ON THE BEHAVIOR OF RATIOS OF SOLUTIONS TO NON-NEGATIVE DIFFERENCE EQUATIONS

MICHAEL ABRAM

ADVISOR: DR. PAUL CULL

ABSTRACT. Difference equations arise in a variety of applications. While closed form solutions are available for some of these equations, often one has to be satisfied with describing the asymptotic behavior of solutions. Fibonacci equations are so special that their solutions may have easy to check properties that hold for all values, not just asymptotically. (For example, Fibonacci-like equations have special rounding properties [Cull][Cu].) Here we look at ratios of solutions. The motivating example is a counting problem associated with the Zeckendorf representation of integers. In that counting problem, the ratio of solutions is monotone increasing [C][Ca][Cc]. We study both numerically and analytically the ratios of solutions to non-negative difference equations. We show that for certain kinds of equations these ratios must oscillate. We devise some necessary conditions for monotonicity. For some very special cases, we show monotone convergence. We investigate various strategies to prove monotone convergence and show why these approaches cannot work.

1. INTRODUCTION

Our goal is to fully understand the behavior of a ratio of any two solutions to a non-negative difference equation. This result would provide qualitative information to researchers who model problems in ecology, robotics, economics, and other fields [Cul][Cull]. We will show that determining the long-term behavior of such a ratio is fairly straightforward, but then the vagueness of "long-term" becomes troublesome. When attempting to extend this result to "short-term" behavior, we are no longer able to prove our result. We thus focus on fully explaining long-term term behavior and showing general characteristics about short-term behavior. In particular, we want to show under what conditions a ratio of solutions to non-negative difference equations is monotonic. We must make some assumptions in order that our process be well-defined and manageable. When we consider a ratio of solutions (which are in fact sequences), we mean the term-by-term ratio for each position in the solution sequence. We only consider solution sequences whose terms are positive real numbers since we want our results to apply to physical phenomena. Finally, to make our work more manageable, we consider solutions to non-homogeneous difference equations whose forcing term is a homogeneous solution to the same recurrence.

In a crash course in difference equations we provide useful theorems and definitions for your understanding of our research. We then consider our motivating counting problem in the fascinating world of Fibonacci numbers. We later formulate more general questions on ratios of solutions to

non-negative difference equations and prove some results about their long-term and short-term behavior. After illustrating why further progress toward fully understanding the short-term behavior has been difficult to achieve, we formulate some conjectures based on numerical experimentation.

2. CRASH COURSE IN DIFFERENCE EQUATIONS

Difference equations are used today in the study of phenomena from many fields including economics and demography [Cul][Cull]. Difference equations are the discrete anaolg of differential equations and in many ways are more practical due to the existence of solutions and the relative ease with which we can find them. Also, many times analytic methods from continuous mathematics can still be used to study difference equations even though our data is discrete [Cl][Cull]. We will first explain difference equations broadly and then narrow our focus to the particular types of difference equations important to our study.

2.1. **Definitions and examples.**

Definition 2.1. *A difference equation is a recurrence relation together with a set of initial conditions. The recurrence is a function that maps vectors to scalars for some field $\mathbb{F}$ denoted by*

$$a_n = \Phi(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n).$$

Here the order of the recurrence is k, the initial conditions are the values of $a_1, a_2, \ldots, a_k$ and the action on $n \in \mathbb{N}$ is the so-called forcing term. Since each element is indexed by a natural number, we can think of our solution as the infinite sequence $(a_1, a_2, \ldots)$.

Before we begin talking about solutions, we need to know whether they exist and are unique.

Theorem 2.2. *Every recurrence relation of the form $a_n = \Phi(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n)$ with k initial conditions has a unique solution.*

Functions like Φ from vectors to scalars are not unfamiliar even to students of elementary mathematics. Consider an example from calculus.

Example 2.3. *An example of a function that maps vectors into scalars is the Euclidean norm in $\mathbb{R}^2$ given by $\|(a,b)\| = \sqrt{a^2 + b^2}$ for some $a, b \in \mathbb{R}^2$. If we consider the Euclidean norm as a recurrence relation with initial conditions 0 and 1, then we can find the third term in our solution sequence via $|(0,1)| = \sqrt{0^2 + 1^2} = 1$. Likewise to find the fourth term in our solution sequence we take $|(1,1)| = \sqrt{1^2 + 1^2} = \sqrt{2}$. We proceed similarly to determine that our solution is the sequence $(0, 1, 1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}, \ldots)$, where each term is the square root of the nth Fibonacci number.*

While this example has a close relationship with the Fibonacci recurrence we will study later, it is in fact much more difficult than those we will be working with. We will now narrow our focus to finite, constant coefficient, linear difference equations over $\mathbb{R}_{\geq 0}$. We will call these non-negative difference equations.

Definition 2.4. *A homogeneous non-negative difference equation is a difference equation whose recurrence relation can be written*

$$X_n = c_1 X_{n-1} + c_2 X_{n-2} + \cdots + c_k X_{n-k}$$

for some $c_1, c_2, \ldots, c_{k-1} \geq 0$ and $c_k > 0$.

Example 2.5. *Consider the famous example attributed to Fibonacci:*

$$f_n = f_{n-1} + f_{n-2}; \; f_1 = 1, f_2 = 1$$

Its solution is the common sequence of Fibonacci numbers $(1,1,2,3,5,8,13,21,34,55,89,\dots)$. *The Fibonacci recurrence is an example of a homogeneous non-negative difference equation.*

We can calculate f_3 directly by substituting the previous two terms, f_2 and f_1, into the recurrence relation. Similarly we can calculate f_n for any $n \in \mathbb{N}$ by direct calculation whenever we know the previous two terms. (Note: Calculating large Fibonacci numbers has been a large area of study recently. Cull, Marukami, and Young have devised methods of calculating large Fibonacci numbers without express knowledge of the previous two terms [Cu].)

Example 2.6. *The solution to* $f_n = f_{n-1} + f_{n-2}; \; f_1 = 2, f_2 = 1$ *is the sequence of Lucas numbers,* $(2,1,3,4,7,11,18,29,47,76,\dots)$, *attributed to Edouard Lucas in the nineteenth century* [K].

Example 2.7. *The Fibonacci equation can actually be generalized to any order k.*

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \dots + f_{n-k}^{(k)}; \; f_m = 2^{m-1} \, for \, m = 1, 2, \dots, k$$

Notice that the $k = 2$ case is the classic version attributed to Fibonacci. (When we make any reference to Fibonacci we will mean the $k = 2$ case. Otherwise we will refer to generalized Fibonacci.) We will study generalized Fibonacci numbers further in our motivating counting problem in Section 3.

Theorem 2.8. *A non-homogeneous non-negative difference equation is a difference equation whose recurrence relation can be written as the sum of a homogeneous non-negative recurrence relation and a function* ψ *of n called the forcing function.*

$$X_n = c_1 X_{n-1} + c_2 X_{n-2} + \dots + c_k X_{n-k} + \psi(n)$$

Observation 2.9. *Every term for a solution to a kth-order non-negative recurrence with k non-negative initial conditions must be non-negative.*

Since our non-negative difference equations are linear, we will make use of the theory of linear operators.

Definition 2.10. *A linear operator is a mapping* $L[\;]$ *from a vector space into itself with the following properties:*
   (1) $L[x+y] = L[x] + L[y]$, *for any vectors x and y*
   (2) $L[\alpha x] = \alpha L[x]$, *for any vector x and any scalar* α

Notation 2.11. *We will write* $L[\;]$ *associated with the difference equation* $X_n = a_1 X_{n-1} + a_2 X_{n-2} + \dots + a_k X_{n-k}$ *as*

$$L[X_n] = X_n - a_1 X_{n-1} - a_2 X_{n-2} + \dots - a_k X_{n-k}.$$

In this manner we can write $L[h_n] = 0$ to mean that h_n is a solution to the homogeneous difference equation given by X_n or $L[s_n] = \psi(n)$ to mean that s_n is a solution to a non-homogeneous difference equation with the particular forcing term given by $\psi(n)$.

Results from algebra encourage us to study the characteristic polynomial associated with $L[\;]$.

Notation 2.12. *The characteristic polynomial of $L[\ ]$ associated with*
$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \cdots + a_k X_{n-k}$ *is given by*

$$ch_L(\lambda) = \lambda^k - a_1 \lambda^{k-1} - a_2 \lambda^{k-2} - \cdots - a_k.$$

Definition 2.13. *A non-negative polynomial is a polynomial of the form*
$x^k - a_{k-1} x^{k-1} - a_{k-2} x^{k-2} - \cdots - a_1 = 0$ *where each $a_i \geq 0$.*

Observation 2.14. *The characteristic polynomial of a non-negative recurrence relation is a non-negative polynomial.*

Theorem 2.15. *The characteristic polynomial of a composition of linear operators is the product of the characteristic polynomials.*

Definition 2.16. *A solution g_n is called a full solution to $L[g_n] = 0$ if whenever $G[g_n] = 0$,*
$deg(ch_L(\lambda)) \leq deg(ch_G(\lambda))$. *(In other words, a full solution of $L[\ ]$ must depend on all of the roots of the characteristic polynomial of $L[\ ]$).*

Theorem 2.17. *Every solution to a homogeneous difference equation can be written as a linear combination of full solutions.*

Corollary 2.18. *If $L[\ ]$ is a kth-order non-negative linear operator such that $ch_L(\lambda)$ has k distinct roots, then solutions to $L^2[X_n] = 0$ are of the form*
$X_n = (\alpha_0 n + \beta_0)\lambda_0^n + (\alpha_1 n + \beta_1)\lambda_1^n + \cdots + (\alpha_{k-1} n + \beta_{k-1})\lambda_{k-1}^n$ *where $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ are the distinct roots of $ch_L(\lambda)$. (More generally, a solution whose roots are not distinct require higher order polynomials multiplied by the corresponding repeated roots.)*

Remark 2.19. *The roots of $ch_L(\lambda)$ may not be real. Even if the coefficients of $L[\ ]$ are are integers, the roots of $ch_L(\lambda)$ may be irrational and/or complex.*

Example 2.20. *The closed form solution for the Fibonacci recurrence is $f_n = \frac{1}{\sqrt{5}}(\lambda_0^n - \lambda_1^n)$ where $\lambda_0 = \frac{-1+\sqrt{5}}{2}$ and $\lambda_1 = \frac{-1-\sqrt{5}}{2}$.*

Lastly, and perhaps most important in our study, is the notion of double non-negativity.

Definition 2.21. *The linear operator $L[\ ]$ is doubly non-negative if $ch_L(\lambda)$ and $\frac{ch_L(\lambda)^2}{\lambda - \lambda_0}$ are both non-negative polynomials.*

Notation 2.22. *For convenience we will define $\hat{L}[\ ]$ to be the linear operator whose characteristic polynomial is equal to $\frac{ch_L(\lambda)^2}{\lambda - \lambda_0}$ for some $L[\ ]$.*

Theorem 2.23. *Every first-order linear operator is doubly non-negative.*

Proof. A general first-order linear operator $L[X_n] = \lambda_0 X_{n-1}$ has $ch_L(\lambda) = \lambda - \lambda_0$. Then $ch_{\hat{L}}(\lambda) = \frac{(\lambda - \lambda_0)^2}{\lambda - \lambda_0} = \lambda - \lambda_0$, which is a non-negative polynomial. Therefore, every first-order linear operator is doubly non-negative. $\square$

Example 2.24. *Let $L[X_n] = X_n - \frac{1}{2}(X_{n-1} + X_{n-2})$. Then $ch_L(\lambda) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = (\lambda - 1)(\lambda + \frac{1}{2})$. Then $ch_{\hat{L}}(\lambda) = (\lambda - 1)(\lambda + \frac{1}{2})^2 = (\lambda - 1)(\lambda^2 + \lambda + \frac{1}{4}) = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}$. Therefore, $\hat{L}[\ ]$ is the third-order non-negative linear operator assciated with the non-negative recurrence $X_n = \frac{3}{4}X_{n-2} + \frac{1}{4}X_{n-3}$.*

2.2. **Properties specific to non-negative polynomials.** Non-negative difference equations are of particular importance because of their applications to mathematical models of physical phenomena that only have meaning for positive quantities. Also, non-negative polynomials are well-understood and have been shown to exihibit very nice properties [Cull]. As we have seen, the characteristic polynomial of a non-negative recurrence relation is a non-negative polynomial, so results from this section can be applied to characteristic polynomials of non-negative recurrence relations.

Theorem 2.25. *A non-negative polynomial has exactly one positive root.*

Proof. Descartes' Rule of Signs states that the number of positive real roots of a constant coefficient polynomial is bounded above by the number of sign changes among its (ordered) coefficients. Non-negative polynomials have a positive leading coefficient and every other coefficient is negative, so the number of positive real roots must be at most one. Furthermore, we know that a non-negative polynomial evaluated at 0 is negative, but evaluated at sufficiently large n is positive. So, by the Intermediate Value Theorem, a non-negative polynomial has at least one positive real root. Thus, non-negative polynomials have exactly one positive real root. $\qquad\square$

Corollary 2.26. *Non-negative polynomials can be normalized so that their single positive root is equal to* 1.

Example 2.27. *The normalized Fibonacci operator has associated characteristic polynomial* $ch_{Norm}(\lambda) = \lambda^2 - \frac{1}{\lambda_0}\lambda - \frac{1}{\lambda_0^2}$. *We can verify that*

$ch_{Norm}(1) = 1 - \frac{1}{\lambda_0} - \frac{1}{\lambda_0^2} = \frac{1}{\lambda_0^2}(\lambda_0^2 - \lambda_0 - 1) = \frac{1}{\lambda_0^2}(ch_{Fib}(\lambda_0)) = 0.$

Theorem 2.28. *Solutions to homogeneous non-negative difference equations must either oscillate or be constant.*

Proof. Let $L[h_n] = 0$ where $L[\ ]$ is a normalized, non-negative linear operator. Suppose there exists a monotonic solution to a homogeneous non-negative difference equation. Then in particular, $h_1 < h_2 < \cdots < h_k < h_{k+1}$. Also, $h_{k+1} = c_1 h_k + c_2 h_{k-1} + \cdots + c_k h_1$ for some $c_i \geq 0$ such that $\sum_{i=1}^{k} c_i = 1$, since our operator is normalized. But since $x_j < x_k$ for all $j < k$,

$$h_{k+1} = c_1 h_k + c_2 h_{k-1} + \cdots + c_k h_1 < c_1 h_k + c_2 h_k + \cdots + c_k h_k = h_k \sum_{i=1}^{k} c_i = h_k.$$

This contradicts that $h_{k+1} > h_k$. Thus, solutions to homogeneous non-negative difference equations cannot be monotone. (Note: There is a case when $c_1 = 1$ where the solution sequence will be constant.) $\qquad\square$

We now begin to determine the relative size of the negative roots.

Observation 2.29. (*Taken from* [Cull]) *If $p(x)$ is a non-negative polynomial with positive root λ_0, then the values of $p(x)$ change from negative to positive at $x = \lambda_0$. From this, we can conclude that for $x > 0$,*

$$ch(x) > 0 \Leftrightarrow x > \lambda_0$$

by showing that $p'(x)$ is positive for all $x > \lambda_0$.

Theorem 2.30. *The single positive real root of a non-negative polynomial is greater than or equal to the modulus of each of its other roots.*

Proof. Let λ_i represent any of the roots of a non-negative polynomial $p(x)$. Then $0 = p(\lambda_i) = \lambda_i^k - a_{k-1}\lambda_i^{k-1} - a_{k-2}\lambda_i^{k-2} - \cdots - a_1$. Therefore, $\lambda_i^k = a_{k-1}\lambda_i^{k-1} + a_{k-2}\lambda_i^{k-2} + \cdots + a_1$. Taking the absolute value of each side, using the triangle inequality, and bringing the exponents outside, we have $|\lambda_i|^k \leq a_{k-1}|\lambda_i|^{k-1} + a_{k-2}|\lambda_i|^{k-2} + \cdots + a_1$. This implies that $p(|\lambda_i|) \leq 0$. By the previous observation, $|\lambda_i| \leq \lambda_0$ where λ_0 is the single positive root of $p(x)$. $\square$

Remark 2.31. *In fact, we can avoid the case where the modulus of a secondary root is equal to the dominant root of our characteristic polynomial by requiring that our linear operator be aperiodic, that is, for $ch(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - a_{k-2}\lambda^{k-2} - \cdots - a_1$, $gcd\{i|a_i \neq 0\} = 1$.*

Theorem 2.32. *The generalized Fibonacci recurrence is doubly non-negative.*

Proof. The characteristic polynomial for the kth-order generalized Fibonacci operator is

$$ch(\lambda) = \lambda^k - \lambda^{k-1} - \cdots - 1.$$

Since this is a non-negative polynomial, it has exactly one positive root, λ_0. Furthermore, because $ch(2) = 2^k - (2^{k-1} + 2^{k-2} + \cdots + 1) = 2^k - \frac{2^k-1}{2-1} = 1 > 0$, $\lambda_0 < 2$. Also, long division yields $\frac{ch(\lambda)}{\lambda - \lambda_0} = g_0\lambda^{k-1} + g_1\lambda^{k-2} + \cdots + g_{k-1}$, where $g_{n+1} = \lambda_0 g_n - 1$ for $n = 0, 1, \ldots, k-1$ and $g_0 = 1$. Suppose that there exists a g_i that is negative. Then $g_i, g_{i+1}, \ldots, g_k$ are all negative. But $g_k = \lambda_0^k - \lambda_0^{k-1} - \cdots - 1 = 0$, a contradiction. Therefore, $g_n \geq 0$ for $n = 0, 1, \ldots, k$. Furthermore, we claim that $g_n \leq 1 = g_0$. We show this inductively. The base case is true since $g_0 = 1 \leq 1$. Assume now that $g_i \leq 1$ for $i = 0, 1, \ldots, m$. Then $g_{m+1} = \lambda_0 g_m - 1 \leq (2)(1) - 1 = 1$. Therefore, $g_n \leq 1 = g_0$, in particular, for $n = 0, 1, \ldots, k-1$. We will now show that $\frac{ch(\lambda)^2}{\lambda - \lambda_0}$ is a non-negative polynomial.

$$\frac{ch(\lambda)^2}{\lambda - \lambda_0} = (\lambda^k - \lambda^{k-1} - \cdots - 1)(g_0\lambda^{k-1} + g_1\lambda^{k-2} + \cdots + g_{k-1})$$

$$= g_0\lambda_0^{2k-1} + \sum_{i=1}^{k-1} \lambda^{2k-1-i}(g_i - g_{i-1} - \cdots - g_0) + \sum_{j=0}^{k-1} \lambda^{k-1-j}(-g_{k-1} - g_{k-2} - \cdots - g_j)$$

Clearly, the coefficient of λ^{2k-1} is positive. The coefficients of the next $k-1$ terms are negative since $g_i \leq g_0$. The coefficients of the last k terms are clearly negative. Thus, $\frac{ch(\lambda)^2}{\lambda - \lambda_0}$ is a non-negative polynomial, so the kth-order generalized Fibonacci operator is doubly non-negative. $\square$

3. MOTIVATING COUNTING PROBLEM

We now turn our attention to our motivating counting problem. We are interested in counting the ratio of the number of zeroes to the total number of digits in the Fibonacci-base binary number system attributed to Edouard Zeckendorf in 1972 [Z][Zt].

3.1. **Zeckendorf representation.**

Theorem 3.1. (*Zeckendorf's Theorem* [Z]) *Every natural number can be written in the form*

$$\sum_{i=2} b_i f_i^{(k)}$$

where $b_i \in \{0,1\}$ and $f_i^{(k)}$ is the ith generalized Fibonacci number of order k. Furthermore, this representation is unique so long as we don't allow any k consecutive b_i's to be equal to 1. (Note: For convenience we avoid using the sigma notation and instead write the string of zeroes and ones (from right to left) that corresponds to our number, much the same as the more common base-two binary system.)

Similar to other number systems, we tend to drop leading zeroes when writting in our Fibonacci-base system. Then for a fixed number of digits, we can only represent a finite amount of numbers. Our counting problem asks us to count the total number of zeroes, N_n, and the total number of digits, W_n, for all of the natural numbers we can represent for a specific number n of digits and a specific order k of the generalized Fibonacci numbers.

$n = 1$	$n = 2$	$n = 3$	$n = 4$
0	00	000	0000
1	01	001	0001
	10	010	0010
		100	0100
		101	0101
			1000
			1001
			1010

FIGURE 1. Binary Strings of length n for $k = 2$ [S]

As we can see from Figures 2 and 3, Zeckendorf representation approaches the efficiency of the base-two binary number system for rather small values of k. Furthermore, for a given k our ratio approaches a limit, meaning the inefficiency of Zeckendorf representation is bounded and won't get worse when representing arbitrarily large natural numbers. Proving the ratio is monotonic for a given order k would mean that Zeckendorf representation never becomes more efficient when representing a larger number.

Capocelli was the first to notice in 1990 that for a given k, we have $\frac{N_1}{W_1} \leq \frac{N_2}{W_2} \leq \frac{N_3}{W_3} \leq \cdots$ [C][Ca]. There is no obvious reason for this behavior. Furthermore, it was noticed that both N_n and W_n are solutions to the same difference equation, namely, $L_{GenFib}^2[N_n] = L_{GenFib}^2[W_n] = 0$ where $L_{GenFib}[\ ]$ is the linear operator associated with the kth-order generalized Fibonacci recurrence. Ratios of solutions to non-negative difference equations then became the target of study and similar monotonic behavior was observed for some particular types of non-negative difference equations. A few conjectures have been proposed since 1990, most notably with Sanders' REU project in 2006 [S], but few results on the short-term behavior of ratios of solutions have been proved. Sanders
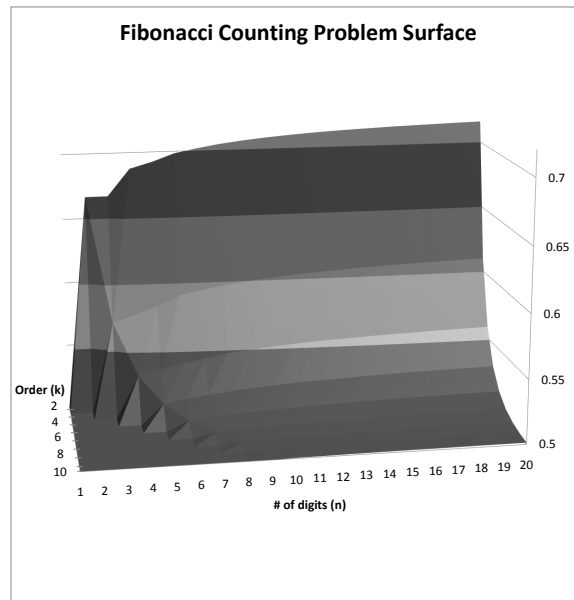
FIGURE 2. Surface of $\frac{N_n}{W_n}$

n	k								
	2	3	4	5	6	7	8	9	10
1	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
2	.6667	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
3	.6667	.5714	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4	.6875	.5769	.5333	.5000	.5000	.5000	.5000	.5000	.5000
5	.6923	.5833	.5379	.5161	.5000	.5000	.5000	.5000	.5000
6	.6984	.5909	.5417	.5191	.5079	.5000	.5000	.5000	.5000
7	.7017	.5944	.5450	.5214	.5097	.5039	.5000	.5000	.5000
8	.7045	.5973	.5481	.5233	.5111	.5049	.5020	.5000	.5000
9	.7066	.5998	.5500	.5249	.5122	.5057	.5025	.5010	.5000
10	.7083	.6016	.5516	.5263	.5131	.5064	.5030	.5013	.5005
11	.7097	.6031	.5530	.5274	.5139	.5069	.5033	.5015	.5006
12	.7109	.6044	.5541	.5282	.5146	.5074	.5036	.5017	.5008
13	.7119	.6055	.5550	.5289	.5151	.5078	.5039	.5019	.5009
14	.7127	.6064	.5558	.5296	.5156	.5081	.5041	.5020	.5010
15	.7134	.6072	.5565	.5302	.5160	.5084	.5043	.5022	.5011
16	.7141	.6079	.5571	.5307	.5164	.5086	.5045	.5023	.5011
17	.7146	.6085	.5577	.5311	.5167	.5089	.5046	.5024	.5012
18	.7151	.6091	.5582	.5315	.5170	.5091	.5048	.5025	.5013
19	.7156	.6096	.5586	.5318	.5172	.5092	.5049	.5025	.5013
20	.7160	.6100	.5590	.5321	.5175	.5094	.5050	.5026	.5014

FIGURE 3. Proportion of 0's [S]

was able to prove the monotonic behavior in the counting problem for the $k = 2$ case [S]. We now take the time to summarize her proof.

3.2. Summary of Sanders' proof for the $k = 2$ case.

Theorem 3.2. *Let N_n and W_n represent the numbers of total zeroes and total digits, respectively, in our counting problem for $k = 2$ (described above). Then $\frac{N_n}{W_n}$ increases monotonically for all $n \in \mathbb{N}$.*

Proof. Through careful computation we can show that $L_{Fib}^2[N_n] = 0$. We also know that a Fibonacci number, f_n, is a full solution of the Fibonacci recurrence because it depends on both roots of the characteristic polynomial associated with the Fibonacci operator. Therefore, we can write N_n in terms of a full solution as $N_n = n(a_1 f_n + a_2 f_{n-1}) + (b_1 f_n + b_2 f_{n-1})$. We then use our knowledge of the first four terms from the counting problem ($N_1 = 1$, $N_2 = 4$, $N_3 = 10$, $N_4 = 22$) to get four equations in four variables. We can then solve for our coefficients a_1, a_2, b_1, b_2 (if the four equations are linearly independent, which they are). We now analyze $\frac{N_n}{W_n} \leq \frac{N_{n+1}}{W_{n+1}}$. With substitution, this becomes

$$\frac{N_n}{W_n} = \frac{n(7f_n + 4f_{n-1}) - 2f_n}{5nf_{n+2}} \leq \frac{(n+1)(7f_{n+1} + 4f_n) - 2f_{n+1}}{5(n+1)f_{n+3}} = \frac{N_{n+1}}{W_{n+1}}$$

Cross multiplication and rearranging yields

$$n(n+1)(f_{n+3}f_{n+1} - f_{n+2}f_{n+2}) \leq 2n(f_{n+2}f_n - f_{n+1}f_{n+1}) + 2f_nf_{n+3}$$

We now use Cassini's identity that $f_{n+1}f_{n-1} - f_nf_n = (-1)^n$ to simplify. More combining of terms finally yields

$$n(n+3)(-1)^{n+2} \leq 2f_nf_{n+3}$$

This must hold for $n > 5$ since $f_n > n > 5$. We can check the cases of $n = 1, 2, 3, 4, 5$ by hand and the inequality holds true. Thus, $\frac{N_m}{W_n}$ is increasing for all $n \in \mathbb{N}$ for the $k = 2$ case. $\qquad\square$

While this is a thorough proof for the $k = 2$ case of our counting problem, we acknowledge that it will not generalize easily. In fact, there is a fundamental difference between the matrix representation for the classic Fibonacci numbers and the matrix representation for higher-order generalized Fibonacci numbers–the matrix representing the second-order Fibonacci recurrence is symmetric whereas the matrix for every higher-order generalized Fibonacci recurrence is no longer symmetric. This could complicate matters in attempts to prove the generalized version our desired result. In particular, Cassini's identity typically does not hold in cases where $k > 2$. We hope to eventually prove the result from our counting problem in a manner that could easily generalize to recurrence relations other than the generalized Fibonacci recurrence.

4. Extending the monotone result

Our objective is to understand both the long-term and the short-term behavior of a ratio of solutions to non-negative difference equations. While we have shown that we can prove a case with specific solutions, we hope to extend this result to arbitrary solutions. As we stated in the introduction, we must have some restrictions on the arbitrary solutions to make our process well-defined and manageable. Throughout this section we will restrict ourselves to working with solution sequences whose terms are positive numbers. Also, we will only consider solutions g_n and h_n such

that $L^2[g_n] = L^2[h_n] = 0$. In other words, we will consider solutions whose forcing terms are homogeneous solutions to the same recurrence. (Note: This assumption is not as strong as requiring each solution to have the same forcing function. For example, in the $k = 2$ case of our motivating counting problem, $L[N_n] = f_{n+2}$ and $L[W_n] = f_{n+2} + f_n$.)

4.1. **Results.** We begin by analyzing the behavior of the simplest type of non-negative difference equation–first-order.

Theorem 4.1. *Let $L[\ ]$ be a linear operator for a general first-order non-negative recurrence. If g_n and h_n are solutions such that $L^2[g_n] = L^2[h_n] = 0$ and $\frac{g_1}{h_1} \leq \frac{g_2}{h_2}$, then $\frac{g_n}{h_n}$ is monotonically increasing for all $n \in \mathbb{N}$.*

Proof. The solutions g_n and h_n have the form $g_n = (an+b)\lambda^n$ and $h_n = (cn+d)\lambda^n$. Furthermore, we have $\frac{g_1}{h_1} = \frac{(a+b)\lambda}{(c+d)\lambda} \leq \frac{(2a+b)\lambda^2}{(2c+d)\lambda^2} = \frac{g_2}{h_2}$. Direct computation shows that $ad - bc \geq 0$. We can now check that the ratio $\frac{g_n}{h_n}$ is in fact increasing.

$$\frac{g_n}{h_n} \overset{?}{\lessgtr} \frac{g_{n+1}}{h_{n+1}}$$

$$\frac{(an+b)\lambda^n}{(cn+d)\lambda^n} \overset{?}{\lessgtr} \frac{(a(n+1)+b)\lambda^{n+1}}{(c(n+1)+d)\lambda^{n+1}}$$

$$(an+b)(cn+c+d) \overset{?}{\lessgtr} (cn+d)(an+a+b)$$

$$acn^2 + acn + adn + bcn + bc + bd \overset{?}{\lessgtr} acn^2 + acn + bcn + adn + ad + bd$$

$$0 \overset{?}{\lessgtr} ad - bc$$

$$0 \leq ad - bc \ \checkmark$$

Thus, $\frac{g_n}{h_n}$ is increasing for all n.   □

We will now prove that a condition on only the coefficients of the dominant eigenvalue in a solution's closed form can guarantee monotonic behavior in the long-term for a ratio of solutions to a non-negative difference equation of any order k.

Theorem 4.2. *Let $L[\ ]$ be a normalized, aperiodic non-negative operator. Let g_n and h_n be any solutions such that $L^2[g_n] = L^2[h_n] = 0$. Then g_n and h_n have the form*
$g_n = (an+b) + p_1(n)\lambda_1^n + \cdots + p_{k-1}(n)\lambda_{k-1}^n$ *and* $h_n = (cn+d) + q_1(n)\lambda_1^n + \cdots + q_{k-1}(n)\lambda_{k-1}^n$.
If $ad - bc > 0$, then $\frac{g_n}{h_n}$ converges asymptotically to $\frac{a}{c}$ like $\frac{1}{n}$ (that is, for sufficiently large n we have $\frac{g_n}{h_n} = \frac{a}{c} - O(\frac{1}{n})$).

Proof. We will show that $\lim\limits_{n \to \infty} n(\frac{a}{c} - \frac{g_n}{h_n})$ exists and is positive.

We have
$\lim\limits_{n \to \infty} n\left(\frac{a}{c} - \frac{g_n}{h_n}\right) = \lim\limits_{n \to \infty} n\left(\frac{a}{c} - \frac{(an+b)+p_1(n)\lambda_1^n+\cdots+p_{k-1}(n)\lambda_{k-1}^n}{(cn+d)+q_1(n)\lambda_1^n+\cdots+q_{k-1}(n)\lambda_{k-1}^n}\right) = \lim\limits_{n \to \infty} n\left(\frac{a}{c} - \frac{an+b}{cn+d}\right)$, since the aperiodicity of $L[\ ]$ ensures $\lambda_i < 1$ for $i = 1, \ldots, k-1$.

Then, $\lim\limits_{n \to \infty} n\left(\frac{a}{c} - \frac{an+b}{cn+d}\right) = \lim\limits_{n \to \infty}\left(\frac{acn+ad-acn-bc}{c^2+(cd/n)}\right) = \lim\limits_{n \to \infty}\left(\frac{ad-bc}{c^2+(cd/n)}\right) = \frac{ad-bc}{c^2} > 0$ by hypothesis.

Thus, $\frac{g_n}{h_n} = \frac{a}{c} - O(\frac{1}{n})$ for sufficiently large $n \in \mathbb{N}$.   □

Determining the long-term behavior is an ideal result, but it is unclear exactly how long the ratio will take to become monotonic. While we have been unable to prove that our ratio is monotonic in the short-term, we now try to at least show that the ratio is somewhat well-behaved relatively close to where it begins.

Theorem 4.3. *Let g_n and h_n be any solutions such that $L^2[g_n] = L^2[h_n] = 0$ for some doubly non-negative linear operator $L[\ ]$. If $\frac{g_n}{h_n} \leq \frac{a}{c}$ for $n = 1, 2, \ldots, 2k-1$, then the ratio $\frac{g_n}{h_n}$ is always less than or equal to $\frac{a}{c}$, its asymptotic value.*

Proof. Recall that since $L[\ ]$ is doubly non-negative, $\hat{L}[\ ]$ is a non-negative linear operator. Now define $Z_n = ah_n - cg_n = h_n c(\frac{a}{c} - \frac{g_n}{h_n})$. Then $\hat{L}[Z_n] = \hat{L}[ah_n - cg_n] = a\hat{L}[h_n] - c\hat{L}[g_n] = a\hat{L}[cn] - c\hat{L}[an] = \hat{L}[acn - acn] = \hat{L}[0] = 0$. Since Z_n is a solution to a homogeneous non-negative recurrence relation of order $2k-1$ and our hypothesis assumes we have $2k-1$ non-negative initial values, we can claim that $Z_n \geq 0$ for every $n \in \mathbb{N}$. Since c and h_n are positive, we can divide to show $\frac{g_n}{h_n} \leq \frac{a}{c}$ for all $n \in \mathbb{N}$. $\qquad\square$

Making extra assumptions sometimes allows for easier results. We now present proofs for some specific cases.

Theorem 4.4. *Let g_n and h_n be any solutions of the form $g_n = an + b + \lambda^n(a_1 n + b_1)$ and $h_n = cn + d + \lambda^n(c_1 n + d_1)$ such that $L^2[g_n] = L^2[h_n] = 0$ and $ac_1 - ca_1 = a_1 d_1 - b_1 c_1 = 0$ for some second-order doubly non-negative linear operator $L[\ ]$. If $\frac{g_n}{h_n} \leq \frac{g_{n+1}}{h_{n+1}}$ for $n = 1, 2, 3$, then $\frac{g_n}{h_n}$ increases for all $n \in \mathbb{N}$. (Note that we have normalized so that $\lambda_0 = 1$.)*

Proof. Direct computation shows that $g_{n+1}h_n - h_{n+1}g_n$ has the form $g_{n+1}h_n - h_{n+1}g_n = \alpha + (\beta n(n+1) + \gamma n + \sigma)\lambda^n + \varepsilon\lambda^{2n}$ for some $\alpha, \beta, \gamma, \sigma, \varepsilon \in \mathbb{R}$. However, the assumption that $ac_1 - ca_1 = 0$ implies that $\beta = 0$. Similarly, the assumption that $a_1 d_1 - b_1 c_1 = 0$ implies that $\varepsilon = 0$. Therefore, $\hat{L}[g_{n+1}h_n - h_{n+1}g_n] = \hat{L}[\alpha + (\gamma n + \sigma)\lambda^n] = 0$. Furthermore, since $L[\ ]$ is doubly non-negative, we know that $\hat{L}[\ ]$ is a third-order non-negative linear operator. We have assumed that three consecutive terms are increasing which implies that $\frac{g_{n+1}}{h_{n+1}} - \frac{g_n}{h_n} = \frac{1}{h_n h_{n+1}}(g_{n+1}h_n - h_{n+1}g_n) \geq 0$ for those three consecutive values. Therefore, $\frac{g_{n+1}}{h_{n+1}} - \frac{g_n}{h_n} = \frac{1}{h_n h_{n+1}}(g_{n+1}h_n - h_{n+1}g_n) \geq 0$ for all $n \in \mathbb{N}$. Thus, $\frac{g_n}{h_n}$ increases for all $n \in \mathbb{N}$. $\qquad\square$

Definition 4.5. *The linear operator $L[\ ]$ is triply non-negative if $ch_L(\lambda)$ and $\frac{ch_L(\lambda)^3}{(\lambda - \lambda_0)^2}$ are both non-negative polynomials.*

Observation 4.6. *Every recurrence relation that is triply non-negative is also doubly non-negative.*

Observation 4.7. *The Fibonacci recurrence is doubly non-negative but not triply non-negative.*

Theorem 4.8. *Let g_n and h_n be any solutions of the form $g_n = an + b + \lambda^n(a_1 n + b_1)$ and $h_n = cn + d + \lambda^n(c_1 n + d_1)$ such that $L^2[g_n] = L^2[h_n] = 0$ and $a_1 d_1 - b_1 c_1 = 0$ for some second-order triply non-negative linear operator $L[\ ]$. If $\frac{g_n}{h_n}$ increases for $n = 1, 2, 3, 4$, then $\frac{g_n}{h_n}$ increases for all $n \in \mathbb{N}$.*

Proof. The proof is similar to the proof of Theorem 4.4 because $\hat{L}[\beta n(n+1)\lambda^n] = 0$ since $L[\ ]$ is triply non-negative. The only difference is that $\hat{L}[\ ]$ is now a fourth-order non-negative linear operator, so we need to assume one more initial condition. $\qquad\square$

Summary 4.9. *If g_n and h_n are solutions to $L^2[g_n] = L^2[h_n] = 0$ where $L[\ ]$ is a doubly non-negative operator, then:*

- *$\frac{g_n}{h_n}$ eventually increases if $ad - bc > 0$ (or similarly, decreases if $ad - bc < 0$)*
- *$\frac{g_n}{h_n}$ stays below its asymptote if $2k - 1$ consecutive terms are below the asymptote*
- *$\frac{g_n}{h_n}$ increases monotonically for all $n \in \mathbb{N}$ if $L[\ ]$ is the operator of a first-order recurrence*
- *$\frac{g_n}{h_n}$ increases monotonically for all $n \in \mathbb{N}$ if $L[\ ]$ is the operator of a second-order recurrence, $\frac{g_n}{h_n}$ increases for $n = 1, 2, 3$, and g_n and h_n are solutions such that $ac_1 - ca_1 = a_1 d_1 - b_1 c_1 = 0$*
- *$\frac{g_n}{h_n}$ increases monotonically for all $n \in \mathbb{N}$ if $L[\ ]$ is the operator of a second-order triply non-negative recurrence, $\frac{g_n}{h_n}$ increases for $n = 1, 2, 3, 4$, and g_n and h_n are solutions such that $a_1 d_1 - b_1 c_1 = 0$*

4.2. **Developing methods.** We will now spend some time delveoping different methods and proof strategies that can be utilized when studying the ratio of solutions to non-negative difference equations.

As is clear from our proofs thus far, the condition of double non-negativity plays a significant role in our research. We would like to understand how double non-negativity affects the relative size of the roots.

Observation 4.10. *The upper bound for the magnitude of the secondary root of a normalized, doubly non-negative second-order recurrence is $\frac{1}{2}$.*

Conjecture 4.11. *For a large enough order, we can make the magnitude of the secondary root of a doubly non-negative recurrence arbitrarily close to 1.*

Since we are determining the conditions we feel are necessary to explain the monotonic behavior of solutions, we should develop a way to easily experiment with and analyze test data. One such method has been by creating Excel spreadsheets that contain tables (or columns) of different values we deemed relevant. Such values include:

- g_n
- h_n
- $\frac{g_n}{h_n}$
- $Z_n = ah_n - cg_n = ch_n\left(\frac{a}{c} - \frac{g_n}{h_n}\right)$
- $D_{x,y} = g_x h_y - h_x g_y = c_1 D_{x-1,y} + \cdots + c_k D_{x-k,y} - \alpha Z_y = c_1 D_{x,y-1} + \cdots + c_k D_{x,y-k} - \alpha Z_x$, where $\alpha = \frac{d}{d\lambda}(ch_{\hat{L}}(1))$, the derivative of $ch_{\hat{L}}$ evaluated at 1
- $D_{x,y}/Z_x$
- $D_{x,y}/Z_y$
- $H_{x,y} = xZ_y - yZ_x - D_{x,y}$
- $D_{x,y}\left(\frac{1}{Z_y} + \frac{1}{Z_x}\right)$

The relevance of the first three values is obvious since $\frac{g_n}{h_n}$ is our ratio. The value Z_n is relevant because its sign corresponds to whether our ratio is greater than (negative) or less than (positive) the asymptotic value $\frac{a}{c}$. Expressions involving $D_{x,y}$ are relevant because $D_{x,x-1} \geq 0$ corresponds to an increasing ratio and $D_{x,x-1} < 0$ corresponds to a decreasing ratio. Our hope is that by combining

these values in an appropriate way we can create an expression that is a solution to our homogeneous non-negative operator $\hat{L}[\ ]$. Then, we need only assume that this expression is positive for $2k-1$ initial terms and we can conclude that the expression is always positive. Furthermore, if we chose an appropriate expression, we can interpret its positivty in terms of the behavior of our ratio.

We now produce some examples of these values for the ratio $\frac{g_n}{h_n} = \frac{2n+2+(2n)(-.5)^n}{n+2+(n+.5)(-.5)^n}$ in Figures 4 through 7. The tables of values should be read like a matrix where the (i, j)-entry is located in the ith row from the top and the jth column for the left. Positive entries in the tables are highlighted for convenience. The tables appear to have nice properties (like entries increasing for a given row, etc.) but as we state in the next section, it is difficult to use this behavior in attempts at proving our ratio is monotonic.

n	g(n) 2n+2+2n(-.5)^n	h(n) n+2+(n+.5)(-.5)^n	g(n)/h(n)	Z(n) = 2+(1)(-.5)^n
1	3	2.25	1.333333	1.5
2	7	4.625	1.513514	2.25
3	7.25	4.5625	1.589041	1.875
4	10.5	6.28125	1.671642	2.0625
5	11.6875	6.828125	1.71167	1.96875
6	14.1875	8.1015625	1.751205	2.015625
7	15.890625	8.94140625	1.777195	1.9921875
8	18.0625	10.03320313	1.800273	2.00390625
9	19.96484375	10.98144531	1.818052	1.998046875
10	22.01953125	12.01025391	1.833394	2.000976563
11	23.98925781	12.99438477	1.846125	1.999511719
12	26.00585938	14.00305176	1.857157	2.000244141
13	27.99682617	14.99835205	1.86666	1.99987793
14	30.00170898	16.00088501	1.875003	2.000061035
15	31.99908447	16.99952698	1.882351	1.999969482
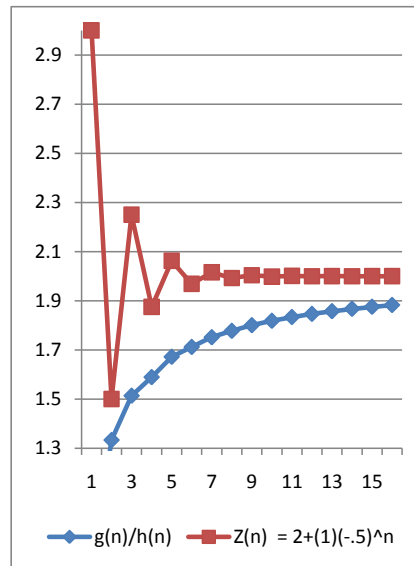
FIGURE 4. Data for $\frac{g_n}{h_n}$ and Z_n

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	-1.88	-2.63	-4.78	-5.81	-7.62	-8.93	-10.5	-12	-13.5	-15	-16.5	-18	-19.5	-21
2	1.875	0	-1.59	-4.59	-6.26	-8.91	-10.9	-13.3	-15.5	-17.8	-20	-22.3	-24.5	-26.8	-29
3	2.625	1.594	0	-2.37	-3.82	-5.99	-7.68	-9.67	-11.5	-13.4	-15.2	-17.1	-19	-20.9	-22.7
4	4.781	4.594	2.367	0	-1.72	-4.05	-5.93	-8.11	-10.1	-12.2	-14.2	-16.3	-18.4	-20.4	-22.5
5	5.813	6.258	3.82	1.717	0	-2.19	-4	-6.07	-7.98	-9.98	-11.9	-13.9	-15.9	-17.8	-19.8
6	7.617	8.906	5.994	4.049	2.187	0	-1.88	-3.99	-5.95	-8	-9.99	-12	-14	-16	-18.1
7	8.93	10.9	7.676	5.928	4	1.883	0	-2.07	-4.01	-6.04	-8.01	-10	-12	-14	-16
8	10.54	13.31	9.669	8.106	6.07	3.988	2.07	0	-1.96	-3.99	-5.98	-7.99	-9.99	-12	-14
9	11.98	15.47	11.47	10.1	7.977	5.947	4.012	1.959	0	-2.02	-4.01	-6.01	-8.01	-10	-12
10	13.51	17.77	13.39	12.2	9.982	7.997	6.035	3.991	2.023	0	-1.99	-4	-5.99	-8	-10
11	14.99	19.99	15.24	14.24	11.93	9.993	8.009	5.978	4.006	1.987	0	-2.01	-4	-6	-8
12	16.5	22.26	17.13	16.32	13.91	12.02	10.01	7.992	6.013	3.996	2.007	0	-2	-4	-6
13	18	24.5	19	18.37	15.87	14.03	12	9.99	8.006	5.992	4.002	1.996	0	-2	-4
14	19.5	26.75	20.88	20.44	17.85	16.05	13.99	12	10.01	7.996	6.004	3.999	2.002	0	-2
15	21	29	22.75	22.5	19.81	18.06	15.98	14	12	9.996	8.002	5.998	4.001	1.999	0

FIGURE 5. Table of $D_{x,y}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	-1.25	-1.75	-3.19	-3.88	-5.08	-5.95	-7.03	-7.98	-9.01	-10	-11	-12	-13	-14
2	0.833	0	-0.71	-2.04	-2.78	-3.96	-4.85	-5.91	-6.87	-7.9	-8.88	-9.89	-10.9	-11.9	-12.9
3	1.4	0.85	0	-1.26	-2.04	-3.2	-4.09	-5.16	-6.12	-7.14	-8.13	-9.14	-10.1	-11.1	-12.1
4	2.318	2.227	1.148	0	-0.83	-1.96	-2.87	-3.93	-4.9	-5.92	-6.9	-7.91	-8.91	-9.91	-10.9
5	2.952	3.179	1.94	0.872	0	-1.11	-2.03	-3.08	-4.05	-5.07	-6.06	-7.07	-8.06	-9.06	-10.1
6	3.779	4.419	2.974	2.009	1.085	0	-0.93	-1.98	-2.95	-3.97	-4.96	-5.96	-6.96	-7.96	-8.96
7	4.482	5.474	3.853	2.976	2.008	0.945	0	-1.04	-2.01	-3.03	-4.02	-5.03	-6.02	-7.02	-8.02
8	5.26	6.64	4.825	4.045	3.029	1.99	1.033	0	-0.98	-1.99	-2.98	-3.99	-4.99	-5.99	-6.99
9	5.994	7.741	5.743	5.054	3.992	2.976	2.008	0.98	0	-1.01	-2	-3.01	-4.01	-5.01	-6.01
10	6.753	8.88	6.692	6.098	4.989	3.997	3.016	1.995	1.011	0	-0.99	-2	-2.99	-4	-5
11	7.498	9.997	7.623	7.122	5.966	4.998	4.005	2.99	2.003	0.994	0	-1	-2	-3	-4
12	8.251	11.13	8.564	8.158	6.954	6.009	5.005	3.995	3.006	1.998	1.004	0	-1	-2	-3
13	8.999	12.25	9.499	9.187	7.937	7.015	5.999	4.995	4.003	2.996	2.001	0.998	0	-1	-2
14	9.75	13.38	10.44	10.22	8.922	8.024	6.996	5.998	5.003	3.998	3.002	1.999	1.001	0	-1
15	10.5	14.5	11.37	11.25	9.906	9.031	7.992	7	6.002	4.998	4.001	2.999	2	0.999	0

FIGURE 6. Table of $D_{x,y}/Z_x$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	-2.08	-3.15	-5.51	-6.83	-8.86	-10.4	-12.3	-14	-15.8	-17.5	-19.3	-21	-22.8	-24.5
2	2.083	0	-1.56	-4.27	-5.96	-8.38	-10.3	-12.6	-14.6	-16.8	-18.9	-21	-23.1	-25.3	-27.4
3	3.15	1.558	0	-2.41	-3.98	-6.17	-7.95	-9.98	-11.9	-13.8	-15.8	-17.7	-19.6	-21.6	-23.5
4	5.506	4.269	2.41	0	-1.7	-3.97	-5.85	-7.98	-9.95	-12	-14	-16.1	-18.1	-20.1	-22.2
5	6.827	5.96	3.978	1.704	0	-2.2	-4.04	-6.11	-8.04	-10.1	-12	-14	-16	-18	-20
6	8.857	8.377	6.171	3.972	2.196	0	-1.88	-3.97	-5.93	-7.96	-9.96	-12	-14	-16	-18
7	10.44	10.32	7.947	5.85	4.04	1.879	0	-2.07	-4.02	-6.05	-8.03	-10	-12	-14	-16
8	12.29	12.55	9.982	7.976	6.112	3.969	2.072	0	-1.96	-3.99	-5.97	-7.98	-9.98	-12	-14
9	13.98	14.62	11.86	9.951	8.044	5.927	4.022	1.958	0	-2.02	-4.01	-6.02	-8.01	-10	-12
10	15.76	16.78	13.83	12.01	10.06	7.964	6.045	3.986	2.024	0	-1.99	-4	-5.99	-7.99	-9.99
11	17.49	18.88	15.75	14.03	12.03	9.955	8.025	5.973	4.008	1.987	0	-2.01	-4	-6.01	-8
12	19.25	21.02	17.7	16.07	14.02	11.97	10.03	7.984	6.016	3.995	2.007	0	-2	-4	-6
13	21	23.14	19.63	18.09	16	13.97	12.02	9.981	8.01	5.991	4.003	1.996	0	-2	-4
14	22.75	25.27	21.57	20.13	17.99	15.99	14.02	11.99	10.01	7.994	6.005	3.998	2.002	0	-2
15	24.5	27.39	23.51	22.16	19.97	17.99	16.02	13.99	12.01	9.993	8.003	5.997	4.001	1.999	0

FIGURE 7. Table of $D_{x,y}(\frac{1}{Z_y} + \frac{1}{Z_x})$

4.3. **Difficulties.** When attempting to prove that the ratio of solutions to non-negative difference equations was monotone in the short term, we ran into numerous difficulties. The theme of our underlying difficulties is that every relevant value oscillates.

Observation 4.12. *The values Z_n, $D_{x,x-1}$, $D_{x,x-1}/Z_x$, and $D_{x,x-1}/Z_{x-1}$ all oscillate around $ad - bc$ and $D_{x,x-1}(\frac{1}{Z_x} + \frac{1}{Z_{x-1}})$ oscillates around $2(ad - bc)$.*

The proof of our desired result seems to beg induction. A difficulty we faced was that our inductive hypothesis was only true for values of y that were reasonably close to x. In other words, we couldn't induct along a particular row or column of our tables because for extreme values of y (or equivalently, x) the typical behavior didn't persist.

Theorem 4.13. $D_{x,y} - D_{x-1,y}$ *can't always be increasing.*

Proof. Assume $Z_n > 0$ for all $n \in \mathbb{N}$ and suppose that $D_{x,y} - D_{x-1,y}$ is always increasing. Notice that $D_{x,y} - D_{x-1,y} \approx xZ_y - yZ_x - ((x-1)Z_y - yZ_{x-1}) = Z_y - y(Z_x - Z_{x-1})$. Since we have shown that Z_x must oscillate, there exists an $N \in \mathbb{N}$ such that $Z_N - Z_{N-1} < 0$ but $Z_{N+1} - Z_N > 0$. Also, we can certainly choose a sufficiently large y such that $y > max\{\frac{Z_y}{Z_{N-1}-Z_N}, \frac{Z_y}{Z_{N+1}-Z_N}\}$. Then

$D_{N,y} - D_{N-1,y} \approx Z_y - y(Z_N - Z_{N-1}) = Z_y + y(Z_{N-1} - Z_N) > Z_y + (\frac{Z_y}{Z_{N-1}-Z_N})(Z_N - Z_{N-1}) = 2Z_y > 0$.

However, $D_{N+1,y} - D_{N,y} \approx Z_y - y(Z_{N+1} - Z_N) < Z_y - (\frac{Z_y}{Z_{N+1}-Z_N})(Z_{N+1} - Z_N) = 0$. Thus, $D_{N+1,y} - D_{N,y} < D_{N,y} - D_{N-1,y}$. Thus, $D_{x,y} - D_{x-1,y}$ is not always increasing. $\square$

The next example illustrates the need for $x \approx y$ in our induction.

Example 4.14. *Let $L[\ ]$ be a normalized, second-order, doubly non-negative linear operator. If g_n and h_n are solutions to $L^2[g_n] = L^2[h_n] = 0$ and $0 \le D_{1,0} \le D_{2,0} \le D_{3,0}$, $0 \le D_{2,1} \le D_{3,1}$, $D_{2,1} \le D_{2,0}$, and $0 \le D_{3,2} \le D_{3,1} \le D_{3,0}$, then we are tempted to conclude that $\frac{g_n}{h_n}$ increases for all $n \in \mathbb{N}$.*

Since $L[\ ]$ is doubly non-negative, $\hat{L}[\ ]$ is a third-order non-negative operator. Furthermore, $\hat{L}_x[D_{x,y} - D_{x-1,y}] = 0$ where we operate with respect to x. Similarly, $\hat{L}_y[D_{x,y} - D_{x,y-1}] = 0$ where we operate with respect to y. Our assumptions give us a four-by-four array

$$
\begin{array}{cccc}
D_{0,0} & D_{0,1} & D_{0,2} & D_{0,3} \\
D_{1,0} & D_{1,1} & D_{1,2} & D_{1,3} \\
D_{2,0} & D_{2,1} & D_{2,2} & D_{2,3} \\
D_{3,0} & D_{3,1} & D_{3,2} & D_{3,3}
\end{array}
$$

in which columns increase in the downward direction and rows decrease in the rightward direction. Our array contains sufficient initial conditions tempting us conclude that a new four-by-four array

$$
\begin{array}{cccc}
D_{1,1} & D_{1,2} & D_{1,3} & D_{1,4} \\
D_{2,1} & D_{2,2} & D_{2,3} & D_{2,4} \\
D_{3,1} & D_{3,2} & D_{3,3} & D_{3,4} \\
D_{4,1} & D_{4,2} & D_{4,3} & D_{4,4}
\end{array}
$$

has the same properties as our beginning array (wrong!). If so, we could repreat this process and show that since each $D_{x,x} = 0$, then $D_{x,x-1} > 0$. However, our mistake is that even though the second array does have the same signs as the first array and we know that $D_{3,1} \le D_{4,1}$, $D_{3,2} \le D_{4,2}$, $D_{3,3} \le D_{4,3}$, and $D_{3,4} \le D_{4,4}$, we no longer know that $D_{4,4} \le D_{4,3} \le D_{4,2} \le D_{4,1}$. So we created a new array that has the desired bi-directional monotonic properties in the upper-leftmost three-by-three region, but doesn't in the fourth column or fourth row. Therefore, our second array does not have the same properties of the first and we cannot invoke our inductive hypothesis. In fact, we have found explicit counterexamples showing that the second array does not necessarily have the same properties of the first array. Stated another way, x and y are not relatively close enough (even in a four-by-four array) for this nice bi-directional monotonic behavior to persist.

4.4. **What isn't the case.** We will now illustrate what we know isn't the case by looking at some specific examples.

Theorem 4.15. *If a ratio of solutions to non-negative difference equations increases for $2k - 1$ terms, then the ratio does not necessarily always increase.*

Proof. Consider $g_n = 2n + 1 + (-.88)^n$ and $h_n = n + 10 + 3(-.88)^n$. As we can see from Figure 8, we have a ratio of two solutions to a second-order non-negative difference equation that increases for the first $4 > 2(2) - 1 = 3$ terms but then begins to oscillate. Since $ad - bc = 2(10) - 1(1) = 19 > 0$, we know that this ratio will eventually be monotonically increasing, but its monontonic behavior does not begin directly after the $2k - 1^{th}$ increasing term. (Note that this example is not doubly non-negative.) $\qquad\square$
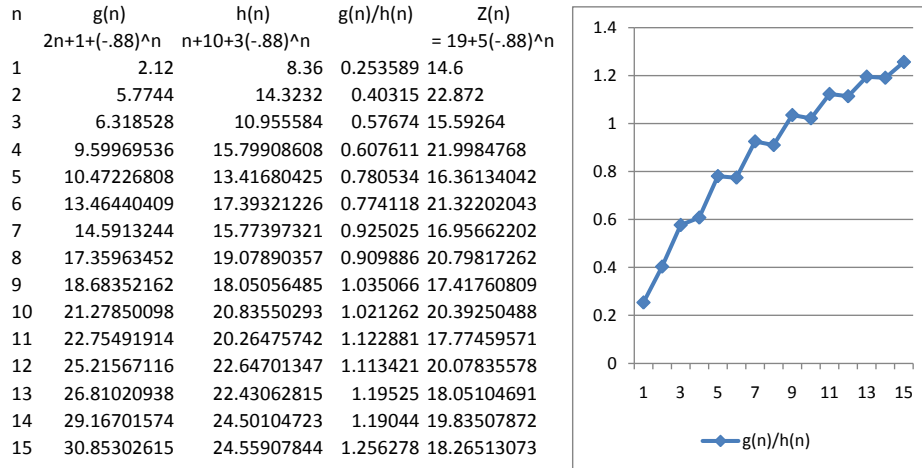
n	g(n) $2n+1+(-.88)^n$	h(n) $n+10+3(-.88)^n$	g(n)/h(n)	Z(n) $= 19+5(-.88)^n$
1	2.12	8.36	0.253589	14.6
2	5.7744	14.3232	0.40315	22.872
3	6.318528	10.955584	0.57674	15.59264
4	9.59969536	15.79908608	0.607611	21.9984768
5	10.47226808	13.41680425	0.780534	16.36134042
6	13.46440409	17.39321226	0.774118	21.32202043
7	14.5913244	15.77397321	0.925025	16.95662202
8	17.35963452	19.07890357	0.909886	20.79817262
9	18.68352162	18.05056485	1.035066	17.41760809
10	21.27850098	20.83550293	1.021262	20.39250488
11	22.75491914	20.26475742	1.122881	17.77459571
12	25.21567116	22.64701347	1.113421	20.07835578
13	26.81020938	22.43062815	1.19525	18.05104691
14	29.16701574	24.50104723	1.19044	19.83507872
15	30.85302615	24.55907844	1.256278	18.26513073

FIGURE 8. Data for Theorem 4.14

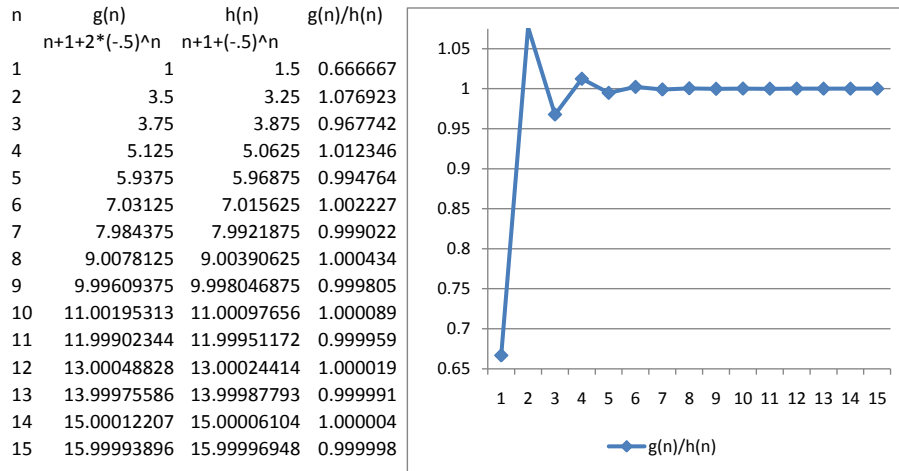| n	g(n) $n+1+2*(-.5)^n$	h(n) $n+1+(-.5)^n$	g(n)/h(n)
1	1	1.5	0.666667
2	3.5	3.25	1.076923
3	3.75	3.875	0.967742
4	5.125	5.0625	1.012346
5	5.9375	5.96875	0.994764
6	7.03125	7.015625	1.002227
7	7.984375	7.9921875	0.999022
8	9.0078125	9.00390625	1.000434
9	9.99609375	9.998046875	0.999805
10	11.00195313	11.00097656	1.000089
11	11.99902344	11.99951172	0.999959
12	13.00048828	13.00024414	1.000019
13	13.99975586	13.99987793	0.999991
14	15.00012207	15.00006104	1.000004
15	15.99993896	15.99996948	0.999998

FIGURE 9. Data for Theorem 4.15

Theorem 4.16. *The ratio of solutions to doubly non-negative recurrence relations does not necessarily increase monotonically.*

Proof. Consider $g_n = n+1+2(-.5)^n$ and $h_n = n+1+(-.5)^n$. As we can see from the Figure 9, the ratio does not increase monotonically. In fact, we can prove analytically that this ratio oscillates for all n. (Note that in this example $ad - bc = 0$.) □

4.5. **Conjectures for future research.** We now present some conjectures that will be the focus of future research. These are based on empirical observations of various data.

Conjecture 4.17.

$$\lim_{n \to \infty} \frac{D_{n,n-i}}{Z_n} = i$$

More importantly, we would like to know how quickly the error between i and our data decreases.

Conjecture 4.18. *The magnitude of a secondary root of a characteristic polynomial for a doubly non-negative recurrence relation can be made arbitrarily close to* 1.

Conjecture 4.19. *There exists a function $g(x,y)$ such that*

    (1) *$\hat{L}[g(x,y)] \leq 0$,*
    (2) *there exists a Y such that for all $y \geq Y$, $g(x,y) < 0$, and*
    (3) *$g(x,x-1) > 0$.*

Conjecture 4.20. *If g_n and h_n are solutions to $L^2[g_n] = L^2[h_n] = 0$ for some k^{th}-order doubly non-negative recurrence relation and $\frac{g_n}{h_n} \leq \frac{g_{n+1}}{h_{n+1}}$ for $n = 1, 2, \ldots, 2k-1$, then $\frac{g_n}{h_n}$ is increasing for all $n \in \mathbb{N}$!!!*

Conjecture 4.21. *Assuming our initial conditions are increasing may be an insufficient assumption for our desired result. We may need to assume additionally that $\frac{g_{n+1}}{h_{n+1}} < \frac{g_n}{h_n} + \frac{\beta}{n}$ for our intitial terms.*

5. CONCLUSION

We have shown under what conditions the long-term behavior of a ratio of solutions to a non-negative difference equation is monotonic. Observations lead us to suspect that this long-term behavior actully begins rather early (about the $2k-1^{th}$ term) for doubly non-negative difference equations. However, as is the case in mathematics, sometimes the simplest statements warrant complex proofs. Despite the fact that determining the exact short-term behavior of a ratio of solutions is difficult, we have managed to prove general trends about the behavior. We have shown that if a ratio of solutions to a doubly non-negative difference equation is on one side of its asymptote for $2k-1$ values, then future terms will always stay on that same side. We have also shown that for first-order non-negative difference equations, only one increasing term is sufficient to conclude that every term in the solution sequence increases. For second-order doubly non-negative difference equations, we can prove our desired result of subsequent monotonic behavior (after monotonic behavior for $2k-1$ terms) for a ratio of *certain* solutions to doubly non-negative difference equations. Extending this result to ratios of *any* solutions to doubly non-negative difference equations of *any* order will be the focus of future research.

REFERENCES

[C] Capocelli,R.M, "A Generalization of Fibonacci Trees," *Applications of Fibonacci Numbers* (Volume 3), G. Bergum, A. Philippou, and A. Horadam (eds.), 37-56, Kluwer; 1990.

[Cp] Capocelli, R.M,de Santis, A, "Regular universal codeword sets, *IEEE Transactions on Information Theory*, 32,1: 129 - 133; Jan. 1986.

[Ca] Capocelli,R.M, G. Cerbone, P. Cull, and J. L. Holloway. "Fibonacci Facts and Formulas," In *Sequences*, 123-137, Springer-Verlag New York, Inc; 1999.

[Cc] Capocelli R.M, Paul Cull. "Rounding the Solutions of Fibonacci-like Difference Equations," *Fibonacci Quarterly*, 41: 133-141, 2003.

[Cl] Cull, Paul, "A Brief Introduction to Algorithms," Chapter 9: Difference Equations, Class Notes; 2009. [website]http://classes.engr.oregonstate.edu/eecs/fall2009/cs325/.

[Cul] Cull, Paul, "Difference Equations as Biological Models," *Scientiae Mathematicae Japonicae*, 64,2: 217-233; Sept. 2006.

[Cu] Cull, Paul, Adam Murakami, and Stephanie Young, "Fast Fibonacci!," Congressus Numeratium, 200: 77-94; 2010.

[Cull] Cull,Paul, Mary Flahive, and Robby Robson, "Difference Equations: From Rabbits to Chaos," Springer, Inc. New York, New York; 2005.

[S] Sanders, Margaret A., "Counting Bits with Fibonacci and Zeckendorf," Proceedings of the Research Experiences for Undergraduates Program in Mathematics, 2006: 177-94. NSF and Oregon State University.

[K] Koshy, Thomas, "Fibonacci and Lucas Numbers with Applications," John Wiley and Sons, Inc. New York; 2001.

[Z] Zeckendorf, E. "Representations des Nombres Naturels par une Somme de Nombres de Fibonacci ou de Nombres de Lucas," *Bulletin de la Societé Royale des Sciences de Liege* 179-182;1972.

[Zt] *Zeckendorf's Theorem*. [website] http://planetmath.org/.

UNIVERSITY OF THE PACIFIC (STOCKTON, CA)

E-mail address: m_abram1@u.pacific.edu