STABILITY AND INSTABILITY IN ONE-DIMENSIONAL POPULATION MODELS

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ABSTRACT. One dimensional difference equations, of the form x_{t+1}=f(x_t), are a commonly used tool in disciplines such as population dynamics, meteorology/climatology, and fluid flow. As we'll see, one dimensional population models can display remarkably complex behavior. Thus, population models of even higher degree will display much wilder behavior. These one dimensional properties also have properties that aren't shared by their higher dimensional cousins. For example, if the one dimensional model has no cycle of period 2, then every starting point, after a finite number of iterations, will reach the fixed equilibrium point. In most of the simple population models, the one key varying factor, besides the population at the previous time increment, is the population growth rate. This growth rate has a tremendous influence on the behavior of the model, and determines much about the stability of the system.

In these population models, as the reproductive rate increases, the model follows the period doubly path to chaos. At low enough rates, however, the model can become locally and even globally stable. Recent work by Cull and Chaffee [2] [1] have given methods which extend local to global stability for simple one dimensional population models by enveloping with linear fractionals.

In this paper, we show that if a model is enveloped by a self-inverse function, then it is globally stable. Later, we show that global stability alone does not imply that the model can be enveloped by a self-inverse function; continuity is also needed. This will be extended to a more general form in which local stability plus continuity is all that's necessary to show local enveloping. Next, the fact that enveloping doesn't imply straight line or even linear fractional enveloping. Lastly, we will look at how stability is affected when the degree of the polynomial is increased, and how the growth rate affects the speed to equilibrium.

1. INTRODUCTION

Simple one dimensional population models have been shown to have the very nice property of being globally stable if they are locally stable. This was shown by Cull [4] using enveloping by linear fractionals. Later in that paper, it was shown that linear fractionals are not necessary, enveloping by a self-inverse function (in other words, $\phi(\phi(x)) = x$) implies global stability.

In this paper, we will show the reverse of what Cull [3] showed; that is, we will show that global stability (plus the extra condition of continuity) implies enveloping. This will then be extended to cover locally stable models as well, the region of local stability - otherwise known as the *local basin*- will be shown to be locally envelopable. Next, the stability and complexity of some simple population models will be explored as the growth rate as well as the degree of the polynomial vary. Lastly, the fact that the enveloping function isn't necessarily a single straight line or a linear fractional will be addressed. The enveloping function could be a curve, or a piecewise function; as

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long as the function is self-inverse, it will be sufficient to imply stability. Pictures will be given to support proofs in the paper.

1.1. **Background Definitions.**

Definition 1.1. *A one-dimensional population model is a function of the form*

$$x_{t+1} = f(x_t)$$

where f is a function that takes the non-negative real numbers to the non-negative real numbers. There exists an equilibrium point, $\bar{x} > 0$, such that:

- *f(0)=0 f(x) $> x$ for $0 < x < \bar{x}$*
- *f(x)=x for x=$\bar{x}$*
- *f(x)$<$x for x$>\bar{x}$*

and if f'(x_m)=0, where x_m is a critical point and $x_m \leq \bar{x}$ then

- *f'(x) > 0 for $0 \leq x < x_m$*
- *f'(x) < 0 for x> x_m such that f(x) > 0.*

- f(f(x)), the second iteration of x, can be denoted by $f^{(2)}$(x). Likewise, the k^{th} iteration will be denoted $f^{(k)}$(x).

Definition 1.2. *A one-dimensional population model of the form $x_{t+1} = f(x_t)$ is* GLOBALLY STABLE *if and only if for all x_o such that $f(x_o) > 0$*

$$\lim_{k\to\infty}(x_k)=\bar{x} \text{ where } k \text{ is the number of iterations.}$$

There are seven standard one dimensional population models, each of which have conditions on it so that they are globally stable.

- **Model 1.** [7] [8] [9] [12] $x_{t+1} = x_t e^{r(1-x_t)}$
  This model is globally stable for $0 < r \leq 2$.
- **Model 2.** [13] $x_{t+1} = x_t[1 + r(1 - x_t)]$
  This model, like model 1, is globally stable for $0 < r \leq 2$.
- **Model 3.** [10] $x_{t+1} = x_t[1 - r\ln(x_t)]$
  This model, like the previous two, is globally stable for $0 < r \leq 2$.
- **Model 4.** [15] $x_{t+1} = x_t(\frac{1}{b+cx_t} - a) where c = \frac{1}{a+1} - b.$
  This model is globally stable when $\frac{a-1}{(a+1)^2} \leq b < \frac{1}{a+1}$.
- **Model 5.** [11] $x_{t+1} = \frac{(1+ae^b)x_t}{1+ae^{bx_t}}$
  This model is globally stable when $a(b-2)e^b \leq 2$.
- **Model 6.** [6] $x_{t+1} = \frac{(1+a)^b x_t}{(1+ax_t)^b}$
  This model is globally stable when a(b-2) $\leq$ 2.
- **Model 7.** [14] $x_{t+1} = \frac{ax_t}{1+(a-1)x_t^b}$
  This model is globally stable when a(b-2) $\leq$ b.

Cobweb Diagrams

Cobweb diagrams are a very simple way of viewing the convergence or divergence of a population model. It allows you to view the "path" a population would take, given an initial population. An initial population will follow a path vertically to where it intersects the model f(x), and will then proceed horizontally to the line y=x, where it will again follow a vertical path to f(x). It will continue this for as many iterations as you tell it to. [5]

Definition 1.3. *A population model model is* LOCALLY STABLE *if and only if for every small enough neighborhood of* $\bar{x}$ *if* x_o *is in this neighborhood, then* x_t *is in this neighborhood for all t, and*

$$\lim_{t\to\infty}(x_t) = \bar{x}$$

Definition 1.4. *The* ***local basin*** *of a locally stable model is the region* (x_1,x_2)*such that* $x_1 < \bar{x} < x_2$*, and for all* $x\in(x_1,x_2)$*,* $\lim_{k\to\infty} f^{(k)}(x) = \bar{x}$.

Definition 1.5. *A function* $\phi(x)$ LOCALLY ENVELOPS *a function f(x) in an interval* (x_1,x_2) *if and only if*

- $\phi(x) > f(x)$ for $x \in (x_1,\bar{x})$
- $\phi(x) < f(x)$ for $x \in (\bar{x}, x_2)$ such that $\phi(x) > 0$ and $f(x)>0$

Theorem 1.6. *If f(x) is differentiable then, a population model is locally stable if* $|f'(\bar{x})|<1$*, and if the model is locally stable then* $|f'(\bar{x})|\leq 1$.

Theorem 1.7. [4]

A continuous population model is globally stable if and only if it has no cycle of period 2. (That is, there is no point except $\bar{x}$ *such that f(f(x))=x.)*

Remark

If a population model happens to hit a cycle of period 2, then that population will oscillate back and forth around the equilibrium, and it will be stuck in that cycle until an outside source perturbs it. We can see this in figure 1.

A way of showing that f(x) has no cycle of period 2 was found by Cull [4] using enveloping by linear fractionals.

Definition 1.8. *A function* $\phi(x)$ ENVELOPS *a function f(x) if and only if*

- $\phi(x) > f(x)$ for $x \in (0,1)$
- $\phi(x) < f(x)$ for $x>1$ such that $\phi(x) > 0$ and $f(x)>0$

Definition 1.9. *A linear fractional function* $\phi(x)$ *is a function of the form:*

$$\phi(x) = \frac{1-\alpha x}{\alpha+(1-2\alpha)x} \text{where} \alpha \in [0,1)$$

with the following properties:

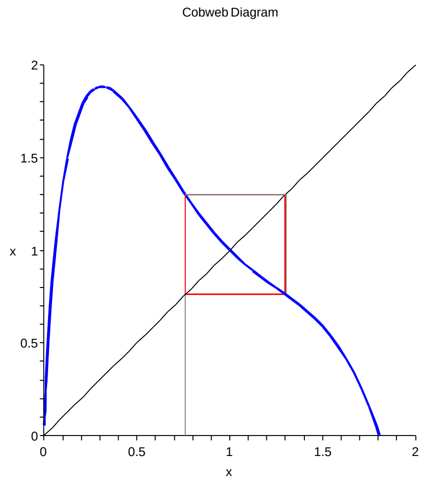

FIGURE 1. Model with a cycle of Period 2

- *ϕ(1) = 1*
- *ϕ'(1) = -1*
- *ϕ(ϕ(x)) = x*
- *ϕ'(x) < 0.*

The following are a couple examples of globally stable population models and their enveloping functions.

Example 1.10. *The function in figure 2 is model 2 with r = 1: f(x) = x(2-x). It is being enveloped by the line g(x) = 2-x, which is a linear fractional with α=1/2.*

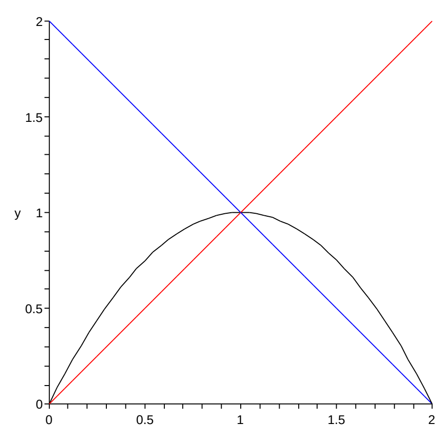

FIGURE 2. Model 2 being Enveloped

Example 1.11. *The function in figure 3 is model 3 with r = 2: f(x) = x(1-2*$\ln x$*). It is being enveloped by* $\frac{3-2x}{2-x}$

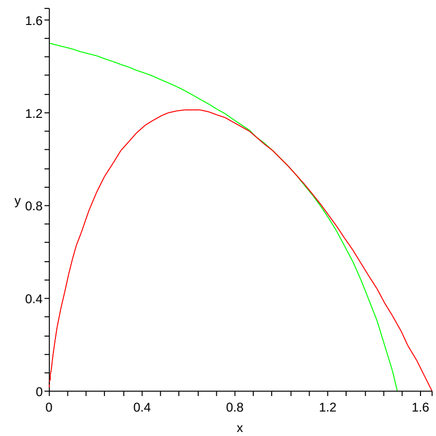

FIGURE 3. Model 3 being Enveloped

Theorem 1.12. [3]

Let $\phi(x)$ be a monotone decreasing function which is positive on $(0,x_-)$ and so that $\phi(\phi(x)) = x$. Assume that f(x) is a continuous function such that:

- $\phi(x) > f(x)$ *on* $(0, 1)$
- $\phi(x) < f(x)$ *on* $(1, x_-)$
- $f(x) > x$ *on* $(0, 1)$
- $f(x) < x$ *on* $(1, x_\infty)$
- $f(x) > 0$ *on* $(1, x_\infty)$

then for all $x \in (0, x_\infty)$, $\lim_{k\to\infty} f^{(k)}(x) = 1$.

The above theorem, when cast in a different light, gives the two following corollaries:

Corollary 1.13. [3]

If $f_1(x)$ is enveloped by $f_2(x)$, and $f_2(x)$ is globally stable, then $f_1(x)$ is globally stable.

Corollary 1.14. [3]

If f(x) is enveloped by a linear fractional function then f(x) is globally stable.

2. STABILITY IMPLIES ENVELOPING

It was previously shown by Cull [3] that enveloping by a self-inverse function implied global stability. Cull [3] was able to show that f(x) did not even need to be continuous; it merely needed to be enveloped and it would be globally stable. It is not, however, readily apparent that one can say global stability implies enveloping. A counter-example is the following:

Example 2.1. *The following model seen in figure 4 is defined as a piecewise function:*

$$\begin{aligned} f(x) &= x(2-x) && on\ 0 \le x \le 1 \\ &= 0.5 && on\ 1 < x < 1.5 \\ &= 1 && on\ x \ge 1.5 \end{aligned}$$

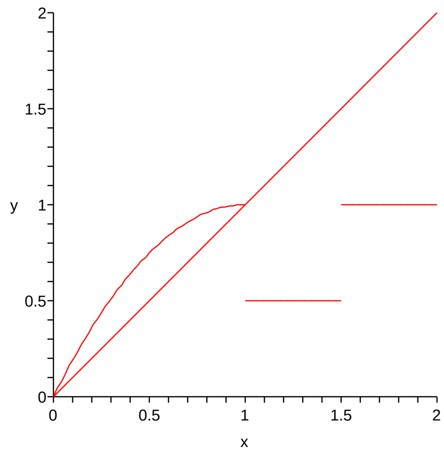

FIGURE 4. Globally Stable Discontinuous Function

This model is globally stable, yet it cannot be enveloped by any sort of self-inverse function. There is no way for a continuous ϕ(x) to be greater than or equal to 1 on the interval [0,1] and be less than 0.5 on the interval (1, x_∞).

So global stability does not imply enveloping for discontinuous models. Continuous models, however, won't be able to do what discontinuous ones can, namely "jump" at x=1. When a population model is plotted with its inverse, one will notice two regions which are symmetric to each other across the line y=x. A construction of an enveloping function can be made, where each point in the first region has a mirror point in the second region, and are joined at the point (1,1).

The picture seen in figure 5 has one of the simple population models plotted alongside its inverse. The uppermost region, enclosed by f^{-1}(x) and f(x), has a mirror image on the right side, where it is again enclosed by f^{-1}(x) and f(x).

Theorem 2.2. *Assume a population model f(x) is continuous and is globally stable. Then, f(x) can and is enveloped by a self-inverse function ϕ(x).*

Proof. Looking at the picture in figure 5, it should be clear from symmetry that any point picked in the upper region which is bounded by f(x) and f^{-1}(x) will have a mirror point in the rightmost region by definition of the inverse. Thus, a monotone decreasing self-inverse function can be constructed point by point until the model is enveloped.

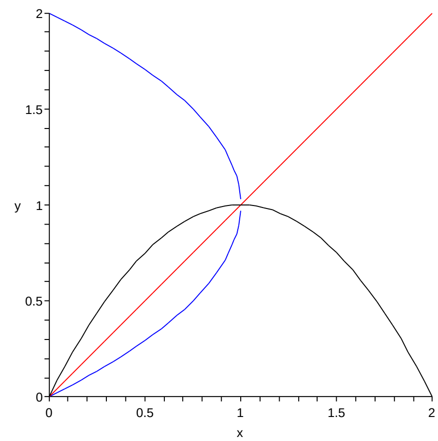

FIGURE 5. f(x) plotted against f^{-1}(x)

If there exists a point (x_0, y_0) in one of the regions, then just define a function $\phi(x)$ such that $\phi(x_0) = y_0$ and $\phi(\phi(x))$ = x for all x.

□

This shows that for one dimensional population models, a continuous model is globally stable iff if it can be enveloped by a self-inverse, monotone decreasing function.

3. LOCAL STABILITY IMPLIES LOCAL ENVELOPING

We will now extend the previous result to models which are locally stable.

What happens outside of the local basin is unpredictable at best. Some of the points may converge to the equilibrium point $\bar{x}$, some may converge to a stable period 2 cycle, yet others may ”crash”.

For example, in a slightly modified version of model 3, even a slight variation in initial population can be the difference between population crash and population convergence.

The models in figures 6 and 7 are

$$f(x) = x(1-(2+2(1-x)+2(1-x)^2)ln(x))$$

In the first picture, seen in figure 6, the initial population is set at x_0 = 0.193, which is well outside of the local basin. It jumps around a bit before jumping inside of the basin and eventually converging.

However, as can be seen in figure 7, changing the initial population to x_0=0.194, which is still outside the local basin, and it crashes.

What happens outside the local basin is unpredictable, but some nice things can be said about inside the basin. Since it took global stability plus continuity to imply enveloping, it would be reasonable to assume that it would take local stability plus continuity to imply local enveloping.

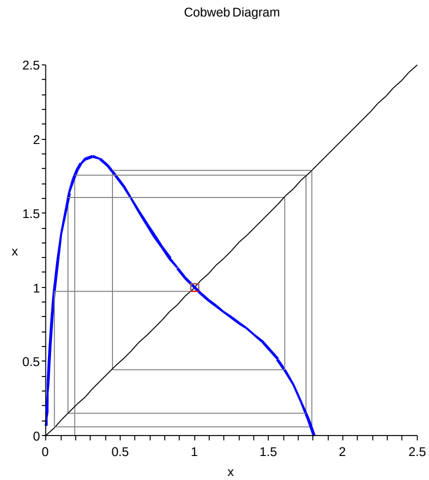

FIGURE 6. Web Plot of a Modified Model 3 with Initial Population of x_0=0.193

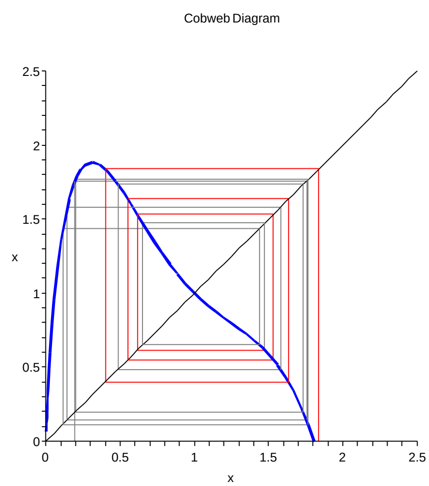

FIGURE 7. Web Plot of a Modified Model 3 with Initial Population of x_0=0.194

Take any f(x) which is locally but not globally stable. Since there is a finite region in which stability occurs, then **Theorem 1.9** and **Theorem 2.2** can be generalized to cover locally stable cases. By replacing 0 with x_1 and x_∞ with x_2, these theorems can be extended to locally stable models.

Theorem 3.1. *If any continuous population model f(x) is locally stable in the local basin region of (x_1,x_2), then that model can be locally enveloped by a monotone decreasing self-inverse function* $\phi(x)$.

Proof. Due to the fact that there exists two regions, which are mirrors of each other, both bounded by f(x) and $f^{-1}(x)$, then it can be trivially said that for any point picked in one of the regions, a mirror point exists in the mirror region. Then simply construct a function $\phi(x)$ such that $\phi(x_0) = y_0$ and $\phi(\phi(x))$ = x for all x for a point (x_0, y_0) in one of the bounded regions.

□

This allows us to talk about stability and enveloping in any case where stability exists for a finite region.

One example of a locally but not globally stable population model is f(x) = x*(x-3/2)*(-2-(x-1)-6*$(x-1)^2$). In this case, one fairly easy method to solve for an enveloping function, assuming it's a linear fractional, is as follows:

- Solve for f(f(x)) = x and find the two points **a** and **b** that are closest to $\bar{x}$=1, but not equal.
- Plug the two points **a** and **b** into the linear fraction for **x** and **y** and solve for α

$$\mathbf{b} = \frac{1-\alpha\mathbf{a}}{\alpha-(1-2\alpha)\mathbf{a}}$$

The enveloping function now merely takes the form of the linear fractional, while using the α that has just been found.

Using the above method on f(x) = x*(x-3/2)*(-2-(x-1)-6*$(x-1)^2$), α is found to be 0.5434746440. Plugging in and plotting, the graph in figure 8 is achieved.

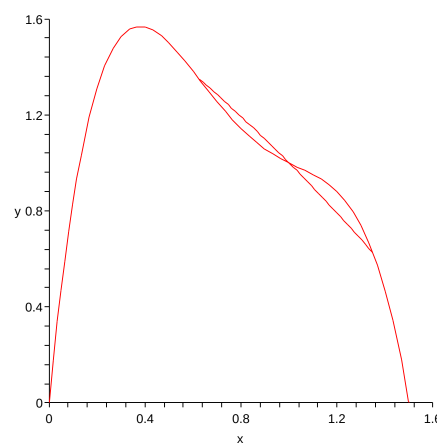

FIGURE 8. Locally Stable Model Being Enveloped by a Linear Fractional

This method won't always work, many times it takes trial and error and/or guesswork. The only constraint on what lines/curves which are picked is that they be monotone decreasing and self-inverse.

4. ENVELOPING AND THE STRAIGHT LINE

It is critical to note that ϕ(x) can not always be a straight line. Specifically, when f'(1)=-1, it is generally impossible to envelop with a straight line. For $\phi(x)$ to be greater than f(x), it would have to have a slope less than -1 for x < 1, thus to be self-inverse it would have to have a slope greater than -1 for x > 1, and straight lines generally do not have these properties.

However, curves and piecewise functions that are self-inverses are perfectly acceptable.

Example 4.1. *Figure 9 is an example of a continuous, globally stable model which can only be enveloped by a piecewise function. The reason for this is because a potential self inverse function* $\phi(x)$ *would have to be above the point (3, 0.5) and pass through the point (1, 1). That implies that the only possible non piecewise function that could do that would be a concave up function such as 1/x. However, due to the fact that the particular population model in figure 9 drops to 0 at x=4 means that a concave up function would not be less than f(x) for all x.*

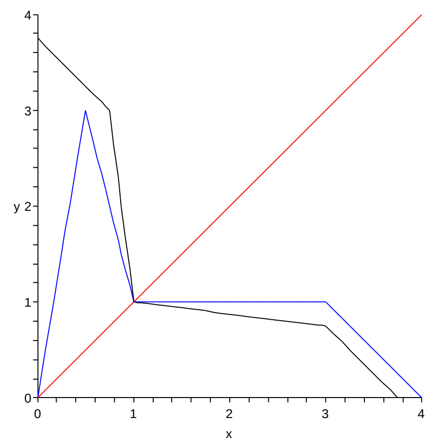

FIGURE 9. Globally Stable Model Being Enveloped by a Piecewise Function

This just goes to show that there's not necessarily one easy formula for what the enveloping curve is, or how many functions it must contain (if piecewise), but that it is quite alright if the enveloping function is piecewise.

5. Speed and Stability vs. Growth and Degree

5.1. **Speed vs. Growth.** Another thing some of the simple population models predict is a negative relation between the speed to equilibrium and the growth rate in the models that depend solely on the growth rate. When I talk about speed, I talk in the loose sense of how many iterations it takes for a given x_0 to reach equilibrium. When the growth rate **r** is very small in simple models, the speed is very fast, and thus it takes very few iterations to reach the equilibrium point. As r gets bigger (while f(x) is still globally stable), the speed slows down.

For most simple models, there is a critical growth rate r_o such that for all $\mathbf{r} \leq r_o$,the model is globally stable, and for $\mathbf{r} > r_o$, the model will lose all stability (global and local).

The number of iterations it takes to reach equilibrium increases slowly with the growth rate, up until the critical growth rate, at which point the convergence of any x_o to $\bar{x}$ becomes very slow.

Example 5.1. *Model 2, which is a simple population model of the form* ***f(x)=x(1+r(1-x))****, is shown below in two time graphs, both of which start with the initial condition of* x_o*=0.04. The first 50 iterations are shown.*

They differ only in their growth rate; the first plot (figure 10) shows model 2 with a growth rate of **r***=1.9. The second plot (figure 11) displays model 2 with the critical growth rate of* $r_o = 2$*.*

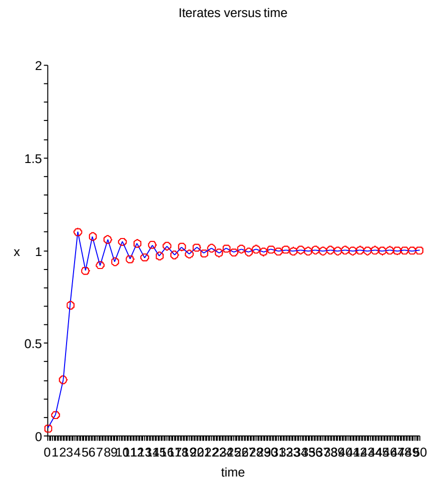

FIGURE 10. Model 2 Time Iterate Graph, **r** = 1.9

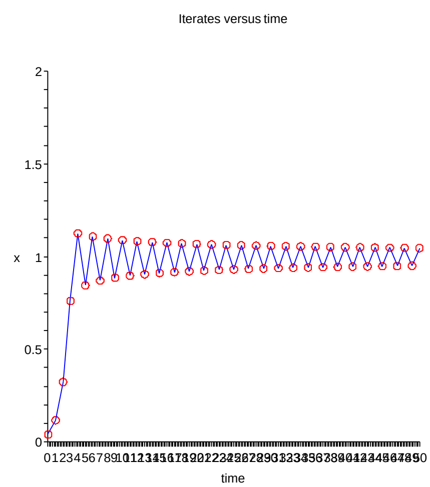

FIGURE 11. Model 2 Time Iterate Graph, **r** = 2.0

One way to see the effect that the growth rate will have on a function is by looking at a bifurcation of the function. Bifurcation plots demonstrate the long term behavior of a population which is affected by a range of reproductive rates and initial population values. Programs that generate the maps will iterate the function an infinite amount of times (in general, a couple hundred represents infinity) but does not plot these population values. This is done in order to eliminate bifurcation transients. It will then plot the population value after an additional few hundred iterations as the vertical axis value, and the reproductive rate as the horizontal axis value. One will see a single line where the population is at equilibrium and then see it fork in two, or bifurcate, representing an oscillation of period 2. It can then bifurcate again into oscillations of period 4 and so on until chaotic behavior is achieved.

The picture seen in figure 12 is a bifurcation plot of simple population model 1. f(x) = x$e^{r(1-x)}$. The horizontal line on the left of the plot is x=1. It runs normal until r=2, when the function loses global stability. It then bifurcates into two curves for an interval of r, and this shows how when the growth rate gets too large, stability is lost. In this case, a period 2 cycle is now apparent in the model (thus the fork in the road).

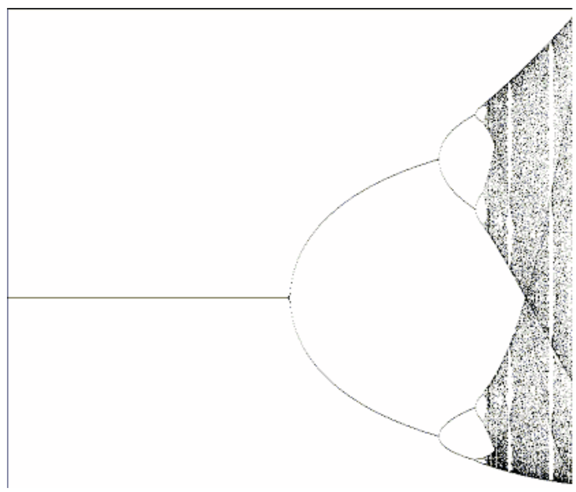

FIGURE 12. Bifurcation Plot of Simple Population Model 1

5.2. **Stability vs. Degree.** One of the things that affects the stability of a model is the degree of the polynomial in the function. Most one dimensional models are of the form f(x)=x*h(x), so increasing the degree of h(x) is one way to increase the complexity of the model.

For example, let's take a look at a modified model 3, which is generally of the form f(x)=x(1-rln$x$). Instead, we will substitute $r_0 + r_1(x-1) + r_2(x-1)^2$ for r.

To make simplify things, we will compare model 3 with r=2 vs. the modified model 3 with r_0=r_1=r_2=2.

The normal model 3 (seen in figure 13), while a bit slow in approaching the equilibrium point, is globally stable.

The modified model 3 (seen in figure 14), however, is not globally stable.

But, when inside the local basin, the speed to equilibrium is no slower than it was with the standard model 3 (figure 15).

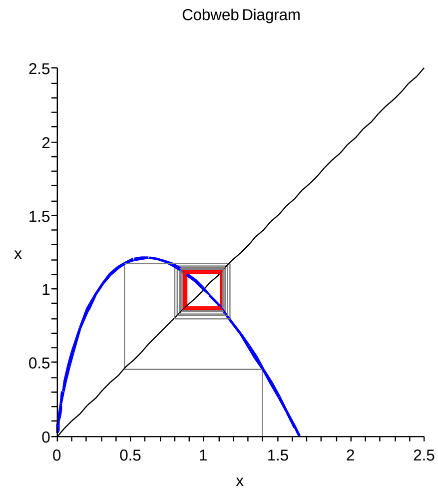

FIGURE 13. Model 3 Cobweb Graph, $x_0 = 1.4$

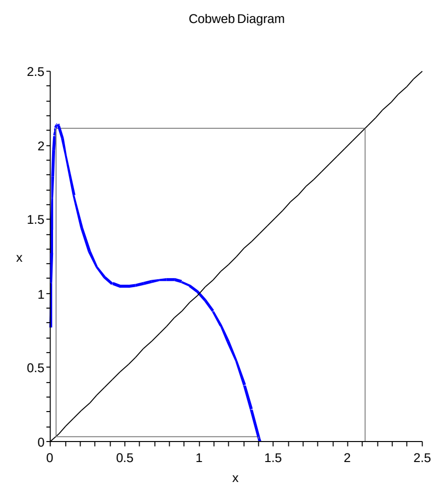

FIGURE 14. Modified Model 3 Cobweb Graph, x_0=1.4

6. CONCLUSION

One dimensional difference equations can display remarkably complex behavior despite being a simplification of what's actually going on.

We have shown that global stability plus continuity implies enveloping by a self-inverse function.

This has been extended to include local stability, meaning that if there exists a finite interval of the function that is locally stable and continuous, that portion of the function can be enveloped by a self-inverse function.

It was then shown that the enveloping function need not be a straight line or a single linear fractional. In other words, the enveloping function may be piecewise as long as it is self-inverse.

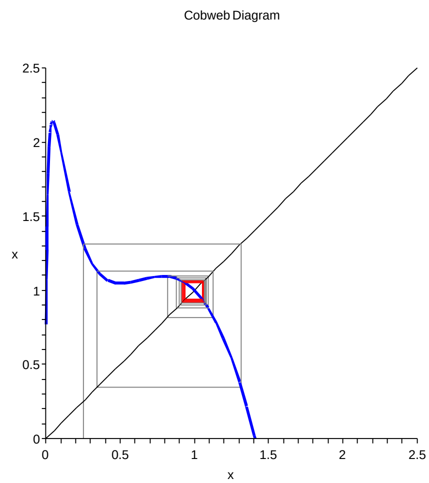

FIGURE 15. Modified Model 3 Cobweb Graph, x_0=0.25

Lastly, we looked at how increasing the growth rate r or the degree of the polynomial affected the stability and complexity of the population model. This was done using iteration plots and a bifurcation plot.

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