EXTINCTION AND CYCLES OF FRACTAL DIMENSION IN ULAM'S CELLULAR AUTOMATON

Rebecca Anderson
Jeremy Smith
Oregon State University

This paper describes the dynamics of a cellular automata system based on a proposal of Stanislaw Ulam and Robert Schrandt. A modification in the form of a computer program allowed for an aging process as well as a birth pattern. Since this structure is a finite space, starting configurations either result in elimination of the colony, or a cycle of finite period. There are several examples of each exhibited in this work. Since a number of the cycles are complicated, and the spaces are too large to enumerate comfortably, windows are used to construct projections that indicate the presence of an attractor or stable point. Two methods of constructing the window are used to eliminate bias; resulting plots are presented here. The Hausdorf dimension of the attractor of the associated continuous set was calculated using the method in Berge et. al. The dimension is found to be fractional.

Ulam and Schrandt created a population model that featured competition between two ageless species for space. The model used in this study is a modification of their model, using the rules for reproduction, but only one species. The program allowed the user to determine the initial configuration of the colony and the lifespan of the organism to be studied. Any organism could have, at most, four offspring in any one generation; actual reproduction depended on the proximity of other organisms. Rules are as follows:

1. Birth can occur if exactly one neighbor cell is occupied.
2. Birth can occur in a cell where death occurs
3. If two units could have been born in the same cell, or in adjacent cells no birth occurs.

Growth and death takes place on tori of varing sizes, which eliminates boundary conditions and accompanying difficulties. For this paper the investigation is restricted to square tori and a two cycle lifespan. Sample generations are shown below(Figure 1).

Figure 1.

6x6 Torus

Generation 1	Generation 2	Generation 3	Generation4	Generation5	Generation 6
000000	010000	020000	010101	020202	000000
010000	121000	202101	101202	202000	000000
000000	010000	020000	010101	020202	000000
000000	000000	010000	121000	202000	000000
000000	000000	000000	000000	000000	000000
000000	000000	010000	121000	202000	000000

Trial runs of the model for varying sizes enable observations concerning the nature of the colonies. The colonies and their progress are represented by matrices, here referred to as boards. Moore proves that a Garden of Eden configuration is a necessary condition for the existance of self-reprooduction. Boards can be divided into regenerative and nonregenerative types. Regenerative boards will be reproduced at some time in the life of some cyclic colony. Nonregenerative boards can be divided into three types: terminal, Garden of Eden, tailboards. A tail is a set of boards leading to a cycle. Cycles can vary in length and it is often not clear if a series of boards is a tail or a set of terminal boards until the appearance of the cycle or death. Any board is made of an arrangement of zeros, ones, and twos; the configuration of the nest board being determined by the rules of the model applied to the previous board. The existance of tails implies the existance of regenerative boards with more than one parent. Some statements concerning the model are:

1. Any configuration with solely ones is a Garden of Eden configuration.
2. Let a be the number of ones in a configuration, b the number of twos. If $a > 4b$, the configuration is a Garden of Eden.
3. Any configuration of two's or zero's has a parent.
4. Any configuration of two's or zero's cannot regenerate itself.
5. All 1x1, 2x2 , 3x3 configurations will always die out due to space restrictions.
6. If the number of ones on a mxm torus exceeds m^2- 4the colony will die.

Simple initial conditions such as one or more organisms arranged linearly resulted in elimination of the colony. In the case of a lone unit with a two cycle lifespan, the lifespan of the colony is of the form 2^n+2 and colonies with the ssame lifespan are grouped in clusters of 2^n. As the number of organisms in the initial state increased, the lifespan of the colony retained the same value, but the cycle of elimination changed by a constant, for example a linear arrangement of three has lifespan 2^n+3.

Figure 2 LIFESPANS OF COLONIES WITH N=1 AT T=0.

M	LIFESPAN	M	LIFESPAN
3	3	18-33	18
4,5	4	34-65	34
5-9	6	66	66
10-17	10		

Several simple 4x4 initial configurations lead to cycling. Most have short tails , these here have tails of one or two, and cycle of period two. A list is given below(Figure 3). Note the element of diagonality in each one. It is possible to eliminate a large number of boards as potentiall tails or cycles by the simple observation that of the 3^{16} possible boards, 2^{16} are not candidates. This still leaves nearly forty-million boards to be eliminated, a formidable task.

Figure 3

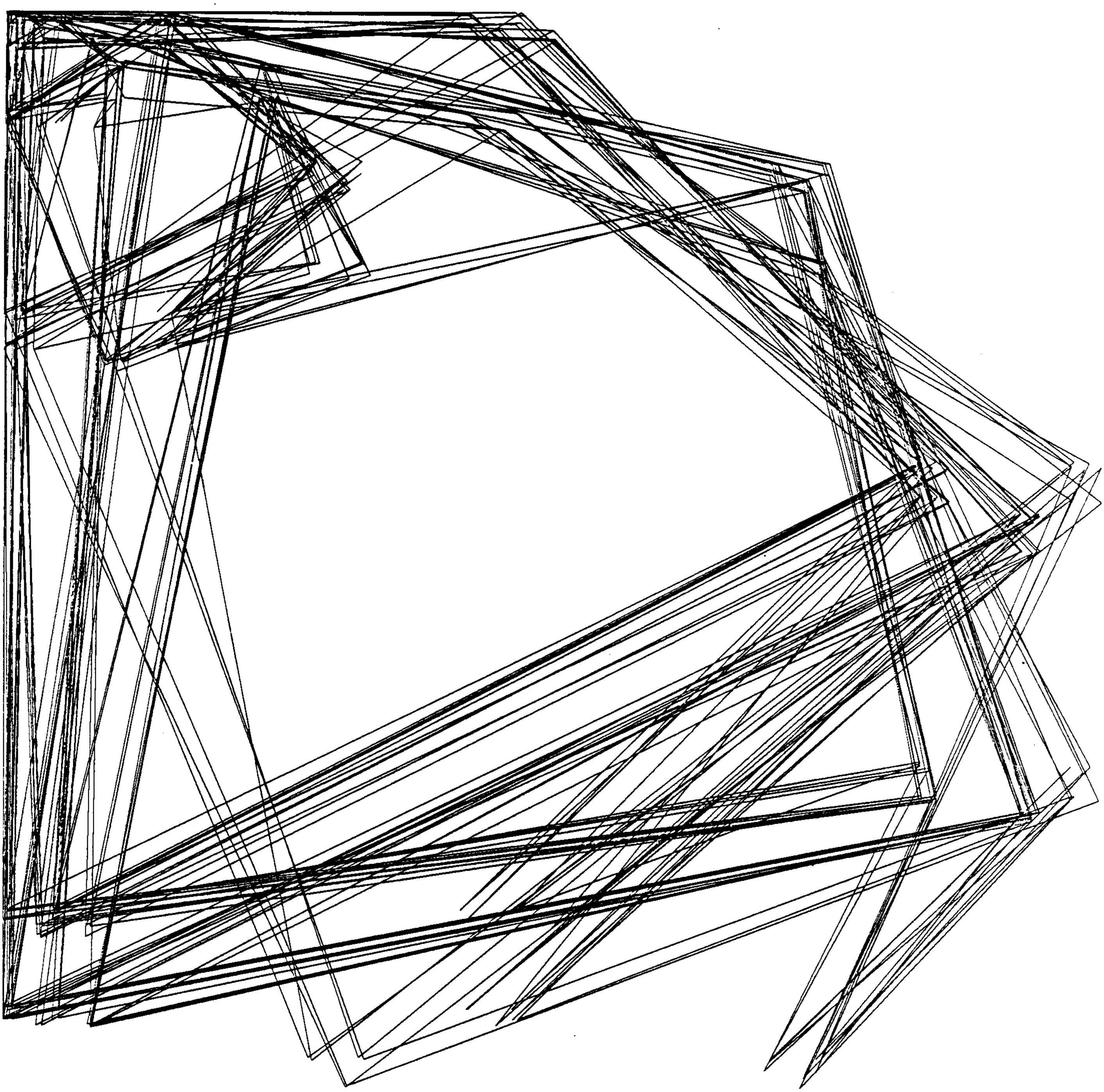
SOME CYCLIC CONFIGURATIONS IN 4X4 TORI

1000	1010	1011	1010	1010	1001	1100	1100	1010	1011	1100
0010	0100	0100	1100	0110	1010	0100	1000	0101	0100	0000
0000	0000	0000	0000	0000	0000	1000	0100	0000	0000	1000
0000	0000	0000	0000	0000	0000	0000	0000	0000	0000	0100

Note: The second configuration above cycles for 5x5 through 12x12 but not for 9x9.

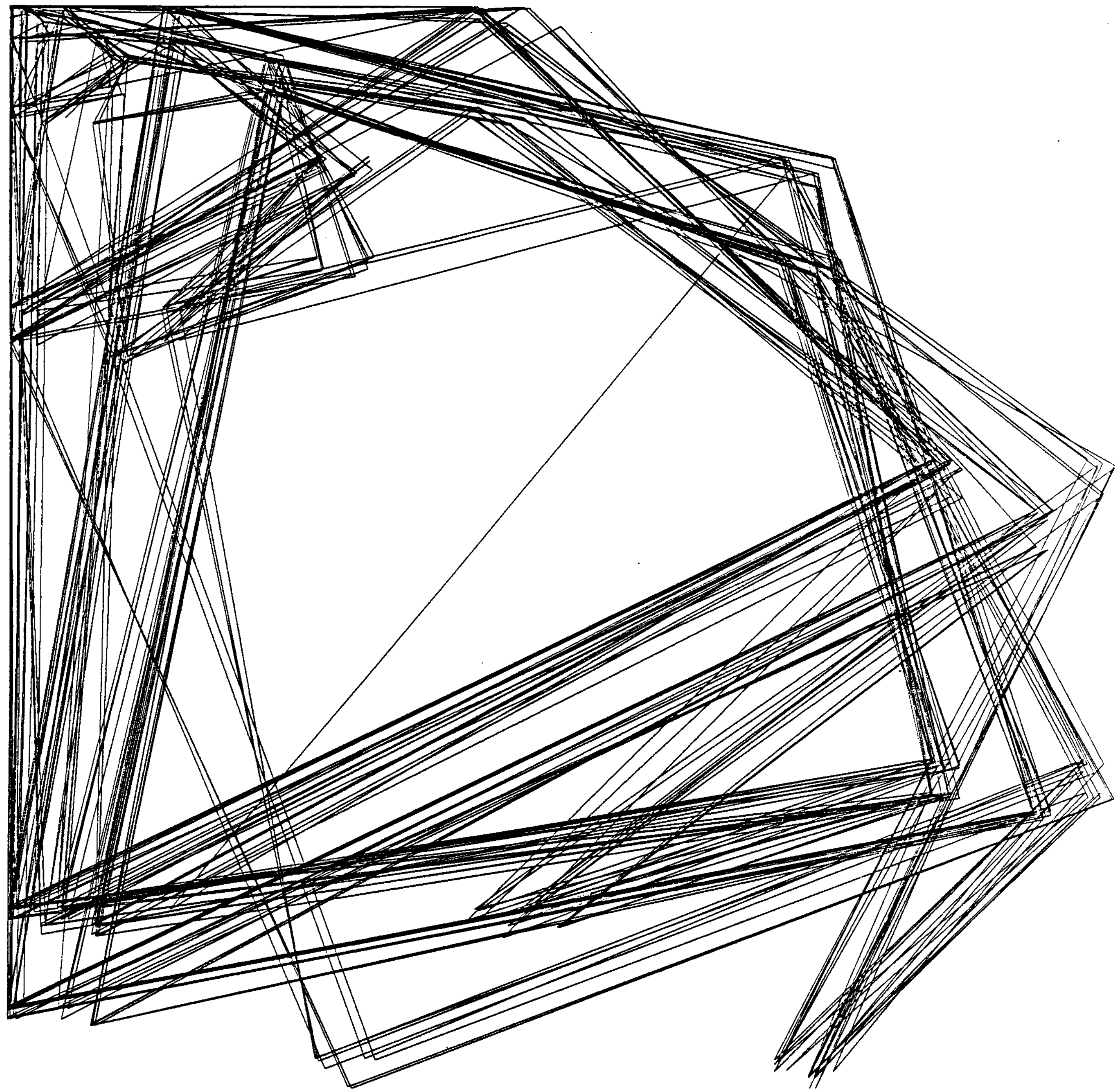
Other interesting results are obtained by examining the results of initial colonies of two units with a diagonal orientation., as well as those for which the two beginning units are a chess horse move apart. For tori of sizes 2x2 through 10x10 (with the exception of 6x6), 11x11, 16x16, 17x17 and 20x20, extinction of the colony occured, although at unexpected places. In this configuration, for other sizes of tori, cycles of varying lengths are observed, from a cycle length of one(constant) and up . The most interesting are those of dimensions14x14, 18x18, 19x19, and 22x22 since these have cycles of very long duration. Often the

18x18 Diagonal , Sample Style 1



18x18 Diag
Window #1

19x19 Diagonal Sample step



19x19 Diag
Window #1

22x22 Diagonal
Sample Style 1
Points only.

boards in the cycle had a strong diagonal orientation, regardless of initial configuration. These last few merit closer attention.

The size of these spaces are very large: $\{0,1,2\}^{n \times n}$ or 3^{196} for the case of the 14x14 tori, and some method of telescoping the space is needed to make conclusions. A good method of gleaning information is to use windows in the space that retain enough of the character while providing some simplification. Two windows into the space are constructed from lines composed of runs with sequential boards as points. The measurement of one involved a square patch on the torus of dimension n/2, the other three parallel lines on the torus of length n/2 unevenly distributed on the torus. In both cases the values of the cell (0,1,2) are read in sequential order to form a Base3 number, which was then converted into a Base10 number and mapped to the interval [0,1]. The values are paired as data points representing the state at time t vs. the state at time t+1. The resulting graphs are not sensitive to sampling method and indicate the presence of attractors; see graphs below. A sample of the graph without connecting lines is also included.

The extension of a finite system to a continuous system as an approximation is considered appropriate due to the large number of cycle points . The formula given by Berge for calculating the attractor dimension is as follows:

$$C(r) = 1/m^2 \sum_{i,j}^{m} H(r - |x_i - x_j|) = Br^a$$

where x_i, x_j are points in the window, r is the distance between them, $a, B \in R$, a is the attractor dimension, and H is the Heavyside function, where

$$H(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases}$$

This method is modified to eliminate measuring pairs of points twice and points where i=j.

$$C(r) = 2/(m^2 - m) \sum_{i=0}^{m} \sum_{j=1}^{i} H(r - |x_i - x_j|) = Br^a .$$

The straight portion of the graph ln(C(r))vs. ln(r) gives the dimension of the attractor. Sample runs are listed below.

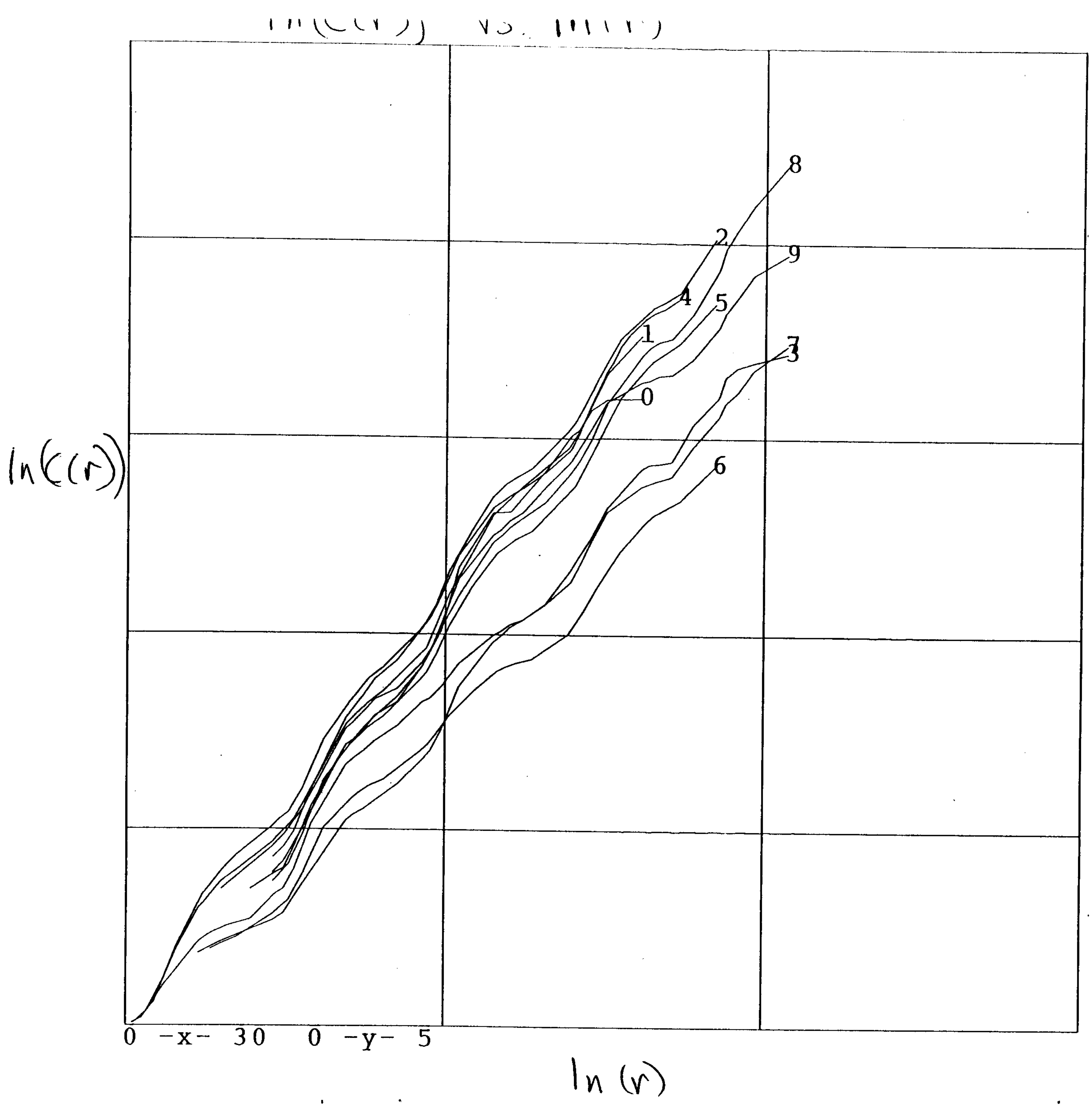

ln(C(r)) vs. ln(r)
ln(C(r))
ln (r)
0 -x- 30 0 -y- 5
0
1
2
3
4
5
6
7
8
9

Figure 4. WINDOWS AND DIMENSIONS

	m	Initial arr.	sample	Dim		m	Initial arr	Sample	Dim
1	14	diag.	1	.238	0	19	diag.	2	.230
3	14	diag	2	.182	6	19	horse	1	.154
2	18	diag	1	.225	7	19	horse	2	. 225
4	18	diag	2	.276	8	22	horse	1	.200
5	19	diag	1	.233	9	22	horse	2	.260

The average dimension is .222. This indicates that only a very small portion of the space is visited during cycles. Thus only a very small portion of the space is regenerative, and this may indicate predicatability.

After examing the results presented here, several questions remain concerning this model. What happens to the dimensionwith larger spaces and other cyclic colonies? Is this model predicatable, as the small dimension indicates? What role dooes the geometry play in determing viable colonies? And that is what we did on our summer vacation.

Bibliography

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