Is the Study of π Chaotic?

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We asked the question "When does $|n \sin n|$, for integer values of n, approach zero?" In this paper we will present how this question led us to the study of continued fractions, why continued fractions led us to chaos theory and Fourier analysis, what conclusions exist, and questions for further study.

Our research started with the fact that $|n \sin n|$ will get close to zero if $\frac{n}{m}$ is a good approximation to π. Since the convergents of continued fractions are "best approximations" of irrational numbers (Niven, 196), we began to study these approximations.

A continued fraction appears in the form

$$x = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}},$$

where the a_k are called partial quotients. The a_k are generated by the formulas (Niven, 194)

$$a_k = \lfloor \xi_k \rfloor,$$
$$\xi_{k+1} = \frac{1}{\xi_k - a_k}.$$

With the formulas $m_{k+1} = a_{k+1}m_k + m_{k-1}$, $n_{k+1} = a_{k+1}n_k + n_{k-1}$ and $|\pi - \frac{n_k}{m_k}| < \frac{1}{m_k m_{k+1}}$ (Niven, 190, 196) we were able to determine a relationship between $|\sin n|$ and the partial quotients (a_k) of the continued fractions. The relationship is derived by the following process:

$$\begin{aligned}
|\sin(m_k\pi - n_k)| &\doteq |m_k\pi - n_k| \quad \text{when } \tfrac{n_k}{m_k} \text{ is close to } \pi \\
|\pi - \tfrac{n_k}{m_k}| &< \tfrac{1}{m_k m_{k+1}} \\
|m_k\pi - n_k| &< \tfrac{1}{m_{k+1}} \\
|m_k\pi - n_k| &< \tfrac{1}{a_{k+1}m_k + m_{k-1}} \\
|m_k\pi - n_k| &< \tfrac{1}{m_k}\tfrac{1}{a_{k+1} + \frac{m_{k-1}}{m_k}} \quad \text{but } m_{k-1} < m_k \text{ so} \\
|m_k\pi - n_k| &< \tfrac{1}{a_{k+1}m_k}.
\end{aligned}$$

To extend this relationship to one with $|n \sin n|$, we multiplied the last equation by n_k and recognized the fact that $\frac{n_k}{m_k} < |\pi - \frac{1}{m_k m_{k+1}}|$. With these manipulations we found the upper bound

$$n_k|m_k\pi - n_k| < \frac{\pi}{a_{k+1}}.$$

Further, this inequality describes a nearly exact approximation, as the a_k are bounded below by approximately $\frac{\pi}{a_{k+1}+1}$ (for large a_k). These bounds confirm experimental data that the magnitude of $|n \sin n|$ was the smallest right before large values of the a_k.

After we established a relation between $|n \sin n|$ and the a_k of π, we began to focus attention on the behavior of the a_k. If the a_k were unbounded then the value of $|n \sin n|$ would become arbitrarily small. On the other hand, if the a_k were bounded, then we would know exactly how small $|n \sin n|$ could be.

The nature of the partial quotients of irrational numbers is unknown except for two instances. First, all quadratic irrationals have periodic partial quotients. Further, any string of partial quotients which are periodic, or eventually periodic, represents a quadratic irrational number (Niven, 204). Secondly, the partial quotients of e are in a regular pattern, almost cyclic, ($[2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$) which has been proven unbounded. No other irrationals are known to have bounded or unbounded partial quotients. (Khinchin, 50)

Since the behavior of the a_k seemed erratic (see fig. 1a), we decided to look at chaos theory. The serious study of continued fractions was at a high point about 50 years ago, long before the development of chaos theory. We hoped that the theories of attactor points for functions with chaotic behavior would tell us something about the boundedness of the a_k.

We used the following defintion of a chaotic function (Devaney, 50):

Definition: Let V be a set. $f : V \to V$ is said to be chaotic on V if

1. f has sensitive dependence on initial conditions,
2. f is topologically trasitive, and
3. periodic points are dense in V.

At first we had a problem with this definition because the function which produces the a_k of the continued fractions maps the real numbers into the integers. We needed a mapping related to the continued fractions which mapped the interval to itself. We then found a new map,

$$f : (0,1) \to (0,1),$$
$$f(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

The relation between the new function and continued fractions is $x_n = \xi_n - a_n$, where x_n is the n^{th} iteration of f and ξ_n is part of the continued fraction formula. We then plotted $f(\pi)$. (see fig. 1b)

The following is a proof that this map is chaotic:

1. The slope along each line in f is greater than one, meaning that the function stretches points apart. For example, let $x = [a_1, \ldots, a_n, b_1, \ldots]$ and $y = [a_1, a_2, \ldots, a_n, c_1, \ldots]$, where $b_1 \neq c_1$ After n iterations of f, $x_n = [b_1, \ldots]$ and $y_n = [c_1, \ldots]$, diverge measurably. Discontinuities also play a part in separating points; after an iteration two points can end up on different sides of the discontinuity. These two facts combine to show that f is sensitive to initial conditions.

2. Take two open intervals U and V. Let x be a point in U with decimal expansion $x = 0.a_0a_1a_2 \ldots a_n \ldots$ and y be a point in V with decimal expansion $y = 0.b_0b_1 \ldots b_na_0a_1 \ldots a_n \ldots$. After n iterations of f acting on y, the first n terms of the decimal expansion of y are removed leaving $0.a_0a_1 \ldots$ or x. Therefore, after n iterations of f, y has moved from V to U and thus is topologically transitive.

3. Since f is related to the continued fractions, the periodic points of the continued fractions are the same as the periodic points of f. For the continued fractions, the periodic points are the quadratic irrationals, so take the quadratic irrationals on $(0,1)$ as the periodic point of f. Let $x \in f$ given $x = [a_1, a_2, \ldots]$ a sequence, $\{x_n\}_{n=0}^{\infty}$, of the form $x_n = [a_1, a_2, \ldots, a_n, a_1, a_2, \ldots]$ (a periodic point of period n) can be constructed. Then $\lim_{n\to\infty} x_n = x$. In other words, it is possible to construct periodic points with large periods which will get arbitrarily close to x.

Part three states that a periodic point of period n can be constructed. This statement implies that all periods are represented.

All fixed and periodic points are repelling points. That is, they "repel" other points from remaining in their neighborhood. We know this for two reasons. First, only quadratic irrationals can have a periodic orbit, thus, non-quadratic irrationals cannot remain in the neighborhood of a periodic point. Secondly, by definition (Devaney, 24-26), since $|(f^n)'(p)| > 1$ they are repelling points. (We are assuming continuity as there are a countably infinite number of discontinuities which have measure zero.)

To see if there was an attractor other than a periodic point, we then looked at the phase portraits of f for 500 iterations each of π, e, five other cubic irrationals and 13 quadratic irrationals. We took the union of phase portaits. (see fig. 2) This new phase portrait looked similar to the graph of the first iteration of f, done with $3,000$ intitial points (see fig. 3).

The only unusual aspect in this graph are the values approaching zero horizontally at .50 on the veritcal axis. (see fig. 2) These represent the iterations of e. It is interesting to note that e has a pseudoperiodic cycle of period 3. The first point is approaching .50, the second 1 and the third zero. Particularly interesting is the fact that the third point approaches zero in such a manner that when iterated, the integer, a_k produced is two graeter than in the previous cycle(a_{k-3}) and $f(x)$ is closer to .50 than in the previous cycle. We realized that a countably infinite number of irrationals would follow this pattern. This family would have partial quotients of the form

$$[b_1, b_2, \ldots, b_n, a_i, b_1, b_2, \ldots, b_n, a_{i+1}, \ldots]$$

where the psuedoperiod is $n+1$ and $a_i < a_{i+1}$ for all i. However, these pseudoperiodic cycles are also not attactors, due to the sensitive dependence of the function.

As a result we know the partial quotients of almost all non-quadratic irrationals will continue to have arbitrary values, over the whole interval, which would make them unbounded (Devaney, 269-270). Also we know from continued fractions that almost all irrationals have partial quotients that are unbounded (Khinchin, 60-62). Looking at the phase portrait of π (see fig. 4) it appears that the partial quotients of π do actually take on all values in the interval. Unfortunately, we have no way of proving that they actually will. That is, we cannot prove that the 431^{nd} partial quotient of π which is $20,776$ is not the largest partial quotient that will ever occur nor that as $k \to \infty$ there will not be a larger partial quotient. We know the global behavior of this mapping, but we do not know the behavior at any specific point.

Next we did a Fourier analysis of the first 512 partial quotients of π. (see fig. 5) The square of the magnitude of the Fourier coefficients falls within a small range. The middle of the range is the square of the largest a_k calculated thus far. For exapmle, the largest a_k calculated for $k = 512$ is at $a_{431} = 20,776$. The magnitude of $a_{431}^2 \doteq 4.4x10^8$ Thus, this is the average value of the square of the magnitude of the Fourier transform.

The meaning of the graph is that all frequencies are represented; the frequencies did not converge to any value. Our graph shows the fact that the a_k are chaotic. It does not tell us anything about the bounded/unbounded nature of the a_k.

When we performed the transform on the partial quotients of e and the partial quotients, $[1,1,2,1,1,2,1,1,2,\ldots]$, these graphs looked very different from π in that they have spikes. (see figs. 6 and 7) The spikes of $[1,1,2,1,1,\ldots]$ show an orbit that is periodic. The graph of e has the same spikes at $\frac{k}{3}$ and $\frac{2k}{3}$, where k is the number of partial quotients used, as the graph of $[1,2,1,1,\ldots]$. This indicates that e has a periodic nature. The other spike at the end of the interval shows that the behavior of e is not quite periodic. (see fig. 6)

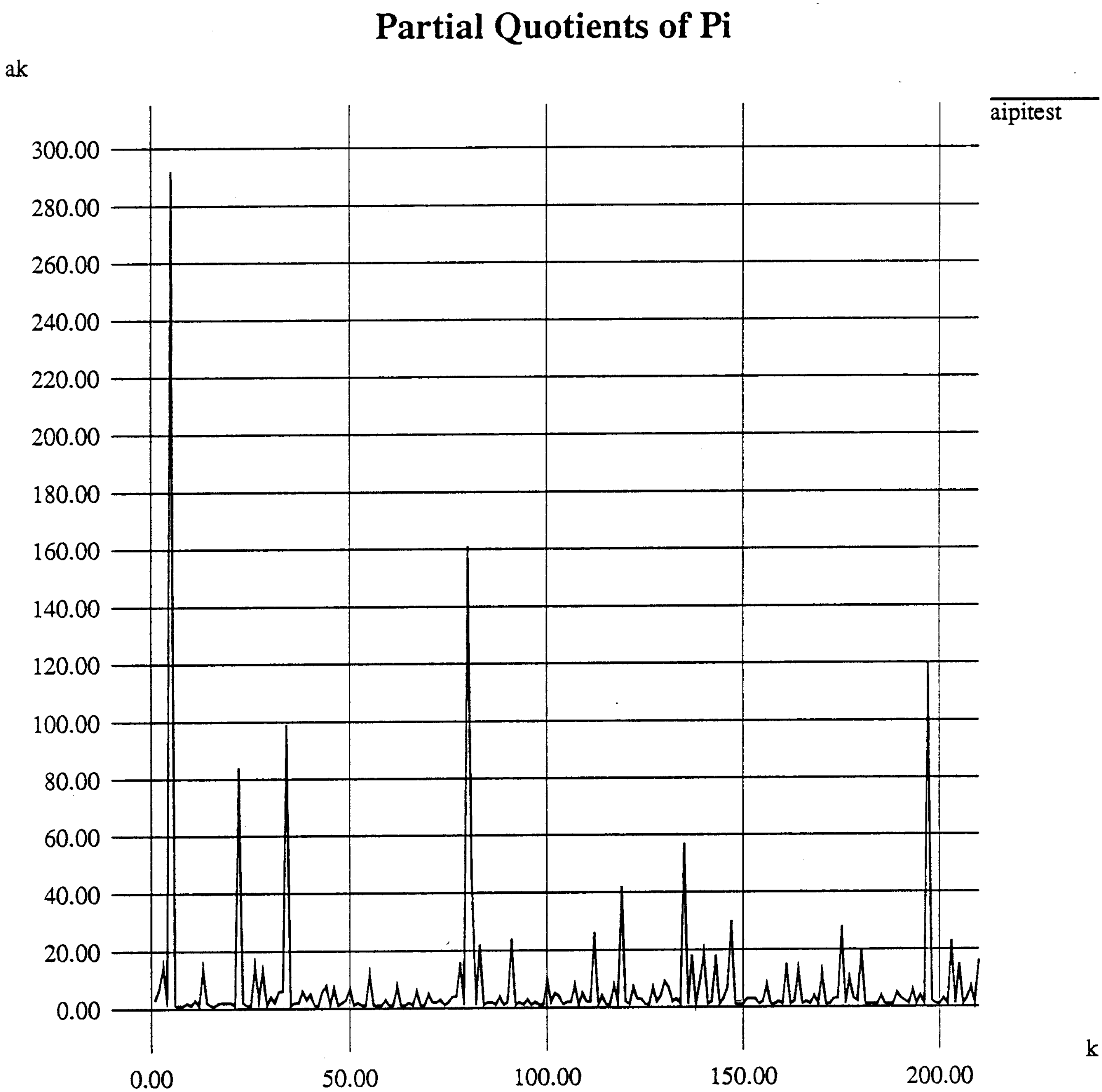
Many open questions have resulted from our study of this problem. A possible approach to take is to look at e as

$$e = \frac{1}{\frac{1}{e}} = \frac{1}{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + (-1)^{k+1}\frac{1}{k!}}$$

and try to find something similar for π.

The final question we have deals with the family of psuedoperiodic cycles we found. Their form is like e, but are they transcendental? Is there some way to link transcendental numbers to their partial quotients?

The smallest value of $|n \sin n|$ that we found happens right before the 430^{th} convergent, c_{430}. The bounds on c_{430} are $1.901870 \times 10^{216} < c_{430} < 1.9801871 \times 10^{216}$ and $a_{431} = 20,776$, thus $|n \sin n| < 0.0001512$. This is the smallest values of $|n \sin n|$ found for the first 4,700 convergents.



Partial Quotients of Pi
ak
aipitest
300.00
280.00
260.00
240.00
220.00
200.00
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
50.00
100.00
150.00
200.00
k

Figure 1a:

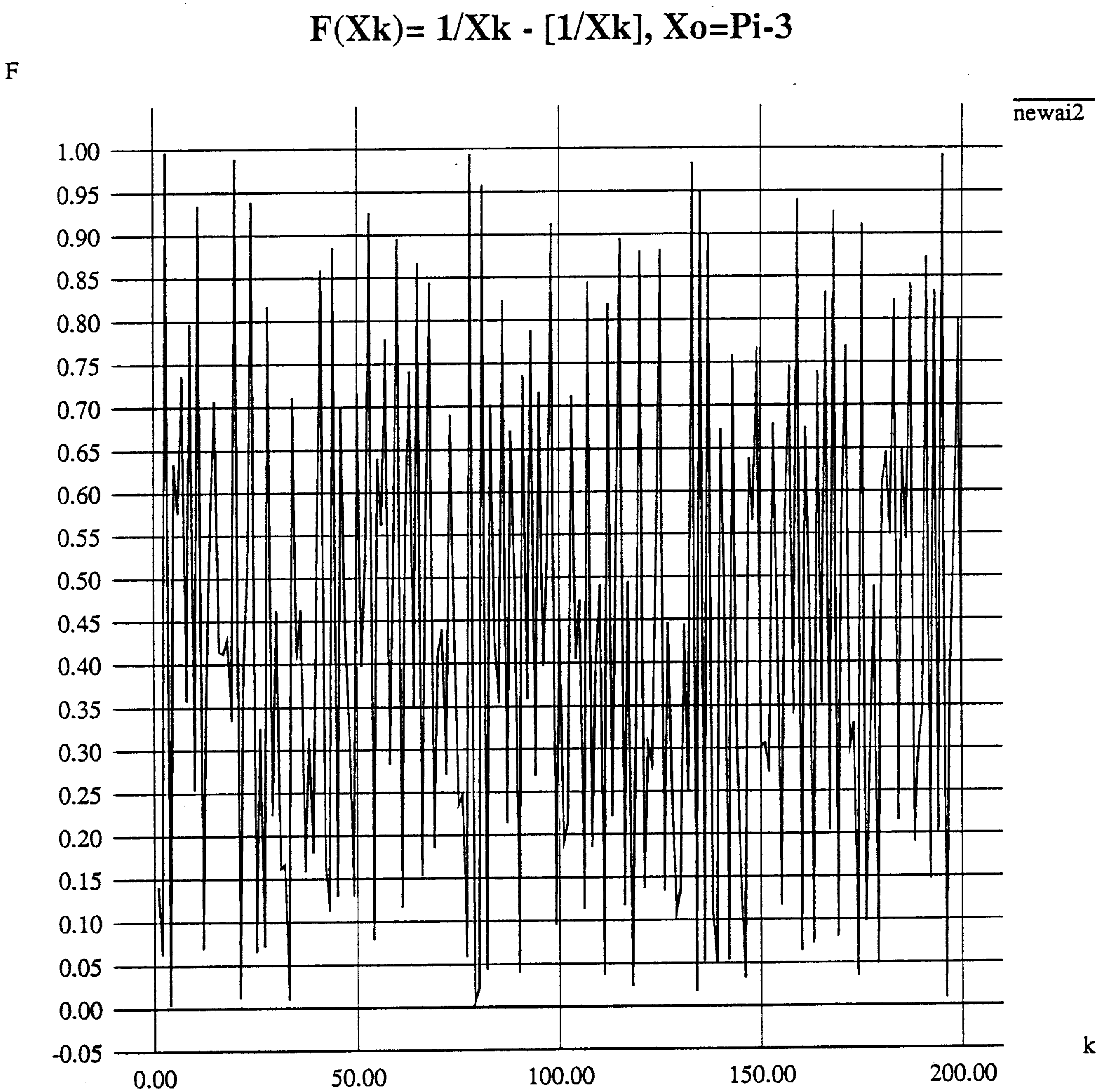
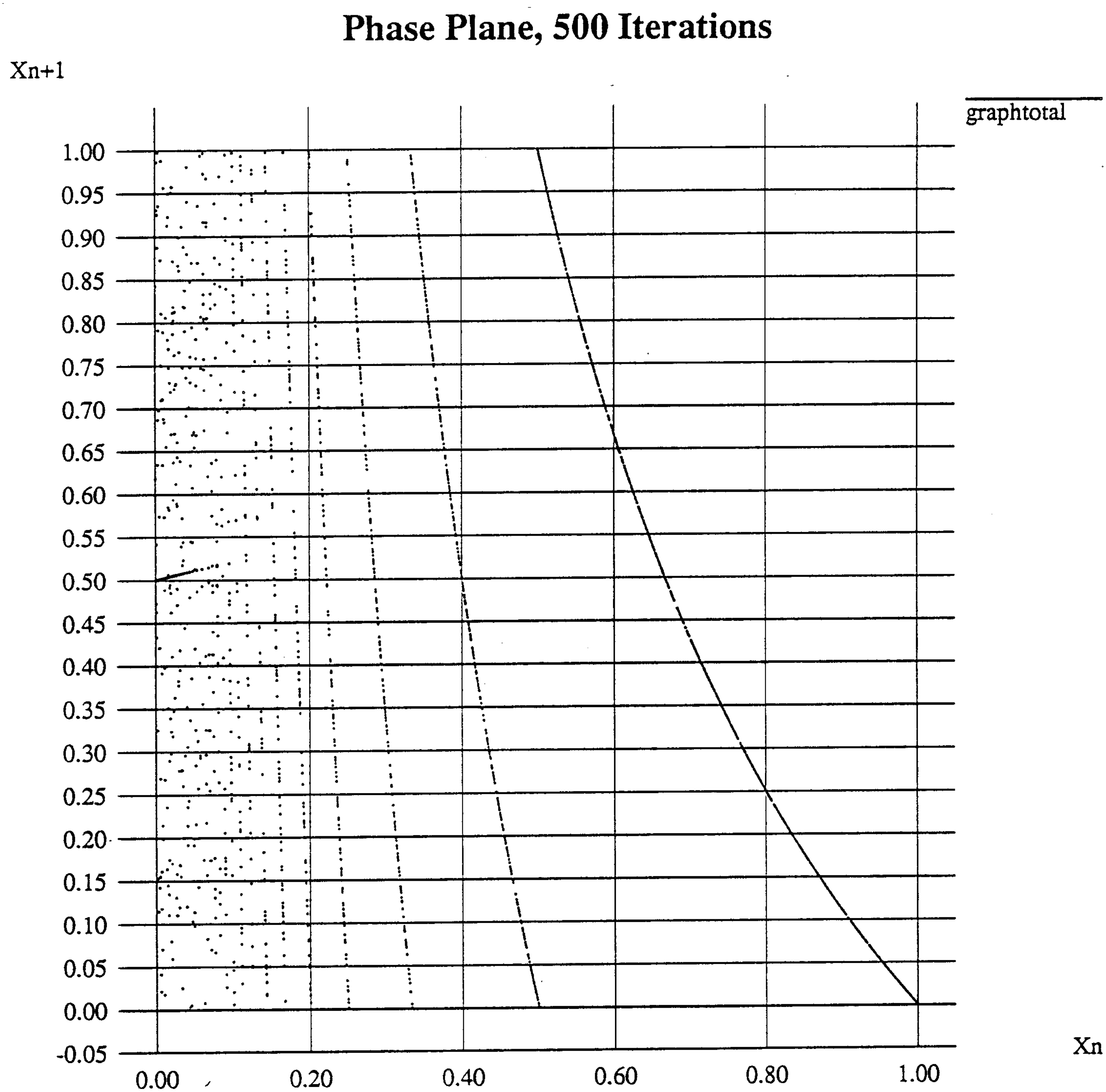

F(Xk)= 1/Xk - [1/Xk], Xo=Pi-3
F
newai2
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0.00
50.00
100.00
150.00
200.00
k

Figure 1b:



Phase Plane, 500 Iterations
Xn+1
graphtotal
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0.00
0.20
0.40
0.60
0.80
1.00
Xn

Figure 2:

First Iteration, X(n+1)=1/X(n)-[1/X(n)]

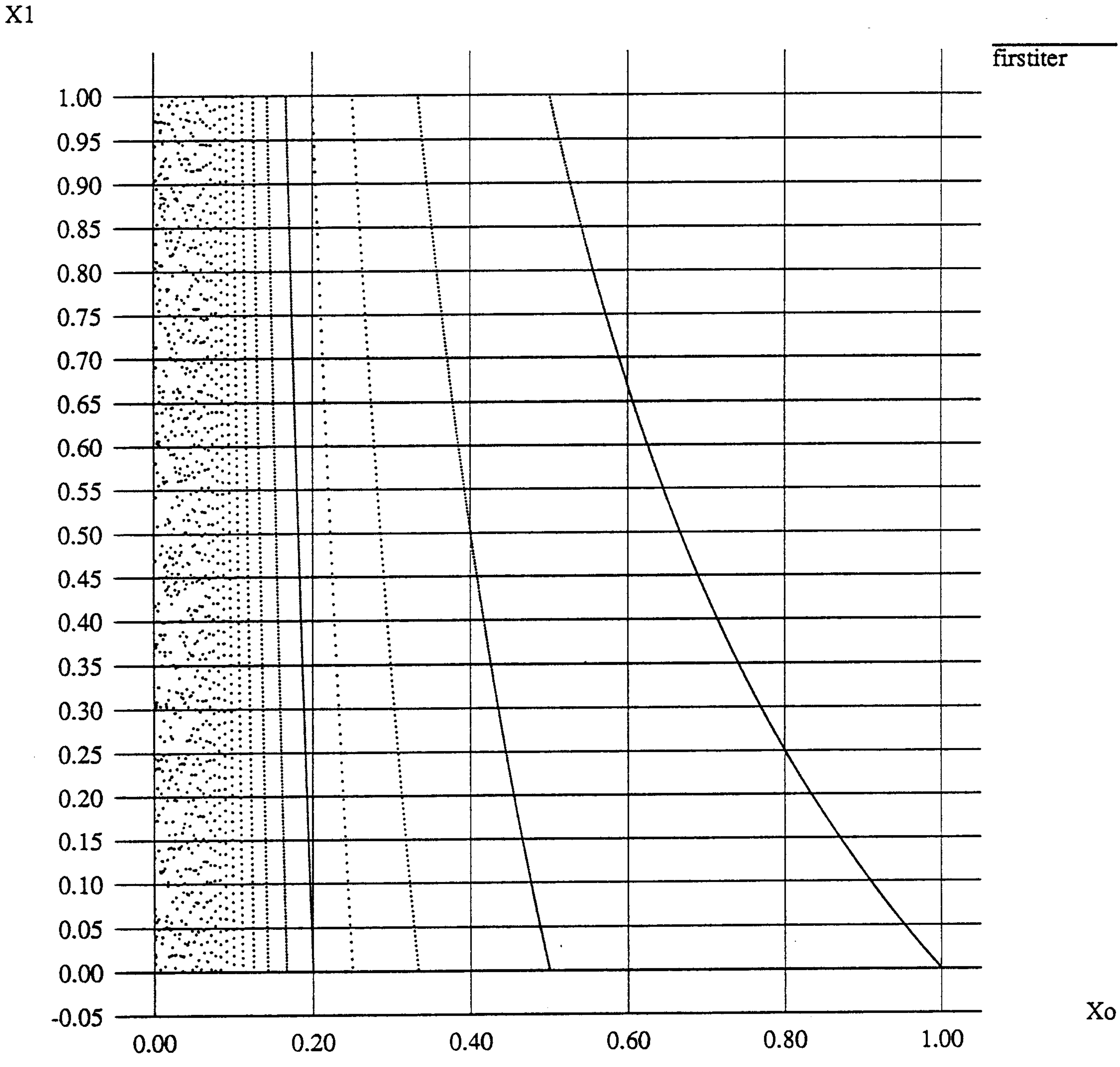


Figure 3:

Phase Plane, 500 Iterations of Pi

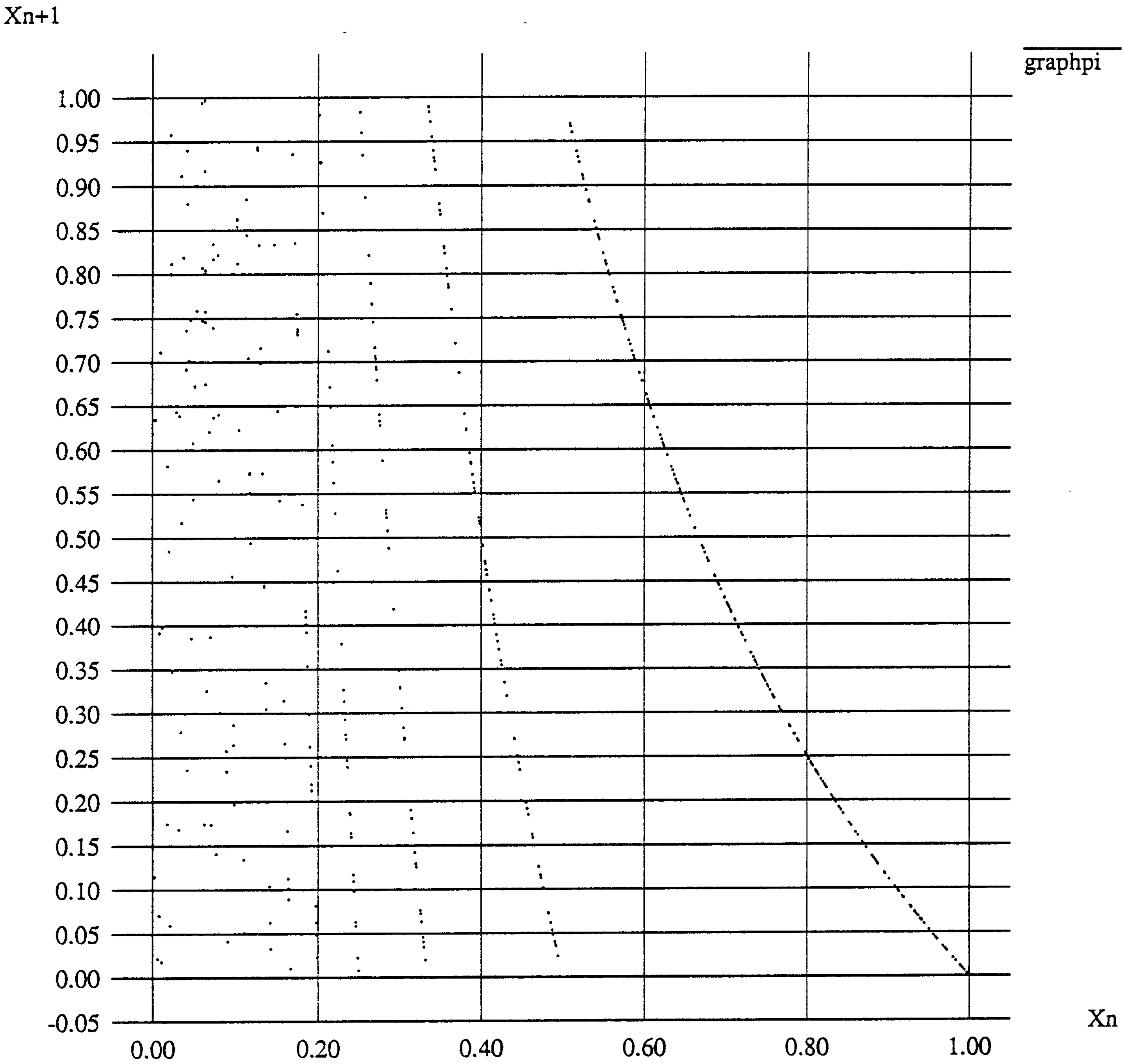


Figure 4:

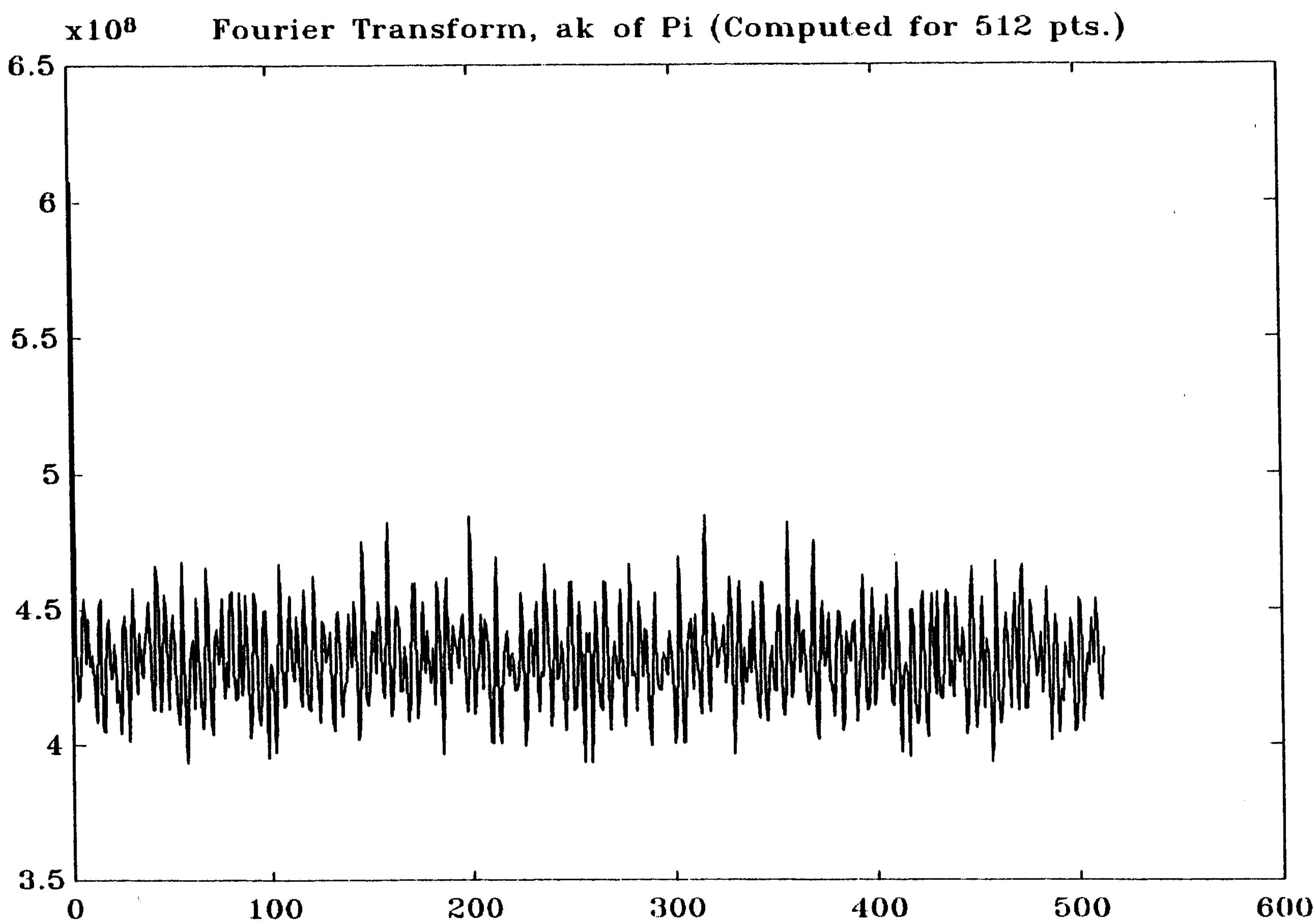

x10^8
Fourier Transform, ak of Pi (Computed for 512 pts.)
6.5
6
5.5
5
4.5
4
3.5
0
100
200
300
400
500
600

Figure 5:

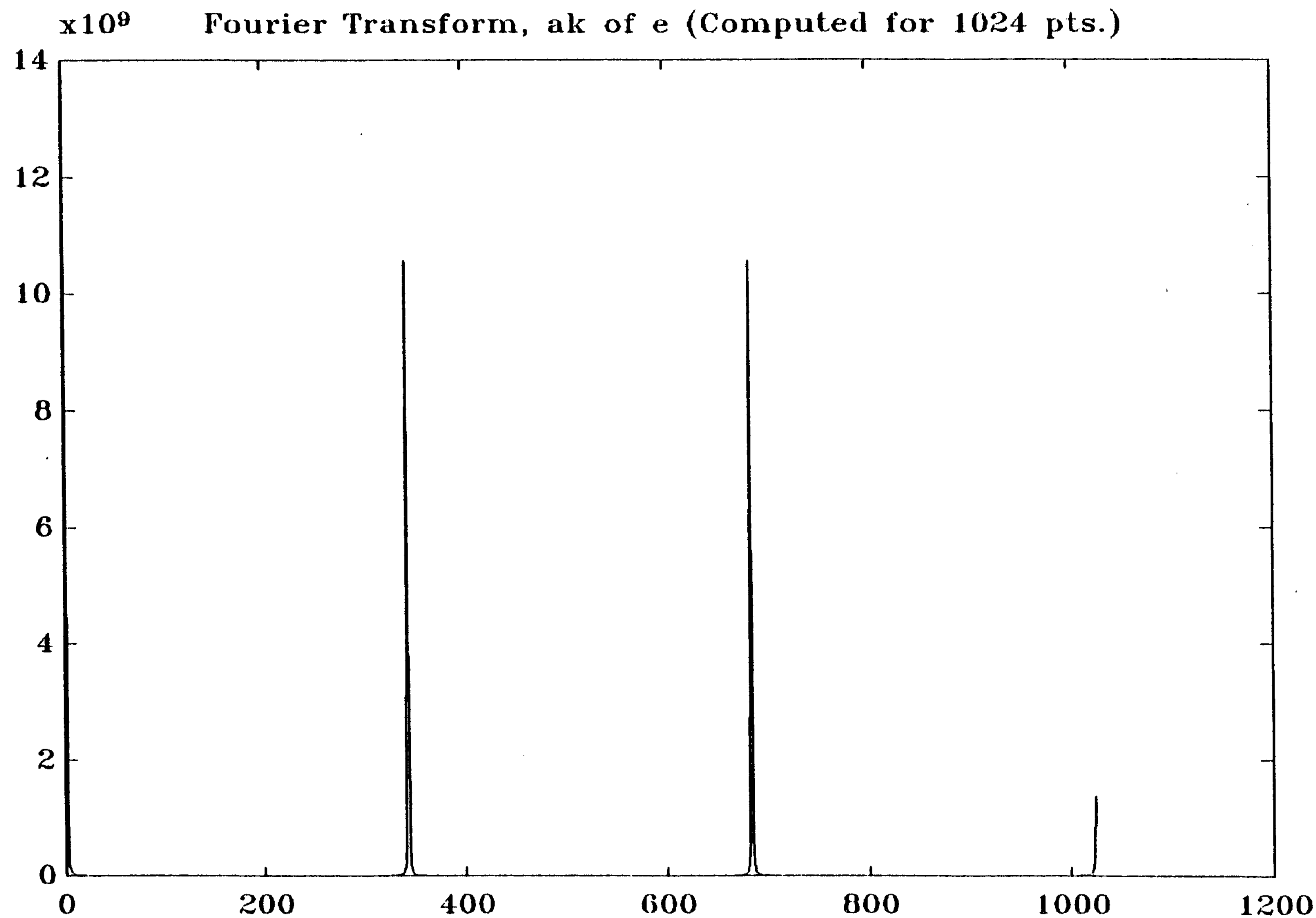

x10⁹
Fourier Transform, ak of e (Computed for 1024 pts.)
14
12
10
8
6
4
2
0
0
200
400
600
800
1000
1200

Figure 6:

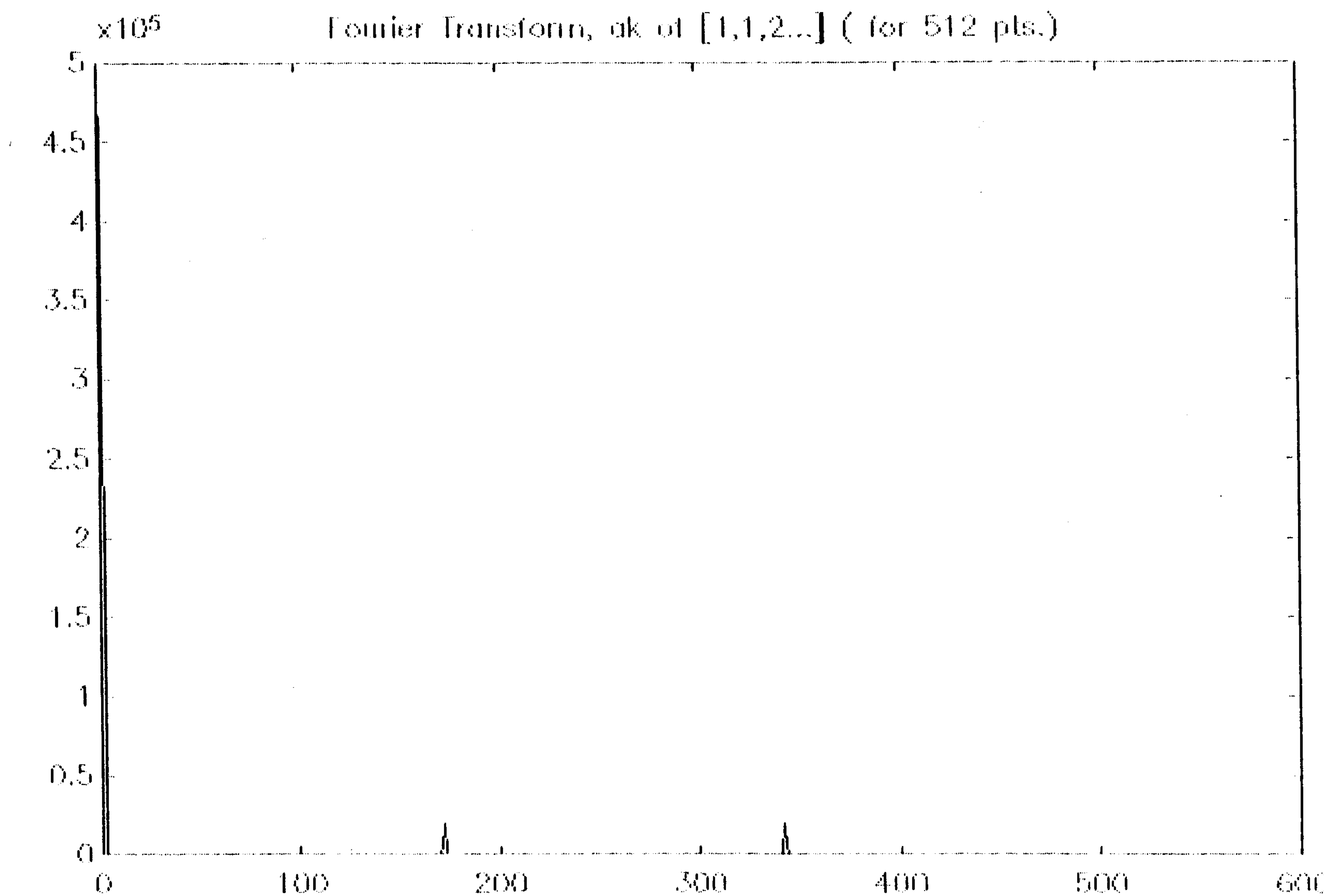

x10^5
Fourier Transform, ak of [1,1,2...] (for 512 pts.)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
100
200
300
400
500
600

Figure 7:

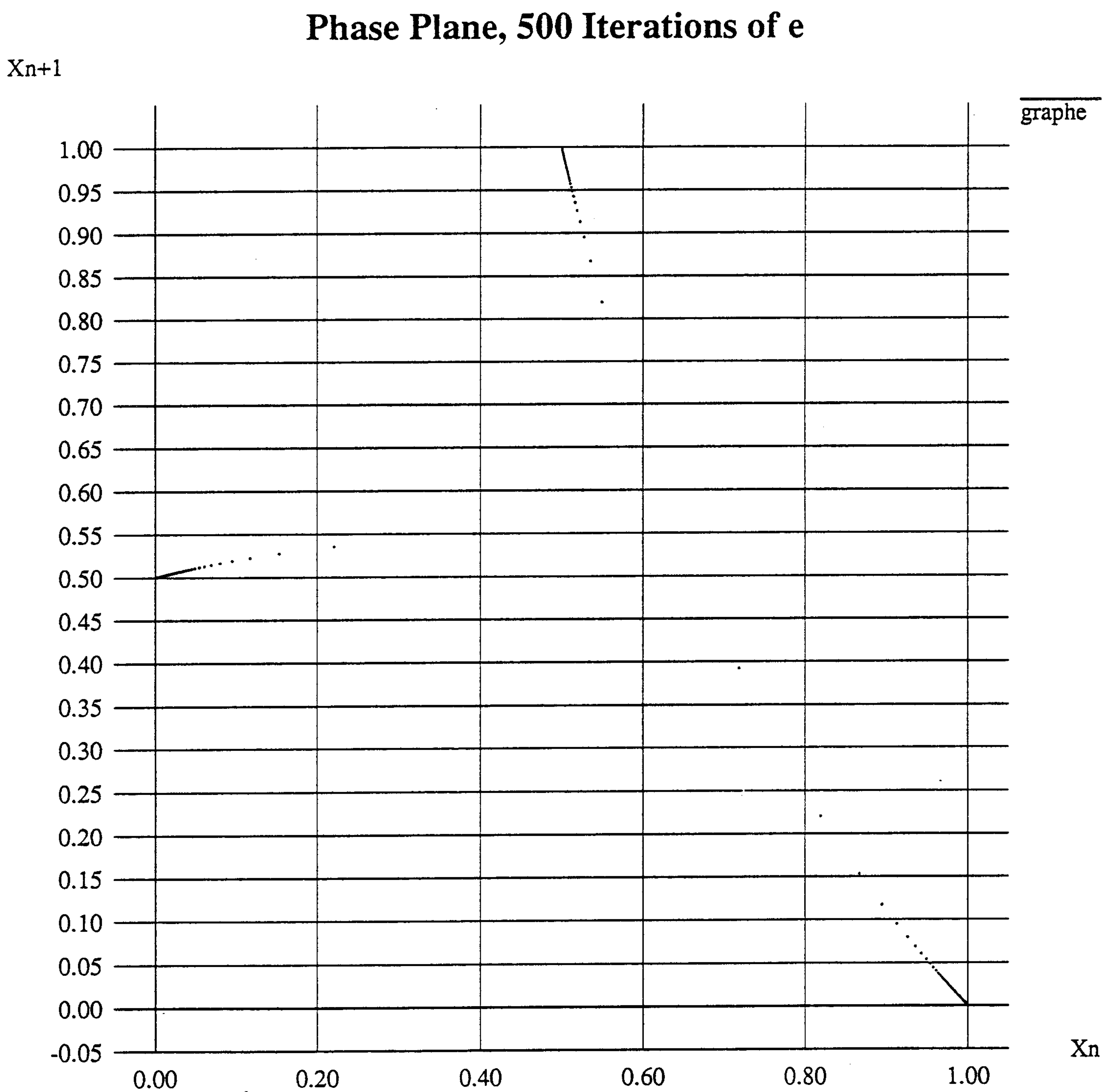

Phase Plane, 500 Iterations of e
Xn+1
graphe
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0.00
0.20
0.40
0.60
0.80
1.00
Xn

Figure 8:

References

Devaney, Robert L. *An Introduction to Chaotic Dynamical Systems,* Addison-Wesley, 1989.

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Munkres, James R. *Topology a first course,* Prentice-Hall, 1975.

Niven, Ivan and Zuckerman, H. S. *An Introduction to the Theory of Numbers,* John Wiley and Sons, 1980.