AN ALGORITHM FOR RECONSTRUCTION OF A CONVEX BODY FROM TWO POINT SOURCES

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ABSTRACT. This paper shows how to, given X-ray data of a convex body from two sources (where the line segment connecting the sources intersects the interior of the convex body), compute tangent lines at the points of intersection. These tangent lines provide us with an approximation of the convex body and we use this as a starting point for the reconstruction algorithm. We show the global error in reconstruction is bounded by the arbitrarily small local error in our initial approximation.

1. INTRODUCTION

Many questions in the field of geometric tomography are still unanswered. We focus on the problem of reconstructing a convex body K from its directed X-ray data from two sources. Previous work has been done on the situation where the line connecting the two point sources intersects the interior of K. It has been proven that in this situation, the two point sources uniquely determine a convex body. A method of finding the location of the two points of intersection has also been determined. One reconstruction algorithm has been published using this knowledge. We build from this previous knowledge to find tangent lines at the points of intersection, as well as create another algorithm for reconstruction. We also look at the error involved in our theory of reconstruction.

1.1. Background and Definitions.

Definition 1.1. *Throughout this paper, a* ***convex body****, K, is assumed to be a compact, convex subset of the plane with non-empty interior. The boundary of K is denoted ∂K.*

Definition 1.2. *Given a convex body K, a source s, and a ray φ from s, the* ***nearside point*** *is the closest point in $\varphi \cap K$ and the* ***farside point*** *is the furthest point in the intersection.*

Definition 1.3. *The* ***nearside function****, $r(\varphi)$, is determined by the set of near side points. Similarly, the* ***far side function****, $R(\varphi)$ is determined by the set of far side points.*

Definition 1.4. *A* ***directed X-ray*** *is the chord length of a convex body along a particular direction φ from a given point s, which is called the* ***point source****. The directed X-ray of K is equivalently defined as*

$$X(\varphi) = R(\varphi) - r(\varphi).$$

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Definition 1.5. *The* ***curvature operator****, $\mathcal{K}f(\varphi)$, characterizes the direction of concavity and is defined as*

$$\mathcal{K}f = f^2 + 2(f')^2 - ff''.$$

where $\mathcal{K}f(\varphi)$ is positive and takes on the sign of the signed curvature of the graph of the function f at φ. This operator is defined when f is C^2. [1]

K will be used throughout this paper to denote an arbitrary convex body. We will focus on the situation where K lies between two point sources, a and b, such that the line connecting a and b intersects the interior of K.

Theorem 1.6. *Suppose a and b are two points in the plane, and H and K are convex bodies with the same chord functions at a and b. If the interiors of H and K meet the line segment ℓ joining a and b, and a and b are exterior to H and K, then $H = K$.* [4]

This was proved by Gardner [4] in 1978 and revised in 1983, but was also independently proven by Falconer [3] in 1981. In Falconer's paper, he produced a formula that finds the intersection of ∂K and the line ℓ connecting our two point sources a and b.

Lemma 1.7. *If a and b are both interior points, then two X-ray functions, X_a and X_b, uniquely determine p and q, the intersection of ℓ, the line connecting a and b, and ∂K. If $a, b \notin K$, X_a and X_b may allow two possibilities for p and q: either p, q lie between a and b or p, q lie to one side of a, b.* [3]

Lemma 1.8. *Suppose that p and q are known and that a and b are either both interior or both exterior points. Then ∂K is determined in some neighborhood of p and q by the chord functions X_a and X_b.* [3]

Using these theorems, Dartmann [2] was able to create a reconstruction algorithm. He assumed the tangent lines at the points of intersection were vertical and found an approximation point of the convex body. From this point he added the X-ray data and found a new approximation. He continued this process until he reconstructed the entire body. He apparently was not aware that the tangent lines can be computed at the two points of intersection. His assumption that the tangent line is vertical limits his reconstruction algorithm.

2. Tangent Lines of K

Before we begin, let us define some notation: Nearside points are denoted $r_a(\varphi_n)$ and $r_b(\psi_n)$. Farside points are denoted $R_a(\varphi_n)$ and $R_b(\psi_n)$. The nearside and farside points at zero are denoted r_a and R_a, r_b and R_b. Nearside approximated points are denoted $\gamma_{a,n}$ and $\gamma_{b,n}$. Farside approximated points are denoted $\Gamma_{a,n}$ and $\Gamma_{b,n}$. Nearside and farside approximated points at zero are denoted γ_a and Γ_a, γ_b and Γ_b.

Since K is convex, ∂K has left and right tangent lines at every point. In particular, these are easily computable at the point of intersection from Falconer's results and some elementary calculations. We have the following theorem:

Theorem 2.1. *Given a convex body K and X-ray sources a and b such that the line passing through a, b intersects ∂K at two points p and q, it is possible to compute upper and lower tangent lines of*

∂K at p and q. In particular,

(1) $$r_a' = \frac{-r_a(R_a X_b' - r_b X_a')}{R_a R_b - r_a r_b},$$

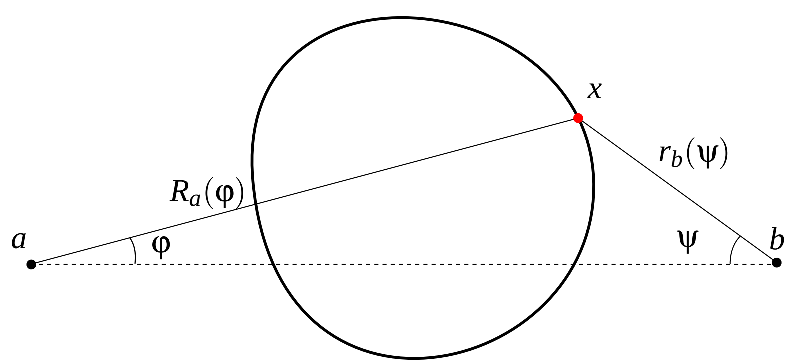
(2) $$r_b' = \frac{-r_b(R_b X_a' - r_a X_b')}{R_a R_b - r_a r_b},$$

(3) $$R_a' = \frac{-r_a(R_a X_b' - r_b X_a')}{R_a R_b - r_a r_b} + X_a',$$

(4) $$R_b' = \frac{-r_b(R_b X_a' - r_a X_b')}{R_a R_b - r_a r_b} + X_b'.$$

Proof. Because K is convex, we know that upper and lower tangent lines must exist at every point on ∂K. Let φ be the positive polar angle measured anticlockwise from point a and ψ be the angle measure clockwise from point b. Let $r_a(\varphi)$ and $R_a(\varphi)$ denote the nearside and farside functions from source a, respectively. Similarly, let $r_b(\psi)$ and $R_b(\psi)$ be the nearside and farside functions from source b. Consider a, b to be on the x-axis and any use of a and b in an equation will refer to the x-coordinate of a and b.

Any point $x \in \partial K$ can be represented in two different ways. Assuming x lies on the farside with respect to source a, x can be described by the coordinates $\big(r_a(\varphi), \varphi\big)$ or $\big(R_b(\psi), \psi\big)$.



With the triangle $\triangle axb$, the law of sines gives us

$$\frac{r_b(\psi)}{\sin(\varphi)} = \frac{R_a(\varphi)}{\sin(\psi)}.$$

By cross multiplying, we find that $r_b(\psi)\sin(\psi) = R_a(\varphi)\sin(\varphi)$. Differentiating this with respect to φ and treating ψ as a function of φ, we obtain

$$\left[r_b'(\psi)\sin(\psi) + r_b(\psi)\cos(\psi)\right]\psi'(\varphi) = R_a'(\varphi)\sin(\varphi) + R_a(\varphi)\cos(\varphi).$$

Solving for ψ' yields

(5) $$\psi'(\varphi) = \frac{R_a'(\varphi)\sin(\varphi) + R_a(\varphi)\cos(\varphi)}{r_b'(\psi)\sin(\psi) + r_b(\psi)\cos(\psi)}.$$

Substituting $\varphi = 0$ gives

$$\psi'(0) = \frac{R_a}{r_b}. \tag{6}$$

Note that R_a and r_b are shorthand for $R_a(0)$ and $r_b(0)$.

Alternatively, the law of cosines tells us

$$r_b(\psi)^2 = (b-a)^2 + R_a(\varphi)^2 - 2(b-a)R_a(\varphi)\cos(\varphi).$$

Differentiating with respect to φ gives us

$$2r_b(\psi)r_b'(\psi)\psi'(\varphi) = 2R_a(\varphi)R_a'(\varphi) - 2(b-a)R_a'(\varphi)\cos(\varphi) + 2(b-a)R_a(\varphi)\sin(\varphi), \tag{7}$$

and solving for $r_b'(\psi)$ yields

$$r_b'(\psi) = \frac{R_a(\varphi)R_a'(\varphi) - (b-a)R_a'(\varphi)\cos(\varphi) + (b-a)R_a(\varphi)\sin(\varphi)}{r_b(\psi)\psi'(\varphi)}.$$

Letting $\varphi = 0$ and substituting equation (6) for ψ', we obtain

$$r_b' = -\frac{r_b}{R_a}R_a'.$$

If we now assume x is on the nearside with respect to a and go through the same computations, we get

$$\varphi'(0) = \frac{R_b}{r_a} \qquad \text{and} \qquad r_a' = \frac{-r_a}{R_b}R_b'.$$

Note that

$$X_a' = R_a' - r_a' \qquad \text{and} \qquad X_b' = R_b' - r_b'.$$

After cross-multiplying our nearside derivatives and substituting our equations above solved for R_a' and R_b', we get the matrix equation

$$\begin{bmatrix} R_b & r_a \\ r_b & R_a \end{bmatrix} \begin{bmatrix} r_a' \\ r_b' \end{bmatrix} = \begin{bmatrix} -r_aX_b' \\ -r_bX_a' \end{bmatrix}.$$

Notice that $\det \begin{pmatrix} R_b & r_a \\ r_b & R_a \end{pmatrix} = R_aR_b - r_ar_b > 0$ because $R_aR_b > r_ar_b$. This ensures there are always solutions for r_a' and r_b'. Elementary linear algebra shows us

$$r_a' = \frac{-r_a(R_aX_b' - r_bX_a')}{R_aR_b - r_ar_b}$$
$$r_b' = \frac{-r_b(R_bX_a' - r_aX_b')}{R_aR_b - r_ar_b}.$$

We now have an expression for r_a' and r_b' involving only known quantities. We add the derivative of our X-ray data to find

$$R_a' = \frac{-r_a(R_aX_b' - r_bX_a')}{R_aR_b - r_ar_b} + X_a'$$
$$R_b' = \frac{-r_b(R_bX_a' - r_aX_b')}{R_aR_b - r_ar_b} + X_b'.$$

□

2.1. **Second Derivative.** If we assume $\partial K \in C^2$, we may calculate r_a'' and r_b''. Recall equation (7) from our first derivative formula.

$$2r_b(\psi)r_b'(\psi)\psi'(\varphi) = 2R_a(\varphi)R_a'(\varphi) - 2(b-a)R_a'(\varphi)\cos(\varphi) + 2(b-a)R_a(\varphi)\sin(\varphi).$$

Differentiating again and substituting $\varphi = 0$, we get

$$r_b'' = \frac{(R_a')^2 + R_aR_a'' - (b-a)R_a'' + (b-a)R_a - (r_b')^2(\psi'(0))^2 - r_br_b'\psi''(0)}{r_b(\psi'(0))^2}.$$

Working from equation (5), we can find ψ'',

$$\psi''(0) = \frac{2r_b^2R_a' - 2R_a^2r_b'}{r_b^3}.$$

Substituting this in gives

$$r_b'' = \frac{\left[(R_a')^2 + R_aR_a'' - (b-a)R_a'' + (b-a)R_a - 2r_b'R_a' + \frac{1}{r_b^2}(r_b')^2R_a^2\right] r_b}{R_a^2}.$$

Combining this with our previous result that $r_b' = -\frac{r_b}{R_a}R_a'$,

$$r_b'' = \frac{\left[(b-a)[R_a^2 - R_aR_a''] + 2(R_a + r_b)(R_a')^2 + R_a^2R_a''\right] r_b}{R_a^3}.$$

Noticing that $(R_a + r_b) = (b-a)$, we can factor out $(b-a)$ and notice our curvature operator leftover. We get a nice formula for r_b'' and through a similar process, r_a'':

$$r_b'' = \frac{r_b(b-a)\mathcal{K}R_a}{R_a^3} + \frac{R_a''r_b}{R_a}$$
$$r_a'' = \frac{r_a(a-b)\mathcal{K}R_b}{R_b^3} + \frac{R_b''r_a}{R_b}.$$

3. Reconstruction Idea

The basic idea behind reconstructing K is to repeatedly push (and sometimes pull) X-ray data from both sources onto the continually-improving nearside approximations until we are arbitrarily close to K.

It is clear that given an approximation of r_a, $\gamma_{a,1}$, an approximation to R_a, $\Gamma_{a,1}$, is

$$\Gamma_{a,1}(\varphi_1) = \gamma_{a,1}(\varphi_1) + X_a(\varphi_1).$$

With the appropriate change of coordinates, $\Gamma_{a,1}$ becomes a nearside approximation from source b called $\gamma_{b,1}$. It is then evident that

$$\Gamma_{b,1}(\varphi) = \gamma_{b,1}(\varphi) + X_b(\varphi).$$

Now change to polar coordinates centered at a and $\Gamma_{b,1}$ becomes our new nearside approximation, $\gamma_{a,2}$. This process gives a sequence of functions $\left\{\gamma_{a,k}(\varphi)\right\}_{k=1}^{\infty}$.

4. Error Estimates in Reconstructing K

In this section, assume $\partial K \in C^2$. Theorem (2.1) shows we can obtain values for r_a' exactly. Let our first approximation, $\gamma_{a,1}(\varphi_1)$, of $r_a(\varphi_1)$ be the point of intersection of the tangent line of ∂K at r_a and the ray φ_1. Then

$$\gamma_{a,1}(\varphi_1) = \frac{r_a^2}{r_a\cos(\varphi_1) - r_a'\sin(\varphi_1)}.$$

A standard measure of error is given by the difference between the actual and our estimate, namely,

$$E_{a,1}(\varphi_1) = r_a(\varphi_1) - \gamma_{a,1}(\varphi_1).$$

Expanding via Taylor Series shows us

$$E_{a,1}(\varphi) = \left(r_a + r_a'\varphi + \frac{r_a''\varphi^2}{2}\right) - \left(\gamma_a + \gamma_a'\varphi + \frac{\gamma_a''\varphi^2}{2}\right) + O(\varphi^3).$$

γ_a is tangent to $r_a(\varphi)$ at $\varphi = 0$, so the first two terms of the Taylor Series for $\gamma_{a,1}$ and $r_a(\varphi_1)$ are equal. This simplifies our equation to

$$E_{a,1}(\varphi) = (r_a'' - \gamma_a'')\frac{\varphi^2}{2} + O(\varphi^3).$$

Focusing just on $r_a'' - \gamma_a''$, multiplying by r_a/r_a and noticing $r_a = \gamma_a$, we see

$$r_a'' - \gamma_a'' = \frac{r_a r_a'' - \gamma_a\gamma_a''}{r_a}.$$

Since $r_a = \gamma_a$ and $r_a' = \gamma_a'$ by (1),

$$\begin{aligned} r_a'' - \gamma_a'' &= \frac{\left(r_a r_a'' - 2(r_a')^2 - r_a^2\right) - \left(\gamma_a\gamma_a'' - 2(\gamma_a')^2 - \gamma_a^2\right)}{r_a} \\ &= \frac{-(\mathcal{K} r_a) + (\mathcal{K}\gamma_a)}{r_a}. \end{aligned}$$

Because $\gamma_{a,1}$ is a line, $\mathcal{K}\gamma_{a,1} = 0$. This means our error is

$$E_{a,1}(\varphi) = \frac{-(\mathcal{K}r_a)\varphi^2}{2r_a} + O(\varphi^3). \tag{8}$$

4.1. **Iterations of Local Error.** In the previous section, we found the local error between our first approximated point and the actual point. In this section we look at how the local error changes with the first two iterations of our proposed reconstruction algorithm. Adding the X-ray data to our first approximation, $\gamma_{a,1}$, we find $\Gamma_{a,1}$. Looking at $\Gamma_{a,1}$ as a point on the ray ψ_1 from the source b, we now call this $\gamma_{b,1}$ and will look at the difference between this approximated point and our actual point $r_b(\psi_1)$

$$E_{b,1}(\psi_1) = r_b(\psi_1) - \gamma_{b,1}(\psi_1).$$

Expanding using Taylor series and canceling the first two terms of each, as we did in the previous error, our error simplifies to

$$E_{b,1}(\psi_1) = (r_b'' - \gamma_b'')\frac{\psi_1^2}{2} + O(\psi_1^3),$$

since $r_b = \gamma_b$ and $r_b' = \gamma_b'$ by (2).

Focusing just on $r_b'' - \gamma_b''$, we remember we have found a formula for r_b'' previously, and we can similarly find a formula for our approximated point γ_b''

$$r_b'' = \frac{r_b(b-a)\mathcal{K}R_a}{R_a^3} + \frac{R_a'' r_b}{R_a} \qquad \gamma_b'' = \frac{\gamma_b(b-a)\mathcal{K}\Gamma_a}{\Gamma_a^3} + \frac{\Gamma_a''\gamma_b}{\Gamma_a}.$$

Note that all of our approximation points and their first derivatives at zero are equal to our actual points and first derivatives at zero. Using our definitions of the second derivatives, our difference becomes

$$r_b'' - \gamma_b'' = \left(\frac{r_b(b-a)\mathcal{K}R_a}{R_a^3} + \frac{R_a'' r_b}{R_a}\right) - \left(\frac{r_b(b-a)\mathcal{K}\Gamma_a}{R_a^3} + \frac{\Gamma_a'' r_b}{R_a}\right).$$

Factoring and combining like terms we see

$$r_b'' - \gamma_b'' = \frac{r_b(b-a)}{R_a^3}(\mathcal{K}R_a - \mathcal{K}\Gamma_a) + \frac{r_b}{R_a}[R_a'' - \Gamma_a'']. \tag{9}$$

Focusing on $R_a'' - \Gamma_a''$, multiplying by R_a/R_a, and remembering that $R_a = \Gamma_a$, we find

$$R_a'' - \Gamma_a'' = \frac{R_a R_a'' - \Gamma_a\Gamma_a''}{R_a}.$$

Adding terms that equal zero we get

$$\begin{aligned} R_a'' - \Gamma_a'' &= \frac{(R_a R_a'' - 2(R_a')^2 - R_a^2) - (\Gamma_a\Gamma_a'' - 2(\Gamma_a')^2 - \Gamma_a^2)}{R_a} \\ &= \frac{-(\mathcal{K}R_a - \mathcal{K}\Gamma_a)}{R_a}. \end{aligned}$$

Plugging this back into our equation (9) and factoring we see

$$\begin{aligned} r_b'' - \gamma_b'' &= \frac{r_b(b-a)}{R_a^3}(\mathcal{K} R_a - \mathcal{K}\Gamma_a) - \frac{r_b}{R_a^2}(\mathcal{K} R_a - \mathcal{K}\Gamma_a) \\ &= \frac{r_b(b-a) - r_b R_a}{R_a^3}(\mathcal{K} R_a - \mathcal{K}\Gamma_a). \end{aligned}$$

Note that

$$\begin{aligned} r_b(b-a) - r_b R_a &= r_b(b-a-R_a) \\ &= r_b(r_b) \\ &= r_b^2. \end{aligned}$$

With this observation our equation now looks like

$$r_b'' - \gamma_b'' = \frac{r_b^2}{R_a^3}(\mathcal{K} R_a - \mathcal{K}\Gamma_a).$$

Expanding the curvature operators, canceling equal terms, and finally factoring, we find

$$\begin{aligned} r_b'' - \gamma_b'' &= \frac{r_b^2}{R_a^3}[(R_a^2 + 2(R_a')^2 - R_a R_a'') - (\Gamma_a^2 + 2(\Gamma_a')^2 - \Gamma_a \Gamma_a'')] \\ &= \frac{r_b^2}{R_a^3}(\Gamma_a \Gamma_a'' - R_a R_a'') \\ &= \frac{r_b^2}{R_a^2}(\Gamma_a'' - R_a''). \end{aligned}$$

Note that

$$\begin{aligned} \Gamma_a'' - R_a'' &= (\gamma_a'' + X_a'') - (r_a'' + X_a'') \\ &= \gamma_a'' - r_a''. \end{aligned}$$

With this we have

$$r_b'' - \gamma_b'' = \frac{r_b^2}{R_a^2}(\gamma_a'' - r_a'').$$

In the previous section, we found a formula for the opposite of $\gamma_a'' - r_a''$, but we restate it as

$$\gamma_a'' - r_a'' = \frac{(\mathcal{K} r_a)}{r_a}.$$

Our final formula becomes

$$r_b'' - \gamma_b'' = \left(\frac{r_b^2}{R_a^2}\right)\frac{(\mathcal{K} r_a)}{r_a}.$$

Inserting this back into our error equation, we see

$$E_{b,1}(\varphi) = \left(\frac{r_b^2}{R_a^2}\right)\frac{(\mathcal{K}\, r_a)\varphi^2}{2r_a} + O(\varphi^3).$$

With a second iteration we can compare the first two approximations that are considered nearside points of the same source, a. Adding the X-ray data of $r_b(\psi_1)$ to the approximated point $\gamma_{b,1}$, we find our new approximated point $\Gamma_{b,1}$, or $\gamma_{a,2}$. The error between this point and the actual point will be

$$E_{a,2}(\varphi_2) = r_a(\varphi_2) - \gamma_{a,2}(\varphi_2).$$

Expanding with Taylor series, as we did with the last calculation, we find

$$E_{a,2}(\varphi) = (r_a'' - \gamma_a'')\frac{\varphi^2}{2} + O(\varphi^3).$$

Focusing on $r_a'' - \gamma_a''$, and following the same steps as the last iteration, we obtain the formula

$$r_a'' - \gamma_a'' = \frac{-r_a^2}{R_b^2}(r_b'' - \gamma_b'').$$

Our previous calculation revealed an equation for $r_b'' - \gamma_b''$. Substituting this in we see

$$r_a'' - \gamma_a'' = \left(\frac{-r_a^2}{R_a^2}\right)\left(\frac{r_b^2}{R_b^2}\right)\frac{(\mathcal{K}\, r_a)}{r_a}.$$

Inserting this back into the error equation, we finally observe

$$E_{a,2}(\varphi) = \left(\frac{r_a^2}{R_a^2}\right)\left(\frac{r_b^2}{R_b^2}\right)\frac{-(\mathcal{K}\, r_a)\varphi^2}{2r_a} + O(\varphi^3). \tag{10}$$

We have now found the coefficient of the φ^2 term of our error. Observe that $r_a < R_a$ and $r_b < R_b$. If we refer back to our first error, equation (8) and compare the coefficient of φ^2 to the coefficient of φ^2 in our iterated error in equation (10), it can be determined that

$$\left(\frac{r_a^2}{R_a^2}\right)\left(\frac{r_b^2}{R_b^2}\right)\frac{-(\mathcal{K}\, r_a)}{r_a} < \frac{-(\mathcal{K}\, r_a)}{r_a} \qquad \text{and therefore} \qquad E_{a,2} < E_{a,1}$$

As far as the φ^2 term we know the local error is decreasing with each iteration. If we continued the algorithm, it would continue to get smaller by a factor of $\left(\frac{r_a}{R_a}\right)^2\left(\frac{r_b}{R_b}\right)^2$ with each iteration of our algorithm. Proceeding inductively we have the following result

Theorem 4.1. *Let $r_a(\varphi_k)$ be the nearside point from source a at angle (φ_k) and $\gamma_{a,k}(\varphi_k)$ be its approximation. With each iteration of the reconstruction algorithm, the formula for the local error on ray (φ_k) is as follows*

$$E_{a,k}(\varphi_k) = r_a(\varphi_k) - \gamma_{a,k}(\varphi_k) = -\left[\left(\frac{r_a}{R_a}\right)^{2(k-1)}\right]\left[\left(\frac{r_b}{R_b}\right)^{2(k-1)}\right]\frac{\mathcal{K}\, r_a\varphi^2}{r_a} + O(\varphi^3), \qquad \forall k \geq 1.$$

We have not yet determined what happens to the terms beyond φ^2.

4.2. μ-error.

Definition 4.2. *μ-measure is defined as*

$$\int_B \frac{f(x,y)}{y}\,dA,$$

where B is some area and $f(x,y)$ is a function that is 1 if $(x,y)\in B$ and 0 otherwise. μ-error has a polar form

$$\int_0^{\varphi_1}\int_0^{\infty}\frac{f(r,\theta)}{\sin(\theta)}\,dr\,d\theta.$$

We will define the μ-error, E_μ to be the μ-measure of the symmetric difference between our approximation of K and K itself. The symmetric difference between B and K is defined as

$$B\backslash K\cup K\backslash B.$$

Note that

$$\int_0^{\infty}\frac{f(r,\varphi)}{\sin(\varphi)}\,dr=\frac{X_a(\varphi)}{\sin(\varphi)}.$$

The local μ-error of $\gamma_{a,1}$ is therefore by (8)

$$\begin{aligned}E_{\mu,\gamma_{a,1}}(\varphi_1)&=\int_0^{\varphi_1}\frac{r_a(\varphi)-\gamma_{a,1}(\varphi)}{\sin(\varphi)}\,d\varphi\\&=\int_0^{\varphi_1}\frac{\frac{-\frac{1}{2}(\mathcal{K}r_a)\varphi^2}{r_a}}{\sin(\varphi)}+\frac{O(\varphi^3)}{\sin(\varphi)}\,d\varphi.\end{aligned}$$

Considering only φ_1 small, $\sin(\varphi)\approx\varphi$ for all $0\le\varphi\le\varphi_1$, therefore

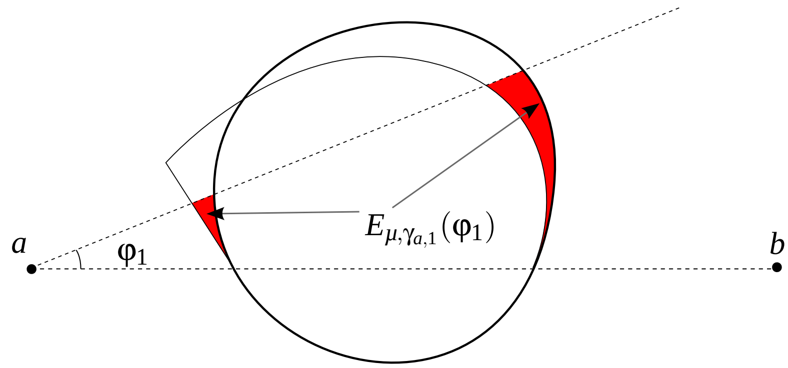
$$\begin{aligned}E_{\mu,\gamma_{a,1}}(\varphi_1)&=\int_0^{\varphi_1}\frac{-\frac{1}{2}(\mathcal{K}r_a)\varphi}{r_a}+O(\varphi^2)\,d\varphi\\&=-\frac{(\mathcal{K}r_a)}{2r_a}\int_0^{\varphi_1}\frac{\varphi}{a}+O(\varphi^2)\,d\varphi,\end{aligned}$$

finally giving

$$E_{\mu,\gamma_{a,1}}(\varphi_1)=-\frac{1}{4r_a}(\mathcal{K}r_a){\varphi_1}^2+O(\varphi_1^3)=O(\varphi_1^2).$$

Now consider the μ-error of $\Gamma_{a,1}=\gamma_{a,1}+X_a$.

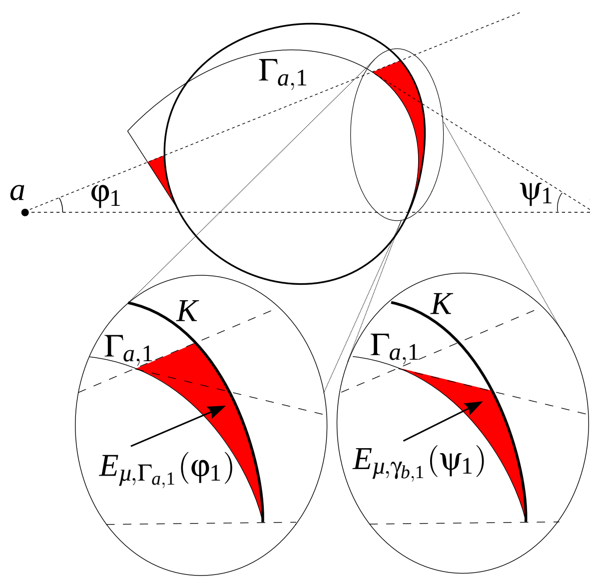
$$E_{\mu,\Gamma_{a,1}}(\varphi_1)=\int_0^{\varphi_1}\frac{R_a(\varphi)-\Gamma_{a,1}(\varphi)}{\sin(\varphi)}\,d\varphi=\int_0^{\varphi_1}\frac{(r_a(\varphi)+X_a(\varphi))-(\gamma_{a,1}(\varphi)+X_a(\varphi))}{\sin(\varphi)}\,d\varphi=E_{\mu,\gamma_{a,1}}(\varphi_1)$$



But, because μ-error can be represented in rectangular coordinates, the μ-error is invariant if we change the origin of our polar coordinates.

Let ψ_1 be the angle needed to meet the point $\Gamma_{a,1}(\varphi_1)$ from source b and let $\gamma_{b,1}$ be the representation of $\Gamma_{a,1}$ in polar coordinates centered at b. It is geometrically clear that

$$E_{\mu,\gamma_{b,1}}(\psi_1) \leq E_{\mu,\Gamma_{a,1}}(\varphi_1).$$



We will exploit this fact when discussing reconstruction.

5. Reconstruction

We intend to show that we can construct an arbitrarily close approximation to K, but first we need a helpful lemma.

5.1. **Intersections of Approximations.**

Lemma 5.1. *Given a continuous near-side approximation,* $\gamma_{a,1}$ *on* $[0,\varphi]$, *that is always "closer" to the source than the actual nearside,* r_a, *i.e.* $\gamma_{a,1} \leq r_a$ *and* $\gamma_{a,1}(0) = r_a(0)$ *and similarly,* $\gamma_{b,1}$, *then if* $\Gamma_{a,1}, \Gamma_{b,1}$ *are the corresponding farside approximations,* $\Gamma_{a,1} \cap \Gamma_{b,1} \in K$.

Proof. First consider the case where $\gamma_{a,1}$ is strictly closer than r_a, i.e. $\gamma_{a,1}(\varphi) < r_a(\varphi)$ for $\varphi > 0$ and $\gamma_{a,1}(0) = r_a(0)$.

Without loss of generality, we need only consider the top half of our convex body, K. Define the angle $\varphi_1 > 0$ such that the ray from source a at this angle is a supporting ray, and suppose $X_a(\varphi_1) = 0$. Similarly define an angle $\psi_1 > 0$ such that the ray at angle ψ_1 from source b is a supporting ray and assume $X_b(\psi_1) = 0$. Let $\text{ray}_a(\varphi_1)$ denote the ray at angle φ_1 emanating from a and $\text{ray}_b(\psi_1)$ denote the ray from b. $\xi = \text{ray}_a(\varphi_1) \cap \text{ray}_b(\psi_1)$, $m = \text{ray}_a(\varphi_1) \cap \partial K$, and $n = \text{ray}_b(\psi_1) \cap \partial K$.

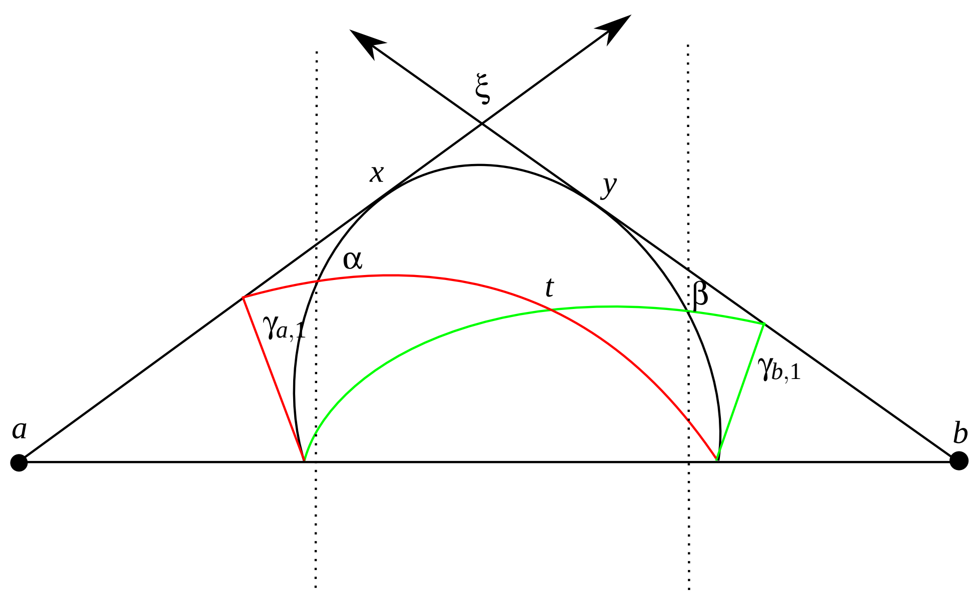
Let p_x denote the x-component of a point p. It is clear that $m_x \leq \xi_x \leq n_x$. Indeed, let $\alpha = \Gamma_{a,1} \cap \partial K$ and $\beta = \Gamma_{b,1} \cap \partial K$. Note that $\alpha_x < m_x < n_x < \beta_x$. This is evident if you consider $\gamma_{a,1}$ is a strictly closer approximation of r_a meaning $\gamma_{a,1} + X_a = \Gamma_{a,1}$ is a strictly closer approximation of R_a. Therefore, $\Gamma_{a,1}(\varphi) \cap R_a(\varphi) = \emptyset$ for $\varphi > 0$. So $\alpha = \Gamma_{a,1}(\varphi) \cap r_a(\varphi)$ for $\varphi > 0$ which by definition, must be on the left side of m. Similarly, $\beta_x > n_x$.

Call $B_{a,1}$ the body with near-side $\gamma_{a,1}$ and farside $\Gamma_{a,1}$ and $B_{b,1}$ the body with near-side $\gamma_{b,1}$ and farside $\Gamma_{b,1}$. The only points of $B_{a,1}$ exterior to K are those to the left of α and the only points of $B_{b,1}$ exterior to K are to the right of β. Considering only the region strictly above the line $\overline{ab}$, label $t = B_{a,1} \cap B_{b,1}$. In order for t to be exterior to K, $\beta_x < t_x < \alpha_x$ must be satisfied. But $\alpha_x < m_x \leq n_x < \beta_x$ yields a contradiction. So $t \in K$.

Now consider the case where $\gamma_{a,1} \leq r_a$. Define a new nearside approximation

$$\gamma_{a,\varepsilon}(\varphi) = \gamma_{a,1}(\varphi) - \varepsilon\varphi,$$

for small $\varepsilon > 0$. Note that $\gamma_{a,\varepsilon} < r_a$ when $\varphi > 0$, so $\Gamma_{a,\varepsilon} \cap \Gamma_{b,\varepsilon} \in K$. Let ε take on values of the convergent sequence $\left\{\frac{1}{w^k}\right\}_{k=1}^{\infty}$ for some appropriate $w > 1$. Because K is compact and all our functions are continuous, $\Gamma_{a,\varepsilon} \cap \Gamma_{b,\varepsilon}$ will remain in K as $\gamma_{a,\varepsilon} \to r_a$.



If $X_a(\varphi_1) > 0$ or $X_b(\psi_1) > 0$, the same proof goes through with obvious modifications. □

Let our first attempt at an approximating body have the nearside, $\gamma_{a,1}$, defined as the tangent line to $r_a(0)$,

$$\gamma_{a,1}(\varphi) = \frac{r_a^2}{r_a \cos(\varphi) - r_a' \sin(\varphi)}.$$

Intuitively, the first farside approximation is

$$\Gamma_{a,1}(\varphi) = \gamma_{a,1}(\varphi) + X_a(\varphi).$$

Identically to Section 4.2, define $\gamma_{b,1}(\psi)$ to be the $\Gamma_{a,1}$ as represented by polar coordinates centered at b. Then

$$\Gamma_{b,1}(\psi) = \gamma_{b,1}(\psi) + X_b(\psi).$$

Switching $\Gamma_{b,1}$ to polar coordinates centered at a, we get our next approximation $\gamma_{a,2}$. Inductively, continue to define $\gamma_{a,n}$ this way. We will show via μ-error that this sequence converges to r_a and therefore reconstructs our body.

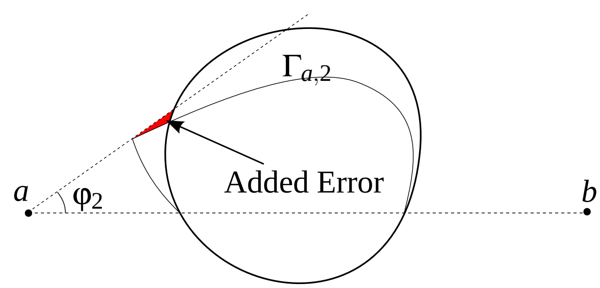
First, consider an arbitrarily small angle $\varphi_1 > 0$. We already showed that

$$E_{\mu,\gamma_{a,1}}(\varphi_1) = O(\varphi_1^2).$$

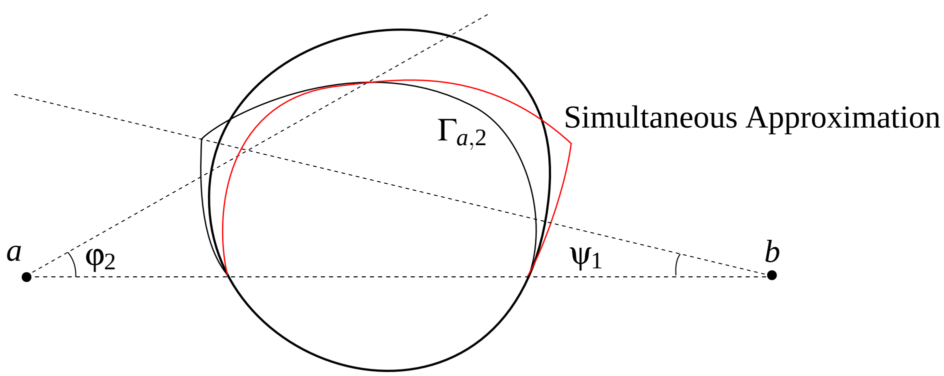
Now consider the corresponding ψ_1. We also showed

$$E_{\mu,\gamma_{a,1}}(\varphi_1) \geq E_{\mu,\gamma_{b,1}}(\psi_1)$$

We are now ready to do the next iteration, this time over a larger angle φ_2, but there is one situation we must avoid if we are to keep a non-increasing sequence of μ-errors.



A simple solution is to use Lemma 5.1. Consider using this iterative approximation method to make approximations starting from both a and b. Then, let φ_2 point to the intersection between some farside approximation from source b and the farside approximation from source a.



Choosing subsequent φ_n in this way ensures

$$E_{\mu,\gamma_{a,1}}(\varphi_1) \geq E_{\mu,\gamma_{a,2}}(\varphi_2) \geq \cdots \geq E_{\mu,\gamma_{a,n}}(\varphi_n).$$

So, μ-error is certainly bounded as our approximation sweeps out a larger and larger area. But, because $E_{\mu,\gamma_{a,1}}(\varphi_1) = O(\varphi_1^2)$, we can make the total μ-error small by choosing φ_1 small.

6. Conclusion

We've shown here how to find the first and second derivatives of the nearside and farside functions at the base points of our convex body from the X-ray data and Falconer's lemma, (Lemma 1.7). Using our theory of reconstruction we should be able to reconstruct a convex body. We have also looked at the local error between the convex body and our reconstruction, and we have shown that the first term in this error is decreasing by a factor of $\left(\frac{r_a}{R_a}\right)^2\left(\frac{r_b}{R_b}\right)^2$ with each iteration of our algorithm.

We have not been able to show complete convergence to the entire convex body, but we can assure convergence on a finite interval of substantial length. We also have not been able to investigate local error terms on the order φ^3.

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