Loss of Global Stability in an Exponential One-Dimensional Iteration.

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Summer Research Program 1988
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What happens to the critical points of the population model $F(x) = x\exp G(x)$, $G(x) = -1.9(x-1) + \alpha(x-1)^3$, as the model loses global stability on the real line? This paper will focus on understanding the above question and my attempts to answer it. I will present a conjecture explaining the resulting behavior; this conjecture is based on Read's work and Devaney's paper. However, I am far from able to prove it at this point in time.

Before I start to explain the population model given above, it is necessary to define some important and relevent terms. For instance, what is a population model? Cull provides an exact definition:

> "A *population model* has the form $x_{t+1} = f(x_t)$ where f is a continuous function from the non-negative reals to the nonnegative reals, and there is a positive number $\mathbf{x}$, the equilibrium point so that $f(0) = 0$
>
> $$f(x) \begin{cases} < x \text{ for } 0 < x < \mathbf{x} \\ = x \text{ for } x = \mathbf{x} \\ > x \text{ for } x > \mathbf{x} \end{cases}$$
>
> and if $f'(x_m) = 0$ and $x_m \leq \mathbf{x}$,
> then $f'(x) > 0$ for $0 \leq x < xm$
> $f'(x) < 0$ for $x > x_m$ such that $f(x) > 0$." (1988, p.2)

This definition describes a population model as a difference equation with the additional characteristics depending on the value of x. These characteristics are more clearly understood when visualized in the graph of a population model (see fig. 1). But there are other definitions that need to be clarified. The equilibrium point $\mathbf{x}$ occurs when $f(\mathbf{x}) = \mathbf{x}$: this is also called the critical or *fixed point*. A *cycle of period n* is a cycle of n distinct points $(x_1, x_2, \ldots, x_n)$ such that $f(x_1) = x_2$, $f(x_2) = x_3, \ldots, f(x_n) = x_1$.

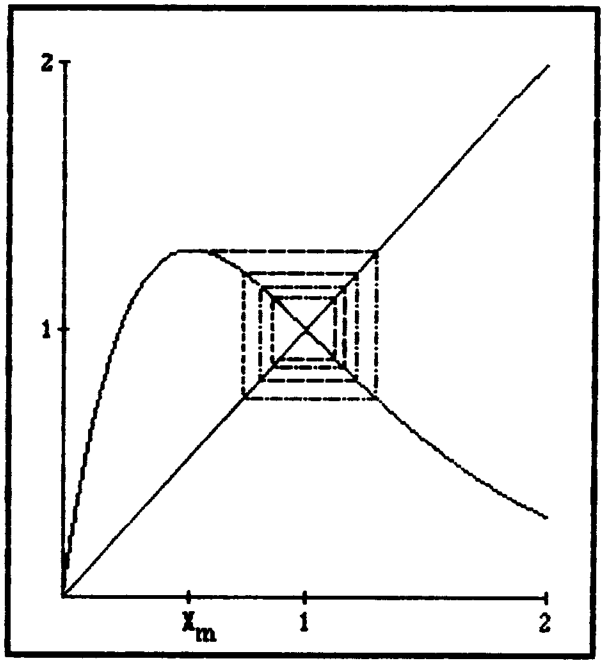

Fig. 1: $F(x) = x\exp[-1.9(x-1)]$ and $H(x) = x.F(x)$ is a population model and satisfies the conditions outlined in the definition. The point of intersection of F(x) and H(x) is the fixed point, in this case **x**=1. The dotted line that spirals in towards the fixed point demonstrates global stability.

In order to understand stability, recall that $x_{t+1} = f(x_t)$ is a difference equation which means that it is an iteration process. For example, the second iterate is $x_{t+2} = f(f(x_t))$ which is the composite operation. Stability depends on the destination of successive iterates. A fixed point is *stable* if the iterates of points tend towards it. This leads to the distinction of global and local stability. A population model is *globally stable* when every initial point x_0, such that $f(x_0) > 0$, has its nth iterate converging to the fixed point as n goes to infinity. To be *locally stable* means that there exists some radius r such that the neighborhood centered at **x** contains only points whose nth iterate goes to **x**. And finally, *instability* occurs if iterates don't approach the fixed point. With these definitions in mind, the given population model, the question, and the conjecture should be easier to understand.

The population model is $F(x) = x\exp G(x)$, $G(x) = -1.9(x-1) + \alpha(x-1)^3$. Cull introduced this equation as an example of a population model that is locally but not globally stable for α nonzero. When $\alpha = 0$ the equation exhibits global stability. This is seen in Fig. 1; the spiraling tends toward the fixed point $\mathbf{x} = 1$. Another fixed point for $F(x)$ is $\mathbf{x} = 0$ but it is unstable. Last summer, Read investigated the convergence scheme for different α. He disregarded $\alpha > 0$ as he "found [no $\alpha > 0$] that would have real cycles" and "since the function is continuous,... there are no positive alpha with real cycles of period 2." (Read, p.4) So, noting that $\alpha = 0$ leads to the global stability of the model on the real line, Read used the remaining α (i.e. $\alpha < 0$) as he addressed the influence of cycles in the complex plane on the real line. He found that cycles appearing on the real line appeared simultaneously in the complex plane. But how does the information presented by Read help us better understand the equation? I think he helps in two ways: he provides useful information on when the real line loses global stability and breaks down into cycles of period 2,4, etc; he extends the model into the complex plane and this extension is invaluable in addressing the question presented at the beginning of this paper. Now we can assume x is a complex number such that $x = u + vi$ where u is the real part and v is the imaginary part of x.

The question centers on the behavior of the fixed points as the population model experiences a loss of global stability. Clearly, the first step to answer this question is to identify the fixed points and to evaluate them for stability. Recall that a fixed point is a point for which $f(\mathbf{x}) = \mathbf{x}$ holds. So the fixed points of the given model must satisfy the equation $\exp G(x) = 1$. From complex analysis we know that $\exp(2\pi i m) = 1$, for integer m, so $\exp G(x) = \exp(2\pi i m)$. Taking the logarithm of both sides, we get $G(x) = 2\pi i m$. For every value of m there are three distinct roots. When $m = 0$, the equation reduces to $-1.9(x-1) + \alpha(x-1)^3 = 0$ and we get the real root $x = 1$ (note that the other two roots are complex). For any other value of m, the fixed points are complex. In other words, $x = 1$ is the only real root for the equation $G(x) = 2\pi i m$. This is verified mathematically in the following way:

> Assume that $\alpha(x-1)^3 - 1.9(x-1) - 2\pi i m$ has a real root r.
> Then $\alpha(x-1)^3 - 1.9(x-1) - 2\pi i m = (x-r)(ax^2 + bx + c)$.
> The other two roots can be found using the quadratic equation but for this proof they aren't necessary.
> Match the coefficients for the x terms:

Term	Coefficients
x^3	$a = \alpha$
x^2	$-ra + b = 3\alpha$ since $a = \alpha$ we have $b = \alpha(3 + r)$
x	$c - rb = 3\alpha - 1.9$
1	$cr = 1.9 + 2\pi i m - \alpha$

> Thus we find that since α is real, a is real. This implies that b is real since α,r are real in the x^2 term. This implies that c is real since α, b, and r are real in the x term. This implies that cr is real but this leads to a contradiction since there is the imaginary part $2\pi i m$, for m nonzero. Thus, for m nonzero, there can never be a real root.

It is interesting to note that $m = 0, \alpha = 0$ is the only case when there is only one root; this case coincides with the initial global stability of the real line.

The identification of the fixed points in the complex plane needs to be accompanied by stability analysis in order to understand any possible influence these points have on the real line. Stability of a point is determined by the following relationship:

$$|F'(\mathbf{x})| \begin{cases} < 1 & \mathbf{x} \text{ is stable} \\ > 1 & \mathbf{x} \text{ is unstable} \\ = 1 & \text{no information can be obtained from this method.} \end{cases}$$

$|F'(\mathbf{x})|$ is the norm of the complex function $F'(x)$ at the point $\mathbf{x}$.

$F'(x) = \exp G(x)[1 + xG'(x)]$ can be reduced to $F'(x) = [1 + xG'(x)]$; since $F'(x)$ is to be evaluated at the fixed point $\mathbf{x}$ we can substitute the known equation $\exp G(\mathbf{x}) = 1$. After some reduction and substitution, $F'(x) = 1 + 3.8x + \dfrac{6\pi im}{x-1}$.

I broke this equation down into real and imaginary parts, R and I respectively: then $R^2 + I^2 = |F'(x)|$. With $x = u + vi$, we have

$$R = \frac{6\pi mv}{(u-1)^2 + v^2} + 3.8u + 1 \text{ and}$$

$$I = \frac{6\pi m(u^2-u+v^2)}{(u-1)^2 + v^2} + 3.8v$$

For $u = 1$, I was able to show that the fixed points were all unstable with the possible exception of one point. But for u not equal to one, my efforts were thwarted by grotesque formulas.

With the above information on complex fixed points, the next step to answering the question is to analyze the influence these points have on the real line and to look at the influence the loss of stability has on the fixed points. At this point it would be useful to discuss the observations and conclusions of Read and Devaney. Read observed that

> "there seems to be a series of 'valleys' of points that diverge, and these valleys get deeper, or closer to the real line... At some point (between $\alpha = -1.14015$ and -1.1402) these valleys stop growing, and then suddenly, for a slightly different value of α, entire areas between valleys 'turn red', that is, cycle instead of converging to a fixed point."(p.9)

These "valleys" may be the basins of the unstable complex fixed points (that is, these basins make up the basin of attraction for infinity). The interaction of these unstable valleys with the stable real line leads to the dramatic change of a region from the unstable and stable parts to an area that cycles.

Meanwhile, Devaney sheds a different light on the dynamics of changing stability through his analysis of Julia Sets and exploding fixed points. He explores the change inflicted on two fixed points, one stable the other unstable, as the parameter lamda is increased. He describes the change as the following: the fixed points merge as lamda increases until the bifurcation point is reached. At that point the merged point splits and enters the complex plane as, it appears, complex conjugates. These new points become repellors. In this way, he provides an explanation of the stability of the fixed points and their changing dynamics as the parameter is increased.

When α is no longer zero, the real line becomes locally but not globally stable. For any α we know that $\mathbf{x} = 1$ remains locally stable. My conjecture is that the fixed points undergo a change similar to that presented by Devaney but in reverse order. The unstable fixed points in the complex plane seem to approach the real line, as noted by Read. Rather then joining to create a new point at the bifurcation point (somewhere between $\alpha = -1.14015$ and $-1,1402$), the dynamics of the unstable meeting the stable leads to cycling in both the real and complex regions.

However, I was unable to prove this mathematically. I had difficulties in proving the fixed points are unstable, and in graphing the interaction of the two regions. I blame my inability to establish whether or not the fixed points were stable on my frustration of working with an unmanageable equation. With the help of Paul Palmer, I was able to produce numerous graphs that fell into three groups. The first set of graphs looked at the second iterates of the function F(x): x_t is on the x-axis and x_{t+2} is on the y-axis. These graphs vary with respect to alpha. As α tends toward negative infinity, these graphs show the breakdown into cycles. The second and third sets of graphics dealt with the complex roots of F(x). In the second set, we fixed α and changed m. This showed the path of the three roots as m varies. In the third set, we fixed m and changed α. Once again, the graphs displayed the path of the three roots but for α varied. After all the kinks were out of the last two sets, the graphs proved to be accurate but difficult to understand. One of these two sets should have indicated the nearness of the fixed points to the real line. I was unable to get passed the unexpected graphs that were produced. Thus, the stability of the fixed points in the complex realm and their encroachment on the real line remained undetermined.

Without the necessary data, it is impossible to prove anything. So what do I need to do in order to be able to answer this question? I need to show the fixed points in the complex plane are unstable by reducing the grotesque equations to something more manageable. Then, the "valleys" in the pictures produced by Read need to be confirmed as the basins of the unstable fixed points. Perhaps Devaney's work on Julia Sets can be extended in this case to illustrate the dynamics of the real and complex regions and to support my conjecture.

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