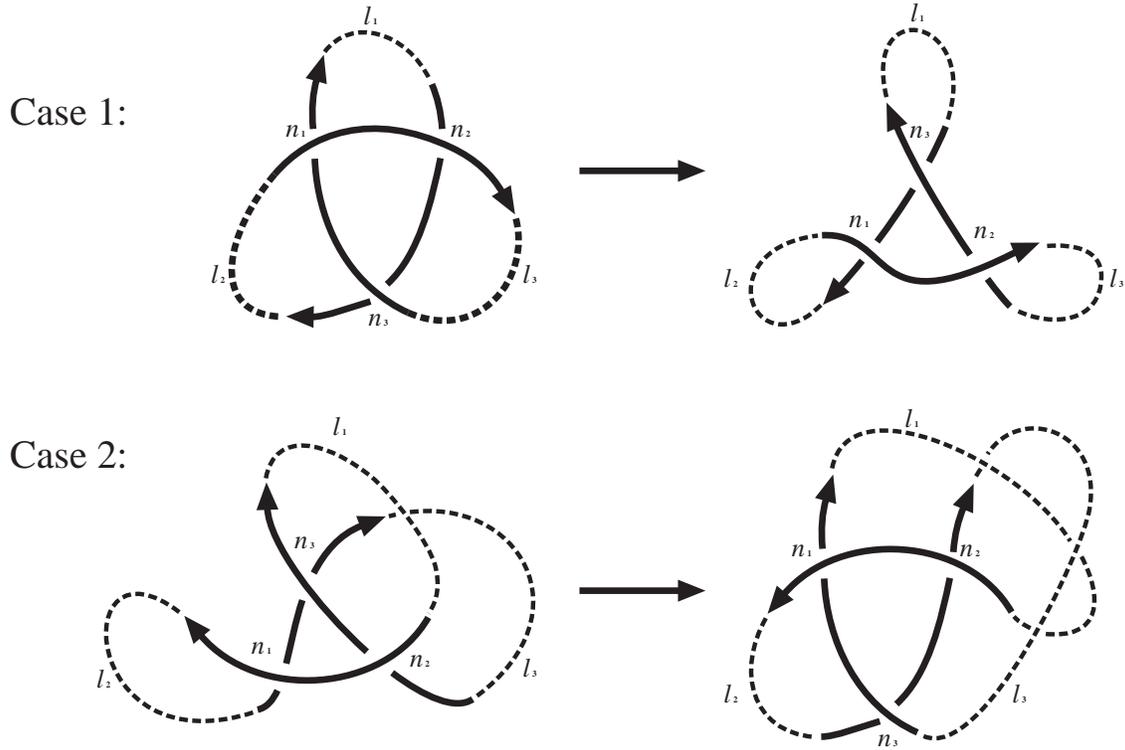


FIGURE 19. Two cases of the type III move.

Case i: by examination, we have the following words, P and P' , which correspond to K before and after the transformation:

$$P = \begin{pmatrix} n_1 & l_1 & n_2 & n_3 & l_2 & n_1 & n_2 & l_3 & n_3 \\ - & & - & - & & + & - & & + \end{pmatrix},$$

$$P' = \begin{pmatrix} n'_1 & l_2 & n'_1 & n'_2 & l_3 & n'_2 & n_3 & l_1 & n_3 \\ - & & - & + & & + & + & & + \end{pmatrix}.$$

We have noted the sign of each crossing below its instance in P . While we tabulate $\sum d_i$, we will shorten our notation in the following manner: let l_i be the sum of the signs of all single instances of a crossing in l_i . And let $n_i = \text{sign}(n_i)$. We will notify the reader when we return to our original notation.

By this designation, we have

$$\sum d_i = n_1(2l_3 + n_2 + n_3) + n_2(2l_2 + n_1 + n_3) + n_3(2l_1 + n_1 + n_2)$$

and

$$\sum d'_i = n'_1(2l_2) + n'_2(2l_3) + n_3(2l_1).$$

By simplification, we find

$$\sum d_i = 2l_1 - 2l_2 + 2l_3 - 2 \quad \text{and} \quad \sum d'_i = 2l_1 - 2l_2 + 2l_3.$$

By subtraction, we find $\Delta \sum d_i = 2$, leaving

$$I'_X = \frac{\sum d_i + 2}{8} + \frac{n}{16}, \quad \text{so} \quad I'_X = I_X + \frac{1}{4}.$$

And we have shown v_2 invariant in this case.

Case ii: We will now return to our original notation to define P and P' for K in the same manner.

$$P = \begin{pmatrix} n_1 & l_2 & n_1 & n_3 & l_3 & n_2 & n_3 & l_1 & n_2 \\ - & & - & - & & - & - & & - \end{pmatrix},$$

$$P' = \begin{pmatrix} n'_1 & l_1 & n'_2 & n'_1 & l_2 & n_3 & n'_2 & l_3 & n_3 \\ - & & - & - & & - & - & & - \end{pmatrix}.$$

We will use our shorthand once more to tabulate $\Delta \sum d_i$. First, we have

$$\sum d_i = n_1(2l_2) + n_2(2l_1 + n_3) + n_3(2l_3 + n_2)$$

and

$$\sum d'_i = n'_1(2l_1 + n'_2) + n'_2(2l_2 + n'_1 + n_3) + n_3(2l_3 + n'_2).$$

By simplification, we see that

$$\sum d_i = -2l_1 - 2l_2 - 2l_3 + 2 \quad \text{and} \quad \sum d'_i = -2l_1 - 2l_2 - 2l_3 + 4.$$

And therefore, we find that $\Delta \sum d_i = 2$ once again, allowing us to make the following substitutions:

$$I'_X = \frac{\sum d_i + 2}{8} + \frac{n}{16}, \quad \text{so} \quad I'_X = I_X + \frac{1}{4}.$$

Therefore, v_2 is invariant under all Reidemeister moves. QED

Corollary 5.35. *Defect is invariant when K_0 is deformed in the plane without changing its chord diagram.*

Proof. From (5.3), we know that

$$I_Y(K_0) = -\frac{2St + J^+}{8} + \frac{n}{16} + \frac{1}{24},$$

where defect is $2St + J^+$. It follows that

$$D = -8I_Y(K_0) + \frac{n}{2} + \frac{1}{3}.$$

By 5.30(2), we know that $I_Y(K_0)$ is invariant under deformations of K_0 that do not alter its chord diagram. Furthermore, those deformations under which $I_Y(K_0)$ is invariant will keep n constant. QED

Corollary 5.35 shows that if two curves have the same chord diagram, then they have the same defect. The following corollary is an extension of this fact.

Lemma 5.36. *Let C be a plane curve. Let c_+ (respectively c_-) be the number of positive (respectively negative) products of signed crossings in the Gauss diagram corresponding to an unknot projection of C . Then $D(C) = -2(c_+ - c_-)$.*

Proof. Let n be the number of double points of C . From [L-W], equations (5.1) and (5.3) yield

$$\frac{n}{8} + \frac{c_+ + c_-}{4} = -\frac{D(C)}{8} + \frac{n}{8}$$

Thus

$$D(C) = -2(c_+ - c_-)$$

QED

Corollary 5.37. *Let C_1 and C_2 be two plane curves whose Gauss diagrams have the same set of intersecting chords. Then $D(C_1) = D(C_2)$.*

Proof. Let G and G' be the Gauss diagrams associated with C_1 and C_2 . Using Lemma 5.36, we know that $D(C_1) = -2(c_+ - c_-)$ where c_+ and c_- are computed from G as defined in Lemma 5.36. This implies that the defect of a plane curve depends only on the intersecting chords in its Gauss diagram. By inserting or deleting nonintersecting chords, we may transform G into G' without changing $D(C_1)$. Using Corollary 5.35, C_2 has the same defect as any other plane curve associated with G' . Thus $D(C_1) = D(C_2)$. QED

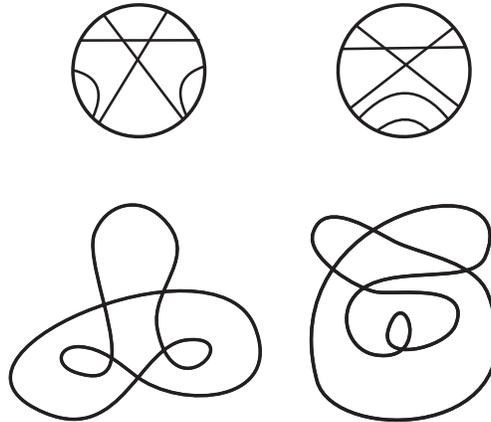


FIGURE 20. Two plane curves with defect 2.

Example 5.38. *Two plane curves with Gauss diagrams that have the same set of 3 intersecting chords both have defect 2. See Figure 20.*

Note that corollary 5.37, in addition to Luo's formulas for J^+ and J^- in Theorem 3.13, gives a relatively simple way of computing the Arnold invariants of a plane curve if the defect associated with a set of intersecting chords is known or calculated using the Definition 5.28.

6. SOME UNFINISHED THOUGHTS ON PLANE CURVES AND GAUSS DIAGRAMS

As mentioned previously, the set of treelike curves is merely a subset of all plane curves with defect 0. The following corollary from Lin and Wang gives two properties of defect 0 curves.

Corollary 6.1. [L-W] *Let C be a plane curve with n double points. Then $D(C) = 0$ if and only if $\alpha(C) = n/8$, and if and only if the signed chord diagram associated with lifting C to an unknot resolution of C has $c_+ = c_-$.*

This section extends results involving chord intersections in Gauss diagrams and the work of Andrew Barker and Ian Biringer [B-B] from the REU program at Oregon State University, Summer 2003. They introduced *bushy* Gauss diagrams which may correspond only to defect 0 plane curves. Here, we give a revision of their definition, which should provide greater workability.

Definition 6.2. A Gauss diagram is bushy if the intersecting chords of the Gauss diagram partition into k pairs of chord sets such that given one pair of chord sets M and N and a chord set Q from another pair, we have

- (1) $|M|, |N| \in 2\mathbb{Z}$;
- (2) For every $m_i \in M$ and $n_j \in N$, m_i intersects n_j ;
- (3) For every $m_i, m_j \in M$, m_i does not intersect m_j ;
- (4) For every $m_i \in M$, $n_j \in N$, and $q_k \in Q$, q_k does not intersect m_i or n_j .

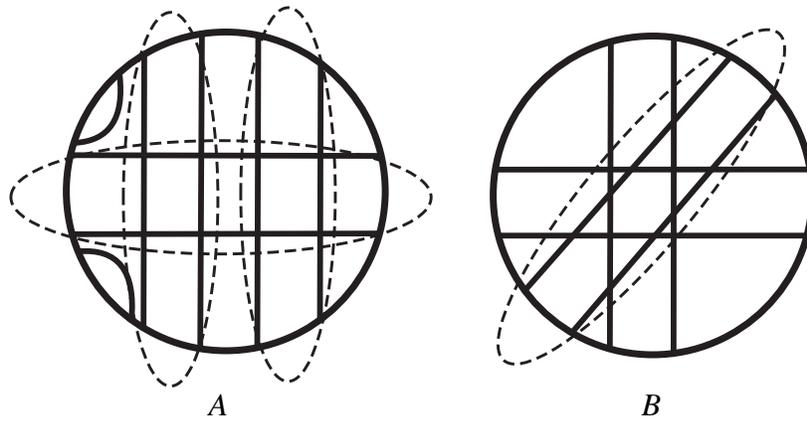


FIGURE 21. Diagram A is bushy while B is not.

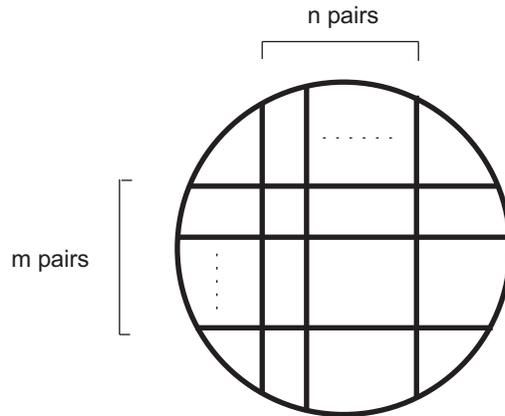


FIGURE 22. G' contains only the set of intersecting chords of G .

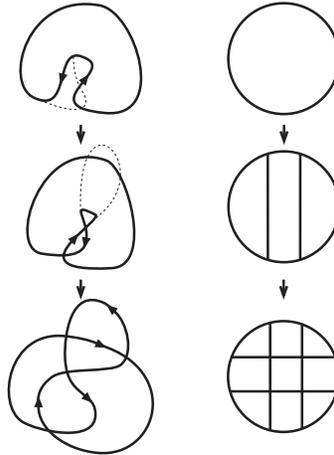


FIGURE 23. Construction of a plane curve associated with a bushy diagram.

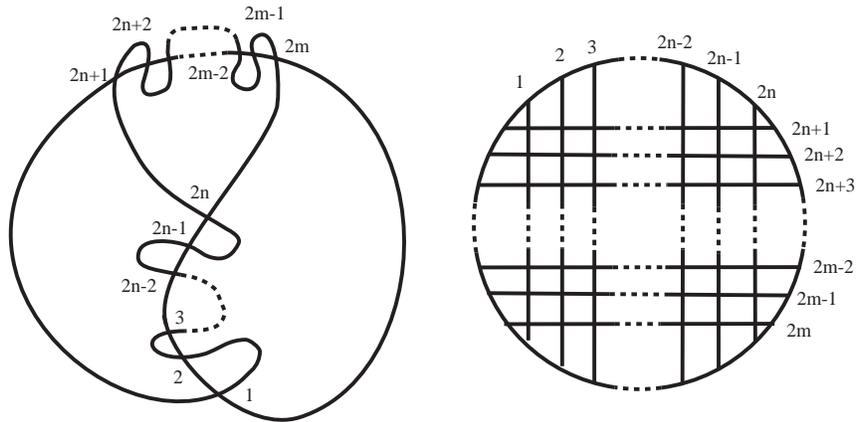


FIGURE 24. Construction of a plane curve associated with G' .

Example 6.3. See Figure 21. The Gauss diagram in A is bushy because the set of intersecting chords may be partitioned into three pairs of adjacent chords. Any two of the pairs are either parallel or perpendicular to each other. Diagram B is not bushy because the circled pair of chords cannot be drawn either parallel or perpendicular to the other two pairs.

When we examine a Gauss diagram for bushiness, we can disregard all chords that are nonintersecting. Any chords with this quality will not be in a partitioned set. Furthermore, we know these chords will not contribute to the defect of the plane curve, which is an important reminder as we introduce the conjecture in [B-B].

Conjecture 6.4. [B-B] A curve has defect 0 if and only if its Gauss diagram is bushy.

In this section, we will provide a proof showing that bushiness implies defect 0; however, we have not proved that 0 defect implies bushiness. In our work, we have not found any examples to invalidate this conjecture.

Theorem 6.5. *Let C be a plane curve. If the Gauss diagram of C is bushy, then $D(C) = 0$.*

Proof. Let G be the Gauss diagram associated with C . Suppose G is bushy and that n pairs of intersecting chords of G may be drawn vertically and m pairs horizontally. By Corollary 5.37 we may compute the defect of C by computing the defect of any curve associated with a Gauss diagram that has the same set of intersecting chords as G . Let G' be the Gauss diagram that contains only the intersecting chords of G , as in Figure 22. As shown in Figure 23, we may construct a curve from the simple circle using only J^- moves that has the bushy diagram shown with four chords. By performing $m - 1$ and $n - 1$ J^- moves in regions M and N , respectively, as shown in Figure 24, we can construct m and n pairs of chords to obtain G' . Because the simple circle has $D = 0$, and J^- moves do not affect D , the defect of the plane curves associated with G' , and thus also with G , have defect 0. QED

We will now provide a summary of our results, which we had hoped would lead to a proof of second half of the conjecture.

6.1. Characterizing Arnold's invariants with Seifert decomposition. First, we studied oriented Seifert cycles to characterize the allowable self-tangency moves for plane curves. We define a function that measures the *insideness* of a Seifert cycle. This idea slightly resembles the t -function defined in [Lu], but we define insideness for all cycles in the decomposition of a plane curve and not over the plane curve entirely.

Definition 6.6. *A bounding cycle is an oriented cycle in a Seifert decomposition, which encloses at least one other cycle within itself.*

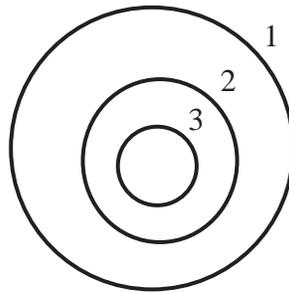


FIGURE 25. A Seifert decomposition illustrating bounding cycles.

Example 6.7. *In Figure 25, 1 is a bounding cycle of 2, and 2 is a bounding cycle of 3. We can also refer to cycle 3 (or 2) as being bound within 2 (or 1). Necessarily, 3 is also bounded by 1; however, it is more important to consider the bounding inside or outside a cycle to which it was formerly connected.*

Definition 6.8. Given a Seifert decomposition of a closed plane curve, let $In(c_i)$ define the degree of insiderness of a given cycle, c_i by the following:

- (1) Given one cycle, c_0 , let $In(c_0) = x$ for some $x \in \mathbb{Z}$.
- (2) For any cycle(s), c_j , connected to and bounded by c_0 , $In(c_j) = x - 1$.
- (3) For any cycle(s), c_k , connected to and bounded outside c_0 , $In(c_j) = x + 1$.

With this relation, we can define all cycles of a Seifert decomposition relative to one another.

The orientation of Seifert cycles will be useful in providing criteria for Arnold's plane curve moves on a Seifert decomposition. In the proof of the following lemma, we will refer to the generic plane curve of index 0 by K_0 , not, as in previous sections, the limiting plane curve.

Lemma 6.9. Given two Seifert cycles, p and q from a series of exteriorly adjacent Seifert cycles, the orientations of p and q will be equivalent if $In(p) = In(q) \pm 2n$ for $n \in \mathbb{Z}$.

Proof. The curve K_0 is composed of two connected Seifert cycles, and its index is 0. Therefore, one cycle's orientation must oppose the other. By composing more cycles onto one another, we perform a connected sum of K_0 plane curves. To perform this operation we must combine two like-oriented cycles. Thus, the next cycle must be oriented oppositely. By continuing this operation, every other cycle will have the same orientation. QED

Theorem 6.10. Given two Seifert cycles, p and q , a J^- move may be performed between them if $In(p) = In(q) \pm 2n$ for $n \in \mathbb{Z}$ and no bounding cycle separates them.

Proof. Let p and q be two oriented cycles bounded within the same cycle, c with $In(p) = In(q)$ for $n \in \mathbb{Z}$ (see Figure 26). Let us perform a self tangency move on both cycles, and by way of contradiction, let us assume that the only possible self tangency is direct. By this assumption, we can force an orientation on the respective cycles, which we see in the Figure 26. But because p and q are connected to the same cycle, we can continue to follow the tangent vectors. By inspection, we will see that this continuation places opposing vectors at either end of l_1 , which contains the remaining portion of the oriented bounding cycle, which can only have one direction, giving us our desired contradiction. Thus, we know that the legal transformation between the two cycles must be a J^- move. Furthermore, we can extend this result to apply to two cycles, p and q for which $In(p) = In(q) + 2n$ with no bounding cycle between them by appealing to Lemma (6.9).

In Figure 26, we have illustrated the case where p and q are bounded within a cycle. The same argument will allow us to prove the case for which p and q are two names for the same cycle and for which they are both bounded outside another cycle. QED

Corollary 6.11. Given two Seifert cycles, p and q , a J^+ move may be performed between them if $In(p) = In(q) \pm (1 + 2n)$ for $n \in \mathbb{Z}$ and no bounding cycle separates them.

Proof. From (6.10), we have conditions by which we can perform a J^- move between two cycles. Using those conditions, let p and r be two cycles, which satisfy those conditions. Consider a third cycle, q , which is exteriorly connected to r but is not bounded from p . By definition, we know that $In(q) = In(r) \pm 1$. From Lemma (6.9), we know that r and q are oriented oppositely. Therefore, the self-tangency move that we perform between p and q must be direct, corresponding, therefore, to a J^+ move. QED

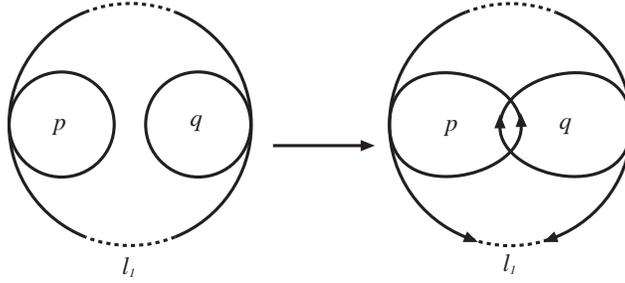


FIGURE 26. Two Seifert cycles and the forced direct self tangency.

6.2. Some final thoughts on Gauss diagrams. By Lemma (5.36), curves of defect 0 have the quality that $c_+ - c_- = 0$. Alternatively, we have $\sum d_i = 0$. By definition, we can consider the explicit summation of all intersecting chords in the Gauss diagram. By Definition (6.2), there is a partition on the chords of the diagram. Therefore, we can compute $\sum d_i/2$ for a bushy Gauss diagram by considering only a subset, N , of all chords in the diagram. Given $n_i \in N$, let a_i (b_i , respectively) be all positive (negative, respectively) chords that intersect n_i . Then

$$\sum \frac{d_i}{2} = \text{sign}(n_1)(a_1 - b_1) + \text{sign}(n_2)(a_2 - b_2) + \dots + \text{sign}(n_n)(a_n - b_n).$$

By showing that all defect 0 plane curves satisfy this identity for $\sum d_i/2$, we will show that there must be a partition on the chords of a Gauss diagram that we define in (6.2).

Another idea for a proof to show that defect 0 implies bushiness involves inductively constructing all Gauss diagrams that satisfies $c_+ - c_- = 0$. Ideally, all of these diagrams would satisfy the criteria for bushiness. However, we can construct Gauss diagrams with pairs of oppositely signed chords that do not correspond to valid plane curves (see Figure 27). Therefore, this method of proof would first involve defining characteristics of Gauss diagrams that provide for valid plane curves, which could easily involve those results connecting Seifert cycles and perestroikas.

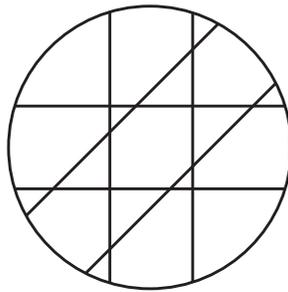


FIGURE 27. A defect 0 Gauss diagram from an impossible plane curve.

7. TWO-COLORABILITY OF PLANE CURVES

The contents of this section came about as a result of attempting to prove every chord in a Gauss diagram that has an associated plane curve crosses an even number of chords. The proof printed

here shows that every plane curve is two-colorable, which is equivalent to what was intended to be proved. It turns out a simpler proof of the statement about chord diagrams is outlined in a paper by Luo.[Lu] Yet a third proof is outlined here using the Oriented Intersection Number.

7.1. Two-Colorability.

Definition 7.1. A connected complement region (CCR) of a plane curve, C , is a connected set in $\{\mathbb{R}^2 \setminus C\}$.

Definition 7.2. A plane curve is two-colorable if every CCR can be assigned one of two colors in a way such that if a curve segment of the plane curve borders two CCRs then those regions are assigned different colors.

Theorem 7.3. Both K_0 and K_1 are two-colorable.

Proof. By inspection of Figure 7.3.

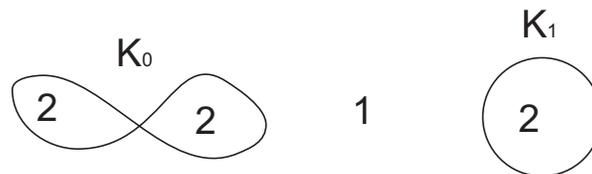


FIGURE 28. Proof of Theorem (7.3)

QED

Theorem 7.4. If C_0 is a two-colorable plane curve corresponding to the alternating knot N_0 and C_1 is a plane curve corresponding to an alternating knot N_1 where N_1 can be obtained from N_0 through one type I Reidemeister move, then C_1 is two-colorable.

Proof. The only difference between C_1 and C_0 is there is an extra loop inside (fully enclosed by) one of C_0 's CCRs. This loop creates another CCR. Since C_0 was two-colorable it only remains to color the new CCR the opposite color from the enclosing CCR for C_1 to be two-colored, as in Figure 29

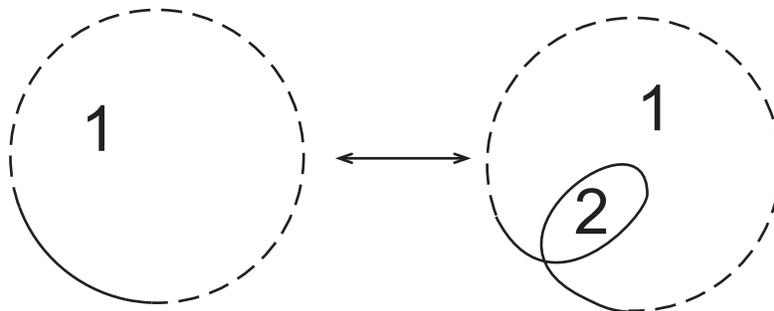


FIGURE 29. Reidemeister type I move two-colorable

QED

Corollary 7.5. K_n is two-colorable.

Proof. K_0 and K_1 are both two-colorable. For $n \in \mathbb{N}$ such that $n \geq 2$, K_n is K_{n-1} after a type I Reidemeister move. Thus by the previous theorem and induction on n , K_n is two-colorable. QED

Theorem 7.6. If C_0 is a two-colorable plane curve and C_1 can be obtained from C_0 through one self-tangency perestroika or one vanishing triangle perestroika move, then C_1 is two-colorable.

Proof. By inspection of Figure 7.6.

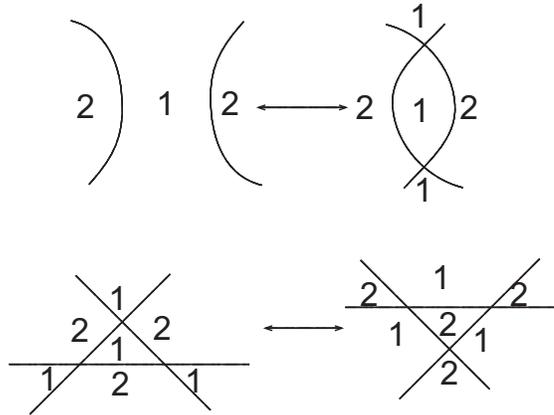


FIGURE 30. Proof of Theorem (7.6).

QED

Corollary 7.7. All plane curves are two-colorable.

Proof. Any plane curve can be obtained from K_n (where n is the Whitney index of the curve) through a series of perestroika moves and ambient homotopies of the plane, which do not change the colorability of the curve. QED

7.2. Oriented Intersection Number.

Definition 7.8. The Oriented Intersection Number is defined as the sum over every intersection between two manifolds, each assigned a ± 1 .

Theorem 7.9. The Oriented Intersection Number is invariant under smooth homotopies of either manifold.

Theorem 7.10. The Oriented Intersection Number of two plane curves is always 0.

Proof. A translation of one of the plane curves is a smooth homotopy. It is possible to translate one of the plane curves so that there are no intersections between the two plane curves. QED

Theorem 7.11. Two plane curves intersect an even number of times.

Proof. The Oriented Intersection Number of two plane curves is 0. In order for a sequence of positive and negative 1's to sum to 0 there must be an even number in the sequence. QED

Theorem 7.12. *Every chord in a Gauss diagram that corresponds to a plane curve intersects an even number of chords.*

Proof. Choosing an arbitrary chord on a Gauss diagram corresponding to a plane curve, begin the Seifert decomposition of the corresponding curve at the intersection that corresponds to that chord. The first Seifert splitting yields two plane curves, and every intersection between them corresponds to a chord that intersects the chosen chord. There must be an even number of intersections, therefore there must be an even number of chords intersecting the chosen chord. QED

8. DEFECT INVARIANT FOR ALL PLANE CURVES CORRESPONDING TO THE SAME KNOT

8.1. Defect Invariant under type I Reidemeister move.

Definition 8.1. *Looping on a plane curve is the analogue to a Type I Reidemeister move on a knot projection as seen in Figure 31.*



FIGURE 31. Effect of a Type I move.

Definition 8.2. *The looping number is obtained by drawing a path from the CCR bounded by the (oriented) plane curve in which the loop is to appear to the CCR exterior to the plane curve without intersecting any of the plane curve's intersections. Orient the path so it begins where the loop will appear and ends exterior to the plane curve. Extend the path from where the loop will appear through the line segment which will be looped, this will orient the path as right (left) handed. At every other intersection between the path and the plane curve assign the intersection +1 if it is right (left) handed and -1 otherwise. Sum along the path to obtain the looping number.*

Example 8.3. *Calculating the Looping Number:*

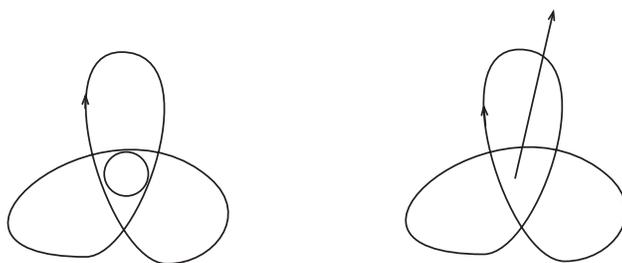


FIGURE 32. Drawing a path from the disc in which the loop will appear to the exterior of the plane curve.

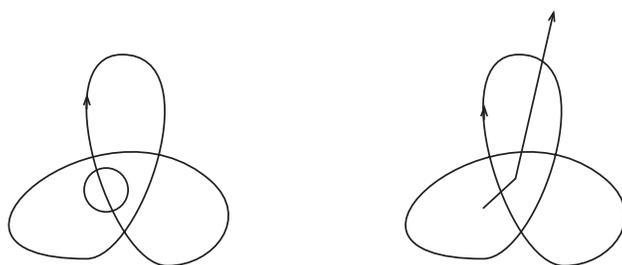


FIGURE 33. Extending the path through the line segment that will loop.

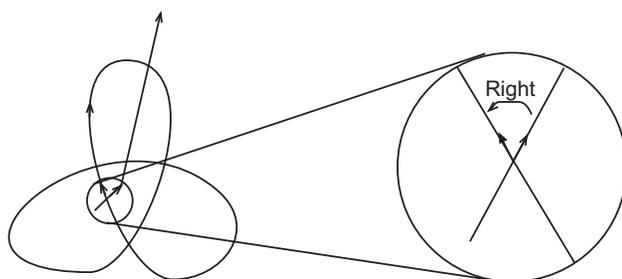


FIGURE 34. Orient the curve.

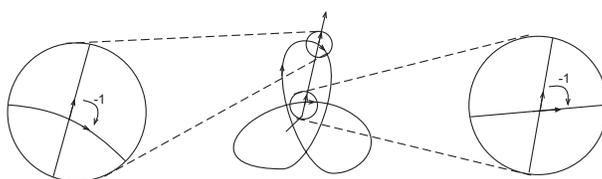


FIGURE 35. The looping number for this path is -2.

Theorem 8.4. *When introducing a loop into a plane curve the Arnold invariants change according to:*

- (1) St decreases by α ;
- (2) J^+ increases by 2α ;
- (3) J^- decreases by $2\alpha + 1$;

where α is the looping number.

Proof. Extend the line segment which will contain the loop along the path until it extends into the exterior. As it passes each +1 (-1) intersection on the path a positive direct (inverse) self-tangency perestroika occurs, thus J^+ increases (J^- decreases) by 2.

Once the line segment is pushed to the exterior of the plane curve take the direct sum of the plane curve with K_0 to obtain the desired loop, this causes J^- to decrease by 1. Note that the loop changes the orientation of the line segment (see Figure 8.4). Thus, when retracting the looped line segment back to the original position, each +1 (-1) intersection on the path signifies a negative inverse (direct) self-tangency perestroika, thus J^- increases (J^+ decreases) by 2. Thus, a +1 intersection cancels a -1 intersection with respect to J^\pm . Summing the intersections along the path, we will obtain α . J^+ will increase by 2α , and J^- will decrease by $2\alpha + 1$.



FIGURE 36. Looping changes the orientation of the line segment.

Because of the loop in the line segment, a vanishing triangle perestroika will also occur at each intersection when returning the line segment. Since St is invariant we may choose base point and orientation of the plane curve arbitrarily. Let us arrange for the loop to provide sides 1 and 2 of the triangle and the intersecting line segment provides side 3 as in Figure 29. Thus a +1 (-1) intersection creates a negative (positive) vanishing triangle and decreases (increases) St by 1. Therefore St will decrease by α

QED

Corollary 8.5. *The looping number is invariant of the path used to calculate it.*

Proof. No matter which path is used, the final plane curve obtained is the same. Since J^\pm and St are plane curve invariants, they will be the same for the final curve no matter which path is used.

QED

Corollary 8.6. *The defect is invariant under inclusion of loops.*

Proof. Prior to looping, the Arnold invariants of the plane curve were J^+ , J^- , and St , so the defect was $2St+J^+$. After looping, the Arnold invariants of the resulting plane curve are $St-\alpha$, $J^++2\alpha$, and

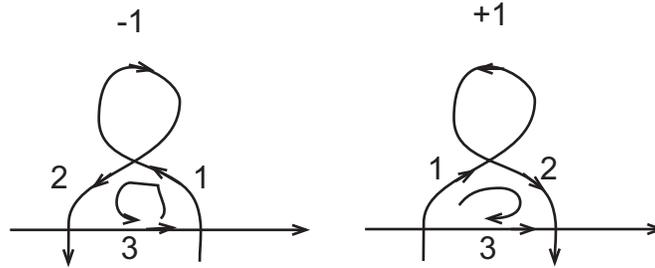


FIGURE 37. The loop leads to vanishing triangle perestroikas.

$J^- - 2\alpha - 1$. The new defect is

$$\begin{aligned} 2(St - \alpha) + J^+ + 2\alpha &= 2St + J^+ - 2\alpha + 2\alpha \\ &= 2St + J^+ \end{aligned}$$

QED

8.2. Defect Invariant under Flype.

Definition 8.7. A plane curve and a knot projection correspond if the knot projection is alternating and the knot projection's shadow on the plane is the plane curve.

Theorem 8.8. If plane curve C_0 corresponds to the alternating knot projection K_0 and plane curve C_1 corresponds to the alternating knot projection K_1 , where K_1 can be obtained from K_0 by performing one flype, then C_0 and C_1 have the same defect.

Proof. In terms of the Gauss diagram, a flype move has the effect of moving one chord, as shown in Figure 38. It is impossible for any chord in the Gauss diagram to anchor both in a tangle arc and a non-tangle arc. We indicate a tangle (respectively, non-tangle) arc in Figure 40 with a T (respectively, a dashed-line). After the flype the displaced chord still intersects a second chord if and only if it intersected the secondary chord prior to the flype. Thus all intersections in the Gauss diagram remain unchanged under a flype. If the Gauss diagram is signed, then the sign of the chord that is moved, as well as the signs of the chords in the tangle, are preserved under the flype (see Figures 39 and 40).

Since the intersections and the signs of the chords are invariant under a flype, c_+ and c_- will both be invariant. Thus, $c_+ - c_-$ is invariant under a flype. From Theorem 5.12, we have

$$v_2 = \frac{D}{8} + \frac{c_+ - c_-}{4} + \frac{1}{24},$$

where v_2 is a knot invariant, D is the defect of the plane curve formed by the knot projection, and $c_+ - c_-$ is obtained from the sign of the intersections on the Gauss diagram of the knot projection. Since a flype does not change the knot, v_2 remains the same, as does $c_+ - c_-$. Thus, the defect must remain the same for the plane curves formed from the knot projection before and after the flype.

QED

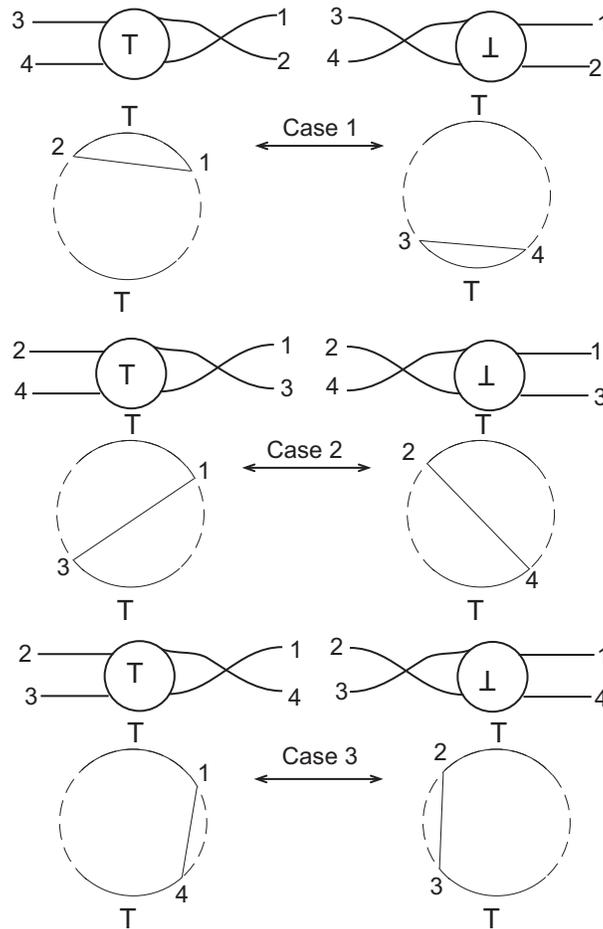


FIGURE 38. The arcs on the Gauss diagrams labelled T correspond to the tangles. No chord can anchor both inside and outside the tangle. The shown chord is the only one changed, it always anchors on the edge of the tangle.

8.3. All Plane Curves corresponding to the same Knot have the same Defect.

Definition 8.9. An alternating knot projection is minimal if there is no type I Reidemeister move that will reduce the number of crossings in the projection.

Theorem 8.10. [Ad] (Tait Flying Conjecture, proved by Menasco and Thistlethwaite in 1990) All minimal projections of an alternating knot can be obtained by a series of flypes and ambient homotopies of the knot on the sphere.

Theorem 8.11. The defect of all plane curves corresponding to the same alternating knot is the same.

Proof. A flype is a defect preserving move. If a knot projection on the plane is transferred into part of the sphere, then an ambient homotopy is performed, finally the new projection is transferred back into part of the plane, the resulting plane curve will have the same Gauss diagram as the original, thus the new corresponding plane curve will have the same defect as the original. Therefore

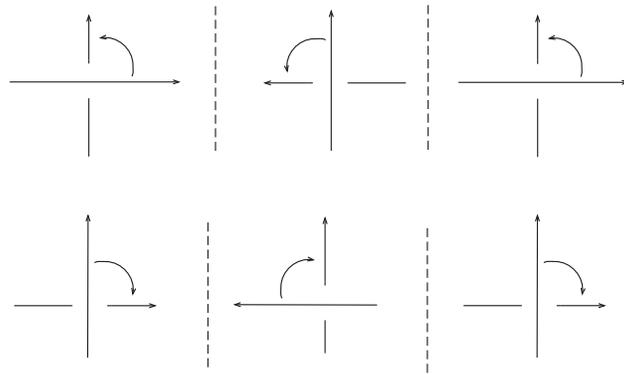


FIGURE 39. A flype rotates all the intersections in the tangle around an axis.

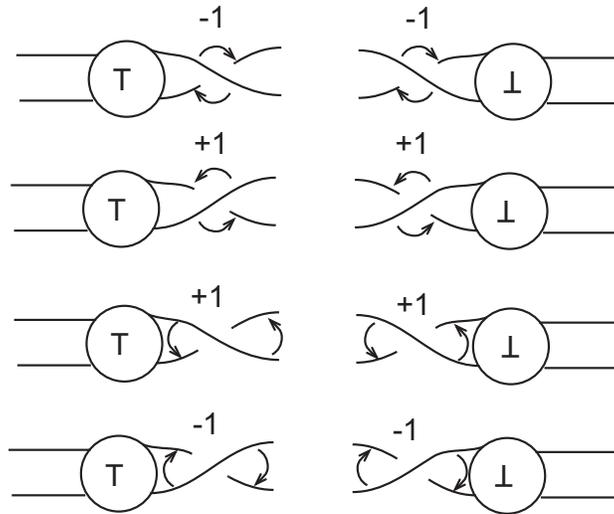


FIGURE 40. The altered chord in the Gauss diagram retains the original sign.

all plane curves corresponding to minimal alternating projections of the same knot have the same defect. All non-minimal alternating projections of a knot can be obtained through a finite number of type I Reidemeister moves that each increase the number of crossings by 1, and defect is invariant under type I moves. Therefore, all plane curves corresponding to alternating projections of the same knot have the same defect.

QED

What follows from this result is a facile proof that defect vanishes for all tree-like curves.

Corollary 8.12. [Ai] *All tree-like curves have defect 0.*

Proof. Tree-like curves all have Gauss diagrams without intersecting chords. Therefore all tree like curves correspond to alternating knot projections that reduce to the unknot when minimized. The unknot corresponds to K_1 which has defect 0.

QED

9. CONCLUSION

We found numerous connections between knots, plane curves and Gauss diagrams, however, some work remains.

Based on the work done in [L-W] we suggest several integrals that might lead to the discovery of new invariants, or at least, new forms of known invariants. Further work is needed to determine whether any of these integrals are actually invariants.

We found one condition that every Gauss diagram that corresponds to valid plane curve must fulfill: each intersecting chord must pass through an even number of other such chords. While this condition eliminates a large number of possible Gauss diagrams from valid Gauss diagrams, it is not a sufficient condition. We suggest that future researchers might be interested in finding other conditions for valid Gauss diagrams.

We were able to prove one direction of a conjecture stated in [B-B]; however, the converse, that all defect 0 plane curves have bushy diagrams, remains to be shown.

REFERENCES

- [Ai] Aicardi, F. *Tree-like Curves* Advances in Soviet Mathematics, Vol. 21, 1994.
- [Ad] Adams, Colin C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. (1994).
- [Ar] Arnold, V. I. *Plane Curves, Their Invariants, Perestroikas and Classifications*. Advances in Soviet Mathematics, Vol. 21. 1994.
- [B-B] Barker, Andrew, and Ian Biringer. *On Defect of Plane Curves* Summer REU Program, Summer 2003.
- [D] Duzhin, S. *Lectures on Vassiliev Knot Invariants* Technical Report, Tokyo, 1999.
- [Lu] Luo, Chenghui. *Proof of Arnold's Conjectures About Plane Curves* Preprint, Brown University Dissertation, 1993.
- [L-W] Lin, Xiao-Song and Zhenghan Wang. *Integral Geometry of Plane Curves and Knot Invariants*. Columbia University and University of Michigan.
- [NE] O'Neill, Barrett. *Elementary Differential Geometry* Academic Press, Inc. San Diego, 1966.
- [R] Reidemeister, K. *Knotentheorie* Springer, reprint, 1974.
- [S] Santalo, L.A. *Integral Geometry and Geometric Probability* Addison-Wesley Publishing Company, 1976.
- [W] Whitney, Hassler. *On Regular Closed Curves in the Plane*.

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