

NON-CONJUGATE, ROOK EQUIVALENT t -CORES

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ABSTRACT. Consider a partition of a natural number n . The partition is called a t -core if each of the hook numbers (one more than the number of squares to the right and below a certain node of n) from its Ferrers board is not divisible by t . [HOS98] conjectured in 1998 that if $t \geq 5$, then there exists a constant N_t such that for every positive integer $n \geq N_t$, there exist two distinct rook equivalent t -cores of n which are not conjugate. In 2003, Anderson proved [And04] that this conjecture is true for $t \geq 12$ with $N_t = 4$. The goal of this 2009 research was to investigate the situation when $5 \leq t \leq 11$. What follows is a collection of lemmas, corollaries, and conjectures that resulted from this research.

1. INTRODUCTION

For a positive integer n , a *partition* P of n is a nonincreasing sequence of positive integers that sum to n . We define the *size* of P to be n , when P is a partition of n , denoted $|P|$. We sometimes write $P = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ for a partition of a positive integer n . In addition, we often represent the partition P by $P = \lambda_1 + \lambda_2 + \dots + \lambda_s$.

Example 1.1. *The following are the partitions of $n = 4$:*

$$\begin{aligned} &4 \\ &3 + 1 \\ &2 + 2 \\ &2 + 1 + 1 \\ &1 + 1 + 1 + 1 \end{aligned}$$

The *Ferrers-Young diagram* of a partition $\Lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ is a collection of n nodes in s rows with λ_i nodes in row i . Thus, rows of dots that are nonincreasing from top to bottom form these diagrams, as illustrated below:

$$\begin{array}{ccccccc} \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \lambda_1 \text{ nodes} \\ \bullet & \bullet & \dots & \bullet & \bullet & & \lambda_2 \text{ nodes} \\ \vdots & & & & & & \\ \bullet & \bullet & \dots & \bullet & & & \lambda_s \text{ nodes} \end{array}$$

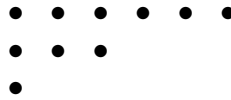
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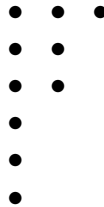
For a partition P , we define $P' = \lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_{s'} > 0$ where $s' = \lambda_1$ to be the *conjugate* of P . The conjugate partition P' can be obtained by reading columns instead of rows in a Ferrers-Young diagram. In other words, the Ferrers-Young diagram of P' is made by reflecting the Ferrers-Young diagram of P over the diagonal passing through the nodes in positions (i, i) . Note that P' is also a partition of n .

Example 1.2. For the partition $P = 6 + 3 + 1$ of $n = 10$, the conjugate is $P' = 3 + 2 + 2 + 1 + 1 + 1$.

P :



P' :

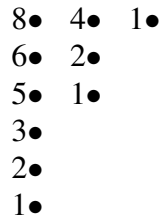


1.1. Hook numbers and t -core partitions. For P , the node in position (i, j) can be assigned a *hook number* $h(i, j)$, which is defined as one more than the number of nodes directly to the right or directly below the node itself. In addition, if we let λ'_r be the number of nodes in column r , then $h(i, j)$ is defined as

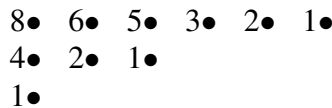
$$h(i, j) = 1 + (\lambda_i - i) + (\lambda'_j - j).$$

Example 1.3. Let $P = 6 + 3 + 1$ and $P' = 3 + 2 + 2 + 1 + 1 + 1$ be the partitions of $n = 10$ from above, and let $Q = 5 + 5$ be another partition of $n = 10$. Then the Ferrers-Young diagrams of P , P' , and Q and the hook numbers of each of the nodes are shown as follows:

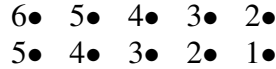
P :



P' :



Q :



Definition 1.4. For a positive integer t , a partition is t -core if each of the hook numbers from its Ferrers-Young diagram is not divisible by t .

In Example 1.3, P , P' , and Q are all 7-core. Also, P and P' both have the same sets of hook numbers. This is true for all pairs of conjugate partitions Y and Y' . Therefore, a partition will be t -core if and only if its conjugate is t -core.

1.2. **Rook equivalence.** A Ferrers board is a subset of the squares of an $N \times N$ chessboard with the property that the rows of the subset are nonincreasing in length.

Example 1.5. Let $P = 6 + 3 + 1$ be a partition of $n = 10$. Then the Ferrers board B_P of P is represented as follows:

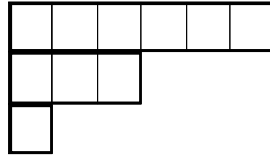


FIGURE 1. The Ferrers board diagram of the partition $P = 6 + 3 + 1$ of $n = 10$.

As in chess, rooks can be placed on the squares of a Ferrers board. The legal placement of k rooks on a board (with at most one rook on any square) is one such that no more than one rook is in any row or column. If we let B be a Ferrers board, then we define $r_k(B)$ be the number of legal placements of k rooks on B .

Definition 1.6. Ferrers boards B_1 and B_2 are called rook equivalent if and only if $r_k(B_1) = r_k(B_2)$ for all $k \in \mathbb{Z}$. In addition, we say two partitions are rook equivalent when their respective Ferrers boards are rook equivalent.

In placing one rook, we note that $r_1(B)$ is equal to the number of squares of B . Thus, two rook equivalent partitions must be the same size. Also, it can be easily seen that a partition is always rook equivalent to its conjugate.

For a given partition $\Lambda = \lambda_1 + \lambda_2 + \dots + \lambda_k$ of a number n , we can extend Λ so that it has n parts, as shown below:

$$\Lambda = \lambda_1 + \lambda_2 + \dots + \lambda_k + \underbrace{0 + 0 + \dots + 0}_{n-k \text{ times}}.$$

Definition 1.7. We can associate to a given partition $B = y_1 + y_2 + y_3 + \cdots + y_k$ a multiset J_B . First, we extend B so it has n parts and then define

$$J_B = \{y_1 + 1, y_2 + 2, y_3 + 3, \dots, y_n + n\}.$$

Example 1.8. Let $n = 8$ and let

$$B = 4 + 3 + 1 + 0 + 0 + 0 + 0 + 0.$$

Then

$$J_B = \{4 + 1, 3 + 2, 1 + 3, 0 + 4, 0 + 5, 0 + 6, 0 + 7, 0 + 8\} = \{5, 5, 4, 4, 5, 6, 7, 8\} = \{4, 4, 5, 5, 5, 6, 7, 8\}.$$

Theorem 1.9. ([And04] pg. 225) Two Ferrers boards B_1 and B_2 are rook equivalent if and only if $J_{B_1} = J_{B_2}$.

The first paper proving significant results about t -core partitions for specific t values was [HOS98], which was the also first to start looking at non-conjugate, rook equivalent t -cores. They were able to show that $\forall n \in \mathbb{Z}^+$ and for $t = 2, 3, 4$, t -cores of size n will only be rook equivalent if they are conjugates of each other. They then produced the following conjecture.

Conjecture 1.10. If $t \geq 5$, then there exists a constant N_t with the property that if $n \geq N_t$, then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

Several years later, [And04] set out to prove the conjecture presented by [HOS98]. She made significant progress, proving the conjecture for $N_t = 4$ and $t \geq 12$. In other words, she was able to prove that for $n \geq 4$ and $t \geq 12$, there exist distinct, non-conjugate, rook equivalent t -cores of size n . She then made a conjecture about the remaining values of t , which is shown below.

Conjecture 1.11. If $t = 6, 7, 8, 9, 10, 11$ and $n \geq 4$, then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

This brings us to the main conjecture that we attempted to prove.

Conjecture 1.12. If $t = 6, 7, 8, 9, 10, 11$ and $n \geq 4$, then there exist two distinct rook equivalent t -cores of size n which are not conjugates. If $t = 5$ and $n \geq N_t$, then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

2. BACKGROUND RESEARCH

In this section, we familiarize ourselves with partition theory presented in [And98]. Next, we go through the methods used by [HOS98] and [And04].

For a partition Λ of n and for all $t \geq 2$, there exists a t -abacus associated to P . This t -abacus consists of w beads on t columns, which are labeled 0 through $t - 1$ and an infinite number of rows, labeled $1, 2, \dots$. Each of the w beads correspond to each part of Λ . In addition, we define the *bead number* b_i by the following:

$$b_i = \lambda_i - i + w.$$

Note that since the λ_i are nonincreasing, the b_i are strictly decreasing. For each b_i , there exists a unique pair of integers (r_i, c_i) where $r_i > 0$ and $0 \leq c_i \leq t - 1$ such that we get the following:

$$b_i = t(r_i - 1) + c_i.$$

Next, each b_i is placed in the position (r_i, c_i) on the t -abacus, where c_i and r_i are the column and row, respectively, where b_i is placed.

Example 2.1. To obtain the 5-abacus for the partition $P = 4 + 2 + 1 + 1$, we first compute the bead number and positions as done below.

$$\begin{array}{lll} b_1 = 4 - 1 + 4 = 7 & b_1 = 7 = 5(2 - 1) + 2 & b_1 = (2, 2) \\ b_2 = 2 - 2 + 4 = 4 & b_2 = 4 = 5(1 - 1) + 4 & b_2 = (1, 4) \\ b_3 = 1 - 3 + 4 = 2 & b_3 = 2 = 5(1 - 1) + 2 & b_3 = (1, 2) \\ b_4 = 1 - 4 + 4 = 1 & b_4 = 1 = 5(1 - 1) + 1 & b_4 = (1, 1) \end{array}$$

We thus obtain the following 5-abacus for P :

$$\begin{array}{cccccc} & 0 & 1 & 2 & 3 & 4 \\ 1 & & b_4 & b_3 & & b_2 \\ 2 & & & b_1 & & \\ 3 & & & & & \end{array}$$

The following theorem of [KR46] classifies t -cores in terms of their corresponding t -abacus.

Theorem 2.2. Let A be a t -abacus of a partition P , and let n_i denote the number of beads in column i . The P is t -core if and only if for every $0 \leq i \leq t - 1$, the n_i beads in column i are the beads in positions

$$(1, i), (2, i), \dots, (n_i, i).$$

Put another way, in any nonempty column of the t -abacus A , the top bead must be in row 1, and any following beads must be placed in the column with no gaps in between the beads. Thus, by looking to the previous example, we see that $P = 4 + 2 + 1 + 1$ is a 5-core. We can use a t -tuple, $(n_0, n_1, \dots, n_{t-1})$, of natural numbers to represent a t -abacus of a t -core. Each n_i of the t -tuple is determined by counting the number of beads in column i of the associated t -abacus. In fact, every t -abacus of a t -core can be represented as one of these t -tuples.

Example 2.3. In Example 2.1, the t -abacus of $P = 4 + 2 + 1 + 1$ has 0 beads in column 0, 1 bead in column 1, 2 beads in column 2, 0 beads in column 3, and 1 bead in column 4. Thus, a t -tuple of the t -abacus of P is

$$(0, 1, 2, 0, 1).$$

Recall that that in [And04], the following theorem was proved.

Theorem 2.4. ([And04] pg. 223)

If $t \geq 12$ and $n \geq 4$, then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

2.1. **The $t = 12$ and 13 case.** In this section, we outline the proof in [And04] of Theorem 2.4 for the cases $t = 12, 13$. She used the t -abacus method to find t -tuple representations, just like the methods used in [HOS98]. Also, she developed several results about t -tuples to help her obtain her results.

Theorem 2.5. ([And04] pg. 225)

Let $t > n \geq 4$ be integers. Then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

To prove Theorem 2.5, [And04] looked at the multisets of partitions. Next, she used the t -abacus to develop partitions which will always be rook equivalent, as shown in the following theorem.

Theorem 2.6. ([And04] pg. 227)

For ≥ 4 and $0 \leq k \leq t - 4$, the t -tuples of natural numbers

$$\begin{aligned} &(n_0, \dots, n, n+1, n+1, n, n_{k+4}, \dots, n_{t-1}) \\ &(n_0, \dots, n+1, n, n, n+1, n_{k+4}, \dots, n_{t-1}) \end{aligned}$$

represent a pair of distinct, rook equivalent t -cores.

Corollary 2.7. ([And04] pg. 227) *Let $t \geq 6$ be an integer and n be a positive integer. If there exist integers x_1, x_2, \dots, x_{t-6} such that*

$$\sum_{i=1}^{t-6} x_i = 0$$

and

$$\sum_{i=1}^{t-6} \left(\frac{t}{2} \cdot x_i^2 + i \cdot x_i \right) + C = n \quad \text{for } C = 4 \text{ or } 5,$$

then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

Using these and the work of Granville et al. and Kiming, Anderson was able to prove cases for $t \geq 14$. However, we focus on her proof of the conjecture of Haglund, Ono, and Sze for $t = 13$ and $t = 12$. Below we will give her proof of these cases since we make use of it in our research.

Lemma 2.8. ([And04] pg. 232)

Let $t \geq 4$ be an integer and x_1, x_2, \dots, x_{t-4} be integers satisfying

$$\sum_{i=1}^{t-4} x_i = 0.$$

Then the t -tuples of integers

$$(1, 0, x_1, \dots, x_{t-4}, -1, 0)$$

and

$$(0, 1, x_1, \dots, x_{t-4}, 0, -1)$$

represent rook equivalent t -cores of size

$$(1) \quad \sum_{i=1}^{t-4} \left(\frac{t}{2} x_i^2 + i x_i \right) + 2.$$

Lemma 2.8 is very useful for finding rook equivalent t -cores. However, trivially, the conjugate P' of a t -core partition P is a rook equivalent t -core of P . Thus, Proposition 2.9 is important for finding non-conjugate, rook equivalent t -cores.

Proposition 2.9. ([And04] pg. 233)

Let t be a positive integer and x_0, x_1, \dots, x_{t-1} be integers such that

$$\sum_{i=0}^{t-1} x_i = 0.$$

The conjugate of the partition represented by the t -tuple

$$(x_0, x_1, \dots, x_{t-1})$$

is represented by the t -tuple

$$(-x_{t-1}, -x_{t-2}, \dots, -x_0).$$

Theorem 2.10. ([And04] pg. 233-235)

Let $t = 12$ or 13 and $n \geq t^3 + 4t + 2$ be an integer. Then there exist two distinct rook equivalent t -cores of size n which are not conjugates.

Proof. Let $n \geq t^3 + 4t + 2$. By Lemma 2.8 and Proposition 2.9, we only need to find integers x_1, x_2, \dots, x_{t-4} such that

$$\sum_{i=1}^{t-4} x_i = 0,$$

$$\sum_{i=1}^{t-4} \left(\frac{t}{2} \cdot x_i^2 + i \cdot x_i \right) + 2 = n,$$

and

$$(1, 0, x_1, \dots, x_{t-4}, -1, 0) \neq (1, 0, -x_{t-4}, \dots, -x_1, -1, 0).$$

Let $N = n - 2$. Then $N \geq t^3 + 4t$ (since $n \geq t^3 + 4t + 2$). By [Kim96], we can write $N = t \cdot m + 2^k \cdot r$ where m, r are odd, $k = 0$ or 1 , and $\|2^k \cdot r\| \leq 2t$. If $t = 12$ and $N \not\equiv 4 \pmod{8}$, then we can write $N = t \cdot m + 2^k \cdot r$ where m, r , are odd, $k = 0, 1$, or 2 , and $\|2^k \cdot r\| \leq t$. If $t = 12$ and $N \equiv 4 \pmod{8}$, then we can write $N = 4(t \cdot m) + 4r$ where m, r , are odd and $\|r\| \leq t$. In each case, since $N \geq t^3 + 4t$, $4m \geq r^2$.

In each case, since m and r are odd and $4m \geq r^2$, $4m - r^2 \equiv 3 \pmod{8}$ and is a positive integer. We see this since m and r are odd, we can write $m = 2a + 1$ and $r = 2b + 1$ for some $a, b \in \mathbb{Z}$.

So $4m = 4(2a + 1) = 8a + 4 \equiv 4 \pmod{8}$, and $r^2 = (2b + 1)^2 = 4b^2 + 4b + 1$. Then $4m - r^2 = 8a + 4 - 4b^2 - 4b - 1 = 8a - 4(b^2 + b) + 3 \equiv 3 \pmod{8}$ since $b^2 + b$ will be divisible by 2 for all $b \in \mathbb{Z}$.

Let $j, k \in \mathbb{Z}$. Then an integer n is a sum of three integral squares if and only if n is not of the form $4j \cdot (8k + 7)$. This is called Gauss' Three Squares Theorem.

If d is odd, then d can be written in the form $d = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Z}$. We see this because clearly, any integer of the form $4j \cdot (8k + 7)$ must be even. Thus, by Gauss' Three Squares Theorem, any odd number cannot be written in the form $4j \cdot (8k + 7)$ and therefore can be expressed as a sum of three integral squares.

By the above corollary, since $4m - r^2 \equiv 3 \pmod{8}$, then $4m - r^2$ is odd. Therefore, there exist integers a, b, c such that

$$4m - r^2 = a^2 + b^2 + c^2.$$

Since $4m - r^2 \equiv 3 \pmod{8}$, then a, b, c must all be odd. By replacing a, b, c by their negatives if necessary, we may assume that $a \equiv b \equiv c \equiv r \pmod{4}$.

Define the integers $\alpha, \beta, \gamma, \delta$ as follows:

$$\alpha = \frac{r + a + b + c}{4}, \quad \beta = \frac{r - a - b + c}{4}, \quad \gamma = \frac{r - a + b - c}{4}, \quad \delta = \frac{r + a - b - c}{4}.$$

Suppose $t = 13$. Let

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \begin{cases} (-\alpha, \alpha, -\beta, \beta, -\gamma, \gamma, -\delta, \delta, 0) & \text{if } k = 0 \\ (-\alpha, -\beta, \alpha, \beta, -\gamma, -\delta, \gamma, \delta, 0) & \text{if } k = 1. \end{cases}$$

Then

$$\sum_{i=1}^{t-4} x_i = -\alpha + \alpha + -\beta + \beta + -\gamma + \gamma + -\delta + \delta + 0 = 0.$$

Also,

$$\sum_{i=1}^{t-4} \left(\frac{t}{2} \cdot \frac{2}{i} + i \cdot x_i \right) + 2 = \sum_{i=1}^{t-4} \left(\frac{t}{2} \cdot \frac{2}{i} \right) + \sum_{i=1}^{t-4} (i \cdot x_i) + 2.$$

We look at each sum separately for $k=0$. We have:

$$\begin{aligned}
\sum_{i=1}^{t-4} \left(\frac{t}{2} \cdot \frac{2}{i}\right) &= \frac{t}{2} ((-\alpha)^2 + (\alpha)^2 + (-\beta)^2 + (\beta)^2 + (-\gamma)^2 + (\gamma)^2 + (-\delta)^2 + (\delta)^2 + (0)^2) \\
&= \frac{t}{2} (2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\delta^2) \\
&= t(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \\
&= t\left(\left(\frac{r+a+b+c}{4}\right)^2 + \left(\frac{r-a-b+c}{4}\right)^2 + \left(\frac{r-a+b-c}{4}\right)^2 + \left(\frac{r+a-b-c}{4}\right)^2\right) \\
&= t\left(\frac{4r^2 + 4a^2 + 4b^2 + 4c^2}{16}\right) \\
&= t\left(\frac{r^2 + (a^2 + b^2 + c^2)}{4}\right) \\
&= t\left(\frac{r^2 + (4m - r^2)}{4}\right) \\
&= t\left(\frac{4m}{4}\right) \\
&= tm
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{t-4} (i \cdot x_i) &= 1(-\alpha) + 2(\alpha) + 3(-\beta) + 4(\beta) + 5(-\gamma) + 6(\gamma) + 7(-\delta) + 8(\delta) + 9(0) \\
&= \alpha + \beta + \gamma + \delta \\
&= \frac{r+a+b+c}{4} + \left(\frac{r-a-b+c}{4}\right) + \left(\frac{r-a+b-c}{4}\right) + \left(\frac{r+a-b-c}{4}\right) \\
&= \frac{4r+0+0+0}{4} \\
&= r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{t-4} \left(\frac{t}{2} \cdot \frac{2}{i} + i \cdot x_i\right) + 2 &= tm + r + 2 \\
&= tm + 2^{(0)} + 2 \\
&= tm + 2^k + 2 \quad (\text{since } k = 0) \\
&= N + 2 \\
&= n.
\end{aligned}$$

This also holds for $k=1$.

Since r is odd and $a \equiv b \equiv r \pmod{4}$, $r \not\equiv -a$ or $-b$. Hence, $\alpha \neq -\delta$ or $-\gamma$. Therefore

$$(1, 0, x_1, \dots, x_{t-4}, -1, 0) \neq (1, 0, -x_{t-4}, \dots, -x_1, -1, 0).$$

Thus, there exist two distinct rook equivalent t -cores of size n which are not conjugates. Now suppose $t = 12$ and $N \not\equiv 4 \pmod{8}$. Let

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \begin{cases} (-\alpha, \alpha, -\beta, \beta, -\gamma, \gamma, -\delta, \delta) & \text{if } k = 0 \\ (-\alpha, -\beta, \alpha, \beta, -\gamma, -\delta, \gamma, \delta) & \text{if } k = 1 \\ (-\alpha, -\beta, -\gamma, -\delta, \alpha, \beta, \gamma, \delta) & \text{if } k = 2 \end{cases}.$$

Then

$$\sum_{i=1}^{t-4} x_i = 0$$

and

$$\sum_{i=1}^{t-4} \left(\frac{t}{2} x_i^2 + i x_i \right) + 2 = tm + 2^k r + 2 = n.$$

Since b, c are odd and $b \equiv c \pmod{4}$, $b \neq c$. Hence, $\alpha \neq \delta$. Then

$$(1, 0, x_1, \dots, x_{t-4}, -1, 0) \neq (1, 0, -x_{t-4}, \dots, -x_1, -1, 0).$$

Thus, there exist two distinct rook equivalent t -cores of size n which are not conjugates.

Finally, suppose $t = 12$ and $N \equiv 4 \pmod{8}$. Let

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-2\alpha, -2\beta, 2\alpha, 2\beta, -2\gamma, -2\delta, 2\gamma, 2\delta).$$

Then

$$\sum_{i=1}^{t-4} x_i = 0$$

and

$$\sum_{i=1}^{t-4} \left(\frac{t}{2} x_i^2 + i x_i \right) + 2 = 4tm + 4r + 2 = n.$$

Since b, c are odd and $b \equiv c \pmod{4}$, $b \neq -c$. Hence $\alpha \neq \delta$. Therefore

$$(1, 0, x_1, \dots, x_{t-4}, -1, 0) \neq (1, 0, -x_{t-4}, \dots, -x_1, -1, 0).$$

Thus there exist two distinct rook equivalent t -cores of size n which are not conjugates.

Using a computer search, it's possible to find natural numbers n_0, n_6, \dots, n_{t-1} such that one of the t -tuples

$$(n_0, n_0 + 1, n_0 + 1, n_0, n_0 + 1, n_0, n_6, \dots, n_{t-1})$$

or

$$(n_0, n_0 + 1, n_0, n_0 + 1, n_0 + 1, n_0, n_6, \dots, n_{t-1})$$

represents a partition of n for all integers n such that $t \leq n \leq t^3 + 4t + 2$. In this search, it is enough to let $0 \leq n_0, n_6, n_7 \leq t$ and $0 \leq n_8, \dots, n_{t-1} \leq 1$. This computer search and the above proofs imply that there exist distinct, non-conjugate, rook equivalent t -cores for $t = 12$ or 13 and $n \geq 4$. □

Unfortunately, Anderson did not have proofs for the cases where $t = 5, 6, 7, 8, 9, 10, 11$. However, she used a computer program to check for the presence of t -cores for $t = 5, 6, 7, 8, 9, 10, 11$ and small values of n (≤ 40). For $t = 6 \dots 11$, the evidence in [And04] suggests that these t values followed the trend of $t \geq 12$. However, for $t = 5$, there do not exist distinct, non-conjugate, rook

equivalent t -cores of size n for many small values of n . It may be the case, however, that there does exist some N_t such that distinct, non-conjugate, rook equivalent t -cores of size n exist for $n \geq N_t$.

With these results and conjectures as a background, we started our research.

3. RESULTS

The first thing we did was examine the proofs of $t = 12$ and $t = 13$ [And04] and try to emulate these for smaller values of t . We immediately ran into problems. First, we tried to use Lemma 2.8 and Proposition 2.9 to assemble 6-tuples that would be distinct, non-conjugate and rook equivalent for all integers n where $n \geq 4$. The following theorem is a result that we obtained while trying to assemble these 6-cores.

Theorem 3.1. *Let $t = 6$ and $n \geq t^3 + 4t + 2$. Then the t -tuple $(x_1, x_2) = (-\alpha, \alpha)$ satisfies $\sum_{i=1}^{t-4} x_i = 0$, but does not satisfy $\sum_{i=1}^{t-2} (\frac{t}{2} \cdot x_i^2 + i \cdot x_i) + 2 = n$. In fact, no value of n where $n \equiv 3 \pmod{5}$ or $n \equiv 4 \pmod{5}$ satisfies the latter sum.*

Proof. Let $t = 6$ and let $n \geq t^3 + 4t + 2$. Then

$$\begin{aligned} \sum_{i=1}^{t-2} (\frac{t}{2} \cdot x_i^2 + i \cdot x_i) + 2 &= \sum_{i=1}^2 (\frac{t}{2} \cdot x_i^2 + i \cdot x_i) + 2 \\ &= \frac{6}{2}x_1^2 + (1)x_1 + \frac{6}{2}x_2^2 + (2)x_2 + 2 \\ &= 3(-\alpha)^2 + (-\alpha) + 3(\alpha)^2 + 2(\alpha) \\ &= 6\alpha^2 + \alpha. \end{aligned}$$

Case 1: Let $\alpha \equiv 0 \pmod{5}$. Then

$$\begin{aligned} 6\alpha^2 + \alpha &\equiv 6(0)^2 + 0 \\ &\equiv 6(0) + 0 \\ &\equiv 0 + 0 \\ &\equiv 0 \pmod{5}. \end{aligned}$$

Case 2: Let $\alpha \equiv 1 \pmod{5}$. Then

$$\begin{aligned} 6\alpha^2 + \alpha &\equiv 6(1)^2 + 1 \\ &\equiv 6(1) + 1 \\ &\equiv 1 + 1 \\ &\equiv 2 \pmod{5}. \end{aligned}$$

Case 3: Let $\alpha \equiv 2 \pmod{5}$. Then

$$\begin{aligned} 6\alpha^2 + \alpha &\equiv 6(2)^2 + 2 \\ &\equiv 6(4) + 2 \\ &\equiv 4 + 2 \\ &\equiv 1 \pmod{5}. \end{aligned}$$

Case 4: Let $\alpha \equiv 3 \pmod{5}$. Then

$$\begin{aligned} 6\alpha^2 + \alpha &\equiv 6(3)^2 + 3 \\ &\equiv 6(4) + 3 \\ &\equiv 4 + 3 \\ &\equiv 2 \pmod{5}. \end{aligned}$$

Case 5: Let $\alpha \equiv 4 \pmod{5}$. Then

$$\begin{aligned} 6\alpha^2 + \alpha &\equiv 6(4)^2 + 4 \\ &\equiv 6(1) + 4 \\ &\equiv 1 + 4 \\ &\equiv 0 \pmod{5}. \end{aligned}$$

Thus, (x_1, x_2) cannot be used for n -values of $3 \pmod{5}$ or $4 \pmod{5}$. □

With this and further study, we determined that it is unlikely that the method used by Anderson for $t = 12, 13$ will work for smaller values of t . The reason for this is that it appears to be necessary to have $-\alpha, \alpha, -\beta, \beta, -\gamma, \gamma, -\delta, \delta$ in the t -tuples created to satisfy both sums for all values of $n \geq 4$. Our next idea was to use these same eight values in the t -tuple, but to try to modify Equation (1) so that we sum up to $t - 2$ instead of $t - 4$. If we could accomplish this, we would be able to possibly prove results for $t = 10, 11$ (since $10 - 2 = 8$ and $11 - 2 = 9$, we would be able to use the eight values that we need). However, when we tried to change this sum, we ran into many more problems. Equation 1 is derived from previous propositions and lemmas in [And04], and trying to change those to be able to sum to $t - 2$ proved to be a formidable task.

To explain what we did next, we first need to define a few terms. A partition P is said to be *self-conjugate* when $P' = P$. Let $sc(n)$ be the total number of self-conjugate partitions of size n , and let $nsc(n)$ be the total number of non-self-conjugate partitions of size n . Similarly, let $sc_t(n)$ be the total number of t -core partitions of size n which are self-conjugate, and let $nsc_t(n)$ be the total number of t -core partitions which are non-self-conjugate.

Definition 3.2. *The rook equivalence class of a partition P of size n is all the partitions of size n that are rook equivalent to P .*

In other words, the rook equivalence class of a partition P of size n is the subset of all the partitions of size n that have the same associated multiset as P . Let $a(n)$ be the total number of rook equivalence classes of partitions of size n , and let $a_t(n)$ be the total number of rook equivalence classes of t -core partitions of size n . Lastly, let's define $c_t(n)$ to be the total number of t -core partitions of size n . It is easy to see from the definition that

$$c_t(n) = sc_t(n) + nsc_t(n).$$

3.1. Generating functions. The next project we undertook was to actually compute partitions, their multisets, hook numbers, t -abaci, and to investigate $a_t(n)$, $sc_t(n)$, and $nsc_t(n)$ for small values of n . We drew out all the partitions of n for $n = 1, 2, \dots, 10$ and examined them to look for certain properties. We focused on $a_t(n)$, $sc_t(n)$, and $nsc_t(n)$ for $t = 5, 6, 7, 8, 9, 10, 11$. The reason we looked at this is explained briefly in [HOS98], but here we explain it in greater detail. Our project

is to look for distinct, non-conjugate, rook equivalent t -cores. In other words, we are looking for non-conjugate t -cores which belong to the same equivalence class. To do this we want to prove that for some n there are more non-conjugate t -cores of size n than there are equivalence classes. If we can do this, then we are guaranteed to have at least two non-conjugate t -cores in at least one equivalence class, and we have found what we were looking for. We know that $sc_t(n)$ is defined as the number of self-conjugate t -cores of n . Also, $\frac{1}{2}nsc_t(n)$ is the number of conjugate pairs of t -cores of n . Clearly, $\frac{1}{2}nsc_t(n) + sc_t(n)$ is the total number of conjugate pairs of t -cores. If we have more conjugate pairs of t -cores than we have equivalence classes of t -cores, then we are guaranteed to have non-conjugate, rook equivalent t -cores. In other words, if

$$\frac{1}{2}nsc_t(n) + sc_t(n) > a_t(n)$$

, then there exist distinct, non-conjugate, rook equivalent t -cores of size n .

Definition 3.3. If $A = \{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}$ is a sequence of real numbers, then

$$f_A(q) = \sum_{n=0}^{\infty} a_n q^n$$

is called the generating function for the sequence A .

The following equations are the beginnings of the generating functions for $sc_t(n)$, $nsc_t(n)$, and $a_t(n)$ for $t = 5, 6, 7, 8, 9, 10, 11$. We also refer to these as the q -series representations of $sc_t(n)$, $nsc_t(n)$, and $a_t(n)$.

The first few terms of the q -series representations for $sc_t(n)$ for $t = 5, \dots, 11$ are shown below.

$$\sum_{i=1}^{\infty} sc_5(n)q^n = 1 + q + q^3 + q^4 + q^7 + q^8 + q^9 + \dots$$

$$\sum_{i=1}^{\infty} sc_6(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \dots$$

$$\sum_{i=1}^{\infty} sc_7(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^8 + 2q^9 + q^{10} + \dots$$

$$\sum_{i=1}^{\infty} sc_8(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \dots$$

$$\sum_{i=1}^{\infty} sc_9(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + \dots$$

$$\sum_{i=1}^{\infty} sc_{10}(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \dots$$

$$\sum_{i=1}^{\infty} sc_t(11)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + \dots$$

Next, the first few terms of the q -series representations for $nsc_t(n)$ for $t = 5, \dots, 11$ are shown below.

$$\sum_{i=1}^{\infty} nsc_5(n)q^n = 2q^2 + 2q^3 + 4q^4 + 2q^5 + 6q^6 + 4q^7 + 6q^8 + 4q^9 + 12q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_6(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 4q^6 + 8q^7 + 10q^8 + 10q^9 + 10q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_7(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 8q^7 + 14q^8 + 14q^9 + 20q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_8(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 12q^8 + 20q^9 + 24q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_9(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 20q^8 + 20q^9 + 32q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_{10}(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 20q^8 + 28q^9 + 30q^{10} + \dots$$

$$\sum_{i=1}^{\infty} nsc_{11}(n)q^n = 2q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 20q^8 + 28q^9 + 40q^{10} + \dots$$

Finally, the first few terms of the q -series representations for $a_t(n)$ for $t = 5, \dots, 11$ are shown below.

$$\sum_{i=1}^{\infty} a_5(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + q^5 + 2q^6 + 3q^7 + 3q^8 + 3q^9 + 3q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_6(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 3q^6 + 4q^7 + 4q^8 + 5q^9 + 6q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_7(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 5q^6 + 3q^7 + 4q^8 + 6q^9 + 6q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_8(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 5q^6 + 5q^7 + 4q^8 + 6q^9 + 8q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_9(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 5q^6 + 5q^7 + 6q^8 + 6q^9 + 8q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_{10}(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 5q^6 + 5q^7 + 6q^8 + 8q^9 + 8q^{10} + \dots$$

$$\sum_{i=1}^{\infty} a_{11}(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 5q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + \dots$$

To be able to prove that non-conjugate, rook equivalent t -cores exist for $t \geq 6$ and for all integers $n \geq 4$, we need to show that $\frac{1}{2}nsc_t(n) + sc_t(n) > a_t(n)$ for all $n \geq 4$. Another way to prove this would be to show that

$$\sum_{i=1}^{\infty} \frac{1}{2}nsc_t(n)q^n + \sum_{i=1}^{\infty} sc_t(n)q^n - \sum_{i=1}^{\infty} a_t(n)q^n$$

has positive coefficients for all q^n (where $n = 4, 5, \dots$). For $n = 4, 5, \dots, 10$ and $t = 6, 7, \dots, 11$,

$$\sum_{i=1}^{10} \frac{1}{2}nsc_t(n)q^n + \sum_{i=1}^{10} sc_t(n)q^n - \sum_{i=1}^{10} a_t(n)q^n$$

has positive coefficients. This supports the conjecture in [And04]. However, for $t = 5$, q^6 , q^8 , and q^{10} did not have positive coefficients, which also matches the work in [And04].

Next, we worked on trying to develop infinite generating functions for $sc_t(n)$, $nsc_t(n)$, and $a_t(n)$. The following generating functions for the $sc_t(n)$ and $c_t(n)$ are found in [GKS90]. They are given below.

$$c_t(n) = \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n}$$

$$sc_t(n) = \begin{cases} \prod_{n=1}^{\infty} (1 + q^{2n-1})(1 - q^{2tn})^{\frac{t}{2}} & \text{if } t \text{ is even} \\ \prod_{n=1}^{\infty} \frac{(1 + q^{2n-1})(1 - q^{2tn})^{\frac{t-1}{2}}}{1 + (q^{2n-1})^t} & \text{if } t \text{ is odd} \end{cases}$$

We know that $c_t(n) = sc_t(n) + nsc_t(n)$, so

$$nsc_t(n) = c_t(n) - sc_t(n)$$

$$= \begin{cases} \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n} - \prod_{n=1}^{\infty} (1 + q^{2n-1})(1 - q^{2tn})^{\frac{t}{2}} & \text{if } t \text{ is even,} \\ \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{1 - q^n} - \prod_{n=1}^{\infty} \frac{(1 + q^{2n-1})(1 - q^{2tn})^{\frac{t-1}{2}}}{1 + (q^{2n-1})^t} & \text{if } t \text{ is odd.} \end{cases}$$

Unfortunately, we were unable to find a way of combining either of the two equations into one infinite product. In addition, we were unable to create or find a generating function for $a_t(n)$. This seems like a promising way to try to solve this problem, so further work in this area is recommended.

3.2. Modular forms. While not being able to find an infinite generating function for $a_t(n)$, we investigated modular forms. Basics about modular forms can be found in [Ono04]. To describe modular forms, we first must look at a few groups. The first is $GL_n(\mathbb{R})$, the group of $n \times n$ matrices with real coefficients and a nonzero determinant. $GL_n(\mathbb{R})$ is called the *general linear group* and is a multiplicative group. Next, we have $SL_n(\mathbb{R})$, which is also called the *special linear group*. It is the multiplicative group consisting of $n \times n$ matrices with real coefficients and determinant 1. Finally, we look at $SL_2(\mathbb{Z})$, which is the set of all 2×2 matrices with integer coefficients and determinant 1. We note that the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

form a basis for $SL_2(\mathbb{Z})$.

$SL_2(\mathbb{Z})$ acts on $H = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ (the upper half of the complex plane) by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Definition 3.4. A modular form is a holomorphic function $f : H \rightarrow \mathbb{C}$ such that

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k \cdot f(z)$$

$\forall z \in H$ and $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and that f is holomorphic at ∞ [Ono04].

The value k in the above equation is called the *weight* of f . As can be seen in the following example, k must be even.

Example 3.5. Let f be a modular form on $SL_2(\mathbb{Z})$. Let $R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\begin{aligned} f\left(\frac{-1 \cdot z + 0}{0 \cdot z + (-1)}\right) &= f(z) = (0 \cdot +(-1))^k (f(z)) \\ &= (-1)^k \cdot f(z) \end{aligned}$$

We define M_k to be the set of all holomorphic (differentiable in neighborhood of a point z_0 for all $z_0 \in \mathbb{C}$) modular forms of weight k on $SL_2(\mathbb{Z})$. It turns out that M_k forms a vector space over H .

Now let $\sigma_{k-1}(n)$ be the divisor function

$$\sigma_{k-1}(n) := \sum_{1 \leq d|n} d^{k-1}$$

where k is a positive integer and $d|n$ are the divisors of n .

Next, we define the Bernoulli numbers B_k as the coefficients of the series

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \dots$$

Definition 3.6. If $k \geq 2$ is even, then the weight k Eisenstein series $E_k(z)$ is given by

$$E_k(z) := 1 - \frac{2k}{B_k} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Proposition 3.7. If $k \geq 4$ is even, then $E_k(z) \in M_k$.

Both Definition 3.6 and Proposition 3.7 give us the following theorem, which is found in [Ono04].

Theorem 3.8. If $k \geq 4$ is even, then M_k is generated by monomials of the form

$$E_4(z)^a \cdot E_6(z)^b,$$

where $a, b > 0$ and $4a + 6b = k$.

In other words, the two Eisenstein series

$$E_4(z) = 1 + 240 \cdot \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(z) = 1 - 504 \cdot \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

generate all of the modular forms on $SL_2(\mathbb{Z})$. This means that $E_4(z)$ and $E_6(z)$ form a basis for the vector space generated by M_k .

We examined modular forms of low weight to determine if the q -series representations for $sc_t(n)$, $nsc_t(n)$, or $a_t(n)$ seemed to match these modular forms. Specifically, we took linear combinations basis vectors and tried to match them to the first few terms of the q -series representations of $sc_t(n)$, $nsc_t(n)$, and $a_t(n)$. Our findings are given in the following theorem.

Theorem 3.9. *The infinite sums $\sum_{n=0}^{\infty} a_t(n)q^n$, $\sum_{n=0}^{\infty} nsc_t(n)q^n$, and $\sum_{n=0}^{\infty} sc_t(n)q^n$ are not modular forms of weight ≤ 36 in $SL_2(\mathbb{Z})$ for $t = 5, 6, 7, 8, 9, 10$, or 11 .*

Proof. This result was determined by comparing the beginnings for the infinite sums for $a_t(n)$, $nsc_t(n)$, and $sc_t(n)$ for $t = 5, 6, 7, 8, 9, 10, 11$ and $n = 1, 2, \dots, 10$ and looking for linear combinations of E_4 and E_6 that match the beginnings of these infinite sums. For weight ≤ 36 , no linear combinations of the basis vectors matched the sums for $a_t(n)$, $nsc_t(n)$, and $sc_t(n)$ for $t = 5, 6, \dots, 11$ and $n = 1, 2, \dots, 10$. □

This doesn't mean that $sc_t(n)$, $nsc_t(n)$, and $a_t(n)$ cannot be represented by modular forms; rather, it means that they cannot be represented by low-weight modular forms on $SL_2(\mathbb{Z})$.

After working with modular forms, we took a new approach. We started looking at partitions and tried to find ways of constructing partitions which would be 6-cores. We thought it would be useful to try to see if 6-cores had a distinct shape that was independent of the n we were working with. Our goal was to find two different ways of building 6-core partitions with the property that the two partitions built would be guaranteed to be rook equivalent but not conjugate.

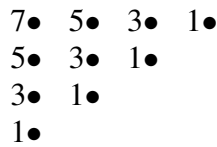
Conjecture 3.10. *A partition Λ of n of the form*

$$\Lambda = \lambda_1 = a \geq \lambda_2 = a - 1 \geq \lambda_3 = a - 2 \geq \lambda_4 = a - 3 \geq \dots \geq \lambda_a = a - (a - 1) = 1.$$

where n is a triangular number $(1, 3, 6, 10, 15, \dots)$ is a t -core for all even, positive t .

We will now refer to this as a *triangular partition*.

Example 3.11. *Let $n = 10$ and let $\Lambda = 4 + 3 + 2 + 1$. The Ferrers-Young diagram of Λ with its associated hook numbers is*



Conjecture 3.12. *A triangular partition of size n is not rook equivalent to any other partition of size n .*

Although we were not able to find infinite families of 6-cores which are rook equivalent, we were able to find two infinite families of partitions which are non-conjugate and rook equivalent which we illustrate in the following theorem.

Theorem 3.13. *Let P_1 be a partition of n with durfee square of size k , a part $B = b_1 \geq \dots \geq b_k$ with k rows and r columns right of the durfee square and a part $A = a_1 \geq \dots \geq a_k$ with r rows and k columns and let $r \leq k$. Let P_2 be a partition of n with durfee square k , the part B below the durfee square, and the part A to the right of the durfee square. Then P_1 and P_2 are rook equivalent.*

Proof. Consider the multisets of P_1 and P_2 . The multiset of P_2 is

$$\{k + a_1 + 1, k + a_2 + 2, k + a_3 + 3, \dots, k + a_r + r, k + (r + 1), k + (r + 2), \dots, \\ k + k, b_1 + (k + 1), b_2 + (k + 2), b_3 + (k + 3), \dots, b_k + (k + k)\}$$

and the multiset of P_1 is

$$\{k + b_1 + 1, k + b_2 + 2, \dots, k + b_k + k, a_1 + (k + 1), a_2 + (k + 2), \dots, \\ a_r + (k + r), 0 + (k + r + 1), 0 + (k + r + 2), \dots, 0 + (k + k)\}.$$

Since the order of the elements in the multisets does not matter, P_1 and P_2 have the same multisets and are therefore rook equivalent. \square

During the time we were trying to construct these different partitions, we started seeing some interesting attributes of the partitions that were working with. We were unable to prove the following things we saw, so they are listed below as conjectures.

Conjecture 3.14. *If a partition P has largest hook number h , then all divisors of h occur as hook numbers.*

4. CONCLUSION

We have presented our results from the 2009 OSU REU. The main conjecture that for $t \geq 5$, there exists a constant N_t such that for every positive integer $n \geq N_t$, there exist two distinct rook equivalent t -cores of n which are not conjugate remains unproved. During the course of our research, we have formulated many other conjectures that remain to be proved. As was stated in [And04] the use of modular forms may still be necessary to prove the main conjecture.

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