COUNTING BITS WITH FIBONACCI AND ZECKENDORF

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Abstract. Fibonacci numbers and difference equations show up in many counting problems. Zeckendorf showed how to represent natural numbers in "binary" Fibonacci bases. Capocelli counted the number of 0 bits and 1 bits in such representation. Here we use the theory of difference equations to try to provide proofs for Capocelli's claims. We also investigate generalization of the Fibonacci difference equation which may show behavior similar to that observed by Capocelli. In particular, we conjecture that "doubly non-negative" difference equations will have solutions whose ratios monotonically approach a limit.

1. Introduction

Eight hundred years ago, Fibonacci used difference equations to count rabbits. Since then, Fibonacci numbers and their generalizations have shown up in a wide variety of counting problems. As Zeckendorf [Z] observed, the Fibonacci numbers can be used to give "binary" representation of natural numbers. In the "limit" there representations lead to the standard binary representation. As Capocelli [C][Cp][Ca] observed, these Zeckendorf representations have additional properties which may make them more suitable than the standard binary for certain computational and communication applications. Capocelli observed, in particular, that the ratio of the number of 0's to the number of bits in the Zeckendorf representation monotonically increased to a limit. Here we attempt to verify and prove that this observation is correct. Further, we try to find a general class of difference equations which display this sort of monotonic convergence. In section 2, we recall some well known results on Fibonacci numbers and difference equations. In section 3, we look specifically at Capocelli's counting problem. In section 4, we analyze the classical ($k = 2$) Fibonacci case. In section 5, we look at convergence of ratios of solutions to difference equations. In section 6, we look at doubly non-negative difference equations. We close with some brief conclusions in section 7.

2. Useful Facts and Definitions

The following are some useful facts and definitions that will help in understanding the rest of this paper. First of all, we shall be analyzing a counting problem that utilizes k^{th} order generalized Fibonacci numbers.

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Definition 2.1. *The* n^{th} Fibonacci number *of order* k *can be found using the following recurrence relation:*

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + ... + f_{n-k}^{(k)}$$

with the initial conditions: $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, *and* $f_n^{(k)} = 2^{n-2}$ *for* $2 \leq n \leq k$.[Ca]

As Zeckendorf showed, these Fibonacci numbers can be used to represent natural numbers.

Theorem 2.2 (Zeckendorf)**.** *Any natural number can be represented as*

$$\sum_{i=2} \alpha_i f_i^{(k)}$$

where $f_i^{(k)}$ *is the* k^{th} *order generalized Fibonacci number and* $\alpha_i \in \{0, 1\}$. *This representation is unique if no* k *consecutive* 1*'s are allowed. Numbers smaller than* 2^{k-1} *can be represented using their standard binary expansions which have no* k *consecutive* 1*'s.* [Z][Zt]

Proof. Let $j = n - f_i$ where by inductive hypothesis j has a unique Zeckendorf representation. Represent n by 1 followed by the Zeckendorf representation of j with an appropriate number of leading 0's. This representation of n has no k consecutive 1's because the representation of j does not and if n's representation had k leading 1's then $n \geq f_i^{(k)}$. □

Counting problems can often be represented by difference equations. The counting problems discussed in this paper will be represented as non-negative difference equations. In fact, we shall only be concerned with non-negative difference equations throughout this paper.

Definition 2.3. *A* non-negative difference equation *is a difference equation of the form*

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + ... + c_k x_{n-k}$$

where $c_1 \geq 0,\ c_2 \geq 0, ...,\ c_k > 0$, *with positive initial conditions.* [Cu]

Throughout this paper, we shall make use of the linear operator L.

Definition 2.4. *The linear operator,* $L[x_n]$ *can be defined as*

$$L[x_n] = x_n - c_1 x_{n-1} - c_2 x_{n-2} - ... - c_k x_{n-k}$$

Therefore, a k^{th} order linear constant coefficient difference equation

$$x_n - c_1 x_{n-1} - c_2 x_{n-2} - ... - c_k x_{n-k} = g(n)$$

can be written in a compact form by using the linear operator, $L[\ ]$. This compact form would be [Cl][Cu]:

$$L[x_n] = g(n)$$

When we write $L[x_n] = 0$, this would correspond to

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + ... + c_k x_{n-k}$$

When discussing difference equations it is often helpful to look at their characteristic polynomial.

Definition 2.5. *The polynomial*

$$ch(\lambda) = \lambda^k - c_1\lambda^{k-1} - ...c_{k-1}\lambda - c_k.$$

is called the characteristic polynomial *of the difference equation. The polynomial is non-negative if* $c_1 \geq 0$,...,$c_{k-1} \geq 0$, *and* $c_k > 0$.

We need to introduce a new characterization of difference equations called doubly non-negative, which will be used later as a condition in conjectures.

Definition 2.6. *A difference equation is* doubly non-negative *if it is non-negative and* $[ch(\lambda)]^2/(\lambda - \lambda_0)$ *is also non-negative, where* $ch(\lambda)$ *is the characteristic polynomial of the difference equation and* λ_0 *is the positive real root of* $ch(\lambda)$.

It is well known that if $ch(\lambda)$ is non-negative, then $ch(\lambda)$ has a unique positive real root λ_0, and $ch(x) < 0$ for $0 \leq x \leq \lambda_0$ and $ch(x) > 0$ for $x > \lambda_0$.

Theorem 2.7. *The Fibonacci operators*

$$L[f_n^{(k)}] = f_n^{(k)} - f_{n-1}^{(k)} - f_{n-2}^{(k)} - .. - f_{n-k}^{(k)}$$

are doubly non-negative.

Proof. The characteristic polynomial for the k^{th} order generalized Fibonacci operator is

$$ch(\lambda) = \lambda^k - \lambda^{k-1} - ... - 1$$

which has positive real root $\lambda_0 < 2$, since $ch(2) = 1 > 0$. Then,

$$ch(\lambda)/(\lambda - \lambda_0) = \lambda^{k-1} + g_1\lambda^{k-2} + ... + g_{k-1}$$

where $g_1 = \lambda_0 - 1$, $g_{i+1} = \lambda_0 g_i - 1$, and each $0 < g_i < 1$. So,

$$[ch(\lambda)]^2/(\lambda - \lambda_0) = (\lambda^{k-1} + g_1\lambda^{k-2} + ... + g_{k-1})(\lambda^k - \lambda^{k-1} - ... - 1) =$$
$$\lambda^{2k-1} + (g_1 - 1)\lambda^{2k-2} + ... + (-g_{k-1})$$

Each $g_i > 0$ because the g_i's are the terms in the partial evaluation of a non-negative polynomial, and if any one of the g's were non-positive, then subsequent g's would also be non-positive. But, $\lambda_0 g_{k-1} - 1 = 0$ and g_{k-1} must be positive. Each $g_i < 1$; $g_1 < 1$ since $g_1 = \lambda_0 - 1 < 2 - 1 = 1$ because $\lambda_0 < 2$. Since $g_{i+1} = \lambda_0 g_i - 1 < 2 * 1 - 1 = 1$, $g_{i+1} < 1$. From earlier calculations, the coefficients (except for the leading coefficient) of $[ch(\lambda)]^2/(\lambda - \lambda_0)$ are all negative. If any coefficient could be positive, it would be a sum of the single positive term g_1 and one or more negative terms which must include -1. But, this sum of terms would be less than $g_i - 1$ which is negative because $g_i < 1$. Hence, for the k^{th} order Fibonacci operator

$$[ch(\lambda)]^2/(\lambda - \lambda_0) = \lambda^{2k-1} - \hat{c}_1\lambda^{2k-2} - ... - \hat{c}_{2k-1}$$

with each $\hat{c}_i > 0$, and for example,

$$\hat{c}_1 = g_1 - 1 = \lambda_0 - 1 - 1 = \lambda_0 - 2.$$

□

The following are theorems and examples involving the L operator and non-negative difference equations.

Theorem 2.8 (Uniqueness)**.** *The equation* $L[x_n] = g(n)$ *with initial conditions* $x_0,\ x_1, ...,\ x_{k-1}$ *has a unique solution.*

Theorem 2.9. *The solution of* $L[x_n] = g(n)$ *with initial conditions* $x_0,\ x_1,\ ...,\ x_{k-1}$ *can be written as*

$$x_n = v_n + \sum \alpha_i h_i(n)$$

where v_n *is any solution to* $L[x_n] = g(n)$ *and* $h_1(n), .., h_k(n)$ *are* k *linearly independent solutions to* $L[x_n] = 0$*, and the coefficients* $\alpha_1, ..., \alpha_k$ *depend on* v_n *and on the initial conditions.*

Theorem 2.10 (Increasing)**.** *If* $x_0 < x_1 < ... < x_k$ *, then* x_n *is an increasing function for* $n \geq 0$*, where* x_n *is a solution to the non-negative difference equation* $x_n = c_1 x_{n-1} + c_2 x_{n-2} + ... + c_k x_{n-k}$*.*

Proof. Assume that $x_{n-k-1} < x_{n-k} < ... < x_{n-2} < x_{n-1}$. By taking these equations,

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + ... + c_k x_{n-k}$$

$$x_{n-1} = c_1 x_{n-2} + c_2 x_{n-3} + ... + c_k x_{n-k-1}$$

you can subtract them to get:

$$(x_n - x_{n-1}) = c_1(x_{n-1} - x_{n-2}) + c_2(x_{n-2} - x_{n-3}) + ... + c_k(x_{n-k} - x_{n-k-1})$$

Then, each of $(x_{n-1} - x_{n-2})...(x_{n-k} - x_{n-k-1}) > 0$ and at least one $c_i > 0$. So, $x_n - x_{n-1} > 0$ and $x_n > x_{n-1}$. Therefore, x_n will always be an increasing function. □

Theorem 2.11. *If* $L[X_n]$ *is an order* k *operator whose characteristic polynomial has* k *distinct roots* $\lambda_0, .., \lambda_{k-1}$*, then the solutions to* $L[X_n] = 0$ *can be written as*

$$X_n = \alpha_0 \lambda_0^n + \alpha_1 \lambda_1^n + ... + \alpha_{k-1} \lambda_{k-1}^n$$

where each α_i *is a constant. The solutions to the double operator* $L^2[X_n] = 0$ *(where the* L *operator is taken twice) can be written as*

$$X_n = (\alpha_0 n + \beta_0)\lambda_0^n + (\alpha_1 n + \beta_1)\lambda_1^n + ... + (\alpha_{k-1} n + \beta_{k-1})\lambda_{k-1}^n$$

where the α*'s and* β*'s are constants.* [Cu]

Example 2.12. In the Fibonacci case, $L[f_n] = f_n - f_{n-1} - f_{n-2}$. So, $L[f_n] = 0$ has solutions of the form

$$f_n = a_0 \lambda_0^n + a_1 \lambda_1^n$$

where λ_0 and λ_1 are the distinct roots of $\lambda^2 - \lambda - 1$. For the equation, $f_n = f_{n-1} + f_{n-2} + g(n)$ where $g(n) = g(n-1) + g(n-2)$, $L^2[f_n] = 0$ and the solution will be of the form

$$f_n = (a_0 n + b_0)\lambda_0^n + (a_1 n + b_1)\lambda_1^n.$$

Example 2.13. The difference equation

$$X_n = X_{n-1} + 2X_{n-2}$$

is non-negative, assuming it has positive initial conditions, and has the characteristic equation

$$ch(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2) * (\lambda + 1).$$

The solutions to this difference equation has the form

$$X_n = a_0(2)^n + a_1(-1)^n$$

For example, if $X_0 = 0, X_1 = 1$, then

$$X_n = 1/3 * (2^n - (-1)^n)$$

while if $X_0 = 1, X_1 = 2$, then

$$X_n = 2^n.$$

For the first set of initial conditions $X_0 < X_1 = X_2$, so X_n is not always increasing. For the second set of initial conditions, $X_0 < X_1 < X_2$ and obviously X_n is always increasing. If

$$L[X_n] = X_n - X_{n-1} - 2X_{n-2}$$

then the characteristic polynomial for L^2 is

$$[(\lambda - 2)(\lambda + 1)]^2$$

and is

$$\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda + 4.$$

So,

$$L^2[X_n] = X_n - 2X_{n-1} - 3X_{n-2} + 4X_{n-3} + 4X_{n-4}$$

and L^2 is not a non-negative operator. The solution to $L^2[X_n] = 0$ can be written in the form

$$X_n = (a_0 n + b_0)2^n + (a_1 n + b_1)(-1)^n.$$

For example, if $X_1 = 2,\ X_2 = 4,\ X_3 = 15,\ X_4 = 50$ then

$$X_n = (n - 1)2^n + (n - 2)(-1)^n.$$

While it is easy to see that this X_n is increasing for $n \geq 1$, this increasing does not follow from Theorem 2.9 because L^2 is not non-negative If you take $[ch(\lambda)]^2/(\lambda - \lambda_0)$ you get

$$[(\lambda - 2) * (\lambda + 1)]^2/(\lambda - 2) = (\lambda - 2) * (\lambda + 1)^2 = \lambda^3 - 3\lambda^2 - 2$$

which is the characteristic polynomial for the non-negative difference equation

$$X_n = 3X_{n-2} + X_{n-3}.$$

Therefore, $X_n = X_{n-1} + 2X_{n-2}$ is a doubly non-negative difference equation. The solution to

$$X_n = 3X_{n-2} + X_{n-3}$$

has the form

$$X_n = a_0 2^n + (a_1 n + b_1)(-1)^n$$

For initial conditions $X_0 = 0,\ X_1 = 1,\ X_2 = 2$ the solution is

$$X_n = 1/9[4 * 2^n + (3n - 4)(-1)^n].$$

By checking that $X_3 = 3$, this solution satisfies the hypotheses of Theorem 2.9 and is always increasing.

For the next theorem, we make use of the condition that the operator is aperiodic.

Definition 2.14. *A non-negative difference equation is* aperiodic *if* $gcd\{i|c_i > 0\} = 1$.

Theorem 2.15. *If $p(\lambda)$ is the characteristic polynomial of a non-negative operator L, then $p(\lambda)$ has a unique positive real root λ_0 and the solution to $L[X_n] = 0$ with positive initial conditions obeys $X_n = O(\lambda_0^n)$. If, in addition, the operator is aperiodic, then there is an $\alpha_0 > 0$ such that*

$$\lim_{n\to\infty} X_n/\lambda_0^n = \alpha_0$$

or similarly

$$\lim_{n\to\infty} X_n/\alpha_0\lambda_0^n = 1$$

3. A Counting Problem

We are now ready to discuss a counting problem involving the ratio of 0's in Zeckendorf strings. Let's consider binary strings of length n that have no k consecutive 1's. These strings represent a unique Zeckendorf representation of a natural number. Because of our restrictions, these strings must start with either 0, 10, 110, ...,or $k-1$ ones followed by a zero. So, in the $k = 2$ case, the string can only start with 0 or 10. The rest of the string will follow the same rules until there are n characters in the string. The first few strings of length n for $k = 2$ can be found in Figure 1.

$n=1$	$n=2$	$n=3$	$n=4$
0	00	000	0000
1	01	001	0001
	10	010	0010
		100	0010
		101	0100
			0101
			1000
			1001
			1010

Figure 1. Binary Strings of length n with k=2

Let us now define $S_n^{(k)}$ to be the number of n character strings with no k consecutive 1's. For the $k = 2$ case, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$, and $S_4 = 8$. In this example, it looks like $S_n^{(k)}$ is related to the Fibonacci numbers, and this would be a good guess to make. $S_n^{(k)}$ satisfies the difference equation

$$S_n^{(k)} = S_{n-1}^{(k)} + S_{n-2}^{(k)} + ... + S_{n-k}^{(k)}$$

with the initial conditions $S_n^{(k)} = 2^n$ if $1 \le n \le k-1$ and $S_k^{(k)} = 2^{k-1}$
because an allowed length n string consists of 0 followed by an allowed length $(n-1)$ string or of 10 followed by an allowed length $(n-2)$ string ...or of 1...10 followed by an allowed length $(n-k)$ string. The order k Fibonacci numbers satisfy the same difference equation as $S_n^{(k)}$, but have different initial conditions. Recall that,

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + ... + f_{n-k}^{(k)}$$

with the initial conditions: $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and $f_n^{(k)} = 2^{n-2}$ for $2 \le n \le k$.
Note that:

$$f_{n+2}^{(k)} = 2^{n+2-2} = 2^n$$

So, the relationship between $S_n^{(k)}$ and the Fibonacci numbers is:

$$S_n^{(k)} = f_{n+2}^{(k)}$$

Now, let $W_n^{(k)}$ be the total number of bits in the binary strings of length n with no k consecutive 1's. Then,

$$W_n^{(k)} = n f_{n+2}^{(k)}$$

since each string of length n has exactly n bits and there are $f_{n+2}^{(k)}$ strings of length n.

The ratio of 0's to the number of bits in our strings will now be the focus of our attention. If our binary strings had no restrictions it would be correct to assume that 1/2 of the bits would be 0's. However, since the number of 1's has been restricted, we expect that there are more 0's than 1's. Therefore, the ratio should be greater than 1/2.

Let $N_n^{(k)}$ signify the total number of 0's in strings of length n with no consecutive k 1's. Then, $N_n^{(k)}/W_n^{(k)}$ will be the proportion of 0's in our strings of length n for order k. As was stated earlier, each string of length n starts with either a 0, 10, or $k-1$ one's with a 0. Consider the strings which start with a 0 to be of length $n-1$ with a 0 in front. The number of 0's in these strings will be $N_{n-1}^{(k)}$, which represents the number of 0's in the strings of length $n-1$, plus the number of strings which start with 0. The strings which begin with 10 have $n-2$ characters remaining in the string, so the number of 0's in these strings is $N_{n-2}^{(k)}$, the number of 0's in strings of length $n-2$, plus the number of strings which start with a 10. The same principle applies to all of the strings until you have only those strings left that begin with $k-1$ ones and then a 0. The number of 0's in these strings is $N_{n-k}^{(k)}$, the number of 0's in strings of length $n-k$, plus the number of strings which start with $k-1$ ones followed by a zero. Recall that there are $f_{n+2}^{(k)}$ total strings of length n.
So, $N_n^{(k)}$ can be computed by using the following difference equation:

$$N_n^{(k)} = N_{n-1}^{(k)} + N_{n-2}^{(k)} + ... + N_{n-k}^{(k)} + f_{n+2}^{(k)}$$

Now, these equations can be used to calculate $N_n^{(k)}/W_n^{(k)}$ for strings of length n of order k. The program used to compute the values in Figure 2 can be found in the appendix. Figure 3 demonstrates pictorially the fact that as k goes to infinity, Zeckendorf representation approaches binary representation. If you look at Figures 2 and 3, there is evidence that $N_n^{(k)}/W_n^{(k)}$ is always a monotonically increasing ratio.

n	k = 2	3	4	5	6	7	8	9	10
1	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
2	.6667	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
3	.6667	.5714	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4	.6875	.5769	.5333	.5000	.5000	.5000	.5000	.5000	.5000
5	.6923	.5833	.5379	.5161	.5000	.5000	.5000	.5000	.5000
6	.6984	.5909	.5417	.5191	.5079	.5000	.5000	.5000	.5000
7	.7017	.5944	.5450	.5214	.5097	.5039	.5000	.5000	.5000
8	.7045	.5973	.5481	.5233	.5111	.5049	.5020	.5000	.5000
9	.7066	.5998	.5500	.5249	.5122	.5057	.5025	.5010	.5000
10	.7083	.6016	.5516	.5263	.5131	.5064	.5030	.5013	.5005
11	.7097	.6031	.5530	.5274	.5139	.5069	.5033	.5015	.5006
12	.7109	.6044	.5541	.5282	.5146	.5074	.5036	.5017	.5008
13	.7119	.6055	.5550	.5289	.5151	.5078	.5039	.5019	.5009
14	.7127	.6064	.5558	.5296	.5156	.5081	.5041	.5020	.5010
15	.7134	.6072	.5565	.5302	.5160	.5084	.5043	.5022	.5011
16	.7141	.6079	.5571	.5307	.5164	.5086	.5045	.5023	.5011
17	.7146	.6085	.5577	.5311	.5167	.5089	.5046	.5024	.5012
18	.7151	.6091	.5582	.5315	.5170	.5091	.5048	.5025	.5013
19	.7156	.6096	.5586	.5318	.5172	.5092	.5049	.5025	.5013
20	.7160	.6100	.5590	.5321	.5175	.5094	.5050	.5026	.5014

FIGURE 2. Proportion of 0's

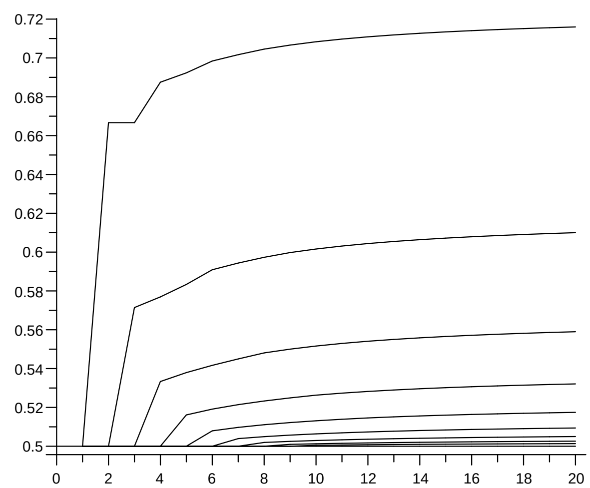

FIGURE 3. Ratio of $N_n^{(k)}/W_n^{(k)}$ for $k = 2$ to $k = 10$ (the top graph is $k = 2$, the next $k = 3$, etc.)

4. Analyzing the Counting Problem

We have shown that

$$N_n^{(k)} = N_{n-1}^{(k)} + ... + N_{n-k}^{(k)} + f_{n+2}^{(k)}$$

but

$$f_{n+2}^{(k)} = f_{n+2-1}^{(k)} + ... + f_{n+2-k}^{(k)}$$

so if we let

$$L[X_n] = X_n - X_{n-1} - ... - X_{n-k}$$

which is the k^{th} order Fibonacci operator, then $L[f_{n+2}^{(k)}] = 0$ and $L[N_n^{(k)}] = f_{n+2}^{(k)}$. So, $L^2[N_n^{(k)}] = L[f_{n+2}^{(k)}] = 0$. Further,

$$\begin{aligned} L[W_n^{(k)}] &= L[nf_{n+2}^{(k)}] \\ &= nf_{n+2}^{(k)} - (n-1)f_{n+2-1}^{(k)} - ... - (n-k)f_{n+2-k}^{(k)} \\ &= f_{n+2-1}^{(k)} + 2f_{n+2-2}^{(k)} + ... + kf_{n+2-k}^{(k)} \\ & \quad Note : f_{n+2}^{(k)} = f_{n+2-1}^{(k)} + ... + f_{n+2-k}^{(k)} \end{aligned}$$

So,

$$L^2[W_n^{(k)}] = L[f_{n+2-1}^{(k)}] + 2L[f_{n+2-2}^{(k)}] + ... + kL[f_{n+2-k}^{(k)}]$$

and since $L[f_{n+2-i}^{(k)}] = 0$ for every i, $L^2[W_n^{(k)}] = 0$.

It has already been pointed out that graphically and numerically the evidence suggests that $N_n^{(k)}/W_n^{(k)}$ is a monotonically increasing ratio. We are now able to state this observation and prove it for the case $k = 2$.

Conjecture 4.1. *If $L[Z_n] = Z_n - Z_{n-1} - Z_{n-2} - .. - Z_{n-k}$ and $L[N_n^{(k)}] = f_{n+2}^{(k)}$ and $L[W_n^{(k)}] = f_{n+1}^{(k)} + ... + kf_n^{(k)}$, then $N_n^{(k)}/W_n^{(k)}$ is increasing in n for $k \geq 2$.*

Theorem 4.2. *If $L[Z_n] = Z_n - Z_{n-1} - Z_{n-2} - .. - Z_{n-k}$ and $L[N_n^{(k)}] = f_{n+2}^{(k)}$ and $L[W_n^{(k)}] = f_{n+1}^{(k)} + ... + kf_n^{(k)}$, then $N_n^{(k)}/W_n^{(k)}$ is an increasing ratio for $k = 2$.*

Proof. For any solution to $L^2[Z_n] = 0$, it can be shown that if f_n is a "full" solution of $L[f_n]$, then

$$Z_n = n[a_1 f_n + a_2 f_{n-1} + ... + a_k f_{n+1-k}] + [b_1 f_n + b_2 f_{n-1} + .. + b_k f_{n+1-k}].$$

For example, if $L[X_n] = X_n - X_{n-1} - X_{n-2}$ (the Fibonacci operator), then

$$Z_n = n[a_1 f_n + a_2 f_{n-1}] + [b_1 f_n + b_2 f_{n-1}].$$

Taking the counting problem $N_0 = 0,\ N_1 = 1,\ N_2 = 4,\ N_3 = 10$. Therefore,

$$\begin{aligned} N_0: \quad 0 &= 0 + b_1 f_0 + b_2 f_{-1} \\ &= b_2 \end{aligned}$$

$$\begin{aligned} N_1: \quad 1 &= [a_1 f_1 + a_2 f_0] + b_1 f_1 \\ &= a_1 + b_1 \end{aligned}$$

$$\begin{aligned} N_2: \quad 4 &= 2[a_1 f_2 + a_2 f_1] + b_1 f_2 \\ &= 2a_1 + 2a_2 + b_1 \end{aligned}$$

$$\begin{aligned} N_3: \quad 10 &= 3[a_1 f_3 + a_2 f_2] + b_1 f_3 \\ &= 3[2a_1 + a_2] + 2b_1 \\ &= 6a_1 + 3a_2 + 2b_1 \end{aligned}$$

Using $b_1 = 1 - a_1$, we get:

$$\begin{aligned} N_2: \quad 4 &= 2a_1 + 2a_2 + (1 - a_1) \\ 3 &= a_1 + 2a_2 \\ 9 &= 3a_1 + 6a_2 \end{aligned}$$

$$\begin{aligned} N_3: \quad 10 &= 6a_1 + 3a_2 + 2 - 2a_1 \\ 8 &= 4a_1 + 3a_2 \\ 16 &= 8a_1 + 6a_2 \end{aligned}$$

Now, we can get

$$7 = 5a_1.$$

So,

$$a_1 = 7/5.$$

And from this,

$$a_2 = 1/2(3 - a_1) = 1/2(15/5 - 7/5) = 4/5$$

and

$$b_1 = 1 - 7/5 = -2/5$$

We can plug these values in to get a general equation of N_n:

$$N_n = 1/5[n(7f_n + 4f_{n-1}) - 2f_n].$$

Now, we want to show that

$$N_n/W_n = 1/5[n(7f_n + 4f_{n-1}) - 2f_n]/nf_{n+2} < 1/5[(n+1)(7f_{n+1} + 4f_n) - 2f_{n+1}]/nf_{n+3}.$$

By cross multiplication, the equation becomes:

$$n(n+1)f_{n+3}(7f_n + 4f_{n-1}) - 2f_n f_{n+3}(n+1) < n(n+1)f_{n+2}(7f_{n+1} + 4f_n) - 2f_{n+1}f_{n+2}(n)$$

The equation can be re-arranged to:

$$n(n+1)[f_{n+3}f_{n+1} - f_{n+2}f_{n+2} < 2n[f_{n+2}f_n - f_{n+1}f_{n+1}] + 2f_n f_{n+3}$$

Note that:

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

$$|A|^n = |A^n| = (-1)^n = f_{n+1}f_{n-1} - f_n f_n$$

So, the equation becomes

$$n(n+1)(-1)^{n+2} < 2n(-1)^{n+1} + 2f_n f_{n+3}$$

which can be reduced to

$$n(n+3)(-1)^{n+2} < 2f_n f_{n+3}$$

Certainly, this inequality holds for $n > 5$ because then $f_n > n$. For $n = 0$ to 5 we get

$$\begin{aligned} n = 0 &\rightarrow & 0 \leq 0 \\ n = 1 &\rightarrow & -4 < 6 \\ n = 2 &\rightarrow & 10 \leq 10 \\ n = 3 &\rightarrow & -18 < 32 \\ n = 4 &\rightarrow & 28 < 78 \\ n = 5 &\rightarrow & -40 < 210 \end{aligned}$$

Therefore, N_n/W_n is an increasing function for all $n \geq 3$ for the case of $k = 2$. □

5. Applying Result to Non-negative Difference Equations

The ratio of 0's in Zeckendorf strips of length n makes a counting problem that can leads to a nice result involving the k^{th} general Fibonacci difference equation. Sometimes what can be said of Fibonacci numbers can be generalized to apply to larger sets of difference equations [Cc]. We tried to generalize this result to come up with conditions to make the ratio of solutions to non-negative difference equations monotonically increasing.

Conjecture 5.1 (First Attempt)**.** *The ratio of two solutions to a non-negative difference equation gives an increasing ratio.*

Refutation of Conjecture 5.1: Fibonacci numbers and Lucas numbers are solutions to the same difference equation

$$Z_n = Z_{n-1} + Z_{n-2}$$

with different initial conditions. The ratio of Fibonacci numbers to Lucas numbers is:

$$f_n/l_n = 0,\ 1,\ 1/3,\ 1/2,\ 3/7,\ 5/11,\ 4/9,\ 13/29....$$

This ratio is not monotonically increasing. In fact, it is not increasing at all. The ratio is oscillating above and below its asymptotic value of $1/\sqrt{5}$ (Figure 4). Note: f_n/l_n is increasing for odd n's and decreasing for even n's. This occurs because λ_1 is negative. More complicated oscillations should be possible, for example, when there are complex eigenvalues.

Conjecture 5.2 (Second Attempt)**.** *If $L[\ ]$ is a non-negative operator that has order k and $L^2[N_n] = L^2[W_n] = 0$, then if N_n/W_n is increasing for k consecutive values of n, N_n/W_n. In particular if $N_1/W_1 < N_2/W_2 < ... < N_k/W_k$, then N_n/W_n is increasing for $n > 0$*

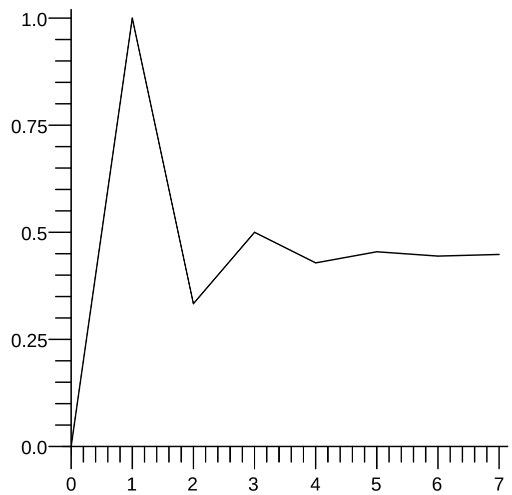

FIGURE 4. Ratio of f_n/l_n

Refutation of Conjecture 5.2: Let

$$N_n = 3N_{n-2} + 2N_{n-3} + 2^n + (2n)(-1)^n$$

with the initial conditions

$$N_0 = 1,\ N_1 = 2,\ N_2 = 3$$

and

$$W_n = 3W_{n-2} + 2W_{n-3} + 2^n + (4n+6)(-1)^n$$

with the initial conditions

$$W_0 = 7,\ W_1 = 9,\ W_2 = 11.$$

Therefore, $L[Xn] = X_n - 3X_{n-2} + 2Xn - 3$ with order $k = 3$ and $L^2[N_n] = L^2[W_n] = 0$. Then,

$$N_n/W_n = 0.1428571429,\ 0.2222222222,\ 0.2727272727,\ 0.3225806452,\ 0.4157303371,$$

$$0.4793388430,\ 0.4893617021,\ 0.5700787402,\ 0.5590027701,\ 0.6190624030,\ 0.6183107672, ...$$

The ratio increases for the first k terms. In fact, it increases for the first $2k+1 = 7$ terms; however, it does not have monotonic increasing because the eighth term is less than the seventh term. Figure 5 has this result demonstrated pictorially. Although $L[X_n]$ is non-negative, it is not doubly non-negative. Perhaps, using doubly non-negativity as a criteria will make a difference. Note: This example also shows that $\lambda_0 \geq 2|\lambda_1|$ is not sufficient for monotonic increasing.

Conjecture 5.3 (Third Attempt)**.** *If $L[\ ]$ is a doubly non-negative operator that has order k and $L^2[N_n] = L^2[W_n] = 0$, then if N_n/W_n is increasing for $2k-1$ consecutive values of n, N_n/W_n. In particular if $N_1/W_1 < N_2/W_2 < ... < N_{2k-1}/W_{2k-1}$, then N_n/W_n is increasing for $n > 0$*

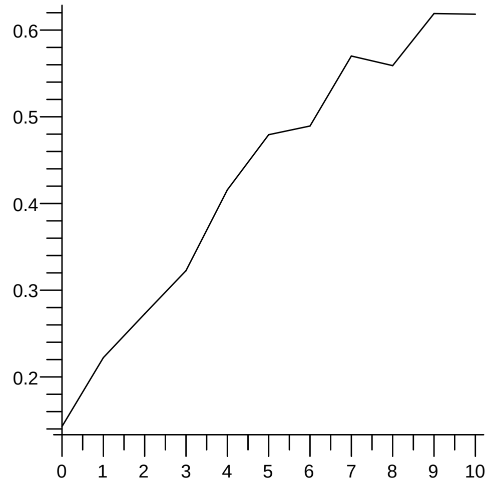

FIGURE 5. N_n/W_n Increases for the 1^{st} Seven Terms

Example 5.4. Let

$$N_n = 3N_{n-1} + 4N_{n-2} + 4^n + 2(-1)^n$$

with the initial conditions

$$N_0 = 0, \; N_1 = 1$$

and

$$W_n = 3W_{n-1} + 4W_{n-2} + 4^n + (-1)^n$$

with the initial conditions

$$W_0 = 1, \; W_1 = 2.$$

Therefore, $L[X_n] = X_n - 3X_{n-1} - 4X_{n-2}$ with order $k = 2$ and $L^2[N_n] = L^2[W_n] = 0$. The characteristic polynomial $ch(\lambda)$ for $L[X_n]$ is

$$ch(\lambda) = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4).$$

Taking $[ch(\lambda)]^2/(\lambda - \lambda_0)$ gives

$$(\lambda - 4)(\lambda + 1)^2 = \lambda^3 - 2\lambda^2 - 7\lambda - 4$$

which is the characteristic polynomial for the non-negative difference equation

$$X_n = 2X_{n-1} + 7X_{n-2} + 4X_{n-3}.$$

Therefore, $L[X_n]$ is a doubly non-negative operator. So,

$$N_n/W_n = 0.5000000000, \; 0.9130434783, \; 0.9214285714, \; 0.9479843953, \; 0.9575835476,$$
$$0.9652921509, \; 0.9703522018, \; 0.9741979942, \; 0.9771417641, \; 0.9794874491.$$

This ratio increases for the first $2k - 1$ terms and continues to increase. Figure 6 has this result represented pictorially.

The example shows evidence that the last conjecture could be true. We have been unable to produce a counter example or a proof of this conjecture.

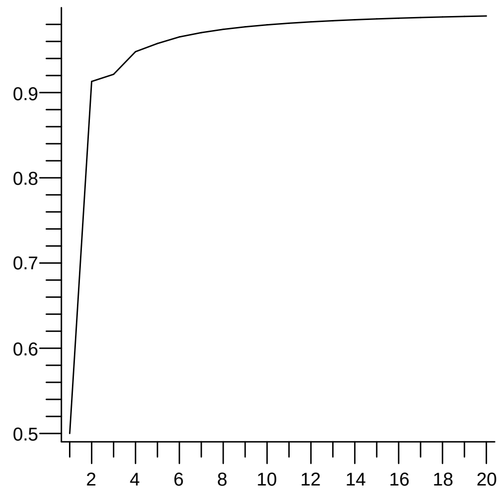

FIGURE 6. N_n/W_n is Increasing

6. Other Observations

While trying to determine conditions for monotonic increasing, other observations were made and should be mentioned. First, the ratio N_n/W_n approaches an asymptotic value. This value depends on N_n and W_n. However, N_n/W_n can approach this asymptotic value in many ways. Stated is a conjecture that says that N_n/W_n will approach the asymptotic value from below that value if certain conditions are met.

Conjecture 6.1. *Let N_n and W_n be two solutions to a doubly non-negative difference equation of order k. If $N_n/W_n < a/c$ where a/c is the asymptotic value for $2k-1$ n's in a row, then $N_n^{(k)}/W_n^{(k)} < a/c$ for all larger n.*

Example 6.2. Let

$$X_n = X_{n-1} + 6X_{n-2}.$$

Then, X_n is a non-negative difference equation. However, it is not doubly non-negative. Now, N_n and W_n are two solutions to X_n. Let

$$N_n = (3/5n)(3^n) + (2/5n)(-2)^n$$

and

$$W_n = (6/5n + 127/125)(3^n) + (1/5n^2 + 41/25n - 127/125)(-2)^n$$

Take N_n/W_n and you get:

$$N_n/W_n = 0.2000000000,\ 0.3255813953,\ 0.4936708861,\ 0.3600654664,\ 0.5057034221,$$

$$0.3960085531,\ 0.4904161342,\ 0.4250069054,\ 0.4807749847,\ 0.4453073066, \ldots$$

The asymptotic value of $N_n/W_n = 0.5$. Therefore, $N_n/W_n < a/c$ for the first $2k-1$ n's, since $k = 2$ and $2k-1 = 3$. However, N_n/W_n is not less than the asymptotic value for all larger n's.

Secondly, an asymptotic increasing result should be mentioned.

Conjecture 6.3. *If*

$$N_n = (an + b)\lambda_0^n + O(\lambda_0^n),$$
$$W_n = (cn + d)\lambda_0^n + O(\lambda_0^n)$$

and

$$ad - bc > 0,$$

then N_n/W_n is increasing for all big enough n.

Double non-negativity is a condition for monotonic increasing in the stated conjecture. So, it would be nice to have an algorithm to determine double non-negativity by a simple means, such as a condition on the coefficients. In attempts to develop such an algorithm conditions for an 2^{nd} order non-negative difference equation to be doubly non-negative were found.

Theorem 6.4. *A second order non-negative difference equation has the characteristic polynomial $ch(\lambda) = \lambda^2 - c_1\lambda - c_2$. If $c_2 \leq 2c_1^2$ (this is equivalent to $\lambda_0 \geq 2|\lambda_1|$), then the difference equation will be doubly non-negative.*

Proof. Take a second order non-negative difference equation. It will have the characteristic polynomial

$$ch(\lambda) = \lambda^2 - c_1\lambda - c_2 = (\lambda - \lambda_0)(\lambda - \lambda_1)$$

where $\lambda_0 > 0$, $\lambda_1 < 0$, $c_1 \geq 0$, and $c_2 > 0$. Then,

$$[ch(\lambda)]^2/(\lambda - \lambda_0) = (\lambda - \lambda_1)ch(\lambda) =$$
$$\lambda^3 - (c_1 + \lambda_1)\lambda^2 - (c_2 - c_1\lambda_1)\lambda + c_2\lambda_1.$$

For double non-negativity, the coefficients need to be ≤ 0.

$$c_2\lambda_1 < 0$$

is true because $c_2 > 0$ and $\lambda_1 < 0$.

$$-(c_2 - c_1\lambda_1) < 0$$

is true because $c_2 > 0$, $c_1 \geq 0$, and $\lambda_1 < 0$. So, the only one left to show is

$$-(c_1 + \lambda_1) \leq 0$$

which can be rewritten

$$c_1 + \lambda_1 \geq 0.$$

Since $c_1 = \lambda_0 + \lambda_1$,

$$\lambda_0 + \lambda_1 + \lambda_1 \geq 0.$$

So,

$$\lambda_0 \geq 2|\lambda_1|$$

By taking $c_1 + \lambda_1 \geq 0$ and multiplying by λ_0 gives

$$c_1\lambda_0 + \lambda_1\lambda_0 \geq 0.$$

Since $\lambda_1\lambda_0 = -c_2$,

$$c_1\lambda - c_2 \geq 0$$

and

$$\lambda_0 \geq c_2/c_1.$$

Note: If $c_1 = 0$, then double non-negativity is not possible. $\lambda_0 \geq c_2/c_1$ if and only if $ch(c_2/c_1) \leq 0$. So,

$$(c_2/c_1)^2 - c_1(c_2/c_1) - c_1 \leq 0$$
$$c_2^2/c_1^2 - c_2 - c_2 \leq 0$$
$$c_2/c_1^2 - 2 \leq 0$$
$$c_2 \leq 2c_1^2.$$

Therefore, a non-negative second order difference equation will be doubly non-negative if and only if $c_2 \leq 2c_1^2$. □

Example 6.5. The non-negative difference equation

$$X_n = 3X_{n-1} + 4X_{n-2}$$

was shown to be doubly non-negative in Example 5.4. Here $c_1 = 3$ and $c_2 = 4$, so

$$4 \leq 2 * 3^2 \leq 18.$$

And double non-negative follows from Theorem 6.4. In this example, $\lambda_0 = 4$ and $\lambda_1 = \lambda_2 = -1$, and as expected $\lambda_0 \geq 2|\lambda_i|$.

7. Conclusion

Capocelli observed that the ratios of two solutions to a generalized Fibonacci difference equation were increasing for some test data and claimed that these ratios always increased. We tried to determine what properties these special equations had that accounted for these increasing ratios. We showed by counterexample that non-negativity and increasing for k n's in a row was not enough. We claim that doubly non-negativity and increasing $2k-1$ n's in a row may be enough, but we have been unable to prove this conjecture. We have also been unable to produce an efficient algorithm to determine double non-negativity, although we have conditions for second order non-negative difference equations.

References

[C] Capocelli,R.M, "A Generalization of Fibonacci Trees," *Applications of Fibonacci Numbers* (Volume 3), G. Bergum, A. Philippou, and A. Horadam (eds.), 37-56, Kluwer; 1990.

[Cp] Capocelli, R.M,de Santis, A, "Regular universal codeword sets, *IEEE Transactions on Information Theory*, 32,1: 129 - 133; Jan. 1986.

[Ca] Capocelli,R.M, G. Cerbone, P. Cull, and J. L. Holloway. "Fibonacci Facts and Formulas," In *Sequences*, 123-137, Springer-Verlag New York, Inc; 1999.

[Cc] Capocelli R.M, Pall Cull. "Rounding the Solutions of Fibonacci-like Difference Equations," *Fibonacci Quarterly*, 41: 133-141, 2003.

[Cl] Cull, Paul, "A Brief Introduction to Algorithms," Chapter 9: Difference Equations, Class Notes; 2005. [website]http://classes.engr.oregonstate.edu/eecs/fall2005/cs325/.

[Cul] Cull, Paul, "Difference Equations as Biological Models," *Scientiae Mathematicae Japonicae*, 64,2; Sept. 2006. (To Appear)

[Cu] Cull,Paul, Mary Flahive, and Robby Robson, "Difference Equations: From Rabbits to Chaos," Springer, Inc. New York, New York; 2005.

[K] Koshy, Thomas, "Fibonacci and Lucas Numbers with Applications," John Wiley and Sons, Inc. New York; 2001.

[Z] Zeckendorf, E. "Representations des Nombres Naturels par une Somme de Nombres de Fibonacci ou de Nombres de Lucas," *Bulletin de la Societé Royale des Sciences de Liege* 179-182;1972.

[Zt] *Zeckendorf's Theorem*. [website] http://planetmath.org/.

Appendix A. MAPLE Program for Calculating $N_n^{(k)}/W_n^{(k)}$

```
kfib:= proc(k,n)
    local a,b,i,j;
    a[0]:=0;
    a[1]:=1;
    for i from 2 to k - 1 do
        a[i]:=2^(i - 2)
    end do;
    if n<1 then
        RETURN(0)
    elif n=1 then
        RETURN(1)
    else
        for j from k to n do
            a[j]:=add(a[j - x],x=1..k)
        end do
    end if;
    RETURN(a[n])
end proc;

kNn:= proc(k,n)
    local a,b,i,j,c;
    a[0]:=0;
    a[1]:=1;
    a[2]:=4;
```

```
    c:=k - 1;
    for i from 3 to k - 1 do
        a[i]:=add(a[i - x],x=1..i)+kfib(i,i+2)
    end do;
    if n=0 then
        RETURN(0)
    elif n=1 then
        RETURN(1)
    elif n=2 then
        RETURN(4)
    else
        for j from k to n do
            a[j]:=add(a[j - x],x=1..k)+kfib(k,j+2)
        end do;
        RETURN(a[n]);
        end if;
    end proc;

kWn:= proc(k, n)
    local a;
    a := n*kfib(k, n+2);
    RETURN(a)
end proc;

kNnWn:= proc (k, n)
    local a;
    a := evalf(kNn(k, n)/kWn(k, n));
    RETURN(a)
end proc;
```

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