SOME CLASSIFICATIONS OF FREE CURVES IN EUCLIDEAN SPACE

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ABSTRACT. We begin by providing a complete classification of second order free homotopy classes in $\mathbb{R}^2$. Following that, we derive a minimum number of $j^{\,th}$ order free homotopy classes of curves in $\mathbb{R}^n$. Finally, we discuss possible methods of completely classifying second through fourth order free curves in $\mathbb{R}^4$.

1. INTRODUCTION

Stephen Smale and Mikhael Gromov have both contributed significantly to the study of homotopies. Their work in regular and free homotopies, respectively, apply directly to the material in this paper. A regular curve in a smooth manifold M is an immersion $K : S^1 \longrightarrow M$ whose derivative never vanishes. Smale classified regular curves on all 2-manifolds by homotopy. A $j^{\,th}$ order free, or non-degenerate, curve K in $\mathbb{R}^n$ is an immersion $K : S^1 \longrightarrow \mathbb{R}^n$ whose curvatures up to the $j^{\,th}$ order do not vanish. More precisely, the first j derivatives of a K are linearly independent. The notion of freedom only makes sense for $2 \leq j \leq n$ because in $\mathbb{R}^n$, any $n+1$ element set of vectors is linearly dependent. In [Gr], Gromov classified free curves in $\mathbb{R}^n$ up to and including $(n-2)^{\,th}$ order freedom by means of his H-Principle. His classification, however, does not provide a homotopy construction and therefore leaves something to be desired. In Section 3, we derive a complete classification of second order free curves in $\mathbb{R}^2$ and provide the construction of a homotopy, due to Whitney, between any two freely homotopic curves in $\mathbb{R}^2$. Inspired by the work of Feldman and Little in classifying free curves in $\mathbb{R}^3$, we provide a minimum classification of $j^{\,th}$ order free curves in $\mathbb{R}^n$ that is possibly the complete classification. Finally, in Section 5, we discuss a possible method for completely classifying free curves in $\mathbb{R}^4$. It will first be necessary to define some terms and establish some notation.

2. DEFINITIONS AND NOTATION

Definition 2.1. *Let X and Y be topological spaces and let $f, g : X \longrightarrow Y$ be continuous functions. A* homotopy *between f and g is a continuous function $H : X \times [0,1] \longrightarrow Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. If there is a homotopy between f and g, we say that f and g are homotopic and write*

$$f \cong g.$$

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Definition 2.2. *Let $H : X \times [0,1] \longrightarrow Y$ be a homotopy. Define the function $H_t : X \longrightarrow Y$ by*

$$H_t(x) = H(x,t).$$

Definition 2.3. *A* regular curve *in $\mathbb{R}^n$ is an immersion $K : S^1 \longrightarrow \mathbb{R}^n$ such that for all $s \in S^1$, $K'(s) \neq 0$. Denote the set of all regular curves in $\mathbb{R}^n$ as $\mathcal{R}^n$.*

Definition 2.4. *A j^{th}* order free curve *in $\mathbb{R}^n$, $1 < j \leq n$, is an immersion $K : S^1 \longrightarrow \mathbb{R}^n$ such that $\{K^{(i)}\}^{1 \leq i \leq j}$ is linearly independent, where $K^{(i)}$ denotes the i^{th} derivative of K. Denote the set of all j^{th} order free curves in $\mathbb{R}^n$ as $\mathcal{F}_j^n$. Note that $\mathcal{F}_j^n \subset \mathcal{R}^n$. For the sake of convenience, let $\mathcal{F}_1^n = \mathcal{F}_0^n = \mathcal{R}^n$.*

Definition 2.5. *Let $H : X \times [0,1] \longrightarrow \mathbb{R}^n$ be a homotopy. We say that H is a j^{th}* order free homotopy *if for every $t \in [0,1]$, $H_t \in \mathcal{F}_j^n$ and H_t is at least j times differentiable on $[0,1]$. Define $\mathcal{F}\,\mathcal{C}_j^n$ to be the set corresponding to the j^{th} order free homotopy classes of $\mathcal{F}_j^n$. That is, let $\mathcal{F}\,\mathcal{C}_j^n$ be a partitioning of $\mathcal{F}_j^n$ such that $K_0, K_1 \in \mathcal{J} \in \mathcal{F}\,\mathcal{C}_j^n$ if and only if there is a j^{th} order free homotopy between K_0 and K_1. We say that H is a* regular homotopy *if for every $t \in [0,1]$, H_t is a regular curve and H is differentiable on $[0,1]$.*

Definition 2.6. *Let $f : X \longrightarrow (\mathbb{R}^n - \{0\})$ be a continuous map. Then the* spherical projection *$f_\varpi : X \longrightarrow S^{n-1}$ of f is defined by*

$$f_\varpi(s) = \frac{f(s)}{|f(s)|}.$$

Definition 2.7. *If $K \in \mathcal{R}^n$, then we define the* tangent indicatrix *of K to be $(K')_\varpi$.*

Definition 2.8. *We define a function $\chi : S^1 \times S^1 \longrightarrow S^1$ by letting $\chi(s_1, s_2)$ be the angle between s_1 and s_2 measured counter clockwise beginning at s_1.*

Definition 2.9. *Let $\mathcal{T} : \mathcal{F}_2^2 \times S^1 \longrightarrow S^1$ be the function defined by*

$$\mathcal{T}(K,s) = \chi((K')_\varpi(s), (K'')_\varpi(s)).$$

Then $\mathcal{T}(K,s)$ is continuous on S^1 for a fixed curve K.

3. A Complete Classification of $\mathcal{F}_2^2$

What follows is a complete classification of the second order free curves in $\mathbb{R}^2$. In this section only, we unambiguously write ”free” in place of ”second order free”. To classify the set, we assign each curve one of two free indices and show that two curves are second order freely homotopic if and only if they are regularly homotopic and share the same free index. Note that the rotation number and the homotopy used to prove the main theorem are both due to Whitney. An explanation of the homotopy may be found in [Wh].

3.1. **Free Index.**

We assign to each free curve K one and only one of two ”free indices”.

Lemma 3.1. *Let $K \in \mathcal{F}_2^2$. Then for all $s \in S^1$, $\mathcal{T}(K,s) \neq 0, \pi$.*

Proof. Without loss of generality, assume $\mathcal{T}(K,s_0) = \pi$. Then the angle between $K'(s_0)$ and $K''(s_0)$ is π and for some $\lambda \in \mathbb{R}$, $K''(s_0) = \lambda K'(s_0)$. Thus K is not free. □

Lemma 3.2. *Let $K \in \mathcal{F}_2^2$. Then $\{\mathcal{T}(K,s)\}_{s\in S^1}$ is contained completely within one of the open intervals $(0,\pi) \subset S^1, (-\pi,0) \subset S^1$.*

Proof. Assume that $\mathcal{T}(K,s_0) \in (0,\pi)$ and $\mathcal{T}(K,s_1) \in (-\pi,0)$. Then, by continuity, for some $s_2 \in S^1$, $\mathcal{T}(K,s_2) = 0$ or π, contradicting freeness of K. □

Definition 3.3. *Let $K \in \mathcal{F}_2^2$. If $\{\mathcal{T}(K,s)\}_{s\in S^1} \subset (0,\pi)$, we assign K the free index of 1. If $\{\mathcal{T}(K,s)\}_{s\in S^1} \subset (-\pi,0)$, we assign K the free index of -1.*

3.2. **The Free Homotopy Theorem.** We now completely classify $\mathcal{F}_2^2$ by utilizing $\mathcal{T}$ and Whitney's regular homotopy.

Lemma 3.4. *Let $K_0, K_1 \in \mathcal{F}_2^2$ have free indices of 1 and -1, respectively. Then there exists no free homotopy between K_0 and K_1.*

Proof. Assume that there exists a free homotopy H between K_0 and K_1. Then for all $s \in S^1$, $\mathcal{T}(H_0,s) \in (0,\pi)$ and $\mathcal{T}(H_1,s) \in (-\pi,0)$. Continuity of H implies that for some $s_0 \in S^1$ and $t_0 \in [0,1]$, $\mathcal{T}(H_{t_0},s_0) = 0$ or π. Then H_{t_0} is not free and H is not a free homotopy. □

Lemma 3.5. *Let $K_0, K_1 \in \mathcal{F}_2^2$ each have the same free index and the same rotation number. Then there exists a free homotopy H between K_0 and K_1.*

Proof. Without loss of generality, assume that K_0 and K_1 have free indices of 1. By the definition of a free curve, K_0 and K_1 are regular curves. By Whitney's Deformation Theorem, K_0 and K_1 are regularly homotopic. We present the proof of the existence of a regular homotopy between K_0 and K_1 found in Whitney's paper and show that such a homotopy is free.

Let g_0 and f_1 be parameterizations of K_0 and K_1 such that for all $s \in S^1$ and for some $L_0, L_1 \in \mathbb{R}$,

$$|g_0'(s)| = L_0, \qquad |f_1'(s)| = L_1.$$

Then the homotopy

$$g(s,t) = g_0(0) + [t\frac{L_1}{L_0} + (1-t)][g_0(s) - g_0(0)]$$

is regular. Since g_0 is free, for all $t \in [0,1]$,

$$g_t'(s) = [t\frac{L_1}{L_0} + (1-t)]g_0'(s)$$

is linearly independent of

$$g_t''(s) = [t\frac{L_1}{L_0} + (1-t)]g_0''(s)$$

and each g_t is free. Thus K_0 is freely homotopic to g_1 and for all $s \in S^1$,

$$|g_1'(s)| = L_1.$$

Let I be the inclusion map $I : S^1 \longrightarrow \mathbb{R}^2$. Let $h : S^1 \times [0,1] \longrightarrow \mathbb{R}^2$ be the homotopy defined by

$$h(s,t) = L_1 I(t(f_1')_{\varpi}(s) + (1-t)(g_t')_{\varpi}(s)).$$

Define $f' : S^1 \times [0,1] \longrightarrow \mathbb{R}^2$ by

$$f'(s,t) = h(s,t) - \int_0^{2\pi} h(r,t)dr.$$

Let

$$f(s,t) = g_1(0) + t[f_1(0) - g_1(0)] + \int_0^s f'(r,t)dr$$

Then f is a regular homotopy between g_1 and f_1. Since g_1 and f_1 both have a free index of 1, for all$s \in S^1$, $\mathcal{T}(g_1,s), \mathcal{T}(f_1,s) \in (0,\pi)$. Therefore $(f_1')_\varpi$ and $(g_1')_\varpi$ are monotonically increasing and for all $t \in [0,1]$,

$$(h_t)_\varpi = t(f_1')_\varpi + (1-t)(g_t')_\varpi$$

is monotonically increasing. Thus for all $s \in S^1$ and for all $t \in [0,1]$, $(f_t')_\varpi$ is monotonically increasing and $\mathcal{T}(f_t,s) \in (0,\pi)$. We see that f_t is free with free index 1. Thus g_1 is freely homotopic to f_1 and there exists a free homotopy between K_0 and K_1. □

Theorem 3.6. *Let $K_0, K_1 \in \mathcal{F}_2^2$. There exists a free homotopy between K_0 and K_1 if and only if K_0 and K_1 have the same free index and the same rotation number.*

Proof. This is directly implied by 3.4 and 3.5. □

4. A Coarse Classification of $\mathcal{F}_j^n$

It is difficult to prove the existence of a free homotopy between two curves $K_0, K_1 \in \mathcal{F}_j^n$, but by defining corresponding curves in different range spaces, we may relatively easily show that certain curves are not freely homotopic. Complete classifications of $\mathcal{F}_2^3$ and $\mathcal{F}_3^3$ are provided in [Fm] and [Li], so this coarse classification shall apply to $\mathcal{F}_j^n$ for $n \geq 4$. Denote by $SO(n)$ the special orthogonal group of $n \times n$ matrices acting on $\mathbb{R}^n$. Let $V_{k,n}$ stand for the Stiefel manifold of orthonormal $k-$frames in $\mathbb{R}^n$. We will project the derivatives of free maps into these spaces and show that when the projections cannot be homotopied, the original curves may not be freely homotopied. We cannot guarantee, however, that the original curves can be freely homotopied if the projections can. There is no promise that there are closed curves in $\mathbb{R}^n$ which correspond to the intermediate curves in the frame homotopies.

4.1. **The Fundamental Group of $SO(n)$.** Fix $n \geq 4$. A well known fact, stated by [Sh], is that for $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$. Representing a point in $SO(n)$ as a column matrix of row n-vectors, the standard generator $\alpha(n) : S^1 \longrightarrow SO(n)$ of $\pi_1(SO(n))$ is defined by

$$\begin{bmatrix} \cos(s) & \sin(s) & 0 & \ldots & 0 \\ -\sin(s) & \cos(s) & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

For $1 \leq i, j \leq n$, let $\gamma_{i,j} : S^1 \longrightarrow \mathbb{R}$ be defined by

$$\gamma_{1,j}(s) = \begin{cases} \cos(s), & \text{if } j = 1 \text{ and } i = 1, \\ \sin(s), & \text{if } j = 2 \text{ and } i = 1, \\ \cos[(1+j)s], & \text{if } j > 2 \text{ and } i = 1, \\ \sin(js), & \text{if } j > 2 \text{ and } i = 1, \\ \gamma'_{i-1,j}(s), & \text{if } i > 1, \end{cases}$$

Then, let $\beta_i : S^1 \longrightarrow \mathbb{R}^n$ be defined by

$$\beta_i(s) = \begin{cases} (\gamma_{i,1}\ ,\ \gamma_{i,2}\ ,\ \cdots\ ,\ \gamma_{i,(n-1)}\ ,\ 0), & \text{if } n \text{ is odd,} \\ (\gamma_{i,1}\ ,\ \gamma_{i,2}\ ,\ \cdots\ ,\ \gamma_{i,(n-2)}\ +\ \gamma_{i,(n)}\ ,\ \gamma_{i,(n-1)}\ ,\ 0), & \text{if } n \text{ is even,} \end{cases}$$

Finally, let $\omega(n) : S^1 \longrightarrow SO(n)$ be the frame curve obtained by applying the Gram-Schmidt process to the rows of the frame curve

$$\begin{bmatrix} \beta_1 \\ \cdots \\ \beta_n \end{bmatrix}$$

It can be shown that $\omega(n)$ is homotopic to $\alpha(n)$. Thus $\omega(n)$ is a generator of $SO(n)$. Let $\psi(n)$: $S^1 \longrightarrow SO(n)$ be defined by

$$\psi(n)(s) = \omega(n)(2s).$$

Then $\psi(n)$ is the curve obtained by traversing $\omega(n)$ twice and is thus homotopic to the curve α^2 obtained by traversing α twice. The generator of $SO(n)$ is α, so α and α^2 are not homotopic. Since ω is homotopic to α and ψ is homotopic to α^2, ω and ψ cannot be homotopic.

4.2. **A Nontrivial Minimum Value of** $|\,\mathcal{FC}^n_{n-1}\,|$**.** Let $i \in \{0,1\}$ and let $f_i \in \mathcal{R}^n$. Then the $(n-1)$-frame

$$A(f_i,s) = \{f_i^{(j)}(s)\}^{1\leq j\leq n-1}$$

corresponding to f_i at s is linearly independent if and only if the Gram-Schmidt process yields a frame $B(f_i,s)$ which we denote by

$$B(f_i,s) = \{g_{i,j}(s)\}^{1\leq j\leq n-1}.$$

Let $E(f_i,s)$ be the unique oriented vector that is orthonormal to $B(f_i,s)$. Then if $f_i \in \mathcal{F}^n_{n-1}$, the frame curve $O_i : S^1 \longrightarrow SO(n)$ defined by

$$O_i(s) = \begin{bmatrix} g_{i,1}(s) \\ \cdots \\ g_{i,n-1}(s) \\ E(f_i,s) \end{bmatrix}$$

corresponds to an element of $\pi_1(SO(n))$. Define

$$f_0 = \begin{cases} (\sin(s),\ \cos(s),\ \frac{\sin(2s)}{2},\ \frac{\cos(2s)}{2},\ \ldots,\ \frac{\sin[2(n-1)s]}{2(n-1)},\ \frac{\cos[2(n-1)s]}{2(n-1)},\ 0), & \text{if } n \text{ is odd,} \\ (\sin(s),\ \cos(s),\ \frac{\sin(2s)}{2},\ \frac{\cos(2s)}{2},\ \ldots\ ,\ \frac{\cos[(n-2)s]}{n-2}\ +\ \frac{\cos(ns)}{n},\ \frac{\sin(ns)}{n},\ 0), & \text{if } n \text{ is even,} \end{cases}$$

$$f_1 = \begin{cases} (\frac{\sin(2s)}{2},\ \frac{\cos(2s)}{2},\ \frac{\sin(4s)}{4},\ \frac{\cos(4s)}{4},\ \ldots,\ \frac{\sin[4(n-1)s]}{4(n-1)},\ \frac{\cos[4(n-1)s]}{4(n-1)},\ 0), & \text{if } n \text{ is odd,} \\ (\frac{\sin(2s)}{2},\ \frac{\cos(2s)}{2},\ \frac{\sin(4s)}{4},\ \frac{\cos(4s)}{4},\ \ldots\ ,\ \frac{\cos[2(n-2)s]}{2(n-2)}\ +\ \frac{\cos(2ns)}{2n},\ \frac{\sin(2ns)}{2n}, & \text{if } n \text{ is even,} \end{cases}$$

Then $f_i \in \mathcal{F}^n_{n-1}$. We see that an $(n-1)^{\,th}$ order free homotopy between f_0 and f_1 induces a homotopy between $B(f_0,*) = \omega(n)$ and $B(f_1,*) = \psi(n)$. Thus there are at least two $(n-1)^{\,th}$ order free homotopy classes in $\mathbb{R}^n$, and

$$|\mathcal{FC}^n_{n-1}| \geq 2.$$

4.3. A Nontrivial Minimum Value of $|\mathcal{FC}_n^n|$. It can be shown that if $\mathcal{J} \in \mathcal{FC}_{n-1}^n$,

$$\mathcal{J} \cap \mathcal{F}_n^n \neq \varnothing.$$

Let $K_0, K_1 \in \mathcal{F}_n^n$ be in distinct $(n-1)^{\,th}$ order free homotopy classes. Then K_0 and K_1 are not $n^{\,th}$ order freely homotopic. Thus

$$|\mathcal{FC}_{n-1}^n| \geq 2.$$

Now, let $i \in \{0,1\}$ and $h_i \in \mathcal{R}^n$. Let $h_0 \in \mathcal{F}_n^n$ and fix h_0 to be $(n-1)^{\,th}$ order freely homotopic to f_0. Let $h_{0r}^{(n)} : S^1 \longrightarrow \mathbb{R}^n$ be defined by letting $h_{0r}^{(n)}(s)$ be the reflection of $h_0^{(n)}(s)$ over the space spanned by $A(h_0, s)$. Then the function h_1 uniquely determined by

$$\begin{aligned} h_1^{(j)}(0) &= h_0^{(j)}(0), && for\ \ 0 \leq j \leq n-1, \\ h_1^{(n)} &= h_{0r}^{(n)}, \end{aligned}$$

is an element of $\mathcal{F}_{n-1}^n$ that, by an application of our classification of $\mathcal{F}_{n-1}^n$, cannot be $(n-1)^{\,th}$ or $n^{\,th}$ order freely homotopic to f_1. Define the function $\varsigma : \mathcal{F}_{n-1}^n \times S^1 \longrightarrow \mathbb{R}^1$ by

$$\varsigma(K, s) = K^n(s) \cdot E(K)(s).$$

Then ς may never take the value 0 and for every $s \in S^1$,

$$\varsigma(h_0, s) > 0 \qquad or \qquad \varsigma(h_0, s) < 0.$$

Without loss of generality, assume $\varsigma(h_0, s) > 0$. Then $\varsigma(h_1, s) < 0$ and h_0 cannot be $n^{\,th}$ order freely homotopic to h_1. For if $H(s,t)$ is an $n^{\,th}$ order free homotopy between the two curves, H_t must equal 0 for some $s \in S^1$ and $t \in [0,1]$, resulting in a contradiction. Thus

$$|\mathcal{FC}_n^n| \geq 3.$$

We may repeat the previous process by letting h_0 be $(n-1)^{\,th}$ order freely homotopic to f_1 to obtain an additional disjoint $n^{\,th}$ order free homotopy class of $\mathbb{R}^n$, yielding

$$|\mathcal{FC}_n^n| \geq 4.$$

4.4. A Trivial Minimum Value of $|\mathcal{FC}_j^n|$ for $j < n-1$. In forming a coarse classification of $\mathcal{F}_j^n$ for $j < n-1$, we cannot construct analogous unique curves in $SO(n)$ corresponding to elements of $\mathcal{F}_j^n$. Instead, we turn to the Stiefel manifold $V_{j,n}$. The space $V_{j,n}$ is the space of orthonormal $k-$frames in $\mathbb{R}^n$. This means that each element in $V_{j,n}$ is a j-element ordered set of orthonormal n-vectors. Let $v : S^1 \longrightarrow V_{j,n}$ and let $v_i(s)$ denote the $i^{\,th}$ element of $v(s)$. Then v is continuous if and only if each $v_i : S^1 \longrightarrow \mathbb{R}^n$ is continuous. Fix $1 < j < n-1$. Then according to [St],

$$V_{j,n} = SO(n)/SO(n-j).$$

Let $i \in \{0,1\}$ and let $f_i \in \mathcal{R}^n$. Then the j-frame

$$C(f_i, s) = \{f_i^{(u)}(s)\}^{1 \leq u \leq j}$$

corresponding to f_i at s is linearly independent if and only if the Gram-Schmidt process yields a frame $D(f_i, s)$ denoted by

$$D(f_i, s) = \{g_{i,u}(s)\}^{1 \leq u \leq j}.$$

Then if $f_i \in \mathcal{F}_j^n$, the frame curve $O_i : S^1 \longrightarrow V_{j,n}$ defined by

$$O_i(s) = \begin{bmatrix} g_{i,1}(s) \\ \dots \\ g_{i,j}(s) \end{bmatrix}$$

corresponds to an element of $\pi_1(V_{j,n})$. We would like to mimic the process we used for curves in $SO(n)$ and rule out homotopies between certain elements of $\mathcal{F}_j^n$, but according to [St], for $k \leq n-2$,

$$\pi_1(V_{k,n}) = \{0\}.$$

Thus O_0 and O_1 are homotopic and the homotopy obstruction we encountered in $\mathcal{F}_{n-1}^n$ and $\mathcal{F}_n^n$ is not present in $\mathcal{F}_j^n$. Thus, our coarse classification is

$$|\mathcal{F}C_j^n| \geq 1.$$

4.5. **In Favor of the Coarse Classification.** These coarse classifications may not seem very useful or specific, especially when one considers the direction of the inequalities. There is significant evidence, however, to suggest that these classifications are indeed the right ones.

Let $K_0, K_1 \in \mathcal{F}_j^n$. If their specific frame projections D_0 and D_1 (in $SO(n)$ or $V_{j,n}$, depending on the value of j) are homotopic through D, there is automatically a corresponding $j^{\,th}$ order free homotopy $H : [0,2\pi] \times [0,1] \longrightarrow \mathbb{R}^n$ whose frame projection is D. This does not ensure, however, that the intermediate curves of H are closed curves. Thus H is not necessarily a $j^{\,th}$ order homotopy between K_0 and K_1.

Additionally, similarities in the structures of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, for $n \geq 3$, hint at the possibility of the equality

$$\mid \mathcal{F}C_j^n \mid = \mid \mathcal{F}C_{j-1}^{n-1} \mid .$$

Noting the base cases of

$$\mid \mathcal{F}C_3^3 \mid = 4 \qquad \textit{and} \qquad \mid \mathcal{F}C_2^3 \mid = 2$$

will complete the conjecture.

5. Concerning the Classification of $\mathcal{F}_j^4$

In classifying $\mathcal{R}^2$, Whitney made use of the exponential representation of S^1. A Similar representation exists for S^3 that may be used in the future to aid in classifying $\mathcal{F}_j^4$.

5.1. **The Quaternions.** Let $SU(2)$ denote the special unitary group of degree 2 over the complex numbers. Let $\mathbb{H}$ denote the quaternions and let

$$\mathbb{SH} = \{q \in \mathbb{H} : |q| = 1\}.$$

Let $q \in \mathbb{H}$ and let $\{\hat{i}, \hat{j}, \hat{k}\}$ be the imaginary basis of $\mathbb{H}$. Then for some $a,b,c,d \in \mathbb{R}$,

$$q = a + b\hat{i} + c\hat{j} + d\hat{k}.$$

Define the function $\Re : Q \longrightarrow \mathbb{R}$ by

$$\Re(q) = a.$$

It is easy to check that the homeomorphism $Q : S^3 \longleftrightarrow \mathbb{SH}$ defined by

$$Q((a,b,c,d)) = a + b\hat{i} + c\hat{j} + d\hat{k}$$

preserves the notion of freedom. It is interesting to note that the function $M : \mathbb{H} \longrightarrow SU(2)$ defined by

$$M(q) = \begin{pmatrix} a-di & -b+ci \\ b+ci & a+di \end{pmatrix}$$

is a group isomorphism. Thus the function $M \circ Q$ introduces the group structure of $SU(2)$ onto S^3. It is believed that this structure may aid in classifying free curves in S^3 and thus help to classify $\mathcal{F}_j^4$. So far, we have left this avenue unexplored. The following, however, outlines an attempt at directly classifying second order free curves in S^3.

It is well known that $\mathbb{H}$ can be represented in an exponential format similar to the exponential representation of the complex numbers. For every $q \in \mathbb{H}$, there exist $R, \theta \in \mathbb{R}$ and $\nu \in \mathbb{SH}$ with $\Re(\nu) = 0$ such that

$$q = Re^{\nu\theta} = R(\cos\theta + \nu\sin\theta).$$

Note that if $q \in \mathbb{SH}$, then $R = 1$. For the sake of convenience, we will identify a pure quaternion $\nu = b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ with $(b,c,d) \in \mathbb{R}^3$. Let $j \in 1,2,3$. Let $K \in \mathcal{R}^4$ and let L be the tangent indicatrix of K. Then $K \in \mathcal{F}_{j+1}^4$ if and only if $L \in \mathcal{F}_j^4$. It is possible to find $\nu : S^1 \longrightarrow \mathbb{R}^3$ and $\theta : S^1 \longrightarrow \mathbb{R}$ such that L may be be expressed as a closed quaternion curve in exponential form by

$$L = \cos\theta + \nu\sin\theta.$$

By explicitly calculating the first and second derivatives of L and setting one to be a multiple of the other, it can be shown that L fails to be second order free if and only if for some $s \in S^1$,

$$(\theta'(s))^3\nu(s) + (\theta''(s))(\sin\theta(s))^2\nu'(s) = (\theta'(s))(\sin\theta(s))^2\nu''(s) + (\theta'(s))^2(\sin\theta(s))(\cos\theta(s))\nu'(s).$$

It is interesting to note that if we fix $\nu = (1,0,0)$ to form a quaternion curve in the complex plane, the quaternion exponential representation of L simplifies to the usual complex representation

$$L = \cos\theta + \imath\sin\theta$$

and the condition on L ceasing to be second order free simplifies to

$$\theta'(s) = 0.$$

If $\theta'(s) = 0$, θ ceases to be monotone. Thus, when restricted, our freeness condition on quaternion curves matches the condition we observed on complex curves.

6. Conclusion

We successfully provided a (perhaps redundant) complete classification of $\mathcal{F}_2^2$ and made a small start at classifying $\mathcal{F}_j^n$. Although no useful quaternion homotopy was found, $\mathbb{H}$ may still prove useful for classifying $\mathcal{F}_j^4$ by means of introducing a group structure on S^3.

A good place for future work to begin would be on free curves in $\mathbb{R}^4$. If a proper analogue between the quaternions and the complex numbers could be drawn, the classification of $\mathcal{F}_4^4$ and $\mathcal{F}_3^4$ might be made simple. Succeeding that, it would be useful to derive homotopies that agree with Gromov's classification of free curves in $\mathbb{R}^n$ up to $(n-2)^{th}$ order freedom.

REFERENCES

[Fm] E.A. Feldman. *Deformations of Closed Space Curves.* Journal of Differential Geometry, 2:67-75, 1968.
[Gr] Mikhael Gromov. *Partial Differential Relations.*Springer, 1986.
[Li] John Little. *Third Order Nondegenerate Homotopies of Space Curves.* Journal of Differential Geometry, 5:503-515, 1971.
[Sh] Unknown Author. http://mathsci.sharif.ac.ir/Files/34/Document/Chapter4.pdf. Available August 2005.
[St] Norman Steenrod. *The Topology of Fibre Bundles.* Princeton University Press, Princeton, 1951.
[Wh] Hassler Whitney. *On Regular Closed Curves in the Plane.* Compositio Mathematica, 4:276-284, 1937.

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