ON THE UNIQUENESS OF A CONVEX BODY GIVEN X-RAY DATA FROM TWO EXTERIOR SOURCES

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ABSTRACT. We investigate under which circumstances a convex shadow body may exist such that its point X-rays, collected from two exterior sources, are equal to those of an original body. Geometric methods are used to determine the basepoints of a shadow body given a particular example. We discuss possible locations of the general shadow body relative to the initial body and sources. Derivations concerning curvature at the basepoints are given to ensure that the shadow body is convex. Finally, we attempt to establish the existence and convexity of a shadow body using recursion.

1. INTRODUCTION

Geometric tomography is a field of mathematics that originated in 1961 when P.C. Hammer [6] posed his general problem: How many X-ray sources are required to uniquely determine a convex body in the plane? In the mathematical sense, an X-ray function simply measures the length of intersection with a given object over all rays emanating from a point source. Here one assumes that the object in question is of constant density equal to one.

Hammer's question was general enough to include X-rays with an assigned direction and those without. In 1983, Falconer [3] and Gardner [5] made much progress in the area. They independently discovered that two directed X-ray sources are sufficient to uniquely determine a convex body, provided that the line passing through the sources intersects the body. This development has been termed "The Uniqueness Theorem". A bit later, Volčič [9] proved that a convex body is uniquely determined by X-rays without a specified direction, or point X-rays, from three non-colinear points. However, nothing conclusive has yet been found concerning point X-rays from two sources.

In this paper, we attempt to prove that The Uniqueness Theorem does not hold when point X-rays are taken as opposed to directed X-rays. Once more, our assumption is that point X-ray data is collected from two sources such that the line passing through the sources intersects the body. Given an original convex body, we wish to establish the existence of a second convex body (which we call a "shadow body") with point X-rays equal to those of the first in order to disprove uniqueness. Ross and Tuite [8] give convincing evidence that a convex shadow body may indeed exist using reconstructions of a particular example. We have chosen to work in the more general setting and have developed necessary conditions concerning existence and convexity of the shadow body.

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2. DEFINITIONS

This section contains basic definitions that are utilized in our analyses.

Definition 2.1. *A **convex body**, K, is a compact, convex subset of the plane with non-empty interior. The boundary of K is denoted ∂K and the interior of K is denoted int K.*

Definition 2.2. *A **source** is a point from which X-ray data is collected for a particular (convex) body. Without loss of generality, we can assume that the sources are located on the x-axis in the Cartesian coordinate system, or the line $\varphi = 0$ in polar coordinates. Throughout this paper, P will be used to indicate the leftmost source and Q will indicate the rightmost source. The **baseline** is defined to be the line passing through P and Q with the given assumptions. Given a convex body K, the **basepoints** are the points at which ∂K intersects the baseline. The leftmost basepoint will be labeled a and the rightmost will be labeled b.*

Definition 2.3. *Given a convex body K, a source P, and an angle $\varphi \in [0, 2\pi)$, the **nearside point (farside point)** is defined as the nearest (furthest) point in $r_\varphi \cap \partial K$ to P, where r_φ is the ray emanating from P at angle φ measured counterclockwise from the x-axis. This definition also holds for the source Q.*

Note: All angles in this paper will be measured counterclockwise from the x-axis (or equivalently, from the baseline).

Definition 2.4. *The **nearside function (farside function)** is defined as the distance from a source P to the nearside (farside) point on the body for each angle φ. The nearside and farside functions for the original body (shadow body) are denoted $r_P(\varphi)$ and $R_P(\varphi)$ ($\mathscr{s}_P(\varphi)$ and $S_P(\varphi)$) respectively.*

Definition 2.5. *Let P be a point (source) in the plane and K a convex body. The **point X-ray** of K at P is a function X_P, defined on $[0, \pi)$, such that for all $\varphi \in [0, \pi)$,*

$$X_P(\varphi) = \lambda_1(K \cap (l_\varphi + P)),$$

where l_φ is the line passing through the origin rotated φ radians from the x-axis and λ_1 signifies length. The directed X-ray can equivalently be defined as $X_P(\varphi) = R_P(\varphi) - r_P(\varphi)$. The point X-ray of K at Q is defined similarly.

Definition 2.6. *Let P be a point (source) in the plane and K a convex body. The **directed X-ray** of K at P is a function $\mathscr{D}_P$, defined on $[0, 2\pi)$, such that for all $\varphi \in [0, 2\pi)$,*

$$\mathscr{D}_P(\varphi) = \lambda_1(K \cap (r_\varphi + P)),$$

where r_φ is the ray from the origin rotated φ radians from the x-axis and λ_1 signifies length. The directed X-ray can equivalently be defined as $\mathscr{D}_P(\varphi) = R_P(\varphi) - r_P(\varphi)$. The directed X-ray of K at Q is defined similarly.

Definition 2.7. *Given a convex body K, a **shadow body**, S, is a second body, $S \neq K$, with X-rays $X_P(\varphi)$ (directed or point) equal to those of K for all angles φ emanating from all sources considered.*

Definition 2.8. *A **lens-shaped body** is given by $C_1 \cap C_2$, where C_1 and C_2 are circular bodies with radii r whose centers c_1 and c_2 lie on the x-axis, $c_1 \neq c_2$. By way of construction, lens-shaped* *bodies are symmetric about the x-axis. The* ***vertices*** *of a lens-shaped body are given by* $\partial C_1 \cap \partial C_2$*. The upper vertex will be denoted* ν*.*

Definition 2.9. *Suppose that* $X_P(\varphi)$ *is an X-ray function for a body* K *from a source* P*. Let* $\alpha = inf\{\varphi : X_P(\varphi) > 0\}$ *and* $\beta = sup\{\varphi : X_P(\varphi) > 0\}$*. Then the* ***support lines*** *of* K *from* P *are the lines emanating from* P *with angles of inclination* α *and* β*. The support lines of* K *from* Q *are defined similarly.*
Note: The vertices of a lens-shaped body will occur where the support lines from P and Q intersect.

Definition 2.10. *The* ***curvature operator****, denoted* $\mathscr{K} f$*, characterizes the direction of concavity and is given by*

$$\mathscr{K} f(\varphi) = f(\varphi)^2 + 2(f'(\varphi))^2 - f(\varphi) f''(\varphi)$$

when f *is* C^2 *at angle* φ*. The curvature operator is positive (negative) when the graph of* f *is concave toward (away from) the source at* $(f(\varphi), \varphi)$*.*

Definition 2.11. *The* ***signed curvature*** *of a function* f*, denoted* κ_f*, is defined as*

$$\kappa_{f(\varphi)} = \frac{f(\varphi)^2 + 2(f'(\varphi))^2 - f(\varphi) f''(\varphi)}{(f(\varphi)^2 + (f'(\varphi))^2)^{\frac{3}{2}}}$$
$$= \frac{\mathscr{K} f(\varphi)}{(f(\varphi)^2 + (f'(\varphi))^2)^{\frac{3}{2}}}$$

when f *is* C^2 *at angle* φ*. Thus,* $\mathscr{K} f(\varphi) = \kappa_{f(\varphi)} \cdot (f(\varphi)^2 + (f'(\varphi))^2)^{\frac{3}{2}}$*.*

3. LOCATION OF THE BASEPOINTS

A natural beginning to our shadow body investigation is location of the basepoints. Once this information is known, it can be built upon to include other conditions concerning shadow body location. This section contains a formula that is later used in basepoint computations.

3.1. μ-measure and Fithian's Triangle Theorem.

First, we define a measurement related to area that is only used in this section.

Definition 3.1. *The* ***μ-measure*** *of some area* T *in the upper half plane is defined as*

$$\mu(T) = \int\int_T \frac{1}{y} dA.$$

The μ*-measure can equivalently be defined in polar form as*

$$\mu(T) = \int_0^{2\pi} \int_0^{\infty} \frac{f(r,\theta)}{\sin(\theta)} dr d\theta,$$

where $f(r,\theta)$ *is* 1 *when* $(r,\theta) \in T$ *and* 0 *otherwise.*

The point X-ray function is used here to obtain μ-measures for various areas. The following theorem is used in the formula derivation, which provides a method for computing the μ-measure of a triangle. The result is due to Fithian [4].

Theorem 3.2. *Fix a ray r emanating from a point w on the x-axis with angle of inclination $\varphi \in (0,\pi)$. Let $x = (w + l_0, 0)$ for some positive l_0. Let y and z be points on r such that $d(w,y) < d(w,z)$. Finally, let $l_1 = d(w,y)$ and $b = d(y,z)$ (see figure 1). If $T = \triangle(x,y,z)$, then*

$$\mu(T) = l_0 \ln(1 + \frac{b}{l_1}). \tag{1}$$

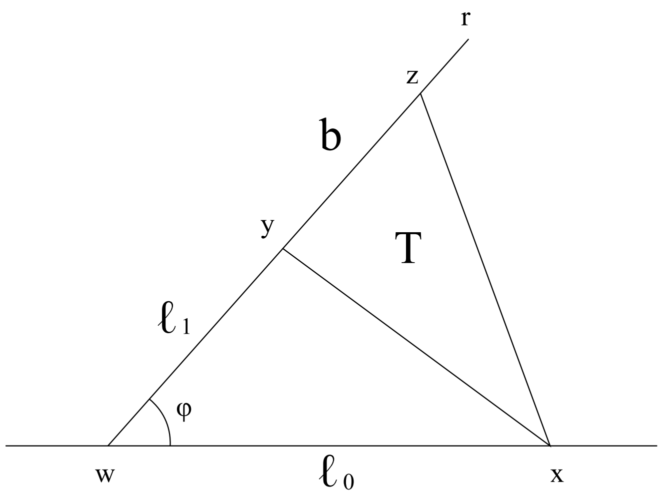

FIGURE 1. Theorem 3.2

3.2. **Falconer's Lemma.** Falconer proved a result in 1983 [3] which can be used to locate the basepoints of both the original body and the possible shadow body. Falconer's Lemma is stated below, which is followed by an alternate proof using the above theorem. In the lemma, K is a convex body.

Lemma 3.3. *Let K have point X-ray functions f_1 and f_2 at P and Q where l, containing P and Q, cuts int$K$. Let a,b be the points of $l \cap \partial K$. Writing p_1, q_1, p_2, q_2 for the (signed) distances Pb, Pa, Qb, Qa respectively, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{2} \left[\int_{\varepsilon}^{\pi-\varepsilon} \frac{f_2(\varphi)}{\sin(\varphi)} d\varphi - \int_{\varepsilon}^{\pi-\varepsilon} \frac{f_1(\varphi)}{\sin(\varphi)} d\varphi \right] = p_1 \ln|p_1| - p_2 \ln|p_2| - q_1 \ln|q_1| + q_2 \ln|q_2|. \tag{2}$$

Proof. There are two cases that we will consider. For simplicity, juxtaposition will be used to denote the signed distance between two points, with left to right being positive and vice versa.

Case 1: First the case in which K lies between the two sources is examined, as shown in Figure 2 below.

Geometrically, the left hand side of equation (2) represents half of the μ-measure of the body above the ray emanating from source P subtracted from the μ-measure above the ray from source Q, both with angle of inclination ε (see Definition 3.1). These methods hold for the portion of the

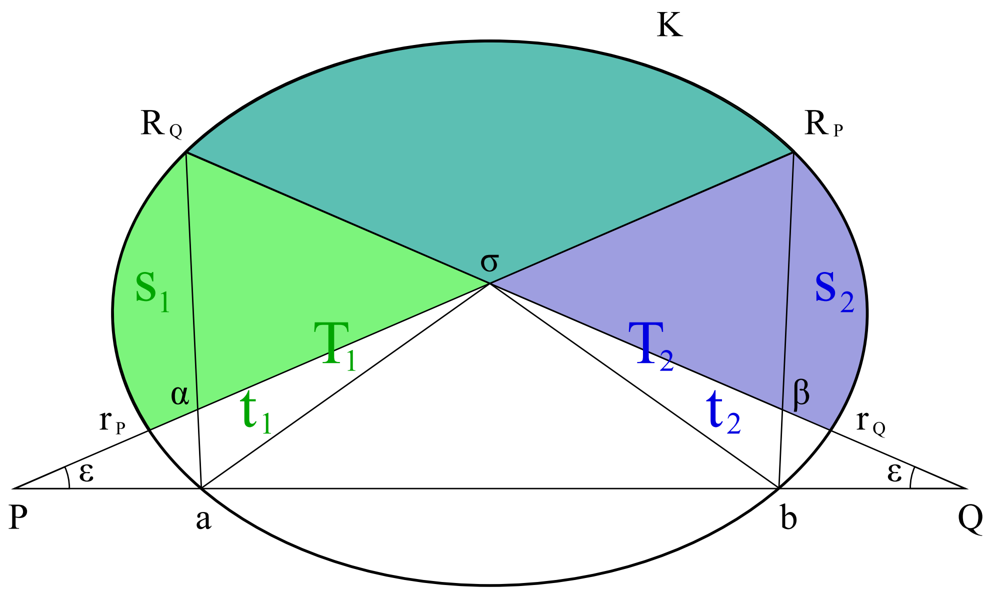

FIGURE 2. K lies between P and Q

body lying beneath the baseline as well. The top center region cancels and the area we are left with is the difference of two regions: $\mu(region(r_Q, \sigma, R_P)$ - $\mu(region(r_P, \sigma, R_Q))$. Let

$$T_1 = \triangle(a, \sigma, R_Q), \qquad t_1 = \triangle(a, \sigma, \alpha),$$
$$T_2 = \triangle(b, \sigma, R_P), \qquad t_2 = \triangle(b, \sigma, \beta),$$
$$s_1 = region(r_P, \alpha, R_Q), \text{ and} \qquad s_2 = region(r_Q, \beta, R_P).$$

Then the left hand side of (2) is equal to

(3) $$\lim_{\varepsilon \to 0} \frac{1}{2}[(\mu(T_2) - \mu(t_2) + \mu(s_2)) - (\mu(T_1) - \mu(t_1) + \mu(s_1))].$$

Theorem 3.2 can be used to compute the μ-measure of the four triangles. Then we let $\varepsilon \to 0$. The μ-measure of the two side regions will be shown to converge to zero. Note that the x-value of σ is always the midpoint of PQ, so the limiting value will be denoted σ for simplicity.

T_1: Applying (1), $\mu(T_1) = aQ\ln(1 + \frac{R_Q\sigma}{\sigma Q})$. But $R_Q\sigma \to a\sigma = aQ - \sigma Q$ as $\varepsilon \to 0$.
So $\mu(T_1) \to aQ\ln(1 + \frac{aQ - \sigma Q}{\sigma Q}) = aQ\ln(aQ) - aQ\ln(\sigma Q)$.

t_1: By (1), $\mu(t_1) = Pa\ln(1 + \frac{\alpha\sigma}{P\alpha})$. But $\alpha\sigma \to a\sigma = P\sigma - Pa$ and $P\alpha \to Pa$ as $\varepsilon \to 0$.
So $\mu(t_1) \to Pa\ln(1 + \frac{P\sigma - Pa}{Pa}) = Pa\ln(P\sigma) - Pa\ln(Pa)$.

T_2: Using (1), $\mu(T_2) = Pb\ln(1 + \frac{\sigma R_P}{P\sigma})$. But $\sigma R_P \to \sigma b = Pb - P\sigma$ as $\varepsilon \to 0$.
So $\mu(T_2) \to Pb\ln(1 + \frac{Pb - P\sigma}{P\sigma}) = Pb\ln(Pb) - Pb\ln(P\sigma)$.

t_2: Once more, $\mu(t_2) = bQ\ln(1 + \frac{\sigma\beta}{\beta Q})$. But $\sigma\beta \to \sigma b = \sigma Q - bQ$ and $\beta Q \to bQ$ as $\varepsilon \to 0$.
So $\mu(t_2) \to bQ\ln(1 + \frac{\sigma Q - bQ}{bQ}) = bQ\ln(\sigma Q) - bQ\ln(bQ)$.

Writing these limits in terms of $p_1, q_1, p_2,$ and q_2 we get

$$\mu(T_1) \to -q_2 \ln(-q_2) + q_2 \ln(\sigma Q),$$
$$\mu(t_1) \to q_1 \ln(P\sigma) - q_1 \ln(q_1),$$
$$\mu(T_2) \to p_1 \ln(p_1) - p_1 \ln(P\sigma), \text{and}$$
$$\mu(t_2) \to -p_2 \ln(\sigma P) + p_2 \ln(-p_2).$$

The natural log function is only defined for positive values, so we must take the absolute value of the appropriate distances. Notice that $P\sigma = \sigma Q$ since σ is the midpoint of PQ. So far, as $\varepsilon \to 0$, we have the limit

(4)
$$[\mu(T_2) - \mu(t_2)] - [\mu(T_1) - \mu(t_1)] \to$$
$$p_1 \ln|p_1| - p_2 \ln|p_2| - q_1 \ln|q_1| + q_2 \ln|q_2| +$$
$$(-p_1 - p_2 + q_1 + q_2) \ln|P\sigma|.$$

But $p_1 - q_1 = p_2 - q_2$, so $-p_1 - p_2 + q_1 + q_2 = -(p_1 - q_1) - (p_2 - q_2) = 0$ and the $\ln|P\sigma|$ term cancels.

It still needs to be shown that $\mu(s_1)$ and $\mu(s_2)$ converge to zero. It suffices to prove that the μ-measure of two triangles containing s_1 and s_2 converges to zero. The following setup will be used, where the line segment from t_a to a is tangent to the body at the point a. (Note that this figure looks slightly different from the first because ε must be small enough to ensure that the triangle completely encloses s_1.

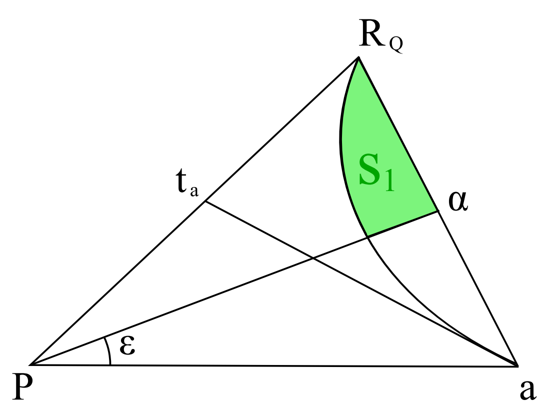

FIGURE 3. Triangles used to compute $\mu(s_1)$

s_1: From (1), $\mu(\triangle(a, t_a, R_Q)) = Pa \ln(1 + \frac{t_a R_Q}{P t_a})$. But as $\varepsilon \to 0$, $t_a \to a$ and $R_Q \to a$, so $t_a R_Q \to 0$. Also, $P t_a \to Pa$. So $\mu(\triangle(a, t_a, R_Q)) \to Pa \ln(1 + \frac{0}{Pa}) = Pa \ln(1) = 0$.

s_2: Applying (1), we have that $\mu(\triangle(b, t_b, R_P)) = bQ \ln(1 + \frac{R_P t_b}{t_b Q})$. As $\varepsilon \to 0$, $R_P \to b$ and $t_b \to b$, so $R_P t_b \to 0$. Also, $t_b Q \to bQ$. So $\mu(\triangle(b, t_b, R_P)) \to bQ \ln(1 + \frac{0}{bQ}) = bQ \ln(1) = 0$.

So the μ-measure of both triangles containing s_1 and s_2 converge to zero as ε approaches zero. Then from (3) and (4), we are left with

$$\lim_{\varepsilon\to 0}\frac{1}{2}[(\mu(T_2)-\mu(t_2)+\mu(s_2))-(\mu(T_1)-\mu(t_1)+\mu(s_1))]=$$
$$\frac{1}{2}[p_1\ln|p_1|-p_2\ln|p_2|-q_1\ln|q_1|+q_2\ln|q_2|].$$

This method can be used for the portion of the body that lies below the baseline to obtain the same results, canceling the $\frac{1}{2}$ coefficient on the right side of the equation. Thus (2) holds.

It should be mentioned that this case has a second figure. It is possible that σ does not lie inside K if one of the sources is sufficiently close to the body and the other far away. However, the proof follows using the same techniques.

Case 2: Next we will use a similar approach to prove the case in which the two sources lie to the same side of K, as shown below.

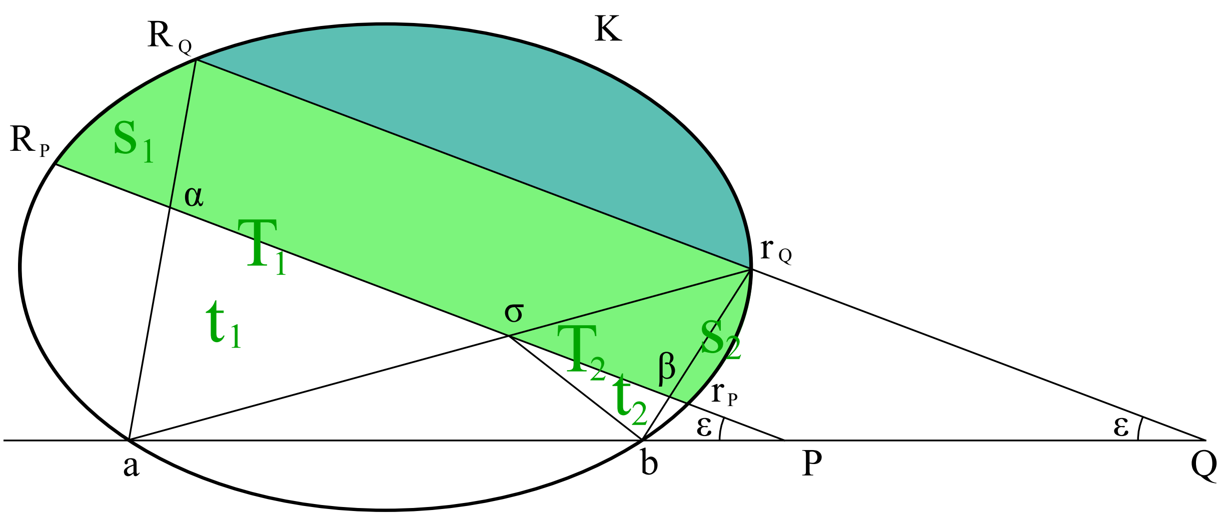

FIGURE 4. K lies to the side of P and Q

The region left in this case is the negative of $\mu(region(R_P,R_Q,r_Q,r_P))$. Let

$$T_1=\triangle(a,R_Q,r_Q),\qquad t_1=\triangle(a,\alpha,\sigma),$$
$$T_2=\triangle(a,r_Q,b),\qquad t_2=\triangle(b,\sigma,\beta),$$
$$s_1=region(R_P,R_Q,\alpha),\text{ and}\qquad s_2=region(\beta,r_Q,r_P).$$

Here, the left hand side of (2) is equal to

$$\lim_{\varepsilon\to 0}\frac{1}{2}\cdot-[(\mu(T_2)-\mu(t_2)+\mu(s_2))+(\mu(T_1)-\mu(t_1)+\mu(s_1))]. \tag{5}$$

Theorem 3.2 is used in the following computations, as before.

$T_1 : \mu(T_1) = aQ\ln|1+\frac{R_Q r_Q}{r_Q Q}| \to aQ\ln|1+\frac{ab}{bQ}| = aQ\ln|1+\frac{aQ-bQ}{bQ}|$
$\quad = aQ\ln|\frac{aQ}{bQ}| = aQ\ln|aQ| - aQ\ln|bQ|$ as $\varepsilon \to 0$.

$t_1 : \mu(t_1) = aP\ln|1+\frac{\alpha\sigma}{\sigma P}| \to aP\ln|1+\frac{a\sigma}{\sigma P}| = aP\ln|1+\frac{aP-\sigma P}{\sigma P}|$
$\quad = aP\ln|\frac{aP}{\sigma P}| = aP\ln|aP| - aP\ln|\sigma P|$ as $\varepsilon \to 0$.

$T_2 : \mu(T_2) = ab\ln|1+\frac{\sigma r_Q}{a\sigma}| \to ab\ln|1+\frac{\sigma b}{a\sigma}| = ab\ln|1+\frac{ab-a\sigma}{a\sigma}|$
$\quad = ab\ln|\frac{ab}{a\sigma}| = ab\ln|ab| - ab\ln|a\sigma|$ as $\varepsilon \to 0$.

$t_2 : \mu(t_2) = bP\ln|1+\frac{\sigma\beta}{\beta P}| \to bP\ln|1+\frac{\sigma b}{bP}| = bP\ln|1+\frac{\sigma P-bP}{bP}|$
$\quad = bP\ln|\frac{\sigma P}{bP}| = bP\ln|\sigma P| - bP\ln|bP|$ as $\varepsilon \to 0$.

In terms of $p_1, q_1, p_2,$ and q_2 we have

$$\begin{aligned}
&\mu(T_1) \to -q_2\ln|q_2| + q_2\ln|p_2|,\\
&\mu(t_1) \to -q_1\ln|q_1| + q_1\ln|\sigma P|,\\
&\mu(T_2) \to ab\ln|ab| - ab\ln|a\sigma|, \text{and}\\
&\mu(t_2) \to -p_1\ln|\sigma P| + p_1\ln|p_1|.
\end{aligned}$$

But $q_2\ln|p_2| = -ab\ln|p_2| + p_2\ln|p_2|$. So as $\varepsilon \to 0$, we have the limit

$$\begin{aligned}
-[(\mu(T_2)-\mu(t_2)) + (\mu(T_1)-\mu(t_1))] \to\\
p_1\ln|p_1| - p_2\ln|p_2| - q_1\ln|q_1| + q_2\ln|q_2| +\\
ab\ln|p_2| + q_1\ln|\sigma P| + ab\ln|a\sigma| - ab\ln|ab| - p_1\ln|\sigma P|.
\end{aligned} \tag{6}$$

To eliminate the remaining terms, first notice that $q_1\ln|\sigma P| = p_1\ln|\sigma P| - ab\ln|\sigma P|$. Then the terms including $p_1\ln|\sigma P|$ cancel. Grouping the ab coefficients, we are left with $ab\ln|\frac{a\sigma\cdot p_2}{ab\cdot\sigma P}|$. A triangle argument can be used to show that this term vanishes as well. The triangles $\triangle(a,\sigma,P)$ and $\triangle(a,r_Q,Q)$ are similar since both share a common angle on the left, have angle ε on the right, and so have equal third angles as well. Hence $\frac{a\sigma}{ar_Q} = \frac{\sigma P}{r_Q Q}$. This equality converges to $\frac{a\sigma}{ab} = \frac{\sigma P}{p_2}$ as $\varepsilon \to 0$. Using this, $ab\ln|\frac{a\sigma\cdot p_2}{ab\cdot\sigma P}| = ab\ln|\frac{\sigma P\cdot p_2}{p_2\cdot\sigma P}| = ab\ln|1| = 0$, yielding the desired result.

It remains to prove that the two side regions converge to zero. A method similar to that in the first case will be used. That is, it will be shown that the μ-measure of triangles containing s_1 and s_2 converge to zero. The construction is as follows.

$s_1 : \mu(\triangle(a,t_a,R_Q)) = aQ\ln|1+\frac{t_a R_Q}{R_Q Q}| \to aQ\ln|1+\frac{0}{aQ}| = aQ\ln|1| = 0$ as $\varepsilon \to 0$.

$s_2 : \mu(\triangle(b,t_b,r_Q)) = ab\ln|1+\frac{t_b r_Q}{at_b}| \to ab\ln|1+\frac{0}{ab}| = ab\ln|1| = 0$ as $\varepsilon \to 0$.

It has been shown that $\mu(s_1)$ and $\mu(s_2) \to 0$ as $\varepsilon \to 0$. Now combining (5) and (6) we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{2}\cdot -[(\mu(T_2)-\mu(t_2)+\mu(s_2)) + (\mu(T_1)-\mu(t_1)+\mu(s_1))] \to\\
\frac{1}{2}[p_1\ln|p_1| - p_2\ln|p_2| - q_1\ln|q_1| + q_2\ln|q_2|].
\end{aligned}$$

Similarly, the lower portion of the body has the same limit. This proves (2).

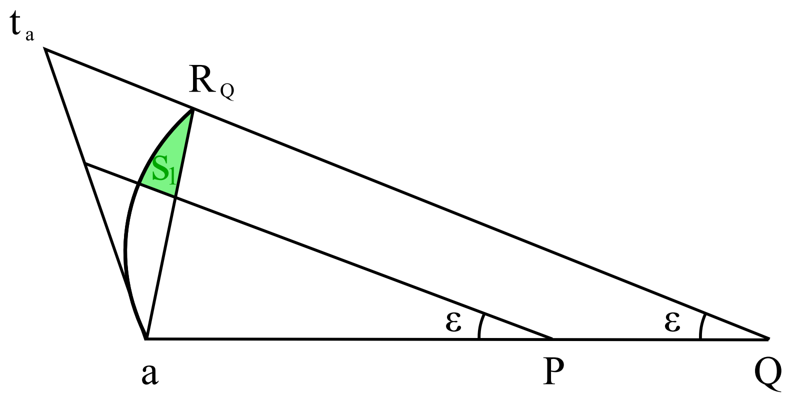

FIGURE 5. Triangles used to compute $\mu(s_1)$

□

4. LOCATION OF THE SHADOW BODY

Now that we have a formula to locate the basepoints of a shadow body for a specific original body, we can generalize these results. That is, we can study the shadow body location with respect to the original body in the general case. In Section 3, we saw that Falconer [3] derived a formula which helps describe the possible location of the shadow body for a given body K. After reparameterization, this formula produces a function, $G(t)$. In this section, we present a geometric interpretation of this $G(t)$ function, including an evaluation of its singularities and extreme values.

4.1. **Function Construction.** The construction of the $G(t)$ function is due to Falconer [3]. We will summarize the construction first, then proceed to the interpretations. The two sources will be denoted P and Q, as usual, and the baseline will be denoted l. Without loss of generality, we can center P at the origin and define A to be the distance from P to Q. Let m be the width of the original body at the baseline, or the distance between the two basepoints a to b. Finally, we use B to label the value of (2). As a reminder, p_1, q_1, p_2, q_2 represent the (signed) distances Pb, Pa, Qb, Qa respectively. In terms of these variables, $m = p_1 - q_1 = p_2 - q_2$ and $A = q_1 - q_2 = p_1 - p_2$ (on account of the distances being signed). Define

$$F(t) = t \ln|t| - (t-m)\ln|t-m|.$$

Then

$$\begin{aligned} F(p_1) - F(p_2) &= p_1 \ln|p_1| - (p_1 - m)\ln|p_1 - m| - p_2 \ln|p_2| + (p_2 - m)\ln|p_2 - m| \\ &= p_1 \ln|p_1| - p_2 \ln|p_2| - q_1 \ln|q_1| + q_2 \ln|q_2| \\ &= B. \end{aligned}$$

This leaves us with the system of equations

$$p_1 - p_2 = A$$
$$F(p_1) - F(p_2) = B.$$

Finally, the G function is obtained by substituting for p_2 and solving for B. This produces solutions of the form

$$G(p_1) = F(p_1) - F(p_1 - A) = B, \text{ or}$$
$$G(t) = t\ln|t| - (A-t)\ln|A-t| - (t-m)\ln|t-m| + (A-t-m)\ln|A-t-m|$$

in terms of t, A, and m, where $t = p_1 = Pb$. Below is an example of a $G(t)$ graph when $A = 10$ and $m = 1$. However, the shape of the graph is similar for all values of A and m.

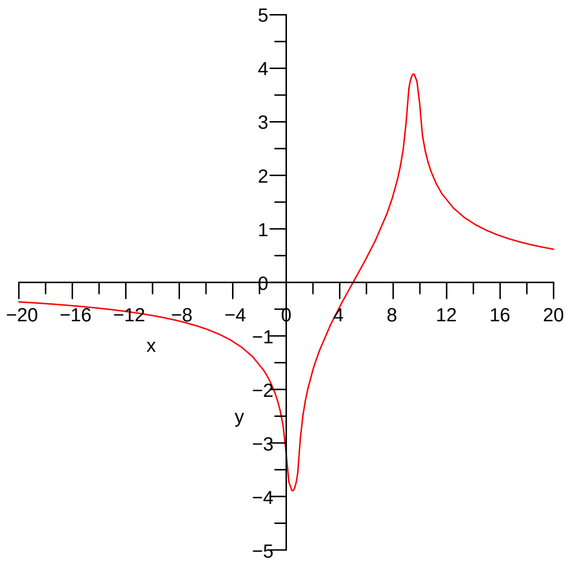

FIGURE 6. $G(t)$ Function

Given A and m values, the location of the shadow body and original body basepoints can be found by graphing $y = B$, which is obtained from (2), against its corresponding $G(t)$ function. The two intersections represent the two $t = p_1$ values for the original and shadow body.

4.2. **Function Singularities.** We are ready to investigate the singularities of $G(t)$ now that the development is complete. This function does not exist for t values of 0, m, A, and $A+m$ since the natural logarithm function is not defined at 0. Without loss of generality, we can assume that $P = (0,0)$ and $Q = (A,0)$. Then, since t corresponds to the distance Pb, these four values are precisely the four possible positions of our original body in which at least one of the sources lies on the boundary. Furthermore, if $m = A$, there is the possibility of both sources lying on the boundary, which is accounted for. This interpretation is compatable with Falconer's original proof of (2) since he assumes that the sources do not lie on the boundary of the body. The geometry is illustrated below.

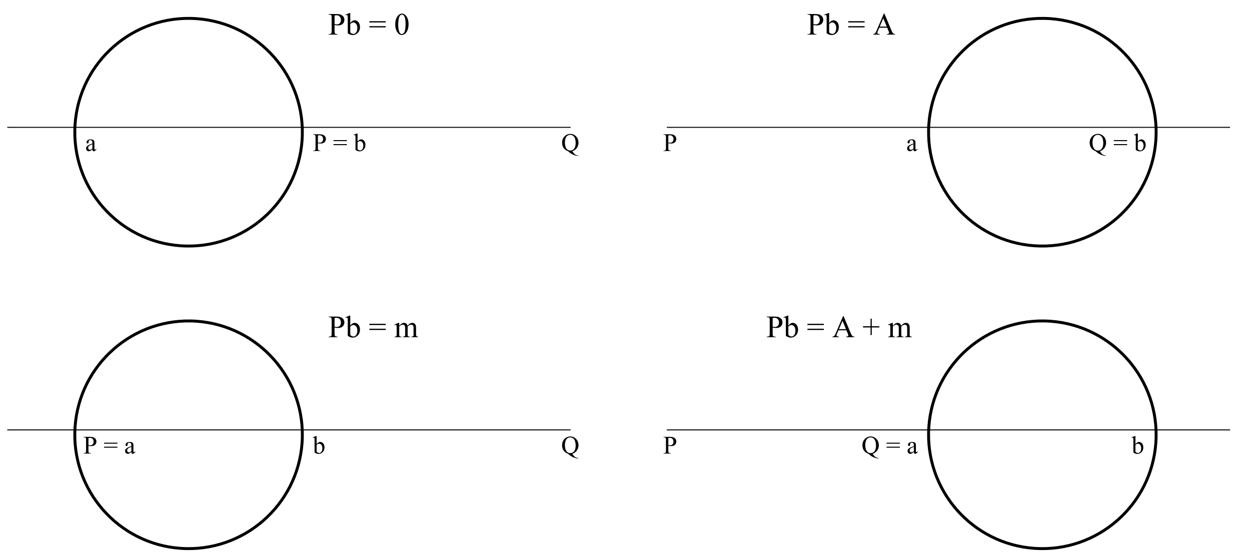

FIGURE 7. Singularities of $G(t)$

It is clear from these figures that $t = Pb$ values less than 0 correspond to bodies to the left of P, values between m and A correspond to bodies between the sources, and values greater than $A+m$ correspond to bodies to the right of Q. Also, t values between 0 and m or between A and $A+m$ correspond to bodies with (at least) one interior source.

4.3. **Function Extremes.** The last characteristic of the function that we study are the extreme points. There are three points at which $G(t)$ only has one solution: the minimum, maximum, and point where it crosses the x-axis. At these points, a shadow body does not exist. The only solution must correspond to the p_1 value of the given original body. From [8], G achieves an absolute minimum at $t = \frac{1}{2}\left(A+m-\sqrt{A^2+m^2}\right)$ and an absolute maximum at $t = \frac{1}{2}\left(A+m+\sqrt{A^2+m^2}\right)$. Also, $G(t) = 0$ if and only if $t = \frac{A+m}{2}$. But what can we make of this geometrically?

The point where $G(t)$ crosses the x-axis is the easiest to interpret, which occurs at $t = \frac{A+m}{2}$. This is simply the midpoint of the distance between the two sources. So if the original body is centered with respect to P and Q, then a corresponding shadow body cannot exist. Our results found concerning the minimum and maximum values are stated below.

Theorem 4.1. *Let A and m be positive values with $A = d(P,Q)$ and $m = d(a,b)$. The function $G(t)$, as defined above, possesses the following properties:*

(i) *The center of the minumim body lies to the left of P and the center of the maximum body lies to the right of Q. That is, $Pb < \dfrac{m}{2}$ at the minimum body and $Pb - A > \dfrac{m}{2}$ at the maximum body.*

(ii) *As the distance between the sources increases, the center of the minimum body tends toward P and the center of the maximum body tends toward Q. That is, as $A \to \infty$, the midpoint of $m \to P$ at the minimum body and the midpoint of $m \to Q$ at the maximum body.*

Proof. First we prove (i). It was assumed that $A, m > 0$. The minimum value occurs at $t = Pb = \frac{1}{2}\left(A+m-\sqrt{A^2+m^2}\right)$. We have that $\sqrt{A^2+m^2} > \sqrt{A^2} = A$. Thus, $A+m-\sqrt{A^2+m^2} < A+m-A = m$ and $\frac{1}{2}\left(A+m-\sqrt{A^2+m^2}\right) < \frac{m}{2}$. This proves that $Pb < \frac{1}{2}$ at the minimum body.

Continuing with the maximum, which occurs at $t = Pb = \frac{1}{2}\left(A+m+\sqrt{A^2+m^2}\right)$, it suffices to prove that $\frac{1}{2}\left(A+m+\sqrt{A^2+m^2}\right) - A > \frac{m}{2}$. This is equivalent to $A+m+\sqrt{A^2+m^2} - 2A > m$, or $m - A + \sqrt{A^2+m^2} > m$. Since $\sqrt{A^2+m^2} > A$, $m - A + \sqrt{A^2+m^2} > m - A + A = m$. Therefore $\frac{1}{2}\left(A+m+\sqrt{A^2+m^2}\right) - A > \frac{m}{2}$, and so $Pb - A > \frac{m}{2}$, proving (i).

Now we prove (ii). Taking the limit of the minimum value,

$$\lim_{A\to\infty}\left[\frac{1}{2}\left(A+m-\sqrt{A^2+m^2}\right)\right] \to \frac{1}{2}(A+m-A) = \frac{m}{2},$$

since m is fixed. So the minimum Pb value converges to $\frac{m}{2}$. This implies that the body is centered at source P.

Lastly, taking the limit of A subtracted from the maximum value,

$$\lim_{A\to\infty}\left[\frac{1}{2}\left(A+m+\sqrt{A^2+m^2}\right) - A\right] \to \frac{1}{2}(2A+m) - A = \frac{m}{2} + A - A = \frac{m}{2}.$$

So the body is centered at soucre Q. This completes the proof.

□

5. Curvature at the Basepoints and Vertices

At this point we change topics and discuss curvature rather than location. Conditions are given that are necessary for the shadow body to be convex. In this section, the original body is assumed to be lens-shaped. Also, the basepoints are used in a majority of the computations.

5.1. **Convexity at the Basepoints.** In [8] a matrix equation was derived which when solved gives the curvature at the basepoints of the shadow body. The system, after correcting for errors, is given below:

$$\begin{pmatrix} \frac{X_P(0)}{S_P(0)} & \frac{-X_P(0)}{\mathscr{s}_P(0)} \\ \frac{X_Q(0)(\mathscr{s}_Q(0))^2}{(S_P(0))^3} & \frac{-X_Q(0)(S_Q(0))^2}{(\mathscr{s}_P(0))^3} \end{pmatrix} \begin{pmatrix} \mathcal{K}S_P(0) \\ \mathcal{K}\mathscr{s}_P(0) \end{pmatrix} = \begin{pmatrix} \mathcal{K}X_P(0) \\ \mathcal{K}X_Q(0) \end{pmatrix}.$$

This equation was used to find the curvature at the basepoints for the reconstruction of a particular example. We have been able to generalize these results to determine necessary and sufficient conditions for convexity at the basepoints of the shadow body for any given original body.

Theorem 5.1. *For a given lens-shaped body K, its corresponding shadow body will be convex at the basepoints if and only if*

$$\frac{\mathscr{s}_P^2}{S_Q^2} \le \frac{R_P^2 + r_P^2}{R_Q^2 + r_Q^2} \le \frac{S_P^2}{\mathscr{s}_Q^2}, \tag{7}$$

where the functions are to be evaluated at angle 0.

Proof. For simplicity, everything in the proof is assumed to be evaluated at 0. Using Cramer's Rule to solve the above matrix, we have

$$\mathcal{K}S_P = \frac{\mathcal{K}X_P\left(\frac{-X_Q S_Q^2}{\mathscr{s}_P^3}\right) - \left(\frac{-X_P}{\mathscr{s}_P}\right)\mathcal{K}X_Q}{\left(\frac{X_P}{S_P}\right)\left(\frac{-X_Q S_Q^2}{\mathscr{s}_P^3}\right) - \left(\frac{-X_P}{\mathscr{s}_P}\right)\left(\frac{X_Q \mathscr{s}_Q^2}{S_P^3}\right)}.$$

Now, since $X_P(0) = X_Q(0)$,

$$\mathcal{K}S_P = \frac{\left(\frac{-X_P}{\mathscr{s}_P}\right)\left(\mathcal{K}X_P\left(\frac{S_Q^2}{\mathscr{s}_P^2}\right) - \mathcal{K}X_Q\right)}{\left(\frac{-X_P^2}{S_P \mathscr{s}_P}\right)\left(\frac{S_Q^2}{\mathscr{s}_P^2} - \frac{\mathscr{s}_Q^2}{S_P^2}\right)}$$

$$= \frac{\mathcal{K}X_P\left(\frac{S_Q^2}{\mathscr{s}_P^2}\right) - \mathcal{K}X_Q}{\frac{X_P}{S_P}\left(\frac{S_Q^2}{\mathscr{s}_P^2} - \frac{\mathscr{s}_Q^2}{S_P^2}\right)}.$$

The curvature operator will be positive when a curve is concave toward the source and negative when it is concave away from the source. So the body will be convex if the curvature of the farside is positive and the curvature at the nearside is negative. Furthermore, it is clear that the above denominator will always be positive. Thus, $\mathcal{K}S_P \geq 0$ if and only if

$$\mathcal{K}X_P\left(\frac{S_Q^2}{\mathscr{s}_P^2}\right) - \mathcal{K}X_Q \geq 0.$$

Since the derivatives of the original lens-shaped body are zero at the basepoints, $\mathcal{K}X_P = X_P\left(\frac{\mathcal{K}R_P}{R_P} - \frac{\mathcal{K}r_P}{r_P}\right)$ (and similarly for $\mathcal{K}X_Q$). Thus, we have

$$X_P\left(\frac{\mathcal{K}R_P}{R_P} - \frac{\mathcal{K}r_P}{r_P}\right)\frac{S_Q^2}{\mathscr{s}_P^2} \geq X_Q\left(\frac{\mathcal{K}R_Q}{R_Q} - \frac{\mathcal{K}r_Q}{r_Q}\right).$$

Again using the fact that the derivatives are zero at the basepoints, we can simplify to get $\mathcal{K}R_P = \kappa_{R_P} R_P^3$ from the definition of signed curvature (simarly for r_P, R_Q, and r_Q). Using this identity and the fact that $X_P = X_Q$ at the baseline, we have

$$S_Q^2(\kappa_{R_P} R_P^2 - \kappa_{r_P} r_P^2) \geq \mathscr{s}_P^2(\kappa_{R_Q} R_Q^2 - \kappa_{r_Q} r_Q^2).$$

Using similar methods, we find that $\mathcal{K}\mathscr{s}_P \leq 0$ if and only if

$$\mathscr{s}_Q^2(\kappa_{R_P}R_P^2 - \kappa_{r_P}r_P^2) \leq S_P^2(\kappa_{R_Q}R_Q^2 - \kappa_{r_Q}r_Q^2).$$

To simplify these equations, notice that since our body is lens-shaped, we have $-\kappa_{r_P} = \kappa_{R_P} = \kappa_{R_Q} = -\kappa_{r_Q}$. So we are left with

$$S_Q^2(R_P^2 + r_P^2) \geq \mathscr{s}_P^2(R_Q^2 + r_Q^2) \quad \text{and}$$
$$\mathscr{s}_Q^2(R_P^2 + r_P^2) \leq S_P^2(R_Q^2 + r_Q^2).$$

Thus, the shadow body has proper curvature at the baseline if and only if

$$\frac{\mathscr{s}_P^2}{S_Q^2} \leq \frac{R_P^2 + r_P^2}{R_Q^2 + r_Q^2} \leq \frac{S_P^2}{\mathscr{s}_Q^2},$$

proving (7).

□

After discovering the above result, we decided to graphically investigate whether the inequalities seemed to hold. Indeed, it appears that the shadow body is convex at the basepoints for most values of A and m. One such graph is shown below.

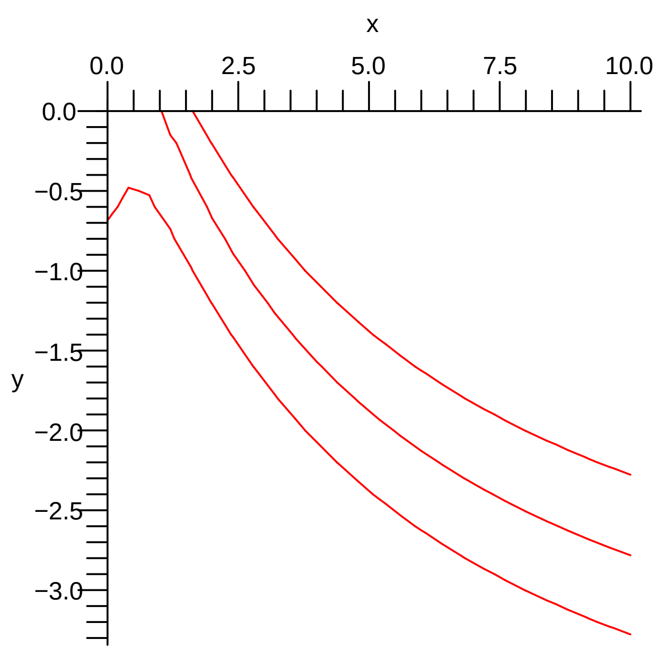

FIGURE 8. Graph of Theorem 5.1 for $m = 1$ and $A = 10$

In Figure 8, the variable x represents the R_Q distance and the variable y represents the $\mathscr{s}_Q$ distance. We reparameterized the $G(t)$ function in terms of the (signed) Qb distance, A, and m. Then using the *implicitplot* command in maple, we graphed $G(x)-G(y)=0$, which yielded the center curve in Figure 8. This curve expresses every possible body - shadow body pair for the case in which the original body is to the right of both sources. Similarly, we then reparameterized the inequality from the above theorem to get

$$\frac{(\mathscr{s}_Q+A-m)^2}{(\mathscr{s}_Q-m)^2} \leq \frac{(R_Q+A)^2+(R_Q+A-m)^2}{R_Q^2+(R_Q-m)^2} \leq \frac{(\mathscr{s}_Q+A)^2}{\mathscr{s}_Q^2}.$$

Finally, we set each individual inequality equal to zero and made the proper substitutions $R_Q=x$ and $\mathscr{s}_Q=y$. We are left with the following functions:

$$F_1(x,y)=\frac{(y+A-m)^2}{(y-m)^2}-\frac{(x+A)^2+(x+A-m)^2}{x^2+(x-m)^2}\leq 0,$$

$$F_2(x,y)=\frac{(y+A)^2}{y^2}-\frac{(x+A)^2+(x+A-m)^2}{x^2+(x-m)^2}\geq 0.$$

By changing the inequalities to equalities, we see that $G(x)-G(y)=0$ must lie below F_2 and above F_1. Indeed, such is the case for most values of A and m, as demonstrated in Figure 8.

5.2. **Lens-Shaped Shadow Body.** Next we investigate when a lens-shaped shadow body could possibly exist. For if a lens-shaped shadow body were to exist, convexity would be guaranteed at all points. The results are given in the theorem below.

Theorem 5.2. *Suppose that K is a lens-shaped body located to the right of two sources P and Q. Also suppose that S is the shadow body of K which lies between the two sources. Then a necessary condition for S to be lens-shaped is given by*

$$\frac{r_P^2+R_P^2}{r_Q^2+R_Q^2}=\frac{\mathscr{s}_P^2+S_P^2}{\mathscr{s}_Q^2+S_Q^2}, \tag{8}$$

where each function is to be evaluated at angle 0.

Proof. All functions are to be evaluated at angle 0 for simplicity. First Cramer's Rule is used to solve the system of equations given previously, which yields

$$\mathscr{K}S_P=\frac{\mathscr{K}X_P\frac{-X_Q(S_Q)^2}{(\mathscr{s}_P)^3}+\frac{X_P}{\mathscr{s}_P}(\mathscr{K}X_Q)}{\frac{X_P}{S_P}\left(\frac{-X_Q(S_Q)^2}{(\mathscr{s}_P)^3}\right)+\frac{X_P}{\mathscr{s}_P}\left(\frac{X_Q(S_Q)^2}{(S_P)^3}\right)} \quad \text{and}$$

$$\mathscr{K}\mathscr{s}_P=\frac{\frac{X_P}{S_P}(\mathscr{K}X_Q)-\mathscr{K}X_P\frac{X_Q(\mathscr{s}_Q)^2}{(S_P)^3}}{\frac{X_P}{S_P}\left(\frac{-X_Q(S_Q)^2}{(\mathscr{s}_P)^3}\right)+\frac{X_P}{\mathscr{s}_P}\left(\frac{X_Q(S_Q)^2}{(S_P)^3}\right)}.$$

To determine conditions under which the shadow body could possibly be lens-shaped, we must set the signed curvature at one basepoint equal to the negative of the other. Using the signed curvature formula, $\kappa_f = \frac{\mathfrak{K}f}{(f^2+(f')^2)^{\frac{3}{2}}}$. In this case, $S'_P = \mathscr{s}'_P = 0$. So we have $\kappa_{S_P} = \frac{\mathfrak{K}S_P}{(S_P)^3}$, and similarly for $\mathscr{s}_P$. Setting $\kappa_{S_P} = \frac{\mathfrak{K}S_P}{(S_P)^3} = -\frac{\mathfrak{K}\mathscr{s}_P}{(\mathscr{s}_P)^3} = -\kappa_{\mathscr{s}_P}$, canceling the denominators, and noticing that $X_P(0) = X_Q(0)$, we have

$$\frac{\mathfrak{K}X_Q\left(\frac{X_P}{\mathscr{s}_P}\right) - \mathfrak{K}X_P\left(\frac{X_Q(S_Q)^2}{(\mathscr{s}_P)^3}\right)}{(S_P)^3} = -\left[\frac{\mathfrak{K}X_Q\left(\frac{X_P}{S_P}\right) - \mathfrak{K}X_P\left(\frac{X_Q(\mathscr{s}_Q)^2}{(S_P)^3}\right)}{(\mathscr{s}_P)^3}\right],$$

$$\mathfrak{K}X_Q(\mathscr{s}_P)^2 + \mathfrak{K}X_Q(S_P)^2 = \mathfrak{K}X_P(\mathscr{s}_Q)^2 + \mathfrak{K}X_P(S_Q)^2, \quad \text{and}$$

$$\frac{\mathfrak{K}X_P}{\mathfrak{K}X_Q} = \frac{(\mathscr{s}_P)^2 + (S_P)^2}{(\mathscr{s}_Q)^2 + (S_Q)^2}. \tag{9}$$

To simplify further, an equation found by Black, Kimble, Koop, and Solmon [1] is useful:

$$\mathfrak{K}X_P = \frac{X_P}{R_P}\mathfrak{K}R_P - \frac{X_P}{r_P}\mathfrak{K}r_P + 2R_P r_P\left[\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right]^2$$

The derivatives are zero in this case, so the last term cancels. Using this and writing the curvature operator in terms of signed curvature, we have

$$\begin{aligned}\mathfrak{K}X_P &= X_P\left[\frac{\mathfrak{K}R_P}{R_P} - \frac{\mathfrak{K}r_P}{r_P}\right] = X_P\left[\frac{\kappa_{R_P}(R_P)^3}{R_P} - \frac{\kappa_{r_P}(r_P)^3}{r_P}\right] \\ &= X_P\left[\kappa_{R_P}(R_P)^2 - \kappa_{r_P}(r_P)^2\right].\end{aligned}$$

We are assuming the original body is lens-shaped, so $\kappa_{R_P} = -\kappa_{r_P}$ and

$$\mathfrak{K}X_P = X_P(\kappa_{R_P})(r_P^2 + R_P^2).$$

Similarly,

$$\mathfrak{K}X_Q = X_Q(\kappa_{R_Q})(r_Q^2 + R_Q^2).$$

Substituting into (9) we have

$$\frac{X_P(\kappa_{R_P})(r_P^2 + R_P^2)}{X_Q(\kappa_{R_Q})(r_Q^2 + R_Q^2)} = \frac{\mathscr{s}_P^2 + S_P^2}{\mathscr{s}_Q^2 + S_Q^2}.$$

Since P and Q are on the same side of the original body, $\kappa_{R_P} = \kappa_{R_Q}$. Canceling, we are left with

$$\frac{r_P^2+R_P^2}{r_Q^2+R_Q^2} = \frac{\mathscr{s}_P^2+S_P^2}{\mathscr{s}_Q^2+S_Q^2}.$$

This proves (8) and gives a necessary condition for a lens-shaped shadow body.

□

The same graphic methods used before apply here. However, no solutions were found. The two functions appeared to be parallel, even after several A and m values were tried. Below is one such graph for specific A and m values.

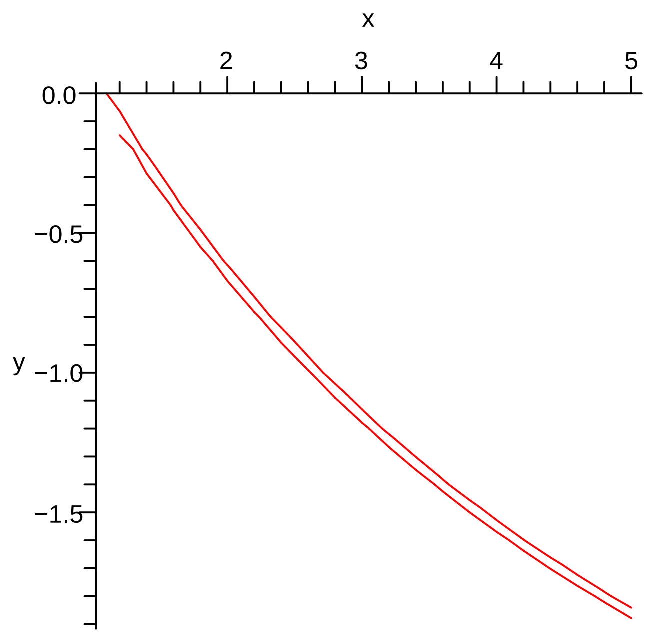

FIGURE 9. Graph of Theorem 5.2 for $m = 1$ and $A = 10$

Further investigation of lens-shaped shadow bodies may produce solutions.

5.3. **Convexity at the Vertices.** Here we hope to find expressions for the curvature of the shadow body at the vertices. If we can find such expressions, we can check the sign to see whether or not the shadow body has proper curvature. In order to do so, we have derived the following. Note that all terms in this subsection will be evaluated at the angle φ near α. Then we will take the limit as φ approaches α.

Lemma 5.3. *Let $X_P(\varphi)$ be an X-ray function for a body from a source P. Then*

(10) $$\frac{\mathscr{K}S_P}{S_P} - \frac{\mathscr{K}\mathscr{s}_P}{\mathscr{s}_P} = \frac{\mathscr{K}R_p}{R_p} - \frac{\mathscr{K}r_P}{r_P} - 2X_P'\left[\frac{r_P'}{R_p} - \frac{\mathscr{s}_P'}{S_P}\right].$$

Proof. From [1], we have

$$\mathscr{K} X_P = \mathscr{K}(S_P - \mathscr{s}_P) = \frac{X_P}{S_P}\mathscr{K} S_P - \frac{X_P}{\mathscr{s}_P}\mathscr{K}\mathscr{s}_P + 2S_P\mathscr{s}_P\left[\frac{S'_P}{S_P} - \frac{\mathscr{s}'_P}{\mathscr{s}_P}\right]^2.$$

Rearranging terms,

$$\frac{\mathscr{K} S_P}{S_P} - \frac{\mathscr{K}\mathscr{s}_P}{\mathscr{s}_P} = \frac{\mathscr{K} X_P}{X_P} - \frac{2\mathscr{s}_P S_P}{X_P}\left[\frac{S'_P}{S_P} - \frac{\mathscr{s}'_P}{\mathscr{s}_P}\right]^2.$$

Realizing that $\mathscr{K} X_P = \mathscr{K}(R_P - r_P)$, we have

$$\frac{\mathscr{K} X_P}{X_P} = \frac{1}{X_P}\left[\frac{X_P}{R_P}\mathscr{K} R_P - \frac{X_P}{r_P}\mathscr{K} r_P + 2r_P R_P\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right)^2\right] = \frac{\mathscr{K} R_P}{R_P} - \frac{\mathscr{K} r_P}{r_P} + \frac{2r_P R_P}{X_P}\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right)^2.$$

Substituting this back in gives us

$$\frac{\mathscr{K} S_P}{S_P} - \frac{\mathscr{K}\mathscr{s}_P}{\mathscr{s}_P} = \frac{\mathscr{K} R_P}{R_P} - \frac{\mathscr{K} r_P}{r_P} + \frac{1}{X_P}\left[2r_P R_P\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right)^2 - 2\mathscr{s}_P S_P\left(\frac{S'_P}{S_P} - \frac{\mathscr{s}'_P}{\mathscr{s}_P}\right)^2\right].$$

Since we will be evaluating these terms as $\varphi \to \alpha$, we know that X_P will converge to 0. So we need to cancel the X_P from the denominator. Observing that $R'_P = X'_P + r'_P$,

$$\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right) = \frac{r'_P + X'_P}{r_P + X_P} - \frac{r'_P}{r_P} = \frac{r_P(r'_P + X'_P) - r'_P(r_P + X_P)}{(r_P + X_P)(r_P)} =$$

(11)
$$\frac{r_P r'_P + r_P X'_P - r_P r'_P - r'_P X_P}{(r_P + X_P)(r_P)} = \frac{r_P X'_P - r'_P X_P}{R_P r_P}.$$

Using similar methods,

$$\frac{S'_P}{S_P} - \frac{\mathscr{s}'_P}{\mathscr{s}_P} = \frac{\mathscr{s}_P X'_P - \mathscr{s}'_P X_P}{S_P \mathscr{s}_P}.$$

At this point it may be easier to work with one term at a time. We start with $\frac{1}{X_P}\left[2R_P r_P\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right)\right]^2$. Substituting in the above identity, we have

$$\frac{1}{X_P}\left[2R_P r_P\left(\frac{R'_P}{R_P} - \frac{r'_P}{r_P}\right)^2\right] = \frac{1}{X_P}\left[\frac{2(r_P^2(X'_P)^2 - 2r_P r'_P X_P X'_P + (r'_P)^2 X_P^2)}{R_P r_P}\right]$$

$$= \frac{2(r'_P)^2 X_P}{R_P r_P} - \frac{2r'_P X'_P}{R_P} + \frac{2r_P(X'_P)^2}{R_P X_P}.$$

Similarly,

$$\frac{1}{X_P}\left[2S_P\mathpzc{s}_P\left(\frac{S_P'}{S_P}-\frac{\mathpzc{s}_P'}{\mathpzc{s}_P}\right)^2\right]=\frac{2(\mathpzc{s}_P')^2X_P}{S_P\mathpzc{s}_P}-\frac{2\mathpzc{s}_P'X_P'}{S_P}+\frac{2\mathpzc{s}_P(X_P')^2}{S_PX_P}.$$

Thus, we have

$$\begin{aligned}\frac{\mathscr{K}S_P}{S_P}-\frac{\mathscr{K}\mathpzc{s}_P}{\mathpzc{s}_P}=\frac{\mathscr{K}R_P}{R_P}-\frac{\mathscr{K}r_P}{r_P}+\frac{2(r_P')^2X_P}{R_Pr_P}-\frac{2r_P'X_P'}{R_P}\\+\frac{2r_P(X_P')^2}{R_PX_P}-\frac{2(\mathpzc{s}_P')^2X_P}{S_P\mathpzc{s}_P}+\frac{2\mathpzc{s}_P'X_P'}{S_P}-\frac{2\mathpzc{s}_P(X_P')^2}{S_PX_P}.\end{aligned}$$

Now working with the term $\frac{2r_P(X_P')^2}{R_PX_P}-\frac{2\mathpzc{s}_P(X_P')^2}{S_PX_P}$ we see that

$$\begin{aligned}\frac{2r_P(X_P')^2}{R_PX_P}-\frac{2\mathpzc{s}_P(X_P')^2}{S_PX_P}&=\frac{2(X_P')^2}{X_P}\left[\frac{r_PS_P-\mathpzc{s}_PR_P}{R_PS_P}\right]+\frac{2(X_P')^2}{X_P}\left[\frac{r_PS_P-R_PS_P+R_PS_P-\mathpzc{s}_PR_P}{R_PS_P}\right]\\&=\frac{2(X_P')^2}{X_P}\left[\frac{S_P(r_P-R_P)+R_P(S_P-\mathpzc{s}_P)}{R_PS_P}\right]\\&=\frac{2(X_P')^2}{X_P}\left[\frac{S_P(-X_P)+R_P(X_P)}{R_PS_P}\right]\\&=\frac{2(X_P')^2(R_P-S_P)}{R_PS_P}.\end{aligned}$$

We are left with

$$\begin{aligned}\frac{\mathscr{K}S_P}{S_P}-\frac{\mathscr{K}\mathpzc{s}_P}{\mathpzc{s}_P}&=\frac{\mathscr{K}R_P}{R_P}-\frac{\mathscr{K}r_P}{r_P}+\frac{2(r_P')^2X_P}{R_Pr_P}-\frac{2r_P'X_P'}{R_P}-\frac{2(\mathpzc{s}_P')^2X_P}{S_P\mathpzc{s}_P}+\frac{2\mathpzc{s}_P'X_P'}{S_P}+\frac{2(X_P')^2(R_P-S_P)}{R_PS_P}\\&=\frac{\mathscr{K}R_P}{R_P}-\frac{\mathscr{K}r_P}{r_P}+2X_P\left[\frac{(r_P')^2}{R_Pr_P}-\frac{(\mathpzc{s}_P')^2}{S_P\mathpzc{s}_P}\right]-2X_P'\left[\frac{r_P'}{R_P}-\frac{\mathpzc{s}_P'}{S_P}\right]+\frac{2(X_P)^2(R_P-S_P)}{R_PS_P}.\end{aligned}$$

If we take the limit as φ approaches α, the terms with X_P in the numerator will cancel since $X_P(\alpha)=0$. This leaves

$$\frac{\mathscr{K}S_P}{S_P}-\frac{\mathscr{K}\mathpzc{s}_P}{\mathpzc{s}_P}=\frac{\mathscr{K}R_P}{R_P}-\frac{\mathscr{K}r_P}{r_P}-2X_P'\left[\frac{r_P'}{R_P}-\frac{\mathpzc{s}_P'}{S_P}\right],$$

proving (10).

□

Using the above lemma, we have derived the following. This theorem gives us a system of equations which can be solved to find the signed curvature at the vertex.

Theorem 5.4. *Let X_P and X_Q be X-ray functions for a body from sources P and Q respectively. Then*

$$\begin{bmatrix} (S_P+(S'_P))^{3/2} & -(\mathscr{s}_P+(\mathscr{s}'_P))^{3/2} \\ \frac{S_P}{\mathscr{s}_Q}\left(\mathscr{s}_Q^2+(\mathscr{s}'_Q)^2\right)^{3/2} & -\frac{\mathscr{s}_P}{S_Q}\left(S_Q^2+(S'_Q)^2\right)^{3/2} \end{bmatrix} \begin{bmatrix} \frac{\kappa_{S_P}}{S_P} \\ \frac{\kappa_{\mathscr{s}_P}}{\mathscr{s}_P} \end{bmatrix} = \begin{bmatrix} \frac{\mathcal{K}_{R_P}}{R_P} - \frac{\mathcal{K}_{r_P}}{r_P} - 2X'_P\left(\frac{r'_P}{R_P} - \frac{\mathscr{s}'_P}{S_P}\right) \\ \frac{\mathcal{K}_{R_Q}}{R_Q} - \frac{\mathcal{K}_{r_Q}}{r_Q} - 2X'_Q\left(\frac{r'_Q}{R_Q} - \frac{\mathscr{s}'_Q}{S_Q}\right) \end{bmatrix}.$$

Proof. We give a proof for source P, but the proof for Q is done in exactly the same manner. We have the following system of equations:

$$\frac{\mathcal{K}_{S_P}}{S_P} - \frac{\mathcal{K}_{\mathscr{s}_P}}{\mathscr{s}_P} = \frac{\mathcal{K}_{R_P}}{R_P} - \frac{\mathcal{K}_{r_P}}{r_P} - 2X'_P\left[\frac{r'_P}{R_P} - \frac{\mathscr{s}'_P}{S_P}\right]$$

$$\frac{\mathcal{K}_{S_Q}}{S_Q} - \frac{\mathcal{K}_{\mathscr{s}_Q}}{\mathscr{s}_Q} = \frac{\mathcal{K}_{R_Q}}{R_Q} - \frac{\mathcal{K}_{r_Q}}{r_Q} - 2X'_Q\left[\frac{r'_Q}{R_Q} - \frac{\mathscr{s}'_Q}{S_Q}\right].$$

Since signed curvature is invariant of parameterization, we can solve for $\mathcal{K}_{S_Q}$ and $\mathcal{K}_{\mathscr{s}_Q}$ in terms of $\mathcal{K}_{S_P}$ and $\mathcal{K}_{\mathscr{s}_P}$ based on the following equalities:

$$\frac{\mathcal{K}_{S_P}}{(S_P^2+(S'_P)^2)^{3/2}} = \kappa_{S_P} = -\kappa_{\mathscr{s}_Q} = \frac{-\mathcal{K}_{\mathscr{s}_Q}}{(\mathscr{s}_P^2+(\mathscr{s}'_P)^2)^{3/2}} \quad \text{and}$$

$$\frac{\mathcal{K}_{\mathscr{s}_P}}{(\mathscr{s}_P^2+(\mathscr{s}'_P)^2)^{3/2}} = \kappa_{\mathscr{s}_P} = -\kappa_{S_Q} = \frac{-\mathcal{K}_{S_Q}}{(S_P^2+(S'_P)^2)^{3/2}}.$$

We are left with two equations and two unknowns, which is expressed by the following system:

$$\begin{bmatrix} 1 & -1 \\ \frac{S_P}{\mathscr{s}_Q}\left(\frac{\mathscr{s}_Q^2+(\mathscr{s}'_Q)^2}{S_P^2+(S'_P)^2}\right)^{3/2} & -\frac{\mathscr{s}_P}{S_Q}\left(\frac{S_Q^2+(S'_Q)^2}{\mathscr{s}_P^2+(\mathscr{s}'_P)^2}\right)^{3/2} \end{bmatrix} \begin{bmatrix} \frac{\mathcal{K}_{S_P}}{S_P} \\ \frac{\mathcal{K}_{\mathscr{s}_P}}{\mathscr{s}_P} \end{bmatrix} = \begin{bmatrix} \frac{\mathcal{K}_{R_P}}{R_P} - \frac{\mathcal{K}_{r_P}}{r_P} - 2X'_P\left(\frac{r'_P}{R_P} - \frac{\mathscr{s}'_P}{S_P}\right) \\ \frac{\mathcal{K}_{R_Q}}{R_Q} - \frac{\mathcal{K}_{r_Q}}{r_Q} - 2X'_Q\left(\frac{r'_Q}{R_Q} - \frac{\mathscr{s}'_Q}{S_Q}\right) \end{bmatrix}.$$

From the definition of signed curvature, $\mathcal{K}_{S_P} = \kappa_{S_P}[S_P^2+(S'_P)^2]^{3/2}$. Making substitutions,

$$\begin{bmatrix} (S_P+(S_P'))^{3/2} & -(\mathscr{s}_P+(\mathscr{s}_P'))^{3/2} \\ \frac{S_P}{\mathscr{s}_Q}\left(\mathscr{s}_Q^2+(\mathscr{s}_Q')^2\right)^{3/2} & -\frac{\mathscr{s}_P}{S_Q}\left(S_Q^2+(S_Q')^2\right)^{3/2} \end{bmatrix} \begin{bmatrix} \frac{\kappa_{S_P}}{S_P} \\ \frac{\kappa_{\mathscr{s}_P}}{\mathscr{s}_P} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\mathcal{K}_{R_P}}{R_P}-\frac{\mathcal{K}_{r_P}}{r_P}-2X_P'\left(\frac{r_P'}{R_P}-\frac{\mathscr{s}_P'}{S_P}\right) \\ \frac{\mathcal{K}_{R_Q}}{R_Q}-\frac{\mathcal{K}_{r_Q}}{r_Q}-2X_Q'\left(\frac{r_Q'}{R_Q}-\frac{\mathscr{s}_Q'}{S_Q}\right) \end{bmatrix}.$$

This completes the proof. □

6. EXISTENCE AND CONVEXITY OF SHADOW BODY

Curvature at the basepoints and vertices has already been examined, so we are ready to turn to the remainder of the body. The methods used in the last section do not apply here, so we need to derive the curvature formulas in a new way. Previously, conditions were found under which the shadow body is convex at the basepoints. In order for this to occur, the nearside basepoint was required to have negative curvature (so that the boundary is concave away from the source) and vice versa. The same idea holds at all other points on the boundary of the body. That is, we must determine when the curvature operator is negative at the nearside points and positive at the farside points for a designated source. To do this, we will use recursion.

Throughout this section we analyze the nearside of the shadow body from source P, given an initial angle φ_0. However, if Q is taken to be the source or if the farside is examined instead, then the methods used could be applied in the same fashion. Also, the shadow body is assumed to lie between the two sources P and Q.

One possibility is to directly prove that the curvature operator must be negative at the nearside for any angle φ_0. A second option is to obtain a contradiction in the following manner. Let's assume that the shadow body S is not convex. The shadow body is convex in a neighborhood of the vertex ν from [8], using the Stable Manifold Theorem (see below). Further, suppose that $\mathscr{s}_P(\varphi_0)$ determines the first point outside of that neighborhood where S is not convex. If, under certain conditions, it can be shown that S is not convex at a point nearer to ν, then by contradiction, our original assumption must be false. Thus S is convex.

A recursive formula is one way to implement the above ideas. It is possible to find an equation for the curvature at the point determined by φ_n from source P in terms of φ_0. But when is this curvature negative?

6.1. **Stable Manifold Theorem.**

The Stable Manifold Theorem guarantees existence of the shadow body in a neighborhood of the vertex ν (and the basepoints). This will be briefly discussed here. See P. 183 of Robinson's book [7] for a proof or more general statement.

Theorem 6.1. *Let $k \geq 1$, $U \subset \mathbb{R}^2$ with U open, $f : U \to \mathbb{R}^2$ with f a C^k function, and v a fixed point of f. Suppose that the eigenvalues of the differential $df(v)$ are μ, λ with $|\mu| < 1$ and $|\lambda| > 1$. Let E^s, E^u be the eigenspaces corresponding to μ, λ respectively. Then there is some neighborhood U' of v, $U' \subset U$, such that the local stable manifold for v in U',*

$$W^s(v, U', f) = \{x \in U' : f^j(x) \in U'\} = \{x \in U' : f^j(x) \to v \text{ as } j \to \infty\}$$

is a C^k curve which is tangent to E^s. (Here, f^j denotes the j-fold composition of f with itself.) More precisely, U' may be chosen so that in U' the stable manifold has the form

$$W^s(v,U',f) = \{v + tu_\mu + s(t)u_\lambda, -r < t < r\}$$

where u_μ and u_λ are unit eigenvectors corresponding to the eigenvalues μ and λ, respectively, and $s = s(t)$ is a C^k function satisfying $s(0) = s'(0) = 0$.

An informal explanation might help with comprehension. After iteration by a function f, points lying on the stable manifold journey toward the fixed point, ν in our case, and points along the unstable manifold move away from the fixed point. The following diagram appeared in [3] by Falconer to help illustrate the theorem. The stable and unstable manifolds are shown in a neighborhood of the fixed point.

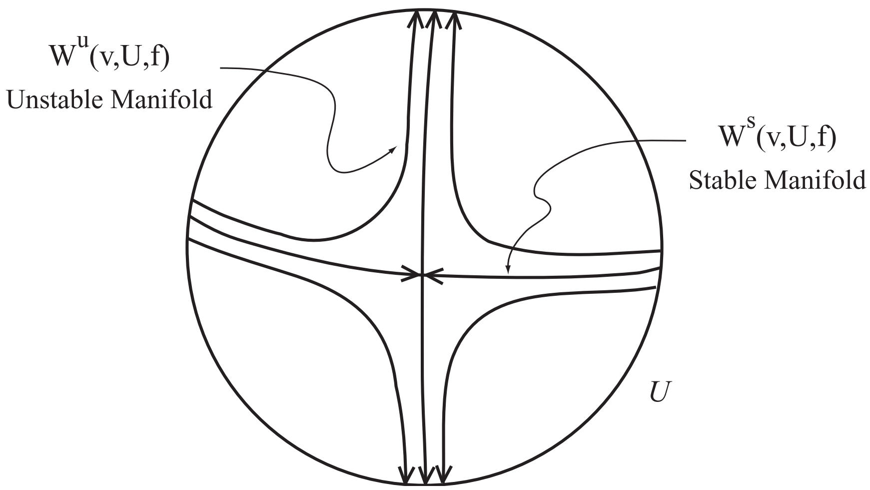

FIGURE 10. Stable Manifold Theorem

In [8] it was shown that for a lens-shaped body, the Stable Manifold Theorem guarantees the existence of the shadow body in a neighborhood of the vertex ν. The proof involved eigenvalue computations that produced the correct quantities. Separate computations (see the previous section or [8]) give conditions under which convexity will be maintained at the vertex. Convexity in a neighborhood of the vertex follows from these curvature computations and continuity. This consequence will be useful later when dealing with recursion formulas, since the points considered are approaching the vertex with iteration.

6.2. **Nearside Function Recursion.** First, we define the angles and nearside and farside functions in such a way that makes it possible (or simpler) to use recursion. The construction is explained here. If φ_0 is the angle of the initial X-ray from source P, then ψ_0 is the angle of the X-ray from source Q that passes through the farside point of $S_P(\varphi_0)$. Similarly, φ_1 is the angle of the X-ray that passes through the farside point of $S_Q(\psi_0)$. This method is used until φ_n is reached. The figure below may help with visualization.

The idea of the operation is to first add X-ray data from P, then add X-ray data from Q, and so forth. Under this operation, the vertex of the shadow body ν is a fixed point and the Stable Manifold

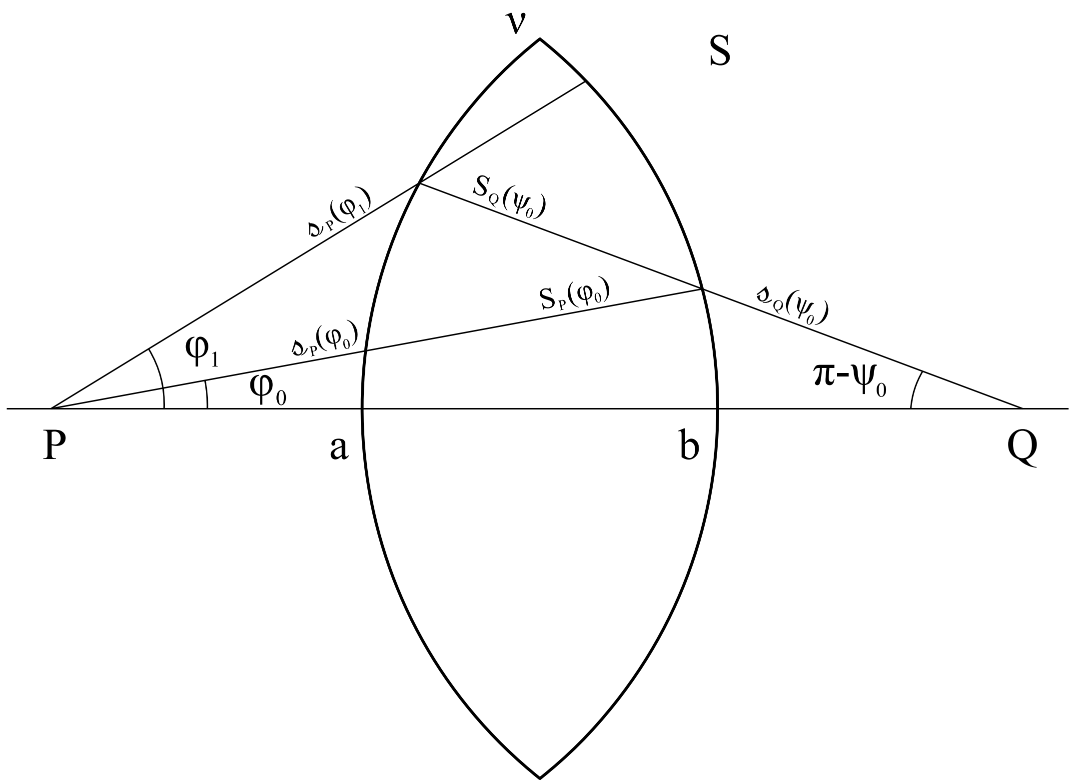

FIGURE 11

Theorem implies there exists a C^∞ curve through ν that is fixed. Suppose that the support lines for S from source P are at angles $\pm\alpha$. Also, suppose that $(\mathscr{s}_P(\varphi_0), \varphi_0)$ with $0 < \varphi_0 < \alpha$ is a point on the stable manifold. Then under iteration, this point converges to the vertex ν. Now, on to the recursion.

Lemma 6.2. *Suppose that the point $(\mathscr{s}_P(\varphi_0), \varphi_0)$ lies on the nearside of the shadow body with respect to source P for some angle φ_0. Then the points $(\mathscr{s}_P(\varphi_j), \varphi_j)$ for $j \geq 0$ lie on the nearside as well and*

$$\sin(\varphi_n)\mathscr{s}_P(\varphi_n) = \sin(\varphi_0)\mathscr{s}_P(\varphi_0) + \sum_{j=0}^{n-1}[\sin(\psi_j)X_Q(\psi_j) + \sin(\varphi_j)X_P(\varphi_j)]. \tag{12}$$

Proof. The Law of Sines can be applied to the two triangles shown in Figure 11. Consider the triangle $\triangle(P, Q, (S_P(\varphi_0), \varphi_0))$. The point $(S_P(\varphi_0), \varphi_0)$ also has polar coordinates $(\mathscr{s}_Q(\psi_0), \psi_0)$ centered at Q. Then from the Law of Sines,

$$\frac{S_P(\varphi_0)}{\sin(\psi_0)} = \frac{\mathscr{s}_Q(\psi_0)}{\sin(\varphi_0)}. \tag{13}$$

Now consider the triangle $\triangle(P, Q, (\mathscr{s}_P(\varphi_1)))$. The point $(\mathscr{s}_P(\varphi_1), \varphi_1)$ also has polar coordinates $(S_Q(\psi_0), \psi_0)$. Again from the Law of Sines,

$$\frac{\mathscr{s}_P(\varphi_1)}{\sin(\psi_0)} = \frac{S_Q(\psi_0)}{\sin(\varphi_1)}. \tag{14}$$

We also know that

$$S_P(\varphi_0) = \mathscr{s}_P(\varphi_0) + X_P(\varphi_0) \text{ and}$$
$$S_Q(\psi_0) = \mathscr{s}_Q(\psi_0) + X_Q(\psi_0).$$

Then using these equivalent definitions, (13), and (14), we have

$$\begin{aligned}
\mathscr{s}_P(\varphi_1) &= \frac{\sin(\psi_0)}{\sin(\varphi_1)} S_Q(\psi_0) \\
&= \frac{\sin(\psi_0)}{\sin(\varphi_1)} \left[X_Q(\psi_0) + \mathscr{s}_Q(\psi_0)\right] \\
&= \frac{\sin(\psi_0)}{\sin(\varphi_1)} \left[X_Q(\psi_0) + \frac{\sin(\varphi_0)}{\sin(\psi_0)} S_P(\varphi_0)\right] \\
&= \frac{\sin(\psi_0)}{\sin(\varphi_1)} X_Q(\psi_0) + \frac{\sin(\varphi_0)}{\sin(\varphi_1)} S_P(\varphi_0) \\
&= \frac{\sin(\psi_0)}{\sin(\varphi_1)} X_Q(\psi_0) + \frac{\sin(\varphi_0)}{\sin(\varphi_1)} X_P(\varphi_0) + \frac{\sin(\varphi_0)}{\sin(\varphi_1)} \mathscr{s}_P(\varphi_0), \text{ or}
\end{aligned}$$

$$\sin(\varphi_1)\mathscr{s}_P(\varphi_1) = \sin(\psi_0)X_Q(\psi_0) + \sin(\varphi_0)X_P(\varphi_0) + \sin(\varphi_0)\mathscr{s}_P(\varphi_0).$$

Proceeding recursively, we obtain the identity

$$\sin(\varphi_n)\mathscr{s}_P(\varphi_n) = \sin(\varphi_0)\mathscr{s}_P(\varphi_0) + \sum_{j=0}^{n-1}\left[\sin(\psi_j)X_Q(\psi_j) + \sin(\varphi_j)X_P(\varphi_j)\right].$$

This completes the proof. □

Corollary 6.3. *If* $\mathscr{s}_P(\varphi_0)$ *is as above and* α *is the angle of the support line for the upper vertex* ν *of* S *from source* P*, then*

$$d(P, \nu)\sin(\alpha) = \sin(\varphi_0)\mathscr{s}_P(\varphi_0) + \sum_{j=0}^{\infty}\left[\sin(\psi_j)X_Q(\psi_j) + \sin(\varphi_j)X_P(\varphi_j)\right].$$

Proof. If we let $n \to \infty$ in (12), then the angle $\varphi_n \to \alpha$ and $\mathscr{s}_P(\varphi_n) \to d(P, \nu)$, the distance from the source P to the vertex ν. Hence we have the final identity stated above, where the series is absolutely convergent by the derivation. □

6.3. **Nearside Function Derivative Recursion.** It would be useful to acquire a recursive formula involving the derivative of $\mathcal{S}_P(\varphi_n)$ as well, since this term will come up in the recursive curvature formula later. As a warning, this subsection and the next get very messy.

This short sidenote is used in the computation. Denote the angle of inclination of the tangent line to the nearside (farside) of the shadow body along the ray with angle φ from a source by η_φ (ω_φ). All angles are measured counterclockwise from the horizontal as usual. One can show [1] that for any source P on the x-axis,

$$\frac{\mathcal{S}_P'(\varphi)}{\mathcal{S}_P(\varphi)} = \cot(\eta_\varphi - \varphi) \quad \text{and} \quad \frac{S_P'(\varphi)}{S_P(\varphi)} = \cot(\omega_\varphi - \varphi). \tag{15}$$

Lemma 6.4. *Suppose that* $(\mathcal{S}_P(\varphi_0), \varphi_0)$ *is as before. Then*

$$\begin{aligned}[\sin(\varphi_n)\mathcal{S}_P(\varphi_n)]' &= [\sin(\varphi_0)\mathcal{S}_P(\varphi_0)]' \frac{d\varphi_0}{d\varphi_n} \\ &\quad + \sum_{j=0}^{n-1}\left[(\sin(\psi_j)X_Q(\psi_j))' \frac{d\psi_j}{d\varphi_n} + (\sin(\varphi_j)X_P(\varphi_j))' \frac{d\varphi_j}{d\varphi_n}\right],\end{aligned}$$

where

$$\frac{d\psi_j}{d\varphi_n} = \left[\frac{\sin(\psi_j)}{\sin(\varphi_n)} \cdot \frac{\sin(\omega_{\psi_j} - \psi_j)}{\sin(\eta_{\varphi_{j+1}} - \varphi_{j+1})}\right] \prod_{i=j+1}^{n-1}\left[\frac{\sin(\omega_{\varphi_i} - \varphi_i)}{\sin(\eta_{\psi_i} - \psi_i)} \cdot \frac{\sin(\omega_{\psi_i} - \psi_i)}{\sin(\eta_{\varphi_{i+1}} - \varphi_{i+1})}\right] \tag{16}$$

and

$$\frac{d\varphi_j}{d\varphi_n} = \left[\frac{\sin(\varphi_j)}{\sin(\varphi_n)}\right] \prod_{i=j}^{n-1}\left[\frac{\sin(\omega_{\varphi_i} - \varphi_i)}{\sin(\eta_{\psi_i} - \psi_i)} \cdot \frac{\sin(\omega_{\psi_i} - \psi_i)}{\sin(\eta_{\varphi_{i+1}} - \varphi_{i+1})}\right]. \tag{17}$$

Proof. First, we may simply take the derivative of (12) with respect to φ_n to get

$$\begin{aligned}\frac{d}{d\varphi_n}[\sin(\varphi_n)\mathcal{S}_P(\varphi_n)] &= \frac{d}{d\varphi_n}[\sin(\varphi_0)\mathcal{S}_P(\varphi_0)] + \frac{d}{d\varphi_n}\left[\sum_{j=0}^{n-1}(\sin(\psi_j)X_Q(\psi_j) + \sin(\varphi_j)X_P(\varphi_j))\right] \\ &= \frac{d}{d\varphi_n}[\sin(\varphi_0)\mathcal{S}_P(\varphi_0)] \\ &\quad + \sum_{j=0}^{n-1}\left[\frac{d}{d\psi_j}(\sin(\psi_j)X_Q(\psi_j))\frac{d\psi_j}{d\varphi_n} + \frac{d}{d\varphi_j}(\sin(\varphi_j)X_P(\varphi_j))\frac{d\varphi_j}{d\varphi_n}\right].\end{aligned}$$

Now that we have a general expression including the derivative of $\mathcal{S}_P(\varphi_n)$, we can determine some of the individual terms more specifically. In particular, $\frac{d\psi_j}{d\varphi_n}$ and $\frac{d\varphi_j}{d\varphi_n}$ can be found by taking the derivative of (13), simplifying, and using recursion. Differentiating with respect to φ_0 gives

$$\frac{d}{d\varphi_0}[S_P(\varphi_0)\sin(\varphi_0)] = \frac{d}{d\psi_0}[\mathcal{S}_Q(\psi_0)\sin(\psi_0)]\frac{d\psi_0}{d\varphi_0}.$$

The next step is to solve for $\frac{d\varphi_0}{d\psi_0}$ and apply (15). Alternative derivative notation is used where $'$ denotes the differential with respect to the argument of the function. We have

$$\begin{aligned}
\frac{d\varphi_0}{d\psi_0} &= \frac{[\mathcal{S}_Q(\psi_0)\sin(\psi_0)]'}{[S_P(\varphi_0)\sin(\varphi_0)]'} \\
&= \frac{\mathcal{S}_Q'(\psi_0)\sin(\psi_0) + \mathcal{S}_Q(\psi_0)\cos(\psi_0)}{S_P'(\varphi_0)\sin(\varphi_0) + S_Q(\varphi_0)\cos(\varphi_0)} \\
&= \frac{\mathcal{S}_Q(\psi_0)}{S_P(\varphi_0)}\left[\frac{\frac{\mathcal{S}_Q'(\psi_0)}{\mathcal{S}_Q(\psi_0)}\sin(\psi_0) + \cos(\psi_0)}{\frac{S_P'(\varphi_0)}{S_P(\varphi_0)}\sin(\varphi_0) + \cos(\varphi_0)}\right] \\
&= \frac{\mathcal{S}_Q(\psi_0)}{S_P(\varphi_0)}\left[\frac{\cot(\eta_{\psi_0} - \psi_0)\sin(\psi_0) + \cos(\psi_0)}{\cot(\omega_{\varphi_0} - \varphi_0)\sin(\varphi_0) + \cos(\varphi_0)}\right].
\end{aligned}$$

Working with the numerator of the last expression and using the trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, we obtain

$$\begin{aligned}
\cot(\eta_{\psi_0} - \psi_0)\sin(\psi_0) + \cos(\psi_0) &= \frac{\cos(\eta_{\psi_0} - \psi_0)}{\sin(\eta_{\psi_0} - \psi_0)}\sin(\psi_0) + \cos(\psi_0) \\
&= \frac{\cos(\eta_{\psi_0} - \psi_0)\sin(\psi_0) + \cos(\psi_0)\sin(\eta_{\psi_0} - \psi_0)}{\sin(\eta_{\psi_0} - \psi_0)} \\
&= \frac{\sin(\psi_0 + \eta_{\psi_0} - \psi_0)}{\sin(\eta_{\psi_0} - \psi_0)} \\
&= \frac{\sin(\eta_{\psi_0})}{\sin(\eta_{\psi_0} - \psi_0)}.
\end{aligned}$$

Similarly for the denominator,

$$\cot(\omega_{\varphi_0} - \varphi_0)\sin(\varphi_0) + \cos(\varphi_0) = \frac{\sin(\omega_{\varphi_0})}{\sin(\omega_{\varphi_0} - \varphi_0)}.$$

Combining the previous work and canceling,

$$\begin{aligned}
\frac{d\varphi_0}{d\psi_0} &= \frac{\mathcal{S}_Q(\psi_0)}{S_P(\varphi_0)}\left[\frac{\sin(\eta_{\psi_0})}{\sin(\eta_{\psi_0} - \psi_0)} \cdot \frac{\sin(\omega_{\varphi_0} - \varphi_0)}{\sin(\omega_{\varphi_0})}\right] \\
&= \frac{\mathcal{S}_Q(\psi_0)}{S_P(\varphi_0)}\left[\frac{\sin(\omega_{\varphi_0} - \varphi_0)}{\sin(\eta_{\psi_0} - \psi_0)}\right].
\end{aligned}$$

Proceeding recursively and applying (13), for $0 \leq i \leq n$,

$$\frac{d\varphi_i}{d\psi_i} = \frac{\eth_Q(\psi_i)}{S_P(\varphi_i)}\left[\frac{\sin(\omega_{\varphi_i}-\varphi_i)}{\sin(\eta_{\psi_i}-\psi_i)}\right] = \frac{\sin(\varphi_i)}{\sin(\psi_i)}\left[\frac{\sin(\omega_{\varphi_i}-\varphi_i)}{\sin(\eta_{\psi_i}-\psi_i)}\right]. \tag{18}$$

Now that we have found $\frac{d\varphi_i}{d\psi_i}$, $\frac{d\psi_i}{d\varphi_{i+1}}$ can be found in a similar manner. Taking the derivative of (14), applying (14), and performing the same simplifications, for $0 \leq i \leq n-1$,

$$\frac{d\psi_i}{d\varphi_{i+1}} = \frac{\eth_P(\varphi_{i+1})}{S_Q(\psi_i)}\left[\frac{\sin(\omega_{\psi_i}-\psi_i)}{\sin(\eta_{\varphi_{i+1}}-\varphi_{i+1})}\right] = \frac{\sin(\psi_i)}{\sin(\varphi_{i+1})}\left[\frac{\sin(\omega_{\psi_i}-\psi_i)}{\sin(\eta_{\varphi_{i+1}}-\varphi_{i+1})}\right]. \tag{19}$$

Then we can multiply the above two quantities to obtain an expression for $\frac{d\varphi_i}{d\varphi_{i+1}}$. Multiplying and canceling,

$$\begin{aligned}\frac{d\varphi_i}{d\varphi_{i+1}} &= \left[\frac{d\varphi_i}{d\psi_i}\right]\left[\frac{d\psi_i}{d\varphi_{i+1}}\right] \\ &= \left[\frac{\sin(\varphi_i)}{\sin(\psi_i)}\cdot\frac{\sin(\omega_{\varphi_i}-\varphi_i)}{\sin(\eta_{\psi_i}-\psi_i)}\right]\cdot\left[\frac{\sin(\psi_i)}{\sin(\varphi_{i+1})}\cdot\frac{\sin(\omega_{\psi_i}-\psi_i)}{\sin(\eta_{\varphi_{i+1}}-\varphi_{i+1})}\right] \\ &= \left[\frac{\sin(\varphi_i)}{\sin(\varphi_{i+1})}\right]\left[\frac{\sin(\omega_{\varphi_i}-\varphi_i)}{\sin(\eta_{\psi_i}-\psi_i)}\right]\left[\frac{\sin(\omega_{\psi_i}-\psi_i)}{\sin(\eta_{\varphi_{i+1}}-\varphi_{i+1})}\right].\end{aligned} \tag{20}$$

Our goal was to find formulas for $\frac{d\psi_j}{d\varphi_n}$ and $\frac{d\varphi_j}{d\varphi_n}$. First $\frac{d\psi_j}{d\varphi_n}$ will be examined. Then $\frac{d\varphi_j}{d\varphi_n}$ will follow. Rewriting and using (19) and (20),

$$\begin{aligned}\frac{d\psi_j}{d\varphi_n} &= \left[\frac{d\psi_j}{d\varphi_{j+1}}\right]\left[\frac{d\varphi_{j+1}}{d\varphi_{j+2}}\right]\cdots\left[\frac{d\varphi_{n-2}}{d\varphi_{n-1}}\right]\left[\frac{d\varphi_{n-1}}{d\varphi_n}\right] \\ &= \left[\frac{d\psi_j}{d\varphi_{j+1}}\right]\prod_{i=j+1}^{n-1}\left[\frac{d\varphi_i}{d\varphi_{i+1}}\right].\end{aligned}$$

When taking the product of $\frac{d\varphi_i}{d\varphi_{i+1}}$, the $\frac{\sin(\varphi_i)}{\sin(\varphi_{i+1})}$ term reduces to $\frac{\sin(\varphi_{j+1})}{\sin(\varphi_n)}$. Thus (16) holds.
The equation for $\frac{d\varphi_j}{d\varphi_n}$ emerges directly from (20):

$$\begin{aligned}\frac{d\varphi_j}{d\varphi_n} &= \left[\frac{d\varphi_j}{d\varphi_{j+1}}\right]\left[\frac{d\varphi_{j+1}}{d\varphi_{j+2}}\right]\cdots\left[\frac{d\varphi_{n-2}}{d\varphi_{n-1}}\right]\left[\frac{d\varphi_{n-1}}{d\varphi_n}\right] \\ &= \prod_{i=j}^{n-1}\left[\frac{d\varphi_i}{d\varphi_{i+1}}\right].\end{aligned}$$

In this case, the $\frac{\sin(\varphi_i)}{\sin(\varphi_{i+1})}$ product term reduces to $\frac{\sin(psi_j)}{\sin(\varphi_n)}$. This proves (17).

□

6.4. **Nearside Curvature Recursion.** We are finally prepared to investigate the curvature of the shadow body. The nearside function from source P for a given angle φ_0 is still used as our foundation for the derivation. The first step in generating a recursive formula is developed here.

Lemma 6.5. *Suppose that $\mathscr{s}_P(\varphi_0)$ is the nearside function from source P for a shadow body S and some angle φ_0. Then the curvature at the point $(\mathscr{s}_P(\varphi_j),\varphi_j)$ for $0 \le j \le n-1$ is given by*

$$\mathscr{K}\mathscr{s}_P(\varphi_j) = a_j\mathscr{K}\mathscr{s}_P(\varphi_{j+1}) + b_j + c_j + d_j, \tag{21}$$

where

$$a_j = \left[\left(\frac{\mathscr{s}_P(\varphi_j)}{S_P(\varphi_j)}\frac{\mathscr{s}_Q(\psi_j)}{S_Q(\psi_j)}\right)\left(\frac{S_P^2(\varphi_j)+(S_P')^2(\varphi_j)}{\mathscr{s}_Q^2(\psi_j)+(\mathscr{s}_Q')^2(\psi_j)}\right)^{3/2}\left(\frac{S_Q^2(\psi_j)+(S_Q')^2(\psi_j)}{\mathscr{s}_P^2(\varphi_{j+1})+(\mathscr{s}_P')^2(\varphi_{j+1})}\right)^{3/2}\right],$$

$$b_j = \left[\left(\frac{\mathscr{s}_P(\varphi_j)}{S_P(\varphi_j)}\right)\left(\frac{S_P^2(\varphi_j)+(S_P')^2(\varphi_j)}{\mathscr{s}_Q^2(\psi_j)+(\mathscr{s}_Q')^2(\psi_j)}\right)^{3/2}\left(\frac{\mathscr{s}_Q(\psi_j)\mathscr{K}X_Q(\psi_j)}{X_Q(\psi_j)}\right)-\left(\frac{\mathscr{s}_P(\varphi_j)\mathscr{K}X_P(\varphi_j)}{X_P(\varphi_j)}\right)\right],$$

$$c_j = \left[-2\left(\frac{\mathscr{s}_P(\varphi_j)}{S_P(\varphi_j)}\right)\left(\frac{S_P^2(\varphi_j)+(S_P')^2(\varphi_j)}{\mathscr{s}_Q^2(\psi_j)+(\mathscr{s}_Q')^2(\psi_j)}\right)^{3/2} S_Q(\psi_j)X_Q(\psi_j)\left(\frac{S_Q'(\psi_j)}{S_Q(\psi_j)}-\frac{X_Q'(\psi_j)}{X_Q(\psi_j)}\right)^2\right],\ \textit{and}$$

$$d_j = \left[2S_P(\varphi_j)X_P(\varphi_j)\left(\frac{S_P'(\varphi_j)}{S_P(\varphi_j)}-\frac{X_P'(\varphi_j)}{X_P(\varphi_j)}\right)^2\right].$$

Proof. From [1],

$$\mathscr{K}X_P(\varphi) = \mathscr{K}(S_P-\mathscr{s}_P)(\varphi) = \frac{X_P(\varphi)}{S_P(\varphi_0)}\mathscr{K}S_P(\varphi) - \frac{X_P(\varphi)}{\mathscr{s}_P(\varphi)}\mathscr{K}\mathscr{s}_P(\varphi) + 2S_P(\varphi)\mathscr{s}_P(\varphi)\left[\frac{S_P'(\varphi)}{S_P(\varphi)}-\frac{\mathscr{s}_P'(\varphi)}{\mathscr{s}_P(\varphi)}\right]^2$$

for some angle φ. Solving for $\mathscr{K}\mathscr{s}_P$ evaluated at φ_0 gives

$$\begin{aligned}\mathscr{K}\mathscr{s}_P(\varphi_0) &= \mathscr{K}(S_P-X_P)(\varphi_0)\\ &= \frac{\mathscr{s}_P(\varphi_0)}{S_P(\varphi_0)}\mathscr{K}S_P(\varphi_0) - \frac{\mathscr{s}_P(\varphi_0)}{X_P(\varphi_0)}\mathscr{K}X_P(\varphi_0) + 2S_P(\varphi_0)X_P(\varphi_0)\left[\frac{S_P'(\varphi_0)}{S_P(\varphi_0)}-\frac{X_P'(\varphi_0)}{X_P(\varphi_0)}\right]^2.\end{aligned}$$

Since $\kappa S_P(\varphi_0) = -\kappa\mathscr{s}_Q(\psi_0)$,

$$\mathscr{K}S_P(\varphi_0) = -\left[\frac{S_P^2(\varphi_0)+(S_P')^2(\varphi_0)}{\mathscr{s}_Q^2(\psi_0)+(\mathscr{s}_Q')^2(\psi_0)}\right]^{3/2}\mathscr{K}\mathscr{s}_Q(\psi_0)$$

from the signed curvature definition. Then

$$
\begin{aligned}
\mathcal{K}\mathscr{s}_P(\varphi_0) = &- \left[\frac{\mathscr{s}_P(\varphi_0)}{S_P(\varphi_0)}\left(\frac{S_P^2(\varphi_0)+(S_P')^2(\varphi_0)}{\mathscr{s}_Q^2(\psi_0)+(\mathscr{s}_Q')^2(\psi_0)}\right)^{3/2}\mathcal{K}\mathscr{s}_Q(\psi_0)\right] \\
&- \left[\frac{\mathscr{s}_P(\varphi_0)}{X_P(\varphi_0)}\mathcal{K}X_P(\varphi_0)\right] + \left[2S_P(\varphi_0)X_P(\varphi_0)\left(\frac{S_P'(\varphi_0)}{S_P(\varphi_0)}-\frac{X_P'(\varphi_0)}{X_P(\varphi_0)}\right)^2\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{K}\mathscr{s}_Q(\psi_0) &= \mathcal{K}(S_Q - X_Q)(\psi_0) \\
&= \frac{\mathscr{s}_Q(\psi_0)}{S_Q(\psi_0)}\mathcal{K}S_Q(\psi_0) - \frac{\mathscr{s}_Q(\psi_0)}{X_Q(\psi_0)}\mathcal{K}X_Q(\psi_0) + 2S_Q(\psi_0)X_Q(\psi_0)\left[\frac{S_Q'(\psi_0)}{S_Q(\psi_0)}-\frac{X_Q'(\psi_0)}{X_Q(\psi_0)}\right]^2
\end{aligned}
$$

and

$$
\mathcal{K}S_Q(\psi_0) = -\left[\frac{S_Q^2(\psi_0)+(S_Q')^2(\psi_0)}{\mathscr{s}_P^2(\varphi_1)+(\mathscr{s}_P')^2(\varphi_1)}\right]^{3/2}\mathcal{K}\mathscr{s}_P(\varphi_1).
$$

Combining these two equations we have that

$$
\begin{aligned}
\mathcal{K}\mathscr{s}_Q(\psi_0) = &- \left[\frac{\mathscr{s}_Q(\psi_0)}{S_Q(\psi_0)}\left(\frac{S_Q^2(\psi_0)+(S_Q')^2(\psi_0)}{\mathscr{s}_P^2(\varphi_1)+(\mathscr{s}_P')^2(\varphi_1)}\right)^{3/2}\mathcal{K}\mathscr{s}_P(\varphi_1)\right] \\
&- \left[\frac{\mathscr{s}_Q(\psi_0)}{X_Q(\psi_0)}\mathcal{K}X_Q(\psi_0)\right] + \left[2S_Q(\psi_0)X_Q(\psi_0)\left(\frac{S_Q'(\psi_0)}{S_Q(\psi_0)}-\frac{X_Q'(\psi_0)}{X_Q(\psi_0)}\right)^2\right].
\end{aligned}
$$

After substituting this equality for $\mathcal{K}\mathscr{s}_Q(\psi_0)$ into our last expression for $\mathcal{K}\mathscr{s}_P(\varphi_0)$ and rearranging terms, we have completed the proof when $j = 0$. However, the methods used apply for any $0 \le j \le n-1$, leaving us with the general form of (21).

□

7. Conclusion

Possible research directions arise out of our results. Several of our findings could be expanded upon, as is the case with most research. One such area that still needs much work is the recursion method in Section 6. We were not able to complete the derivations here. Though messy, this procedure does have potential. Once convexity is shown to hold for an arbitrary initial angle, existence follows. Since we have already determined existence of the local stable manifold (in neighborhoods of the vertices and basepoints), global existence would be established through iteration. Thus a continuation of our results will presumably prove both convexity and existence of the shadow body.

This paper provides a beginning to the shadow body search. Geometric interpretations, which are important to this field, have been a common theme throughout. We have also found conditions under which a convex shadow body may or may not exist. Many of our conditions include curvature formulas. Further research will likely lead to solutions to these formulas, proving that The Uniqueness Theorem does not hold when point X-rays are taken, solving the problem once and for all.

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