POPULATION DYNAMICS IN ONE DIMENSION

KATIE WALSH AND JON WHERRY

ADVISOR: PROFESSOR CULL
OREGON STATE UNIVERSITY

ABSTRACT. In many simple one dimensional population models, local stability implies global stability. We look at population models that are locally stable but not globally stable. We also examine why different classes of functions behave in different manners, e.g. while some models have initial populations that go to zero, other models have initial populations that map to stable and/or unstable cycles. We also investigate how complex a model has to be before instability arises. In particular, we determine why the cubic is always globally stable if locally stable, while this does not hold for the quartic. We also look at modifying other commonly studied population models to see when instabilities arise. Finally, we use fractals as a tool to determine where instabilities occur.

1. INTRODUCTION

Biological populations vary, displaying both periods of growth and decay. These seemingly complex populations can be described using relatively simple one dimensional nonlinear difference equations with a single positive equilibrium point. Cull has shown that in the seven standard biological models local stability implies global stability. This idea was originally noted by biologists, who could tell if a model was globally stable if locally stable but lacked a way of analytically showing it. Mathematicians have come up with several ways to explain this behavior. One method of determining global stability is through the process of enveloping by a simple self-inverse function (e.g. a straight line or a ratio of linear functions). This not only shows global stability, but it also easily verified both visually and analytically.

This paper will focus primarily on why certain functions are globally stable if locally stable while others are not. Can we control when a period two oscillation occurs? What simple changes to the seven population models preserve local stability implying global stability? We will examine these questions to determine what conditions need to be met to have local without global stability.

After providing background information, we will manipulate several population models that we know have local stability implying global stability and examine the affects of the manipulations. The changes in these models, we feel, will be minute, allowing us to hopefully gain insight as to why these seven population models behave as they do.

We will also create a simple piecewise population model that has a period three cycle (a cycle whose existence implies the existence of cycles of all other periods). This simple model with

Date: August 13th, 2007.
This work was done during the Summer 2007 REU program in Mathematics at Oregon State University.

existence of complex period cycles will serve as a contrast to the simple models implying global stability.

Finally, material will be presented analytically, when possible, along with various visual aids. These include sequence plots, webplots, and fractals, among others. We will use these as tools to helping us better understand how population models change when altered.

2. BACKGROUND INFORMATION

The following definitions will help the reader to better understand what a population model is and what it means for that model to be locally stable, globally stable, or chaotic:

Definition 2.1. [6] *A* ONE-DIMENSIONAL POPULATION MODEL *is a difference equation of the form*

$$x_{t+1} = f(x_t)$$

where f if a continuous function that maps non-negative reals to non-negative reals with a unique positive equilibrium point, $\bar{x} > 0$, such that:

(1) $f(0) = 0$
(2) $f(x) > x \ for \ 0 < x < \bar{x}$
(3) $f(x) = x \ for \ x = \bar{x}$
(4) $f(x) < x \ for \ x > \bar{x}$
and if $f'(x_m) = 0$ where x_m is a maximum point and $x_m \leq \bar{x}$ then
(5) $f'(x) > 0 \ for \ 0 \leq x < x_m$
(6) $f'(x) < 0$ *for* $x > x_m$ *such that* $f(x) > 0$
(7) $f(x) > 0$ *for* $x \in (0, x_\infty)$
(8) $f(x) = 0$ *for* $x \geq x_\infty$ *where* x_∞.

Definition 2.2. [6] *A population model is* GLOBALLY STABLE *if and only if for all x_0 such that $f(x_0) > 0$*

$$\lim_{k \to \infty} f^{(k)}(x_0) = \bar{x}$$

where $f^{(k)}$ is the value of x after k iterations of f.

Definition 2.3. [6] *A population model is* LOCALLY STABLE *if and only if there exists a δ such that*

$$\lim_{k \to \infty} f^{(k)}(x_0) = \bar{x} \, \text{for all } x_0 \in (\bar{x} - \delta, \bar{x} + \delta).$$

Theorem 2.4. [3] *A population model is locally stable if $|f'(\bar{x})| < 1$ and if a population is locally stable then $|f'(\bar{x})| \leq 1$.*

Definition 2.5. [3] *A function is* CHAOTIC *if it has cycles of every length.*

The following theorem provides us with a way of seeing whether or not a function is locally stable by looking for period oscillations:

Theorem 2.6. [17] TWO RESULTS FROM SARKOVSKII'S THEOREM*: A continuous population model is globally stable if and only if it has no cycle of period 2, or in other words, if there is no other point except $\bar{x}$ and 0 such that $f(f(x)) = x$. If a continuous population model has a period three cycle, then it has cycles of every other integer length.*

The next set of definitions and theorems provide the background information necessary to understand the process of enveloping and what it means to be *doubly positive*:

Definition 2.7. [6] *A* LINEAR FRACTIONAL FUNCTION, $\phi(x)$, *is a function of the form:*

$$\phi(x) = \frac{1-\alpha x}{\alpha + (1-2\alpha)x} \quad where \quad \alpha \in [0,1)$$

with the following properties:

(1) $\phi(1) = 1$
(2) $\phi'(1) = -1$
(3) $\phi(\phi(x)) = x$
(4) $\phi'(x) < 0$.

Theorem 2.8. [6] *If f(x) is enveloped by a linear fractional function then f(x) is globally stable.*

Definition 2.9. [6] *A function h(z) is* DOUBLY POSITIVE *if and only if*

(1) *$h(z)$ has a power series $\sum_{i=0}^{\infty} h_i z^i$*
(2) $h_o = 1$, $h_1 = 2$
(3) *for all $n \leq 1$, $h_n \geq h_{n+1}$*
(4) *for all $n \leq 2$, $h_n - 2h_{n+1} + h_{n+2} \leq 0$*

Theorem 2.10. [6] *Let $x_{t+1} = f(x_t)$ and $f(x) = xh(1-x)$ where $h(z)$ is doubly positive, then f is enveloped by the linear fractional function*

$$\phi(x) = \frac{1-\alpha x}{\alpha + (1-2\alpha)x}$$

where $\alpha = \frac{3-h_2}{4-h_2} \leq \frac{1}{2}$, and $x_{t+1} = f(x_t)$ is globally stable.

Theorem 2.11. [6] *If $f_1(x)$ is enveloped by $f_2(x)$ and $f_2(x)$ is globally stable, then $f_1(x)$ is globally stable.*

3. INSTABILITY AND STABILITY IN GENERAL FORM POLYNOMIALS

In polynomials of degree four it is possible to have local without global stability. This was first introduced in Cull, with the example $f(x) = x(x-3/2)(-2-(x-1)-6(x-1)^2)$ [6]. The graph of this function is seen in Figure 1.

A cobweb plot is an easy way to visualize how a given population model behaves over a series of iterations given an initial population value. In these plots, we draw a vertical line from our initial population value on the x-axis to the point where this value gets mapped (i.e. from $(x_0, 0)$ to $(x_0, f(x_0))$). Then, we draw a horizontal line to the line $y = x$, i.e. to the point $(f(x_0), f(x_0)$.

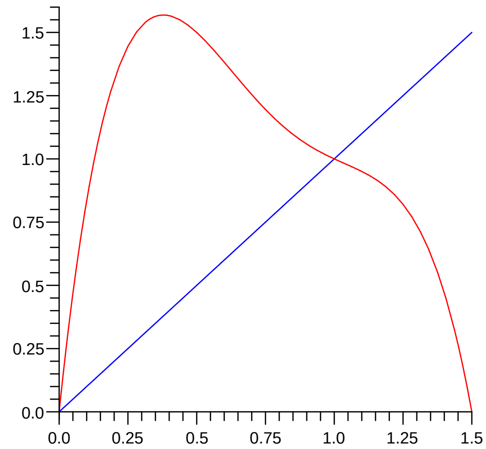

FIGURE 1. Graph of $f(x) = x(x-3/2)(-2x-(x-1)-6(x-1)^2)$ plotted with $y = x$. The point where the two curves cross is the equilibrium point $\bar{x}$.

Then, a vertical line to the function $y = f(x)$, the point $f(x_0), f(f(x_0)))$. Since $f(x_0) = x_1$, this is the same as drawing a line to the point $(x_1, f(x_1))$. We continue this pattern–alternating vertical and horizontal lines going between $y = x$ and $y = f(x)$.
An example of a cobweb plot using the previously defined $f(x)$ with initial population of .7 is shown in Figure 2. Here we can see that when $x_0 = .7$, the population ends up at the equilibrium point, $\bar{x} = 1$.

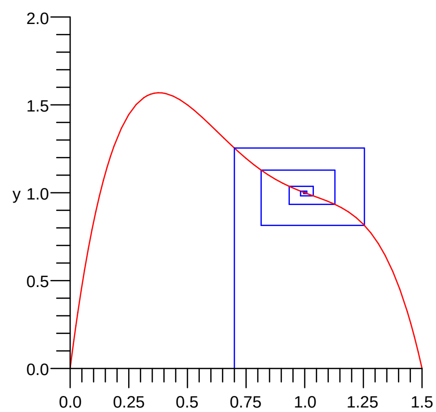

FIGURE 2. Cobweb Plot for $f(x)$ with $x_0 = .7$ we see that the cobweb spirals in to the point $(1,1)$, our equilibrium point.

To see where different initial population values tend to after many iterations, we can look at *sequence plots*. In a *sequence plot*, the initial population values are along the x-axis, and then

evaluated for 300 iterations (discarding the first 100) and then plotted.
Figure 3 is the sequence plot for our previously defined $f(x)$.

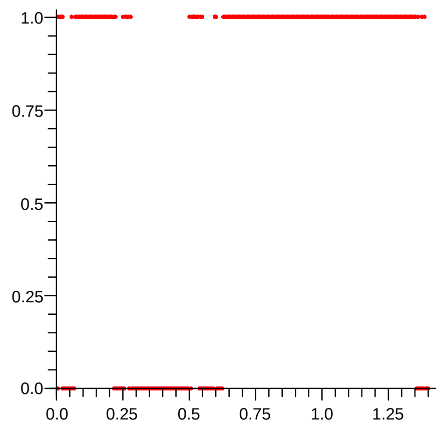

FIGURE 3. Sequence Plot for $f(x) = x(x-3/2)(-2-(x-1)-6(x-1)^2)$. The x-values that map to 1 are points that, after a large number of iterations, tend to our equilibrium point. Since not all x-values tend to the equilibrium point, this model is not globally stable.

From this plot, we can see that there are various intervals where initial population values do not go to 1 after multiple iterations. Since 1 is our equilibrium point, we can see that this is not a globally stable model.

Another way to visualize instabilities in a model is through the application of Sarkovskii's Theorem. According to the theorem, if a function is globally stable, then it has no period-two cycles. In other words, if it has any cycles of any length, then it necessarily has a cycle two point. To easily check for cycle two points, we can graph $y = f(f(x))$ and the line $y = x$. Any points of intersection between these two lines are either equilibrium points or cycle two points. If there are other points of intersection besides the equilibrium, then the function is not globally stable. Figure 4 shows the graph of $f(f(x))$ for $f(x) = x(x-3/2)(-2-(x-1)-6(x-1)^2)$.

From the graph of $f(f(x))$ we can see that $f(x)$ has two cycles of period two, one that oscillates between approximately .25 and 1.45 and another that oscillates between approximately .625 and 1.35. Since these cycles do not show up in our sequence plot, we can conjecture that they are both unstable cycles, that is, there is no small neighborhood around these points such that the points in the interval map to the cycles.
In order to see where points between the different regions map, we look at the cobweb plots for different initial populations values. A few cobweb plots for different starting values are shown in Figures 5, 6, 7, and 8.

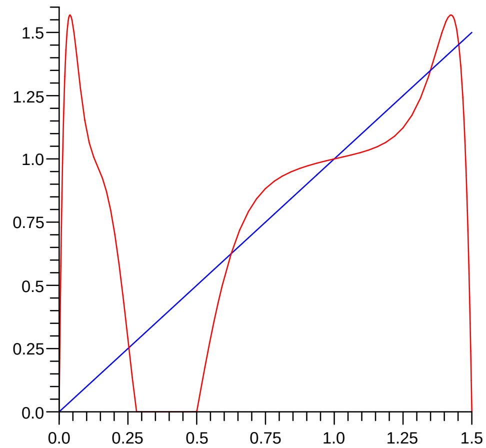

FIGURE 4. Graph of $f(f(x))$ against $f(x) = x$. Since the function $f(f(x))$ crosses the function $f(x) = x$ at more points than just 0 and $\bar{x}$ this model is not globally stable.

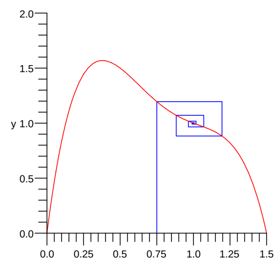

FIGURE 5. Cobweb plot with $x_0 = .75$. When our initial value, x_0 is .75, the function tends to the equilibrium point after a large number of iterations.

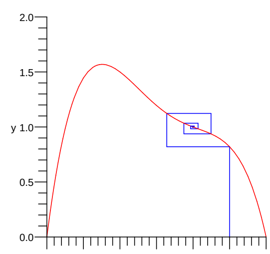

FIGURE 6. Cobweb plot with $x_0 = 1.25$. When our initial value, x_0 is 1.25, the function tends to the equilibrium point after a large number of iterations.

From Figures 5 and 6, we can see that 1 is a locally stable point, since values slightly above and slightly below 1 tend towards 1. On the other hand, both period two cycles are unstable since

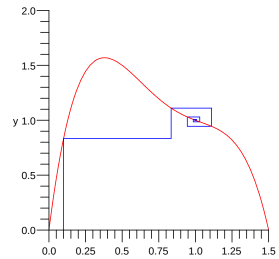

FIGURE 7. Cobweb plot with $x_0 = .1$. When our initial value, x_0 is .1, the function tends to the equilibrium point after a large number of iterations.

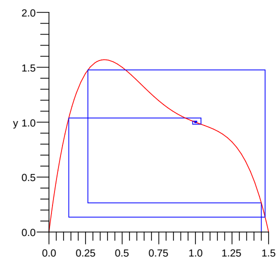

FIGURE 8. Cobweb plot with $x_0 = 1.45$. This initial value is near to the period two point detected by plotting f(f(x)) against x. Notice how it tends to 1. This means that the period two cycle is unstable.

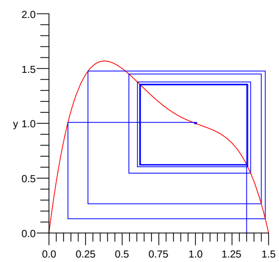

FIGURE 9. Cobweb plot with $x_0 = 1.35$. This, like our cobweb plot for $x_0 = 1.45$, is a cobweb plot for an initial value close to a detected period two point. Notice how the cobweb forms a dark box, showing that the function maintains a period two cycle for a short while. However, after a large number of iterations, this function tends to 0.

for values slightly above or below the points that map to the cycle two tend towards 1 after many iterations. In Figure 9, we can see that the population stays near the cycle two point for a short while, but does end up going to 1 after several iterations.

This quartic raises some interesting questions. If a period two oscillation can occur in the quartic, can it arise in lower degree polynomials as well? The claim in the paper by Cull [6] was that this was not the case.

CLAIM: A continuous polynomial, $f(x)$, must have degree ≥ 4 for local stability to not imply global stability.

PROOF: We want to show that a polynomial of degree less than or equal to three has the property that if it is a population model displaying local stability then it must be globally stable. In other words, we want to show that if $f(x) = ax^3 + bx^2 + cx + d$ is a locally stable population model, then it must be globally stable. Since lower degree polynomials can be obtained by setting a, b, c, or d to zero, it is sufficient to show that these claims hold true only for the cubic.

By the definition of a population model, we know that $f(0) = 0$. For our cubic, $f(x) = ax^3 + bx^2 + cx + d$. This means that $f(0) = a(0)^3 + b(0)^2 + c(0) + d = d$. Therefore $f(0) = 0$ implies $d = 0$.

This leaves our general cubic in the form $f(x) = ax^3 + bx^2 + cx$.

To place even more constraints on our function, let's assume that our population model has been normalized. This is a reasonable assumption since for any population model that is not normalized, (i.e. $f(\bar{x}) = \bar{x}$, where $\bar{x} \neq 1$), we can divide f by $\bar{x}$ to normalize $f(x)$. Thus, without loss of generality, we assume f is normalized (i.e. $f(1) = 1$ and $\bar{x} = 1$). Therefore, $f(1) = a(1)^3 + b(1)^2 + c(1) = a + b + c = 1$.

The third condition for population models gives us that $f(x) > x$ for $0 < x < 1$. Thus, between 0 and 1, $ax^3 + bx^2 + cx > x$. Since $x > 0$, dividing both sides by x yields $f(x) = ax^2 + bx + c > 1$ for $0 < x < 1$. Similarly, the condition $f(x) < x$ for $1 < x$ gives us that $ax^2 + bx + c < 1$ for $1 < x$. From this, we notice that $c \geq 1$.

Further, since we are looking at a locally stable population model, $|f'(1)| \leq 1$. Thus $|f'(1)| = |3a + 2b + c| \leq 1$. Since $a + b + c = 1$

$$|3a + 2b + 1 - a - b| = |2a + b + 1| \leq 1.$$

Splitting the inequality and rearranging terms gives us

$$-2 - 2a \leq b \leq -2a \tag{1}$$

CLAIM: If $b = -2 - 2a$ then f is globally stable.

PROOF: We have that $b = -2a - 2$ and thus $c = a + 3$. We know that if $f(x) = x(ax^2 + bx + c)$ has the form $f(x) = xh(1 - x)$ where $h(z)$ is doubly positive then $f(x)$ is globally stable.

$$\begin{aligned} f(x) &= x(a^2 + bx + c) \\ &= x(ax^2 + (-2 - 2a)x + (a + 3)) \\ &= x(1 + 2(1 - x) + a(1 - x)^2) \end{aligned}$$

CLAIM: $a \leq 1$

PROOF: If $a > 0$, then $f(x) = x(1 + 2(1 - x) + a(1 - x)^2) > 0$ for large x. And, if $a < 0$, then $f(x) = x(1 + 2(1 - x) + a(1 - x)^2) < 0$ for large x. Either way, in order for $f(x)$ to be a population model, we need $f(x_\infty)$ to be zero (since $f(x)$ doesn't approach zero as x approaches infinity).

This means that $f(x)$ must have a real root. Solving for the roots of the equation of the equation $f(x) = 1+2(1-x)+a(1-x)^2$, we get

$$\begin{aligned} 1-x &= \frac{-2 \pm \sqrt{4-4a}}{2a} \\ &= \frac{-1 \pm \sqrt{1-a}}{a} \end{aligned}$$

Therefore, for a real root to exist, $1-a \geq 0$. And so, $a \leq 1$ Since $a \leq 1 < 2$, $f(x) = xh(1-x)$, where $h(z) = 1+2z+az^2$ is doubly positive. Therefore, with $b = -2a-2$ and $c = a+3$, $f(x)$ is globally stable by Theorem 2.10.

Now we look at the case when $b > -2a-a$ and $c > a+3$.
CLAIM:If $b > -2a-a$ and $c > a+3$, then $f(x)$ is enveloped by

$$g(x) = ax^3 - (2a+2)x^2 + (a+3)x.$$

PROOF: To show that $f(x)$ is enveloped by $g(x)$, we show that $f(x) < g(x)$ for $0 < x < 1$ and $f(x) > g(x)$ for $x > 1$. First we will deal with the case where $x < 1$. Since $b > -2a-2$,

$$b+2a+2 > 0,$$

multiplying both sides of $x < 1$ by $b+2a+2$ yields

$$(b+2a+2)x < (b+2a+2).$$

Now, since $x > 0$, we can multiply both sides by x, giving us:

$$(b+2a+2)x^2 < (b+2a+2)x,$$

through multiplication and rearrangement of terms we obtain

$$bx^2 + (1-a-b)x < -(2a+2)x^2 + (a+3)x$$

and we can substitute c for $1-a-b$ giving us

$$bx^2 + cx < -(2a+2)x^2 + (a+3)x,$$

adding ax^3 to both sides we are left with

$$ax^3 + bx^2 + cx < ax^3 - (2a+2)x^2 + (a+3)x$$

which is what we wanted to show. Beginning with $x > 1$, we can go through the same process, just reversing all the inequalities to obtain that $f(x) > g(x)$ for $x > 1$. Thus $f(x)$ is enveloped by $g(x)$ when $b < -2a-a$.

We saw earlier that $b \geq -2a-2$ when f is locally stable . We have just shown that when $b = -2a-2$ or $b > -2a-2$, f is globally stable. Thus we can conclude that in a general third degree polynomial, local stability implies global stability.

4. THE EXPONENTIAL FUNCTION AS A POPULATION MODEL

It is also possible to have an exponential function that displays local but not global stability. One such function, which we will examine next, is the exponential function $f(x) = e^{-q(x)}$ where

$$q(x) = 1.9(x-1) - (7.6 - 8ln3)(x-1)^3).$$

The graph of the function is in Figure 10

Since this function does not go to zero, as the last one did, we will not have the same type of

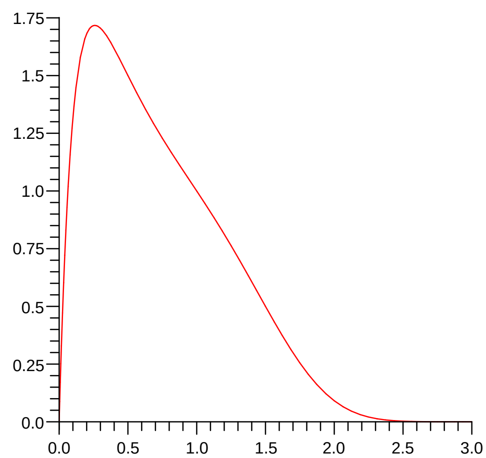

FIGURE 10. Plot of $f(x) = e^{-q(x)}$ where $q(x) = 1.9(x-1) - (7.6 - 8ln3)(x-1)^3)$.

instabilities as the previous function. This function, however, is also not globally stable. We can easily see this from looking at a sequence plot.

In Figure 11, we see that not only is this graph not globally stable (i.e. not every initial population value tends to 1), but also that there is a stable period two. If we look at the plot of $y = f(f(x))$ and $y = x$, we can see that in fact this function has two period two cycles. One of these cycles is stable, while the other is not. In Figures 13-16, we can see where different initial population values tend to by looking at the different cobweb plots.

Again, there was a claim in Cull [6] that the function $f(x) = xe^{-q(x)}$ has the property that local stability implies global stability for all polynomials $q(x)$ of degree less than three. Below is the proof of this claim.

CLAIM: If $q(x)$ is a polynomial with real coefficients less than degree three, then if $xe^{-q(x)}$ is locally stable it is globally stable.

PROOF: According to Definition 1.10, we know that all functions enveloped by a globally stable function are globally stable. Therefore, if we can show that for the set of locally stable functions in the form $xe^{-q(x)}$, where $q(x)$ is a polynomial with degree less than three, there exists a family of globally stable functions that envelop all other locally stable functions then we know that local stability implies global stability in all models of this form.

According to the definition of a population model, $f(x)$ needs to satisfy the following properties:

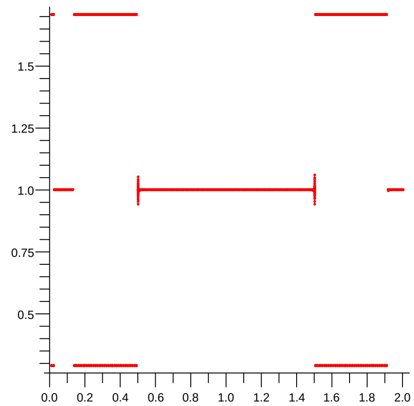

FIGURE 11. Sequence plot of $f(x)$. Notice how in addition to the places that tend to 1 after several iterations, there are initial values that tend to two values after several iterations. These are period two cycles.

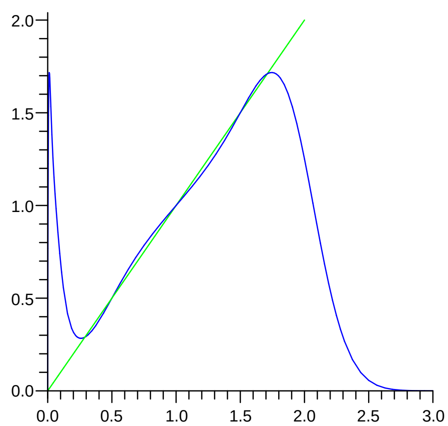

FIGURE 12. Plot of $f(f(x))$ against $f(x) = x$. Since the graph of $f(f(x))$ crosses the graph of $f(x) = x$ in places other than 0 and the equilibrium point, we can tell that f(x) is not globally stable.

(1) $f(1) = 1$
(2) $f(0) = 0$
(3) $f(x) > x$ if $0 < x < 1$
(4) $f(x) < x$ if $1 < x$

Let's look at the restrictions this places on $q(x)$ for our function $f(x) = xe^{-q(x)}$:

(1) $f(1) = (1)e^{-(a(1)^2+b(1)+c)} = 1$. Therefore, $\ln e^{-(a+b+c)} = a+b+c = \ln(1) = 0$. This yields $a+b+c = 0$.

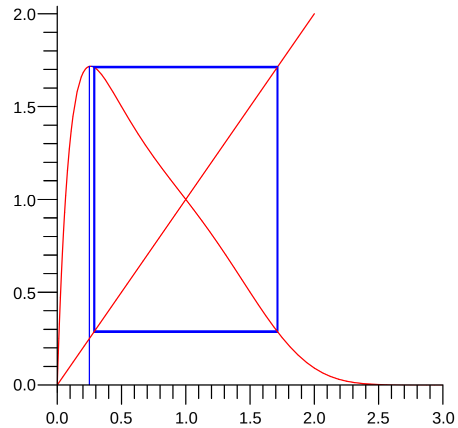

FIGURE 13. Cobweb Plot for $f(x)$ with $x_0 = .25$. This initial value was in the region of the sequence plot that mapped to two values. Notice how it gives us a period two cycle.

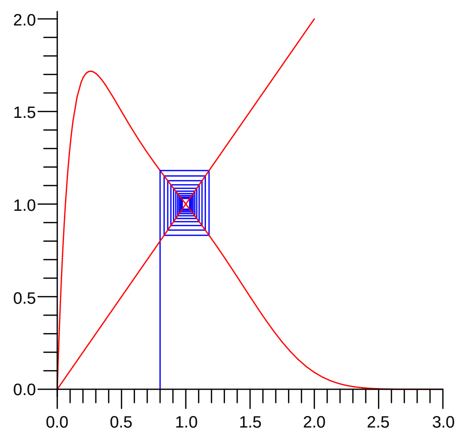

FIGURE 14. Cobweb Plot for $f(x)$ with $x_0 = .80$. This initial value was in the region of the sequence that mapped to 1. Notice how it spirals in to one equilibrium point.

(2) $f(0) = (0)e^{-q(0)} = 0$, which places no more constraints on $q(x)$
(3) If $f(x) > x$ when $0 < x < 1$, then $xe^{-(ax^2+bx+c)} > x$. Which yields $e^{-(ax^2+bx+c)} > 1$. Taking the natural log yields $-(ax^2+bx+c) > 0$. This implies $ax^2+bx+c < 0$ when $0 < x < 1$.
(4) If $f(x) < x$ when $1 < x < \infty$, then $xe^{-(ax^2+bx+c)} < x$. This yields $e^{-(ax^2+bx+c)} < 1$. Taking the natural log yields $-(ax^2+bx+c) < 0$. This implies $ax^2+bx+c < 0$ when $1 < x < \infty$.

From (3) and (4), we can show that $c \leq 0$ and $a > 0$.

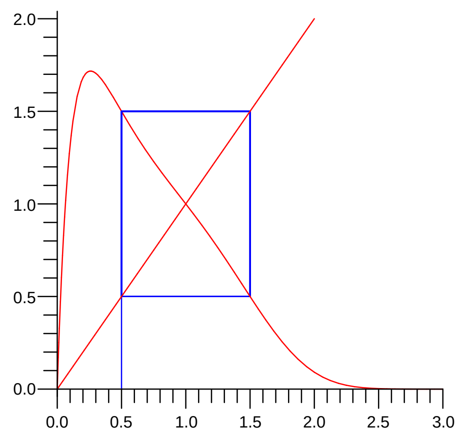

FIGURE 15. Cobweb Plot for $f(x)$ with $x_0 = .50$. This is an initial value for a value slightly before the part of the sequence plot where it goes from tending to two points to tending toward one point.

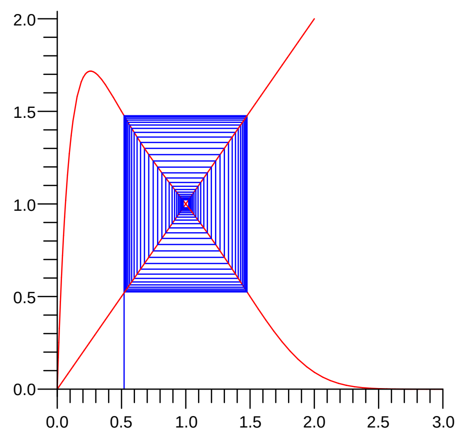

FIGURE 16. Cobweb Plot for $f(x)$ with $x_0 = .52$. This initial value is slightly after the part of the sequence plot where it changes from tending toward two points to tending toward one. The above cobweb plot shows that it tends towards the equilibrium, but the darkness of the plot indicates it takes a long time to converge.

CLAIM: $c \leq 0$.
PROOF: Assume c is positive. Since $q(x) = ax^2 + bx + c$ is continuous and $q(x) < 0$ for all $0 < x < 1$, we know that there exists an $x_0 \in (0,1)$. Therefore, since we have $q(0) = c > 0$ and $q(x_0) < 0$, we know by the Intermediate Value Theorem that there must exist a x_1 such that $q(x_1) = 0$, with $0 < x_1 < x_0$. This is a contradiction. Therefore c cannot be positive.

CLAIM: $a \geq 0$.

PROOF: We know from (4) that the function $q(x)$ must be greater than zero for all values of x greater than the equilibrium point 1. Since $q(x)$ is a parabola, this forces a to be nonnegative.

We now know that in order for $f(x) = xe^{-q(x)}$ to be a population model, we must have $q(x) = ax^2 + bx + c$ where $a \geq 0$ and $c \leq 0$ and $a+b+c = 0$.

Let's assume that $f(x)$ is locally stable. In order to be locally stable, we know that $|f'(\bar{x})| \leq 1$ by Theorem 1.12.
Using the chain rule, we get $f'(x) = e^{-(ax^2+bx+c)} + xe^{-(ax^2+bx+c)}(-2ax-b) = (1-2ax^2-bx)e^{-(ax^2+bx+c)}$.

Therefore, since $\bar{x} = 1$ when normalized, $f'(\bar{x}) = f'(1) = (1-2a(1)^2 - b(1))e^{-(a(1)^2+b(1)+c)} = (1-2a-b)e^{-(a+b+c)}$. However, since we know this is a population model, a+b+c=0. This substitution gives us $f'(1) = (1-2a-b)e^0 = (1-2a-b)$.

This gives us the constraint that $|f'(x)| = |1-2a-b| \leq 1$. Which can be rewritten as

$$-1 \leq 1-2a-b \leq 1$$

$$-2 \leq -2a-b \leq 0$$

$$2 \geq 2a+b \geq 0$$

Which, by our condition that $a+b+c = 0$, is equivalent to

$$0 \leq 2a+(-a-c) = a-c \leq 2$$

$$c \leq a \leq 2+c.$$

Since $a \geq 0$, we know that $0 \leq a \leq 2+c$. Since $c \leq 0$, the largest c can be is 0. This means that $0 \leq a \leq 2$. The smallest a can be is 0, this means that $c \leq 0 \leq 2+c$. This implies that $-2 \leq c \leq 0$.

This gives us the triangle of potential solutions, which we can see in Figure 17, where a is plotted along the x-axis.
Before we proceed, we know that $a+b+c = 0$, let's rewrite our general form equation, $e^{-(ax^2+bx+c)}$.

$$\begin{aligned} e^{-(ax^2+bx+c)} &= e^{(-ax^2-(-a-c)x-c)} \\ &= e^{-ax^2+ax+cx-c} \\ &= e^{ax(-x+1)-c(-x+1)} \\ &= e^{(1-x)(ax-c)} \end{aligned}$$

Now, we claim that when $c = 2-a$, i.e. along the bottom diagonal of the triangle, we have a globally stable model. We see this by looking at a 3-dimension graph of $y = f(f(x))$ and $y = x$., which is in Figure 18. As displayed in this figure, $f(f(x))$ is greater than x for all $x \in (0,1)$, thus by using Sarkovskii's Theorem, we have a globally stable model. However, we have not been able to prove this given any of the other theorems that prove global stability. We will look at where these theorems fail, but first we will look at why proving that the bottom diagonal of the triangle is globally stable is sufficient.

CLAIM: For a given value of a, any function with a c value in the triangle is enveloped by the

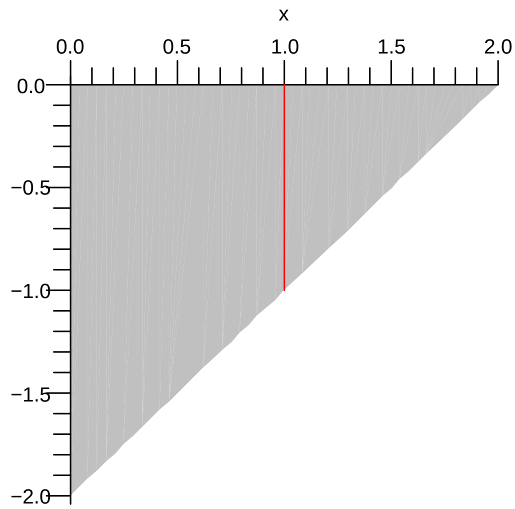

FIGURE 17. Potential values for a and c such that the model $f(x)$ is locally stable. The values for a are plotted along the x-axis, while the values for c are plotted along the y-axis.

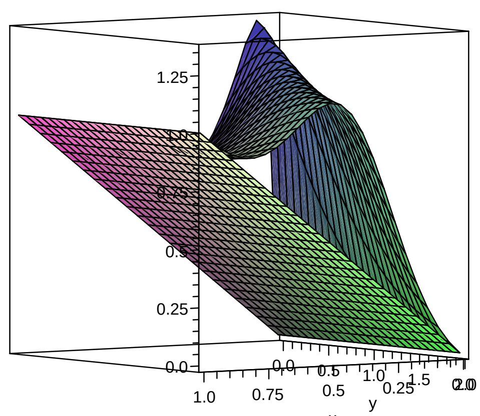

FIGURE 18. 3-D Plot of $y = f(f(x))$ and $y = x$

function with c value along the lower diagonal.
PROOF: First we will deal with $0 < x < 1$. For a fixed a, let $c^* = a - 2$ thus c^* is the value along

the diagonal. We need to show that for $c > c^*,\ f_{c^*} > f_c$.

$$\begin{aligned} c > &\quad c^* = a-2 \\ -a+2 > &\quad -c \\ ax-a+2 > &\quad ax-c \\ (1-x)(ax-a+2) > &\quad (1-x)(ax-c) \\ e^{(1-x)(ax-a+2)} > &\quad e^{(1-x)(ax-c)} \\ e^{(1-x)(ax-c^*)} > &\quad e^{(1-x)(ax-c)} \end{aligned}$$

Now, for the case when $x > 1$, $(1-x) < 0$, the inequality in step 4 will change and we will get that $f_{c^*} < f_c$, which is the second condition we need for enveloping.

So if we show the lower diagonal is globally stable, since functions along this line envelop the remainder of these functions, we will be able to conclude that for functions of the form $e^{-q(x)}$, where $q(x)$ is at most a third degree polynomial, local stability implies global stability.

Now we will look at different methods for proving global stability, and why they fail.

4.1. **Cull's Enveloping Method.** We know from Theorem 2.10 that if $f(x)$ is of the form $f(x) = xh(x-1)$, where $h(z)$ is doubly positive, then $f(x)$ can be enveloped by a linear fractional. We will have $h(z)$ be a general exponential with a second degree power. Thus,

$$h(z) = e^{z(\alpha z+\beta)}.$$

Before we begin to look at the conditions for doubly positive, we will look at the relationship between α and β and the a,b, and c we used in our previous general equation $f(x) = ex^{-ax^x-bx-c}$. If $h(z) = e^{z(\alpha z+\beta)}$, then

$$\begin{aligned} f(x) &= xe^{(1-x)(\alpha(1-x)+\beta)} \\ &= xe^{\alpha-2\alpha x+\alpha x^2+\beta-\beta x} \\ &= xe^{\alpha x^2+(-2\alpha-2)x+(\alpha+2)} \end{aligned}$$

Thus, $a = -\alpha$, $b = 2\alpha+2$ and $c = -\alpha-2$.

Now we will look at the conditions for doubly positive. When we expand $h(z)$ through power series expansion we obtain:

$$1+z(\alpha z+\beta)+\frac{(z(\alpha z+\beta)^2}{2!}+\frac{(z(\alpha z+\beta)^3}{3!}+\dots$$

When we expand this to find the coefficients, we obtain:

$$\begin{aligned} &1+\beta z+\alpha z^2 \\ &\qquad +\beta^2 z^2 \qquad +2\alpha\beta z^3+\alpha^2 z^4 \\ &\qquad\qquad +\beta^3 z^3+3\alpha\beta^2 z^4 \qquad +3\alpha^2\beta z^5+\alpha^3 z^6+\dots \end{aligned}$$

Since we want $h(z)$ to be doubly positive, $\beta = 2$, so $h_1 = 2$. Also, since $h_n \geq h_{n+1}$, when we let $n = 1$, we get $2 \geq \alpha + 2$, thus $\alpha \geq 0$.

Now in order to determine if h is doubly positive, we need to make sure conditions three and four for doubly positive hold. To do this we come up with a generating function for the coefficients of x^n in h.

CLAIM: The generating function for the coefficients of x^n is:

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{n-2i}\alpha^i}{i!(n-2i)!}.$$

PROOF: We want to find the coefficient of z^n in

$$\begin{aligned} e^{z(\alpha z+\beta)} &= \sum_{n=0}^{\infty} \frac{(\alpha z^2+\beta z)^n}{n!} \\ &= 1 + (\alpha z^2 + \beta z) + \frac{(\alpha z^2+\beta z)(\alpha z^2+\beta z)}{2!} + \frac{(\alpha z^2+\beta z)(\alpha z^2+\beta z)(\alpha z^2+\beta z)}{3!} + \cdots \end{aligned}$$

We look at how we can get z to have a power of n. If there are i αz^2 terms, and we want z^n, then there must be $n-2i$ β terms. Since there are $n-2i+i = n-i$ terms total, we divide by $(n-i)!$. (We can see in the generating function for $e^{z(\alpha z+\beta)}$, for a set value of n, we have n terms in our expansion, then divide by $n!$). Now we look at the number of different ways to choose which i terms will be az^2 out of our $(n-i)$ total terms. There are $\binom{n-i}{i} = \frac{(n-i)!}{i!(n-2i)!}$ ways to do this. There can be anywhere from 0 to $\lfloor n/2 \rfloor$ az^2 terms that contribute to our z^n, thus the coefficient of z^n is

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\beta^{n-2i}\alpha^i(n-i)!}{(n-i)!i!(n-2i)!},$$

which, after cancelation of $(n-i)!$ is the same as originally claimed.

In order for h to be doubly positive,

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{n-2i}\alpha^i}{i!(n-2i)!} - \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{2^{n+1-2i}\alpha^i}{i!(n+1-2i)!} \geq 0$$

for all α and n. This is not true however. In Figure 19, we can see the general trend of the coefficients for $\alpha = -.25$, plotted versus the corresponding n values. As we can see, the coefficients decrease after $n = 2$, but then once they go negative, they begin to increase. For example, take $\alpha = -.25$ and $n = 7$, then

$$x^7 = \sum_{i=0}^{3} \frac{2^{7-2i}(-.25^i)}{i!(7-2i)!} = -.017882$$

and

$$x^8 = \sum_{i=0}^{4} \frac{2^{8-2i}(-.25^i)^i}{i!(8-2i)!} = -.004812.$$

Thus, $x^8 > x^7$, and we can see that h is not doubly positive. Since h is not doubly positive, we

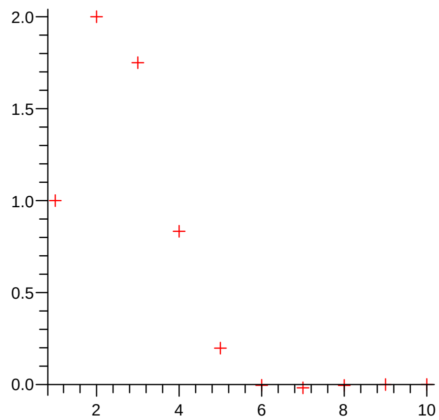

FIGURE 19. Coefficients of x^n for $\alpha = -.25$

cannot use this method to determine whether f is enveloped by a linear fractional. However, since doubly positive is not a necessary condition, this does not mean that f is enveloped. In fact, we observed that for α values where the doubly positive condition failed, f was still enveloped by the linear fraction in Theorem 2.10.

4.2. **Therorem A.**

Theorem 4.1. [4] *If a population model has a maximum x_M in $(0, \bar{x})$ and satisfies additional conditions:*

1. *$g(x)$ has no change of concavity in $(x_M, \bar{x})$*
2. *if g has a change of concavity at x_I then $g''(x)$ is nondecreasing in (x_M, x_I)*

then $\bar{x}$ is globally stable if and only if it is locally stable.

Again, we will look at the general polynomial $xe^{(1-x)(ax-c)}$, where $a = c-2$. Here we will show that we can't use this theorem for all $0 < a < 2$ by showing the conditions don't hold for $a = .25$. Figure 20 shows the plot of f, f' and f''. We can see that between x_M and $\bar{x}$, i.e. between where f' crosses the x-axis and 1, f'' goes from negative to positive, thus there is a change in concavity in $(x_M, \bar{x})$, and therefore the conditions of Theorem A do not hold.

4.3. **Theorem B.**

Theorem 4.2. [5] *A population model with $f'(x) = -1$ is globally stable if*

1. *$k'(x) \leq 2$ on $[x_M, \bar{x})$ where $k(x) = x/f(x)$*
2. *$g(x) \geq 0$ on $[x_M, f(x_M)]$ where $k(x)/k'(x) = g(x) + Bx$ and B is a constant chosen to make g(x) nonnegative.*
3. *$g'(x) \leq 0$ on $[x_M, f(x_M)]$*
4. *$g''(x) \geq 0$ on $[x_M, f(x_M)]$*

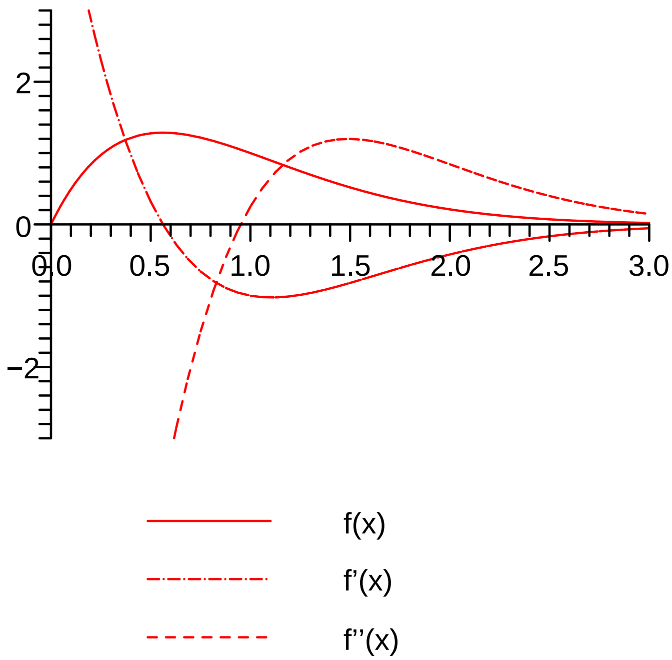

FIGURE 20. Plots of f, f' *and* f''

Since we want $f'(x) = -1$, we will look at the case when $c = a-2$, again the bottom diagonal of the triangle. So our general equation is $xe^{(1-x)(ax-a+2)}$. In order to show the conditions for this theorem do not hold for all a that give a locally stable population model, we will look at the specific case $a = 1$. First, we get

$$k(x) = x/f(x) = \frac{x}{xe^{(1-x)(ax-a+2)}} = e^{-(1-x)(ax-a+2)}$$

and

$$k'(x) = (2ax-2a-2)e^{-(1-x)(ax-a+2)}.$$

So

$$g(x) = k(x)/k'(x) - Bx = \frac{e^{-(1-x)(ax-a+2)}}{(2ax-2a-2)e^{-(1-x)(ax-a+2)}} = \frac{1}{2ax-2a-2} + (-B)x.$$

Since $(-B)$ is added to make $g(x)$ positive, B needs to be negative. Since we want $g'(x)$ to be negative, that means $g(x)$ needs to be deceasing, so we want B to have the smallest absolute value possible (If B were larger, then we mould be adding more for larger values of x, so our function would be more likely to be increasing). To find what the smallest value possible for B is, we look at where $\frac{k(x)/k'(x)}{x}$ is the most negative. Figure 21 shows the graph of this function. We can see that this is most negative at the left most point, which is at $x_M = \sqrt{2}/2$. At this point $k(x)/k'(x) = \frac{\sqrt{2}}{\sqrt{2}-4}$, we let $B = \frac{\sqrt{2}}{\sqrt{2}-4}$, then $g(x) = \frac{1}{2ax-2a-2} - \frac{\sqrt{2}x}{\sqrt{2}-4}$. We then look at $g(x)$ to make sure it is decreasing. Figure 22 shows the plot of $g(x)$ which we can see is not decreasing. Thus, we can't use Theorem B.

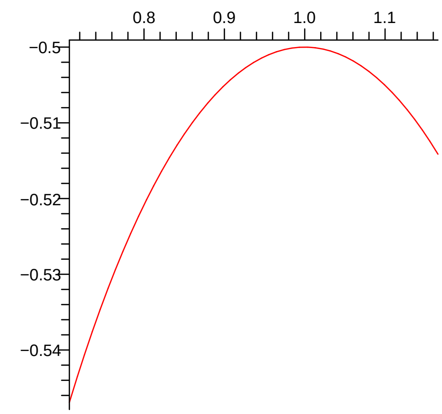

FIGURE 21. Plot of $\frac{k(x)/k'(x)}{x}$. We can see that the smallest value for B is at $x_M = \sqrt{2}/2$

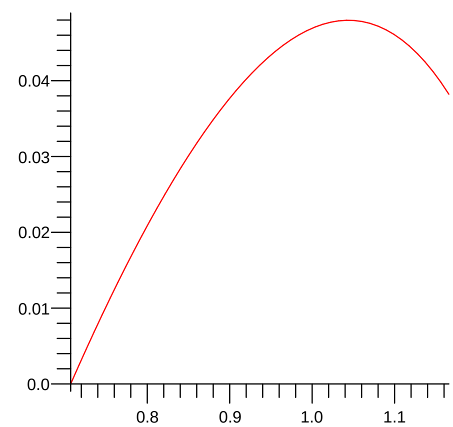

FIGURE 22. The Plot of $g(x)$. Notice that it is not decreasing.

4.4. **Modified Theorem S.**

Theorem 4.3. [1] *Let G be the set of all endomorphisms which satisfy:*

(1) $f(0) = f(1) = 0$;
(2) f *has a unique critcal point in* $[0,1]$; *and*
(3) $S(f,x) < 0$ *for* $x \in [x_M, f(x_M)]$,

where $S(f,x) = \frac{f^{(3)}(x)}{f'(x)} - \frac{3}{2}(\frac{f''(x)}{f'(x)})^2$. *Then for any* f *in G there is at most one stable orbit in* $(0,1)$, *namely* $\{\bar{x}, \bar{x}, \bar{x}, \dots\}$

We can use this theorem to show that local stability implies global stability, because we know since we have local stability, we have the stable fixed point $\bar{x}$, and this theorem says there are no other stable orbits, so there are no period two cycles. Thus, we can attempt to use Sarkovskii's Theorem to show that this is a globally stable model.

In order to use this theorem, we have to re-scale our general function so that $f(0) = f(1) = 0$. Since our function never goes to 0, we assign $f(f(x_M)) = f(x_\infty) = 0$ Then we scale our f such that $x_\infty = 1$. In order to skew f, we will multiply each x in our equation by x_∞. Our scaled f, which we will call w is

$$w(x) = xx_\infty e^{(1-xx_\infty)(axx_\infty c)}.$$

We need the first second and third derivative to compute the Schwarzian Derivative. We have

$$w'(x) = x_\infty e^{(1-xx_\infty)(axx_\infty - c)} + xx_\infty(-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty)e^{(1-xx_\infty)(axx_\infty - c)}$$

$$\begin{aligned} w''(x) = {} & 2x_\infty(-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty)e^{(1-xx_\infty)(axx_\infty - c)} \\ & - 2xx_\infty^3 ae^{(1-xx_\infty)(axx_\infty - c)} + xx_\infty((-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty))^2 e^{(1-xx_\infty)(axx_\infty - c)} \end{aligned}$$

$$\begin{aligned} w^{(3)}(x) = {} & -6x_\infty^3 ae^{((1-xx_\infty)(axx_\infty - c))} \\ & + 3x_\infty((-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty))^2 e^{(1-xx_\infty)(axx_\infty - c)} \\ & - 6xx_\infty^3 a(-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty)e^{(1-xx_\infty)(axx_\infty - c)} \\ & + xx_\infty((-x_\infty(axx_\infty - c) + (1-xx_\infty)ax_\infty))^3 e^{(1-xx_\infty)(axx_\infty - c)} \end{aligned}$$

Now, $S(w,x) = \frac{w^{(3)}(x)}{w'(x)} - \frac{3}{2}(\frac{w''(x)}{w'(x)})^2$. The $e^{(1-xx_\infty)(axx_\infty - c)}$ terms will cancel out, but what we have left is still really complicated and we cannot easily determine the sign of S(w,x), thus this theorem does not help us to determine if our function, f, is globally stable.

5. Logarithmic Function

After investigating the polynomial and exponential functions, we looked into another one of the population models, the logarithmic model. [13] This model is of the form $x(1 - r\ln x)$. For this function, we decided to alter it by adding on terms of the form $c(\ln x)^n$ where c is a constant and n is a natural number. Altering the original function by adding on a quadratic term, gave us a model in the form $x(1 - r\ln x + \alpha(\ln x)^2)$. However, this model, if α is nonzero, is not even a population model.

Claim: $x(1 - r\ln x + \alpha(\ln x)^2)$, where α is nonzero, is not a population model.

Proof of Claim: To be a population model, $f(x) > x$ for $0 < x < 1$. This implies, since $\ln x < 0$

for $0 < x < 1$ and α is nonzero, that

$$\begin{aligned}
x(1 - r\ln x + \alpha(\ln x)^2) &> x \\
1 - r\ln x + \alpha(\ln x)^2 &> 1 \\
-r\ln x + \alpha(\ln x)^2 &> 0 \\
(\ln x)(-r + \alpha(\ln x)) &> 0 \\
-\alpha\ln x + r &> 0 \\
\ln x &< \frac{r}{\alpha}
\end{aligned}$$

Also, in order to be a population model, $f(x) < x$ for $1 < x$. This implies, since $\ln x > 0$ for $1 < x$ and α is nonzero, that

$$\begin{aligned}
x(1 - r\ln x + \alpha(\ln x)^2) &< x \\
1 - r\ln x + \alpha(\ln x)^2 &< 1 \\
-r\ln x + \alpha(\ln x)^2 &> 0 \\
(\ln x)(-r + \alpha(\ln x)) &> 0 \\
-\alpha\ln x + r &> 0 \\
\ln x &< \frac{r}{\alpha}
\end{aligned}$$

However, $\ln x$ is not bounded either above or below by a constant y, for all positive x. Therefore, $x(1 - r\ln x + \alpha(\ln x)^2)$ is not a population model.

6. Piecewise

From the previous sections, the question arose as to how simple a model has to be in order to show local stability but not global stability. An attempt was made to make a piecewise function displaying a period three cycle.

From Sarkovskii's Theorem [17], we know that if we have a continuous function that is a population model with a period three cycle, then it has cycles of every other integer length. Here, we look at creating a piecewise population model that has a cycle of period three and then see where the periods of other lengths appear. Our function, which is shown in Figure 23, is

$$h(x) = \begin{cases} 2x & \text{if } 0 < x \le .25, \\ 3x - .25 & \text{if } .25 < x \le .75, \\ 1/2x + 1.5 & \text{if } .75 < x \le 1.15, \\ -6.75x + 8.6875 & \text{if } 1.15 < x \le 1.25, \\ e^{\frac{-ln(.2)(1-x)}{.25}} & \text{if } x < 1.25. \end{cases} \tag{2}$$

This equation was created such that $h(.5) = 1.25$, $h(1.25) = .25$ and, $h(.25) = .5$. Figure 24 shows

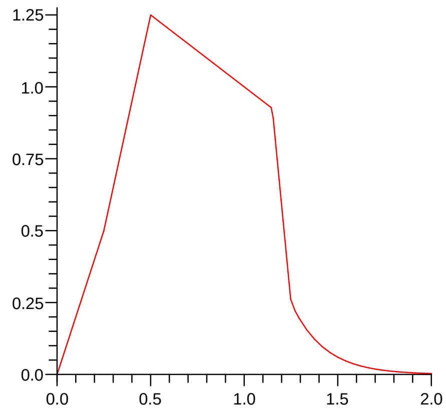

FIGURE 23. Plot of the piecewise function, $y = h(x)$.

the cycle three.

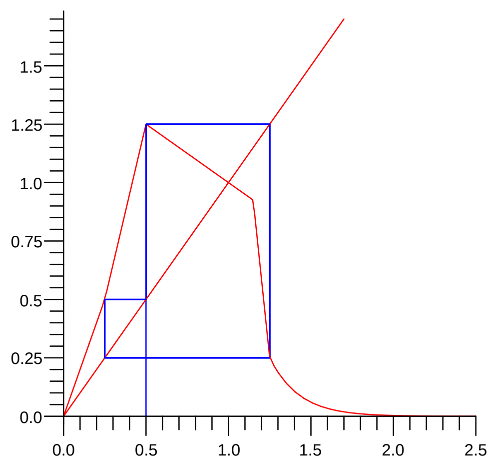

FIGURE 24. Cobweb plot for $x_0 = .5$. When the initial value is $x_0 = .5$, the function has a period three cycle, which can be through the three points where the cobweb meets the graph of $h(x)$.

We can look for that location of other cycles by plotting $y = x$ and $y = h(h(h(\ldots h(x)\ldots)))$ where the number of h's is equal to the cycle that was are looking for. For example, if we wanted to find a cycle two, we would plot $y = x$ and $y = h(h(x))$. This plot is in Figure 25. In Figure 26, we can see the location of the cycle of length 5.

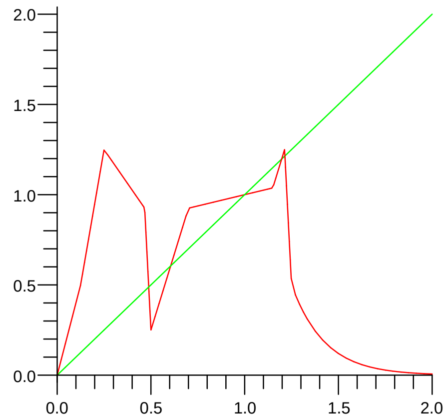

FIGURE 25. Plot of $y = h(h(x))$. There are intersection points besides 0 and 1. Therefore, there is a period 2 cycle.

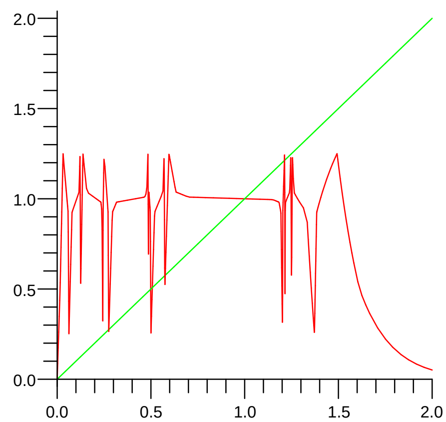

FIGURE 26. Plot of $y = h(h(h(h(h(x)))))$. Since there are points where the graph crosses the line $f(x) = x$, we can conclude that there are cycles with period 5.

7. FRACTALS

Another way of visualizing how models will behave is through the use of fractals. These plots are made by varying initial values along the x-axis and altering an imaginary component on the y. Colors are assigned according to the time it takes to diverge. Dark red means it never diverges. By plotting in this manner, when a convergent region decreases in size with respect to a changing parameter, we will be able to see it. This allows us to further see when a possible cycle might arise. (The figures for this section are located in the Appendix, due to technical difficulties. They are best viewed in full color.)

First we examine the fractals for the quartic, $f(x) = x(x-3/2)(-2x-(x-1)-a(x-1)^2$. In

Figure 27 we see that when $a = 1$, there is a large region that converges, i.e. a large red region between 0 and 2.5 on the real axis. When $a = 3$, however, the convergent region starts to pinch in on the left hand side of the real axis, as we can see in Figure 28. This tells us that we are possibly approaching an a value that is not stable. As we continue to increase a we see that this pinch point gets closer to the real axis. At $a = 4.705$ we get a series of touch points, as we can see in Figure 29, which tells us that the model has developed instabilities at these initial values. As we further increase a, we obtain nonconvergent regions within the region of convergence, as in Figure 30. This model corresponds with the unstable period four polynomial that we discussed earlier.

Next, we examine the fractals for the exponential function $f(x) = e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$. We start with $a = 1$. In Figure 31 and Figure 32, we can see that there is a region of convergence all along the real axis. As a decreses to -2, pictured in Figure 33 and Figure 34, we can see that there are numerous touch points along the real axis. When $a = -3$, as in Figure 35 and Figure 36, we see convergent regions seperated by nonconvergetn regions, implying that this model is not globally stable.

Finally, we look at another standard population model, $f(x) = x*(1+a(1-x))$ [10] [18]. When $a = 1$, you can see a large convergent region, as in Figure 37. This is expected since we know that this model is globally stable for $0 < a < 2$. When $a = 2.1$ we can see touch points, as we see in Figure 38, which is in agreeance with the fact that we know this a value does not result in a globally stable model. For $a = 2.5$ and $a = 3$, which are in Figure 39 and 40, the convergent region gets closer to the real axis and there are more touch points, illustrating the fact that the model becomes less and less stable as we increase a.

Overall, fractals gives a way to see where instabilities will occur before they acually occur in a real model.

8. Conclusion

Throughout the course of this paper, we studied the affects of alterations to some of the seven standard population models where local implied global stability. Occasionally, this resulted in a population model that preserved local implying global, while other times modifications resulted in a model no longer being globally stable. For example, general polynomials of degree three or less had the property of local implying global stability. This was shown through enveloping by linear fractionals. However, local did not imply global for the quartic model.

For the exponential function with polynomial exponent, local implied global for all models with polynomial exponents of degree two or less. However, the application of Theorem A, Theorem B, and doubly positive conditions all failed to show this. Theorem S also proved impractical to use due to the complexity of the Schwaritzan Derivative. But, by graphing and the application of Sarkovskii's Theorem, we can see that local stability does imply global stability, despite the failures at showing so analytically.

The logarithimic function, with powers of $ln(x)$ added, failed to be a population model for all functions of the form $x(1-r\ln x+\alpha(\ln x)^2)$. Different modifications to this model may yield different results, but our method of alteration did not preserve local stability implying global stability.

We created a simple piecewise function with an unstable period three cycle. This shows that even "simple" population models can have complex behavior (since the existence of a period three cycle implies the existence of cycles of all other integer lengths).

In addition to making alterations and studying these functions analytically, we created webplots, sequence plots, plots of iterated functions, and fractals to visualize how these functions behaved. It was through these that we could see whether a model was globally stable and if not where a cycle should appear. These tools and visual tactics for analyzing models are of note due to their simple nature. Since biologists forming the major models were able to "see" when a model was globally stable if locally stable, this fusion of mathematics and figures is exciting. It allows for a visual way to check a model in a manner that is founded in rigorous mathematics.

We also began work on, though drew no major conclusions on, the quotient of polynomials, examining when that quotient resulted in a population model, a locally stable population model, and a globally stable population. Further research could be done on this, as well as working more in depth with the fractals, and determining what the alterations that preserve the property of local implying global stability have in common. Finally, it might be worth trying to manipulate the Schwarzian for the exponential function, with the third degree polynomial as its exponent, to prove analytically that functions of this form have local stability imply global stability only when the exponent is of degree three or less.

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9. Appendix: Maple Code for Cobwebs, Sequence Plots and Fractals

9.1. Cobwebs.

```
> f:=x->x*exp(-1.9*(x-1)+(7.6-8.788898)*(x-1)^3
> print();
>                            a := 0.7
>                          pop0 := 0.3
> POP := proc(n::nonnegint)  ...  end;; POPS := proc(n::nonnegint)  ...  end;;
>                s := seq(POPS(n), n = 0 .. 500)
>               L := [[s[1], 0], [s[1], f(s[1])]]
> for i to 500 do L1 := [[f(s[i]), f(s[i])], [f(s[i]), f(s[i + 1])]];
>   L := [op(L), op(L1)];
>   od;
>              with(plots);
>                p1 := plot(f, 0 .. 2, y = 0 .. 2);
>                p2 := plot(d, 0 .. 2, y = 0 .. 2);
>                p3 := plot(L);
>                display(p1, p2, p3);
```

9.2. Sequence Plots.

```
> restart;
>                     /
>   f := x -> piecewise|0 <= x and x <= 0.25, 2 x, 0.25 < x and x <= 0.5,
>                     \
>                                     1
>   3 x - 0.25, 0.5 < x and x <= 1.15, - - x + 1.5, 1.15 < x and x <= 1.25,
>                                     2
>                                     /(-1) ln(0.2) (1 - x)\\
>   (-1) 6.75 x + 8.6875, 1.25 < x, x exp|--------------------||;
>                                     \        0.25        //
> print();
>                           pop0 := a
> L := [];
>                       1
>   for i to 400 do a := --- i;
>                      200
>     POP := proc(n::nonnegint)  ...  end;;
>     POPS := proc(n::nonnegint)  ...  end;;
>     s := seq(POPS(n), n = 50 .. 75);
>     for j to 25 do L1 := [[a, s[j]]]; L := [op(L), op(L1)]; od;
>     od;
> print();
>                     plot(L, style = point)
```

9.3. **Oprea's Fractals [14].**

```
> restart; with(plots):
 Julia sets

 written by John Oprea, oprea@math.csuohio.edu

> julia := proc(c,x, y)
>    local z, m;
>    z := evalf(x+y*I);
>    for m from 0 to 30 while abs(z) < 3 do
>       z := z^2 + c
>    od;
>    m
> end:

> J := proc(d)
>    global phonyvar;
>    phonyvar := d;
>    (x, y) -> julia(phonyvar, x, y)
> end:

> plot3d(0, -2 .. 2, -1.3 ..1.3, style=patchnogrid,
>    orientation=[-90,0], grid=[250, 250],
>    scaling=constrained, color=J(-1.25));
```

10. APPENDIX:FRACTAL PICTURES

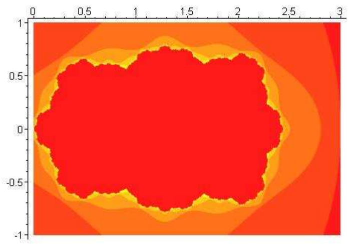

FIGURE 27. Fractal for $f(x) = x(x-3/2)(-2x-(x-1)-a(x-1)^2$ where $a=1$.

GEORGE WASHINGTON UNIVERSITY
E-mail address: kwalsh3@gmail.com

OREGON STATE UNIVERSITY
E-mail address: wherryj@onid.orst.edu

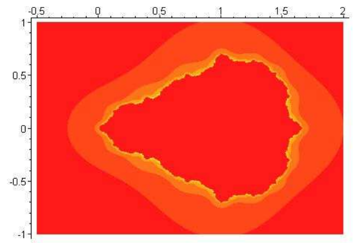

FIGURE 28. Fractal for $f(x) = x(x-3/2)(-2x-(x-1)-a(x-1)^2$ where $a = 3$.

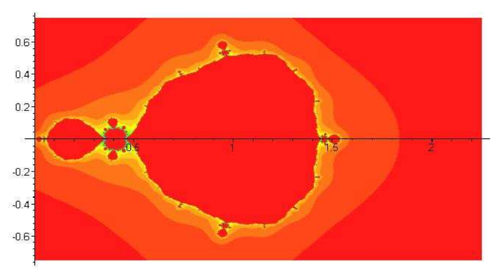

FIGURE 29. Fractal for $f(x) = x(x-3/2)(-2x-(x-1)-a(x-1)^2$ where $a = 4.705$.

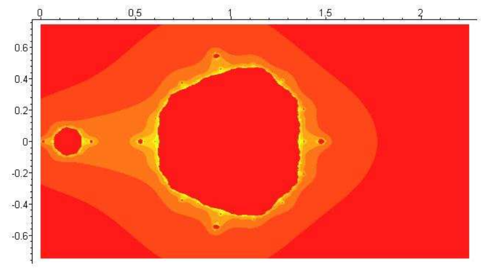

FIGURE 30. Fractal for $f(x) = x(x-3/2)(-2x-(x-1)-a(x-1)^2$ where $a = 6$.

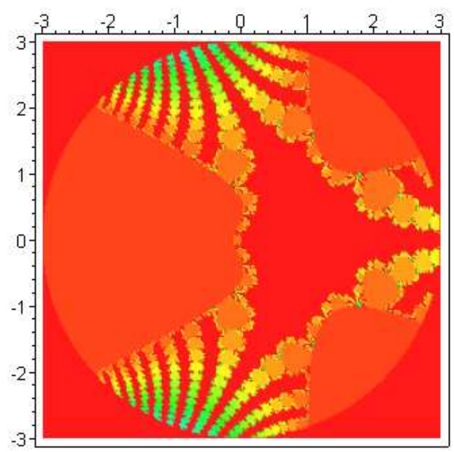

FIGURE 31. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1)-a(x-1)^3$ where $a = -1$.

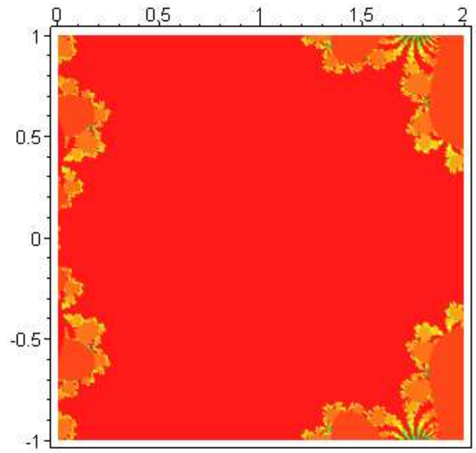

FIGURE 32. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$ where $a = -1$, zoomed in.

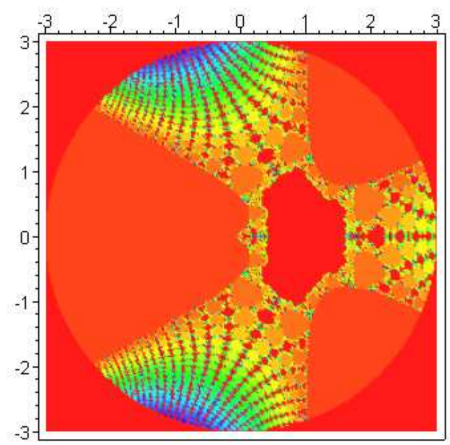

FIGURE 33. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$ where $a = -2$.

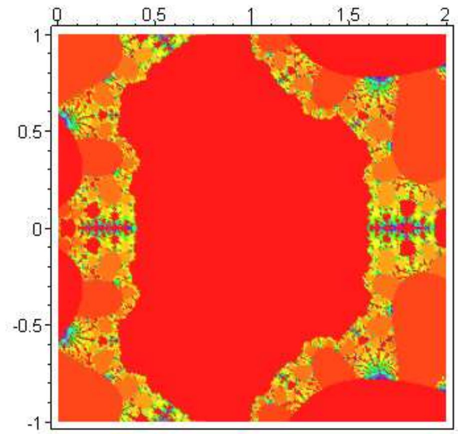

FIGURE 34. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$ where $a = -2$, zoomed in.

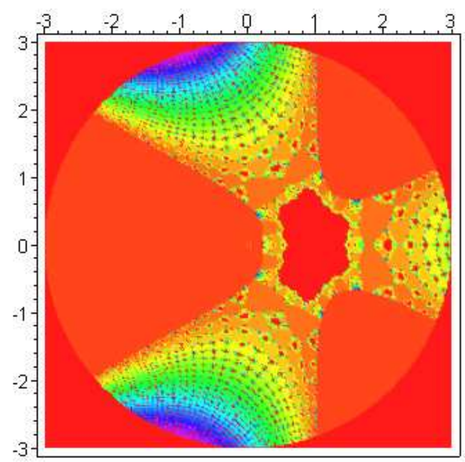

FIGURE 35. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$ where $a = -3$.

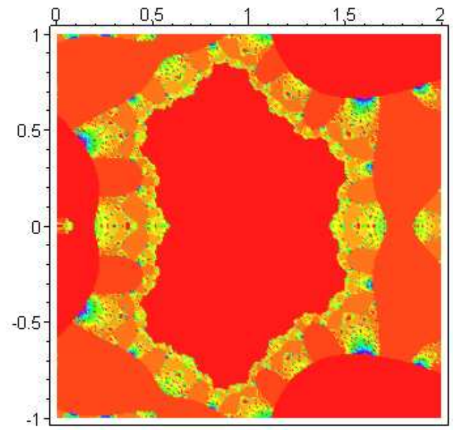

FIGURE 36. Fractal for $e^{-q(x)}$ where $q(x) = 1.9(x-1) - a(x-1)^3$ where $a = -3$, zoomed in.

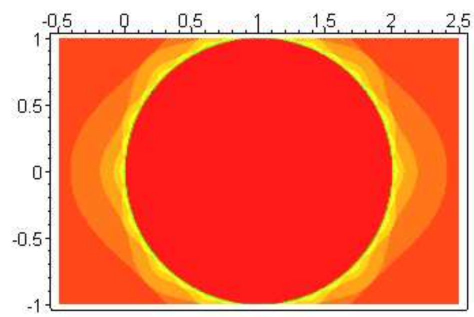

FIGURE 37. Fractal for $f(x) = x * (1 + a(1-x))$ where $a = 1$.

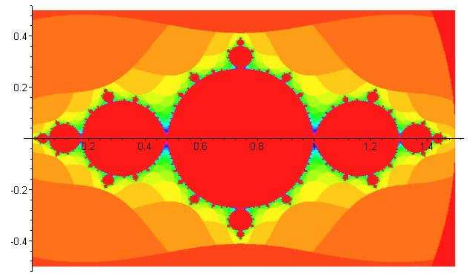

FIGURE 38. Fractal for $f(x) = x * (1 + a(1 - x))$ where $a = 2.1$.

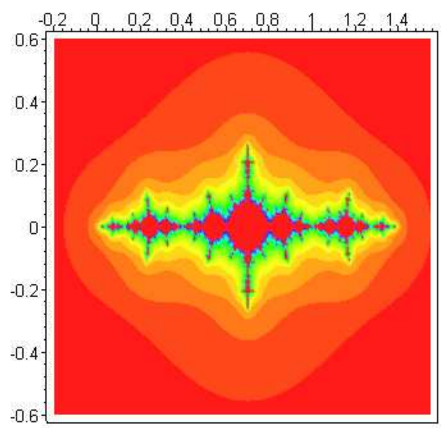

FIGURE 39. Fractal for $f(x) = x * (1 + a(1 - x))$ where $a = 2.5$.

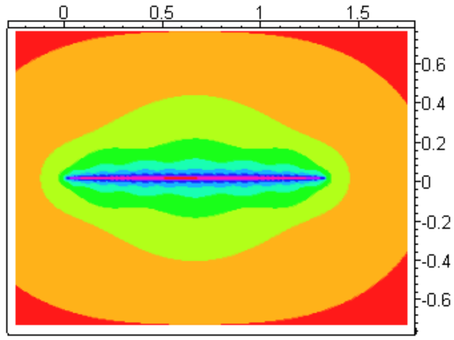

FIGURE 40. Fractal for $f(x) = x * (1 + a(1 - x))$ where $a = 3$.