GAPS: THE STRATEGY BEHIND A SPEEDY COUPLING OF CARD SHUFFLING

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ABSTRACT. Using methods of coupling we analyze two models of card shuffling, the random transposition and the random-to-random process, to find minimal coupling times. Disregarding the rank and suit on the standard playing cards we are familiar with, we focus only on the color; specifically, we consider decks where half the cards are white and half are black. We define the *total gaps* on a pair of decks, which proves to be a valuable distance and technique to a quick coupling. Using the total gaps and taking advantage of the color scheme of the decks, we bound the mixing time of the random-to-random shuffle to $O(n \log n)$. Along the way, we prove useful properties unique to a deck with only two colors.

1. INTRODUCTION

The stationary distribution π of an irreducible, aperiodic Markov chain satisfies $\pi = \pi P$ for transition matrix P and is the eigenvector corresponding to the eigenvalue $\lambda = 1$ for the matrix P. To couple two Markov chains (X_t, Y_t), take the original Markov chain X_t that has some initial probability distribution μ and introduce a second Markov chain Y_t that follows the stationary distribution. Both chains will use the same transition matrix, P.

It is useful to have some way to measure the distance between distributions. Thus, we define the **total variation distance** between any two distributions μ and ν over the same state space Ω, to be

$$\|\mu - \nu\|_{TV} = max_{A \in \Omega} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

By the Convergence theorem in [1], we know that an irreducible, aperiodic Markov chain with some initial distribution will converge to its stationary distribution. Often when dealing with Markov chains, we care about how long this convergence will take, which we call the mixing time. The **mixing time** of a Markov chain is precisely the time until any initial distribution gets ε-close to the stationary distribution:

$$\tau_{mix}(\varepsilon) = \|\mu_0 - \pi\|_{TV} < \varepsilon.$$

And since we will be using the coupling method to find upper bounds on mixing times, we define the coupling time. Given a coupling of two Markov chains (X_t, Y_t) the **coupling time** is the time until the chains meet for the first time:

$$\tau_{couple} = min\{t : X_t = Y_t\}.$$

Essential to relating coupling times and mixing times is the following theorem [2], which we call the coupling inequality,

$$\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} \leq \mathbf{P}_{x,y}\{\tau_{couple} > t\},$$

where $\{(X_t, Y_t)\}$ is a coupling that stays together once they first meet, with $X_0 = x$ and $Y_0 = y$. An immediate corollary is the following, using Markov's inequality [1]

$$\max_{x \in \Omega} \|P^t(x,\cdot) - \pi\|_{TV} \leq \max_{x,y \in \Omega} \mathbf{P}_{x,y}\{\tau_{couple} > t\} \leq \frac{\max_{x,y \in \Omega} \mathbb{E}(\tau_{couple})}{t}.$$

Thus, finding upper bounds of the coupling time by analyzing the worse case scenarios will give us upper bounds on the mixing time.

If we let A_i be a deck of n cards at time i then $A_0, A_1, \cdots$ is a Markov chain which is essentially a random walk on the set of unique permutations of a deck of n cards. Thus, we can produce a coupling (A_i, B_i) where B_i follows the stationary distribution and we simply wait for the two chains to meet. As long as our coupling is successful ($\tau_{couple} < \infty$), any initial distribution will converge to the stationary distribution; i.e. there exists some time t such that $\tau_{couple} = t$, which makes $\mathbf{P}_{x,y}\{\tau_{couple} > t\} = 0$, and thus the distribution of the chain at time t is equal to π.

1.1. Shuffling with a black and white deck.

We will consider a deck where exactly half of the cards are black and half are white. This requires different methods of coupling than a deck with all unique cards, as we only distinguish between the color of the card. Also, we expect the convergence to stationary to occur much quicker since each card has many potential "partners" in the second deck that it can be coupled with. Analyzing a deck of this type is useful for several reasons. In some card games, it suffices to have only the colors of the cards in the deck mixed, and considering each suit and rank would be unnecessary. More importantly, we can reduce any shuffling problem using a deck with n unique cards to a problem with only 2 unique cards, with $\log_2(n)$ different arrangements via binary representation. The shuffling methods we discuss in this paper are the random transposition and the random to random processes. We perform one shuffle at each unit of time until the two decks are completely coupled, and only at that point do we stop. The time until then is our coupling time.

2. RANDOM TRANSPOSITIONS USING DISCREPANCIES

Given a deck of n cards where $n/2$ cards are black and $n/2$ cards are white, we shuffle the deck using random transpositions: randomly choose two cards from the deck with replacement, and transpose them. The stationary distribution is uniform, with each of the $\binom{n}{n/2}$ arrangements equally likely; recall, we do not distinguish between cards of the same color.

In order to proceed we must introduce a second deck of cards of the same size that follows the stationary distribution; call this deck B and the original deck A. The simplest method of coupling involves choosing a card at random from deck A (with equal probability of being black and white), then locating a card of the same color in deck B (randomly choose one of the n cards of that color). Next, choose a random location in the decks, say $1 \leq k \leq n$, and the chosen cards in decks A and B will be transposed with that card at location k in their respective decks. However, this coupling is inefficient and can be improved by eliminating the possibility of increasing the number of discrepancies as a result of any one shuffle.

Let A_i denote the card chosen from deck A at position i, and B_j the card we decide to choose from deck B at position j; $(1 \leq i, j \leq n)$. Once location k is chosen, we transpose the cards A_i with A_k and B_j with B_k. Any pair of cards at the same location (A_i, B_i) will be called *coupled* if they are the same color, and *uncoupled* otherwise. Any single card that is not coupled is called a discrepancy. Lastly, we will use the variable d to denote the number of cards that are discrepancies.

Lemma 2.1. *In any shuffling problem with a deck of half black cards and half white cards, the number of coupled black cards must always equal the number of coupled white cards.*

Proof. Suppose there is not an equal number of coupled cards of both colors; let b be the number of black couples and w be the number of white couples, with $b \neq w$. Given that the deck has a total of n cards, there will be $n - b$ uncoupled black cards and $n - w$ uncoupled white cards. But the only way to have an uncoupled card is if it is paired with the opposite color, so it must be true that $n - b = n - w$, and $b = w$, a contradiction. $\square$

Using a somewhat simple method of coupling, we can achieve a coupling time of $n \log n$. The method we use is described below.

- Randomly choose card A_i.
  - If A_i is coupled, choose card B_j such that $j = i$.
  - If A_i is not coupled, locate card B_j from the cards that are uncoupled and the same color as A_i.
- Randomly choose a position k, and transpose A_i with A_k and B_j with B_k.

First, we must check that our method is "legal" in both decks; that is, when looking at the decks separately, each one must be undergoing the random transposition process. It is obvious that card A_i and location k are chosen randomly with uniform probability, so we just need to justify the randomness of choosing card B_j. Suppose there are currently d cards that are discrepancies, then any of the coupled cards in deck B will get chosen to be card B_j with probability $((n-d)/n)(1/(n-d)) = 1/n$. Similarly any of the discrepancies in deck B will get chosen as card B_j with probability $(d/n)(1/d) = 1/n$. Thus, we can see that each card in deck B has an equal chance of getting picked.

We start by analyzing the probability that a transposition results in an increase of coupled cards, which only occurs when A_i is a discrepancy. We must only consider the possible cards at position k because when A_i is a discrepancy, B_j is also a discrepancy by our method of coupling. Below are the three possible card arrangements at position k and their respective probabilities of occurring (see Figure 1 for visuals of the three cases).

Case 1. The cards are both the same color as A_i and B_j; $p = \frac{(n-d)/2}{n} = \frac{n-d}{2n}$

Case 2. The cards are both the opposite color as A_i and B_j; $p = \frac{(n-d)/2}{n} = \frac{n-d}{2n}$

Case 3. The cards are not the same color; $p = \frac{d}{n}$

Of these three cases, only case 2 and 3 will result in a decrease of two discrepancies. Case 1 results in no change of d.
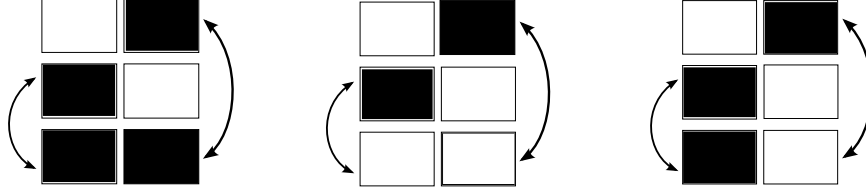
FIGURE 1. The three possible card arrangements when A_i and B_j are discrepancies. We distinguish between the three cases by the colors of the cards at location k.

$$\mathbb{P}(d-2) = \mathbb{P}(d-2|A_i \text{ coupled})\mathbb{P}(A_i \text{ coupled}) + \mathbb{P}(d-2|A_i \text{ not coupled})\mathbb{P}(A_i \text{ not coupled})$$

$$= (0)\left(\frac{n-d}{n}\right) + \left(\frac{n-d}{2n} + \frac{d}{n}\right)\left(\frac{d}{n}\right) = \left(\frac{n+d}{2n}\right)\left(\frac{d}{n}\right) = \frac{nd+d^2}{2n^2}$$

Observe that as d approaches 0, the probability above approaches 0; this makes sense because the likelihood of choosing a discrepancy gets smaller as the number of discrepancies decrease. We let $d = 2x$ to preserve the fact that d must always be even (from Lemma 2.1). The time between successive increases of couplings is a geometric random variable with probability of success $p = {nd+d^2}/{2n^2}$. To find the total expected time to achieve all couplings in the deck, we sum over the value of $d = 2x$ and approximate using an integral.

$$\mathbb{E}[\tau_{couple}] = \sum_{x=1}^{n/2} \frac{n^2}{2x^2+nx} \approx \int_1^{n/2} \frac{n^2 dx}{2x^2+nx}$$

which we evaluate using partial fractions,

$$\frac{1}{2x^2+nx} = \frac{1}{(x)(2x+n)} = \frac{A}{x} + \frac{B}{2x+n}$$

$$\frac{(x)(2x+n)}{(x)(2x+n)} = \frac{A(x)(2x+n)}{x} + \frac{B(x)(2x+n)}{2x+n}$$

$$1 = A(2x+n) + B(x)$$

with solutions $A = {1}/{n}$ and $B = {-2}/{n}$. Our integral becomes

$$\mathbb{E}[\tau_{couple}] = n^2 \int_1^{n/2} \frac{1/n}{x} - \frac{2/n}{2x+n}dx = n \int_1^{n/2} \frac{1}{x} - \frac{2}{2x+n}dx$$

$$= n(\log(x) - \log(2x+n))\Big|_1^{(n/2)}$$

$$= n(\log(n/2) - \log(2(n/2)+n) - \log(1) + \log(2+n))$$

$$\approx n\log(n) + O(n).$$

3. Random to Random with Discrepancies and Gaps

Given a deck of n cards that is half black and half white, we shuffle using the random-to-random process. A random card is removed from the deck and inserted into a random location in the deck, both locations chosen with uniform probability. Once again, we introduce a second identical deck of cards following the stationary distribution where each of the $\binom{n}{n/2}$ permutations of the cards is equally likely. For simplicity in explaining our strategies in this section, we imagine the locations of insertion to be the spaces between the cards (including both ends) rather than the cards them self. Since we know the cover time for random to random shuffling is $O(n \log n)$ by the coupon collector problem [1], our ultimate goal is to find a coupling argument that also produces this time.

This shuffle process presents different challenges than that of the random transposition. Particularly, some cards may become misaligned when a removal and insertion occurs. We begin with the obvious coupling method that gives a coupling time of n^2. However, we continue our search for a more efficient method and develop a different method of measuring the distance between any two distributions.

3.1. Discrepancy coupling.

There is a simple coupling method for this shuffle that is not as quick as we'd like, but produces a constantly decreasing number of discrepancies, thus a straightforward analysis. First, we randomly choose card A_i to remove from deck A and remove card B_i from deck B, both at location i, $1 \leq i \leq n$. Next choose a random location k to insert card A_i. Let l denote the location we insert card B_i in deck B. If A_i and B_i are the same color then we choose $l = k$. Otherwise, if k is next to a discrepancy of the opposite color, then we choose $l = k + 1$ or $l = k - 1$ such that the discrepancy cancels. If k is not next to a discrepancy, then we choose $l = k$ as to not disturb any other couples. See Figure 2 for an example of an insertion that decrease the discrepancies.
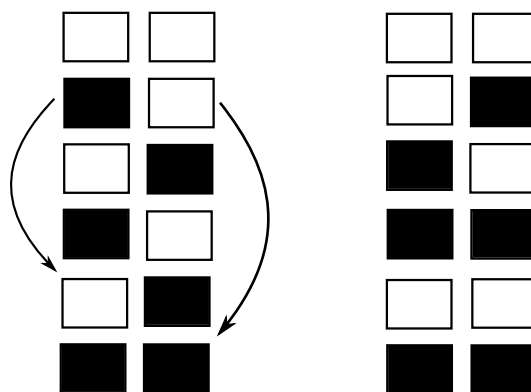

FIGURE 2. Removing a discrepancy and inserting next to a discrepancy of opposite color. This shuffle results in a decrease of two discrepancies, seen in the arrangement on the right.

It is easy to check that this coupling method performs its shuffling uniformly random. All removals and all insertions where $l = k$ are random; we must only check the case where $l = k + 1$

or $l = k - 1$. However, when k is next to a discrepancy, it could be either above or below it; if it is above we choose $l = k + 1$, and if it is below we choose $l = k - 1$.

The idea of this method is that we are not misaligning the cards in the decks unless it will result in a cancellation of discrepancies. Every time we remove a pair of cards and insert in a non-ideal location, we choose $l = k$ so we will simply be moving two cards in the decks such that the number of discrepancies do not change. To analyze the probability of a successful shuffle, we look at the worse case arrangement where all the discrepancies of the same color are next to each other (this gives us the least number of choices of locations next to a discrepancy of the opposite color). Thus, if there are d discrepancies in the deck, we will remove one with probability d/n. At worse, and insert next to a discrepancy of opposite color with probability $(d/2)+1/n$. And since removal and insertion are done independently, the probability of a decrease in discrepancies is $(d/n)(d+2/n) = d^2+2d/2n^2$. The time between each successful shuffle is geometrically distributed with probability of success $\frac{d^2+2d}{2n^2}$ and expected value $2n^2/d^2+2d$, so we can sum over the values of d to find the expected (worse case) coupling time. Again, we let $d = 2x$ since d is always even, and approximate with an integral.

$$\mathbb{E}[\tau_{couple}] = \sum_{x=1}^{n/2} \frac{2n^2}{4x^2 + 4x} = \frac{1}{2}n^2 \sum_{x=1}^{n/2} \frac{1}{x^2 + x} \approx \frac{1}{2}n^2 \int_1^{n/2} \frac{dx}{x^2 + x}$$
$$= \frac{1}{2}n^2 \int_1^{n/2} \frac{1}{x} + \frac{-1}{x+1} dx = \frac{1}{2}n^2 (\log(x) - \log(x+1))_1^{n/2}$$
$$= \frac{1}{2}n^2 (\log(n/2) - \log((n/2) + 1) - \log(1) + \log(2)) = O(n^2)$$

3.2. Gap coupling. We will couple the decks using a more clever method. Randomly remove a card from deck A and determine its position in the deck out of only those cards of the same color (the 3rd white card, the 6th black card, etc.); suppose it is the xth black card. Remove the xth black card from deck B. Then, randomly choose a location and insert both cards in that same location in their respective decks. For this method, we will use the term "pair" as follows: the pair of the xth black card in A is the xth black card in B, not necessarily at the same location.

Definition 3.1. *The **total gaps** is the sum of the space between each pair of cards when we couple them as defined in our coupling method. If the card at position a is the ith card of color x in deck A, and the card at position b is the ith card of color x in deck B, then*

$$g := \sum_{a=1}^{2n} |a - b|.$$

See Figure 3 for an example of how to count the gaps in a deck.

Example 3.2. *Consider the arrangement with six cards in each deck in Figure 4. The initial configuration has $g = 8$, and we see that g decreases to either 6 or 4 depending on the location of insertion. Note also that if we count the number of discrepancies, it increases as a result of the shuffle.*
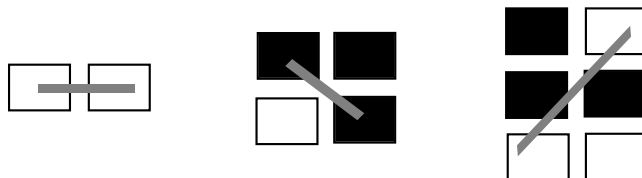
FIGURE 3. Example of how to count gaps. From left to right, pairings of cards that have gap values of 0, 1, 2.
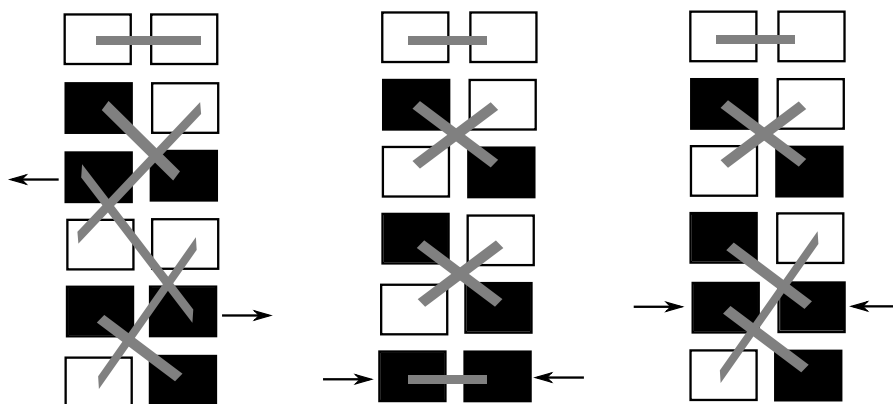

FIGURE 4. Example of determining the total gaps of a deck. Paired cards are indicated by the grey line connecting them. From left to right: the starting configuration, $g = 8$; one possible resulting configuration, $g = 4$; another possible resulting configuration, $g = 6$.

This technique of counting gaps instead of discrepancies seems to be effective because each time a pair of cards is inserted, it will break up any longer gaps that stretch over it since we re-pair the cards after each shuffle. Observe that our goal is to have the total gaps equal zero, since then all the cards would be paired. To get a peek at the potential of this new measure of distance, we wrote a program to simulate random-to-random shuffling using our coupling method. We can compare the behavior of the number of discrepancies and the number of gaps at each unit of time up until the coupling time, see Figure 5. Observe that the curve for the gaps has a very nice shape; specifically, it decreases very quicky. However, we also realize that this method of couting gaps also has times where the gaps will increase, leading to a difficult analysis. The specific arrangement of cards and shuffle that causes this increase is shown in Figure 6.

Since our goal is to find a way to normalize or smooth the curve in our simulation, we can do an average-case analysis of our coupling method using gaps to find the expected coupling time. Before proceeding to analyze this measure, we prove a few properties about the gaps in a deck that will be helpful in the future.

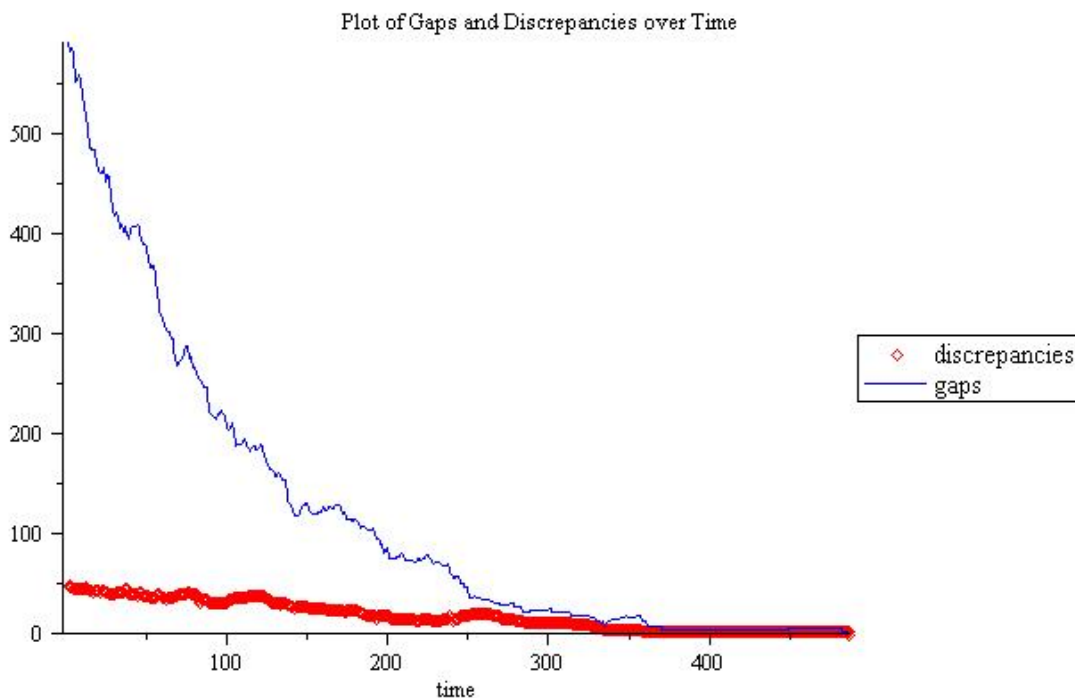**Lemma 3.3.** *The maximum value for the total gaps over a pair of deck of size n is $n^2/2$.*

FIGURE 5. Simlation of random-to-random shuffling using a deck of 100 cards. Gaps and discrepancies calculated after each shuffle until the coupling time.
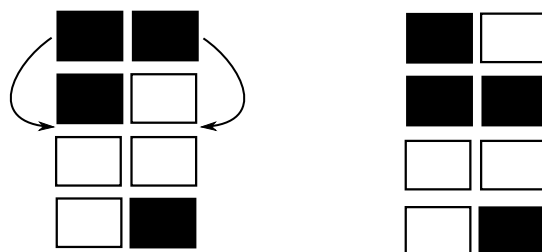

FIGURE 6. After the shuffle indicated by the arrows, the gaps increase from 4 to 6.

Proof. Each card can have a gap size of at most $n/2$, so with n cards, the total gap size is $n^2/2$. This occurs only when all cards are discrepancies and all discrepancies of the same color are next to each other; i.e. deck A has order $(B, \cdots, B, W, \cdots, W)$ and deck B has order $(W, \cdots, W, B, \cdots, B)$, or vice versa. Thus, there are 2 unique arrangements that will have the maximum number of gaps. Also, the minimum number of gaps is 0, when all cards are coupled.   □

Lemma 3.4. *In any given transposition of neighboring cards in only one deck, the total black gap and total white gap will each change by either* $1, 0,$ *or* -1.

Proof. When two neighboring cards are transposed, they will remain paired with the same cards in the other deck unless the transposing cards are the same color. In the former case, the cards

are changing their location by 1, and since the cards are not changing partners, the gap length will either be stretched by 1 or compressed by 1. In the latter case the pairs will be swapped, but we disregard noting the change since we do not distinguish between cards of the same color.   □

Lemma 3.5. *For any pair of neighboring cards in a deck, if one has a gap of zero, and the other has a gap greater than zero, the lines illustrating their gaps (like those shown in Figure 4) cannot be crossing.*

Proof. Suppose there is a black card with a gap of zero, and a white card below it with a gap greater than zero, whose partner is above the black card; i.e. their gap lines cross. Then the two black cards in that pair must be the ith black card in deck A and the ith black card in deck B, which means there are $i - 1$ black cards above the pair in deck A and in deck B. Also, the only way the white card is paired with a white card above the black pair is if there is a different number of white cards, say a and b with $a \neq b$, above the black pair in deck A than deck B. So the total number of cards above the black pair in deck A is $i - 1 + a$ and that in deck B is $i - 1 + b$; these values must be equal since the black pair has a gap of zero, so $i - 1 + a = i - 1 + b$ and $a = b$, a contradiction.   □

Lemma 3.6. *The total number of gaps for black cards is equal to the total number of gaps for white cards in any given arrangement of two decks of (half black, half white) cards.*

Proof. We proceed by induction on the number of transpositions, beginning with decks that have a gap total of 0. Our transpositions will be done by swapping any two neighboring cards in any one deck. We will only consider those transpositions that involve switching a black and a white card, and ignore transposing two black or two white cards since they are redundant. Note that any permutation of n cards can be achieved via a series of transpositions of neighboring cards (a product of two-cycles). Let t_i be the time that we perform the ith transposition in our process that is not redundant.

In our base case, $i = 1$, we start with identically arranged decks and transpose any two neighboring cards in one of the decks. It is obvious that the gap total for black cards increases by 1 and the gap total for white cards increases by 1. Thus, our gap total for the two decks increases from 0 to 2 and the black and white gap totals are equal.

Now assume for all $i \leq n$ the number of black gaps is equal to the number of white gaps. For $i = n + 1$, we will switch any two neighboring black and white cards. Using Observation 3.4, we know what possible values the change in total gaps will have. Thus, we must just determine if the change will be positive, negative, or zero. We base this decision on whether the gap lines (as shown in Figure 4) of neighboring cards are crossing or not.

- If both cards have individual gaps of zero, then the black gaps will increase by 1 and the white gaps will increase by 1.
- If one has a gap of zero and one had a gap greater than zero, then by Lemma 3.5 their gap lines cannot be crossing, and so both will increase by 1.
- If both have gaps greater than zero and:
    - their gap lines are crossing each other then both gaps will decrease by 1;
    - their gap lines are not crossing then the gaps will both increase by 1.

In any case, the two cards being swapped will behave in the same way; that is, they will both either increase by 1 or decrease by 1. Thus, after one transposition, the black gaps will still equal the black gaps.   □

Lemma 3.7. *Without loss of generality, consider only the gaps of the black cards. Suppose we just removed a pair of cards from our decks and there are now g_t black gaps. When we insert the pair back into the deck at a location that has x gaps crossing it, then the black gap total will increase by x.*

Proof. Suppose we have a location k with x gaps crossing over it. By inserting two cards into k, we are essentially turning one location into two locations, say k_1 and k_2 (one on each side of the cards that we inserted). Of the gaps crossing k, $x-1$ of them will still cross both location k_1 and k_2; one of the gaps will get broken into two smaller gaps, creating one additional gap crossing k_1 and one additional gap crossing k_2. Thus, when counting gaps after the insert, k_1 will have $x-1+1$ gaps crossing over it and k_2 will have $x-1+1$ gaps crossing over it, which is an increase of x gaps. See Figure 7 for examples of how the gaps increase. □

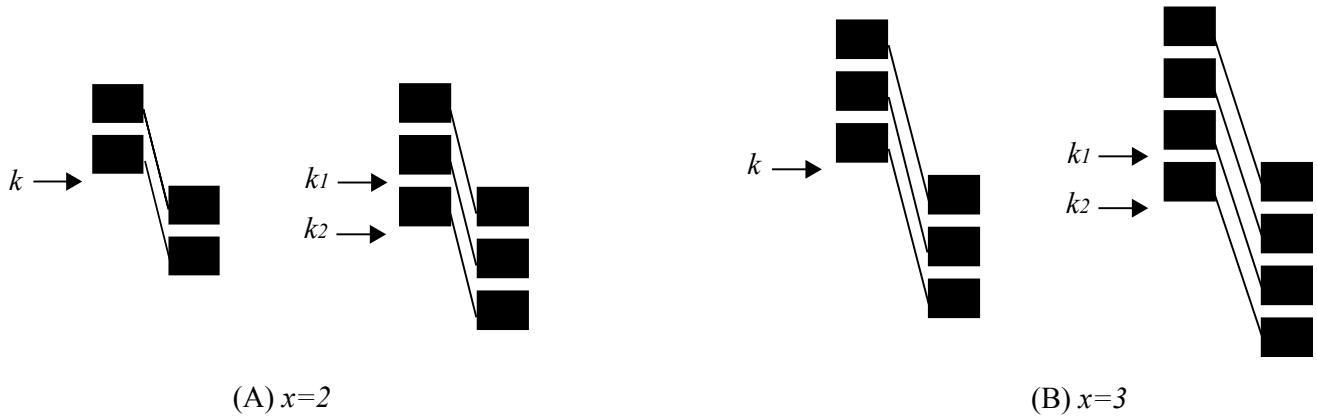(A) *x=2*   (B) *x=3*

FIGURE 7. Examples of how an insertion increases the gaps by the number of gaps crossing that location. (A) two gaps crossing k increases the black gap total by 2; (B) three gaps crossing k increases the black gap total by 3.

Theorem 3.8. *The upper bound on the mixing time of the random to random shuffle is $2n\log n + O(n)$ using a coupling argument with a deck of half black, half white cards.*

Proof. We will use the coupling method as described in the beginning of Section 3.2, and measure the distance between the two decks with the total gaps. To simplify the analysis we will only consider the black cards in the deck, which will not have any significant effect since we proved that the total white gaps is equal to the total black gaps in Lemma 3.6 (all shuffles will involve the removal of only black cards). Generalizing to the entire deck will be simple.

Suppose at time t we have g_t black gaps in a deck of n cards, half black and half white. First, remove a pair of cards using the coupling method; at this point the gaps will have decreased or at worse, remained the same. We calculate the expected value that the gaps will decrease by finding the average gap size at time t, which is $g_t/(n/2)$. We subtract this from g_t to get the expected number of gaps in the deck after the removal,

$$g_t - \frac{g_t}{n/2} = \left(1 - \frac{2}{n}\right)g_t.$$

Next, we insert the cards at some location, determined at random. We can find the expected amount that the gaps will increasing just by determining how many gaps were crossing that location; for each gap that is crossing the location, the black gap total will increase by 1, by Lemma 3.7. We find the expected gap crossings at each location, which is

$$\frac{\text{number of gaps}}{\text{number of locations}} = \frac{\left(1 - \frac{2}{n}\right) g_t}{n}.$$

If we take the expected number of gaps after the removal and add it to the expected number of gaps per location, we have the expected number of gaps after one shuffle,

$$\mathbb{E}\left(g_{t+1}\right) = \left(1 - \frac{2}{n}\right) g_t + \frac{\left(1 - \frac{2}{n}\right) g_t}{n}$$

$$= \left(1 - \frac{2}{n}\right)\left(1 + \frac{1}{n}\right) g_t$$

$$= \left(1 - \frac{1}{n} - \frac{2}{n^2}\right) g_t.$$

We know the maximum number of gaps in a deck of size n is $n^2/2$ from Lemma 3.3 and we are only considering the black cards, so we have the bound $g_0 \leq n^2/4$. We want to find the smallest t such that

$$\mathbb{E}(g_t) = \left(1 - \frac{1}{n} - \frac{2}{n^2}\right)^t g_0 \leq \varepsilon,$$

but since our function is decreasing over time, this will occur when $\mathbb{E}(g_t) = \varepsilon$. Using the initial (worse case) condition $g_0 = n^2/4$ and using the approximation $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$, we have

$$\left(1 - \frac{1}{n} - \frac{2}{n^2}\right)^t \left(\frac{n^2}{4}\right) = \varepsilon$$

$$t \log(1 - \frac{1}{n} - \frac{2}{n^2}) = \log(4\varepsilon) - 2\log(n)$$

$$t\left(-\frac{1}{n} + O\left(1/n^2\right)\right) = \log(4\varepsilon) - 2\log(n)$$

$$-\frac{1}{n}t\left(1 + O\left(1/n\right)\right) = -2\log(n) + \log(4\varepsilon)$$

$$t = 2n\log(n)\left(1 + O\left(1/n\right)\right) - n\log(4\varepsilon)\left(1 + O\left(1/n\right)\right)$$

$$= 2n\log(n) + O\left(n\right).$$

Thus, we can apply Markov's inequality and the coupling inequality to our analysis of the expected number of gaps at some time t since

$$\|\mu^t - \pi\|_{TV} \leq \mathbf{P}\{\tau_{couple} > t\} = \mathbf{P}\{g_t \geq 1\} \leq \frac{\mathbb{E}(g_t)}{1} \leq \varepsilon.$$

By bounding the expected gaps by ε, we are also bounding the distance between our distribution and the stationary distribution by ε. Therefore, the time we found is an upper bound to the mixing time of the random-to-random shuffle with a deck of black and white cards. $\qquad\square$

Recall, our proof involved only shuffling the black cards in the deck, so what about the white cards? Since we know that after every shuffle, the total black gaps is equal to the total white gaps, by just following the black gaps we are also following the value of the white gaps. Thus, when the black gaps reaches zero (all the black cards are coupled), it must also be true that the white gaps is zero. Therefore, our argument holds even if we were to pull both black and white cards from the deck.

We can also check that this theoretical analysis fits our simulation of a real set of decks being coupled, see Figure 8.
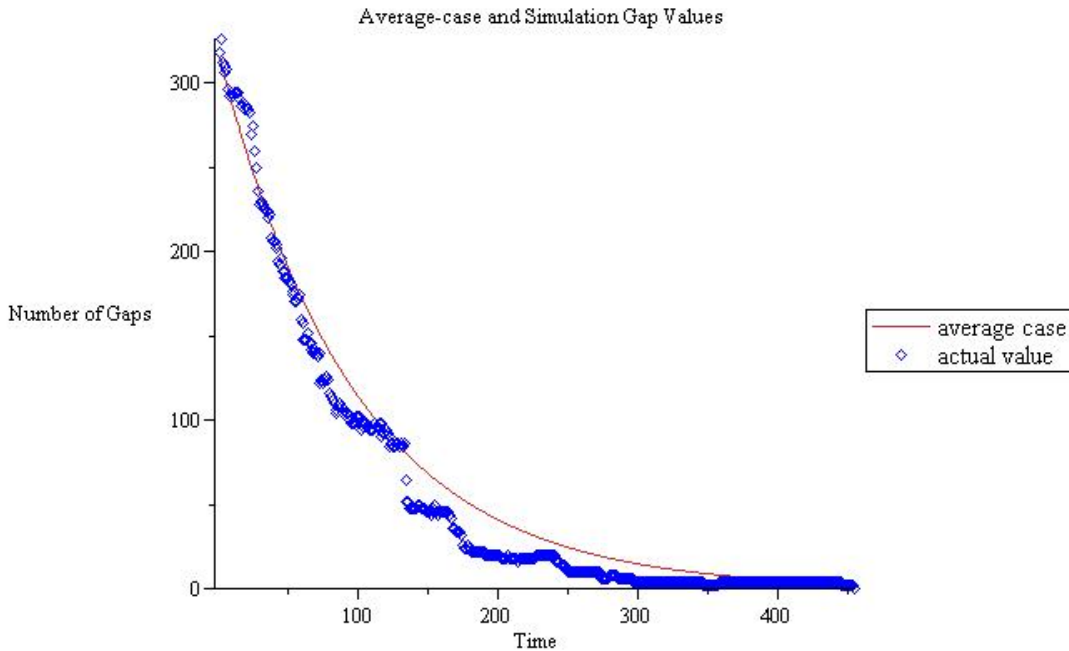

FIGURE 8. Average case analysis on plot of simulation.

4. CONCLUSION AND POSSIBLE IMPROVEMENTS

Using coupling techniques, we found upper bounds for the mixing time of two different card shuffling problems. For the random transposition shuffle, we proved a $O(n\log n)$ bound using a simple coupling method and counting discrepancies. The random-to-random shuffle proved to be more difficult; we were able to find a $O(n^2)$ upper bound on the mixing time with discrepancies,

but needed a new measure on distance to improve this bound. Our roadblock to a quicker coupling time using discrepancies led to the discovery of the total gaps, which gave us an $O(n \log n)$ bound on mixing. Also, recall that the key strategy to approaching this problem relied on the fact that the deck was half black and half white; were were able to consider the shuffle that only removed black cards (essentially ignoring the white cards), which is an exact model to the shuffle that removes both black and white cards.

Finding improvements to the bounds we discussed in this paper is an open problem. Also, one might consider other shuffling models using the black and white deck, taking advantage of the special properties to this deck. More research should be done on the "toy model" that we use in our main result, which considers shuffling that removes only one color.

APPENDIX A. PROGRAM TO SIMULATE COUPLING IN MAPLE

```
GDsimulation := proc(n::posint)
> # simulates the entire coupling of two decks and plots the
> # discrepancies and gaps after  every shuffle
> # n is the number of cards in the deck (even integer)
> # to execute: GDsimulation(100); produces plot of 100-card simuation
>
> local A,B,                   # two decks of cards-half 0, half 1
>                                 # 0=Black, 1=White
> i,j,t,                       # counters in for loops
> time,                        # time keeps track of time
> cup,                         # tells whether the decks are coupled
> numgaps, numdiscr,           # total number of gaps and discr
> C,                           # array of two decks, pass from the shuffle method
> T,                           # array to plot time
> G,D,                         # array to keep gaps & discr
> L1,L2;                       # lists to prepare data to plot
> AVG,temp                     # to plot theoretical curve
>
> # initialize local variables
> time := 0; cup := true; numgaps := 0; G := []; D := [];
> A := initarray(n); B := initarray(n);
>
> # randomize the decks using shuffle method
> with(Statistics);
> A := Shuffle(A); B := Shuffle(B);
>
> # check if deck is coupled initially
> for i to n do if A[i] <> B[i] then cup := false end if end do;
>
> # main while loop that does shuffling and counts gaps and discrepancies
> while cup = false do
>   time := time+1; C := shuffledeck(A, B, n);
>
>   # separate the decks and count gaps/discrepancies
```

```
>   A := C[1]; B := C[2];
>   numgaps := 0; numgaps := countgaps(A, B, n);
>   numdiscr := 0; numdiscr := countdiscr(A, B, n);
>
>   # add nim gaps and discr to array to plot later
>   G := [op(G), numgaps]; D := [op(D), numdiscr];
>
>   # again, check if decks are coupled
>   cup := true;
>   for j to n do if A[j] <> B[j] then cup := false;
>   end if; end do;
> end do;
>
> # OPTION 1: plot of gaps & discr versus time
> T:=[];
> for t from 1 to time do T:= [op(T), t]; od;
>
> L1 := zip( (X,Y)->[X,Y], T,D );
> L2 := zip( (X,Y)->[X,Y], T,G );
>
> # OPTION 2: plot of gaps and theoretical curve
> T := []; AVG := [];
> for t to time do T := [op(T), t];
>   temp := convert((1-1/100-2/100^2)^t*G[1], float, 100000);
>   AVG := [op(AVG), temp] end do;
>
> L3 := zip(proc (X, Y) options operator, arrow; [X, Y] end proc, T, AVG);
> L4 := zip(proc (X, Y) options operator, arrow; [X, Y] end proc, T, G);
>
> with(plots):
>
> # use either L1,L2 or L3,L4 to plot
> return plot( [L1,L2], color=[red,blue], style=[point,line]);
> end:

initarray := proc (n)
> # called by GDsimulation()
> # initializes the arrays to contain half 1, half 0
> # n is the number of cards in the deck
>
> local i, x;
>
> # initialize local variable
> x := [];
>
> for i to n do
>   if i <= (1/2)*n then x := [op(x), 0] else x := [op(x), 1] end if;
```

```
> end do;
>
> return x;
> end:

shuffledeck:=proc(deckA,deckB,numCards);
> # called by GDsimulation()
> # A, B are the two decks we wish to couple
> # procedure takes a random card and inserts it to a random location
> # for both decks
>
> local A,B,n,          # the two decks and deck size
> generateRand,         # variable to generate random number
> rem, ins,             # location to remove from deck A and insert
> bA,bB,wA,wB,          # the locations of b & w cards in deck A & B
> i,j,k,l,x,y,q,z,      # counters in for loops
> removeFromB,          # the location of the card to remove from deck B
> isBlack,              # boolean value saying if a card is black
> C;                    # array to return both decks to the main method
>
> # initialize local vairables
> A:=deckA;  B:=deckB;  n:=numCards;
> bA:=0; wA:=0; bB:=0; wB:=0; removeFromB:=0;
> generateRand:=rand(1..n);  rem:=generateRand();   ins:=generateRand();
>
> # set the value of isBlack
> if A[rem] = 0 then isBlack := true else isBlack := false end if;
>
> # locate the card in deck A to remove and determine its
> # position in the deck (our of the same color cards)
> if isBlack then
>   for i to rem do
>     if A[i] = 0 then bA := bA+1 end if;
>   end do; # we know that rem is the (bA)th black card in the deck
>
>   # find the (bA)th black card in deck B and store its position
>   for j to n do
>     if B[j] = 0 then bB := bB+1 end if;
>     if bB = bA then removeFromB := j; break end if;
>   end do; # we know that j is the index of the (bA)th card in deck B
>
> else # the card is white
>   for k to rem do
>     if A[k] = 1 then wA := wA+1 end if;
>   end do; # we know that rem is the (wA)th white card in the deck
>
>   # find the (wA)th white card in deck B and store its position
```

```
>   for l to n do
>      if B[l] = 1 then wB := wB+1 end if;
>      if wB = wA then removeFromB := l; break end if;
>    end do; # know that j is the index of the (wA)th card in deck B
> end if;
>
> # remove and insert cards in deck A
> if (rem <ins) then
>    for x from (rem) to (ins-1) do A[x]:=A[x+1];  od;
>    if isBlack then A[ins]:=0; else A[ins]:=1; fi;
> else
>    for y from (rem) to (ins+1) by (-1) do A[y]:=A[y-1];  od;
>    if isBlack then A[ins]:=0; else A[ins]:=1; fi;
> fi;
>
> # remove and insert cards in deck B
> if (removeFromB <ins) then
>    for q from (removeFromB) to (ins-1) by 1 do B[q]:=B[q+1];  od;
>    if isBlack then B[ins]:=0; else B[ins]:=1; fi;
> else
>    for z from (removeFromB) to (ins+1) by (-1) do B[z]:=B[z-1];  od;
>    if isBlack then B[ins]:=0; else B[ins]:=1; fi;
> fi;
>
> C:=[A,B];
> return C;
> end:


countgaps := proc (deckA, deckB, numCards)
> # called by GDsimulation()
> # counts the gaps for the given arrangement
>
> local A, B, n, # the two decks and number of cards
> i, j, k,       # counters in for loops
> b, w,          # the number of b & w cards alread looked at
> g;             # running total of gaps
>
> # initialize the local variables
> A := deckA; B := deckB; n := numCards;
> b := 0; w := 0;
> g := 0;
>
> # iterate through deck A and keep track of the wth white card
> # and bth black card
> for i to n do
>    #find the ith white and black card and compute their gap size
>    if A[i] = 0 then # the card at location i is black
```

```
>      for j from b+1 to n do
>        if B[j] = 0 then b := j; g := g+abs(j-i); break; end if;
>      # update the value of b and add the gap value to the total count
>      end do;
>    else # the card is white
>      for k from w+1 to n do
>        if B[k] = 1 then w := k; g := g+abs(k-i); break; end if;
>      # update the value of w and add the gap value to the total count
>      end do;
>    end if;
> end do;
>
> return g; # return the total number of gaps
> end:


countdiscr := proc (deckA, deckB, numCards)
> # called by GDsimulation()
> # counts the discrepancies for the given arrangement
>
> local A, B, n,    # the two decks and number of cards
> i,                # counter in for loop
> d;                # running total of discrepancies
>
> # initialize the local variables
> A := deckA; B := deckB; n := numCards; d := 0;
>
> # iterate through deck A and determine if the two cards are discrepancies
> for i to n do
>   if A[i] <> B[i] then d := d+1 end if;
> end do;
>
> return d; # return the total number of discrepancies
> end:
```

REFERENCES

[1] D. A. Levin, Y. Peres, and E. Wilmer, *Markov Chains and Mixing Times*. Providence, RI: American Mathematical
     Society, 2009.
[2] T. Lindvall, *Lectures on the Coupling Method*. John Wiley and Sons, Inc, 1992.

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