Some Observations for Pseudo-linear Recurrence Relations of the Partition Type

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Perhaps the most famous identity in the theory of partitions is Euler's pentagonal numbers theorem, which gives a formal power series identity for a product which is fundamental to the theory:

$$\prod_{n=1}^{\infty}(1-x^n) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\ldots = \sum_{k=-\infty}^{\infty}(-1)^k x^{\omega(k)}.$$

Here $\omega(k)$ denotes the pentagonal number $1/2(3k^2-k)$. Euler used this identity to find a recurrence relation for the partition function $p(n)$, which may be defined by the generating function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}\frac{1}{(1-x^n)}.$$

By multiplying the above expression through by the product in the denominator and using the pentagonal numbers theorem, we get the following recurrence relation for $p(n)$:

(1)

$$p(n) = \sum_{k=1}^{\infty}(-1)^{k+1}(p(n-\omega(k)) + p(n-\omega(-k)),$$

where we define $p(n) = 0$ for $n < 0$ and $p(0) = 1$. Thus we have, for example,

$$p(15) = p(14) + p(13) - p(10) - p(8) + p(3) + p(0).$$

Hardy and Ramanujan found an asymptotic (divergent) series for $p(n)$ which, when truncated appropriately, gives $p(n)$ exactly. Hans Rademacher altered their analysis slightly and arrived at an exact, however complicated, formula for $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\frac{sinh(\frac{C\lambda_n}{k})}{\lambda_n},$$

where

$$C = \pi\sqrt{\frac{2}{3}},\ \lambda_n = \sqrt{n-\frac{1}{24}},\ A_k(n) = \sum_{(h,k)=1,h=1}^{k} e^{\pi is(h,k)-2\pi inh/k},$$

and $s(h,k)$ is the famous Dedekind sum defined as follows, with $((x)) = x - \lfloor x \rfloor - 1/2$ for nonintegral x, and $((x)) = 0$ for integral x:

$$s(h,k) = \sum_{\mu mod k} ((\frac{\mu}{k}))((\frac{h\mu}{k})).$$

We may regard Rademacher's expression as a closed-form solution to the recurrence relation (1) for $p(n)$. Thus the main question of this paper presents itself: if we were to replace $\omega(k)$ in (1) by an arbitrary quadratic which maps integers to integers, could we solve the corresponding recurrence relation? Trying to get exact solutions to arbitrary recurrence relations is a notorious problem, so of course a more feasible question is: what can be said asymptotically about the generated sequence? As will be noticed later, the recurrence relation (1) for $p(n)$ seems to occupy a special place amongst recurrence relations of the type

$$s_\alpha(n) = \sum_{k=1}^{\infty}(s_\alpha(n-\alpha(k)) + s_\alpha(n-\alpha(-k))).$$

Once again, we need to define $s_\alpha(n) = 0$ for $n < 0$. If we had a generating function for the sequence s_α, we could of course use it to find asymptotic estimates via standard analyses such as the circle method (due to Hardy and Littlewood) and the so-called saddle point (or steepest descent) method. In his book Ramanujan, Hardy gives a simple demonstration that $p(n)$ grows with $e^{C\sqrt{n}}$, while he and Ramanujan (and independently, Sierpinski) showed that the correct order of magnitude is $\frac{e^{C\sqrt{n}}}{4n\sqrt{3}}$. Given an arbitrary quadratic $\alpha(k)$, it is no simple matter to produce the generating function, and it is possible that no "nice" generating function even exists. Given only methods accessible in an undergraduate curriculum, the problem may seem hopeless. Thus I back off a little and ask more reasonable questions. One (naive but helpful) attack on the general problem is utilizing standard facts about linear recurrence relations. These facts are perhaps helpful since the recurrence relation for $s_{\alpha(n)}$ is linear for n between consecutively increasing $\alpha(k)$'s. For example, the linear recuurence relation

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7)$$

may be solved explicitly given $p(1)$ through $p(7)$, and the solution will give $p(8)$ through $p(11)$ correctly since 12 is the least pentagonal number greater than the pentagonal number 7.

If we are to obtain useful asymptotic information by considering the linear recurrences that arise, we are of course interested mainly in the roots of the characteristic polynomials corresponding to the linear relations. These eigenvalues, especially the ones with maximal moduli, dictate the growth of the sequence. Since the characteristic polynomials are monic polynomials with all coefficients 0, 1, or -1, it is natural to expect that the spectrum, the set of complex roots, will approach the unit circle as the degree of the polynomial increases. This idea will be important in the observations which follow. The mathematical software Mathematica was used to generate lists and plots of the spectra arising from characteristic polynomials for various $\alpha(k)$'s. A natural place to begin looking for patterns in the spectra is the characteristic polynomials corresponding to the recurrence (1) for $p(n)$. For example, the linear relation given last paragraph has characteristic polynomial

$$x^n - x^{n-1} - x^{n-2} + x^{n-5} + x^{n-7},$$

and this is useful for n between 7 and 11. On the following pages are plots of the spectra of the linear recurrences arising for $\omega(k)$ (the pentagonal numbers); i.e., they represent the complex roots of the characteristic polynomial of degree $\omega(k)$ for the recurrence (1) with $n = \omega(k)$. Patterns are not very evident for small pentagonal numbers, but by the time we have plotted the roots of the polynomials of degrees greater than $26=\omega(-4)$ or so, a very definite pattern begins to form. I show two diagrams– the first for the polynomial of degree $\omega(7) = 70$, and the second for degree $301 = \omega(-14)$. Given the memory I had available to run Mathematica on, this was about the limit of such computations; of course, one does not need any more roots to predict what the spectra are going to "look like" as $\omega(k)$ increases!

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Do[ListPlot[roots[pent[n]], AspectRatio->1],{n,7,8}]
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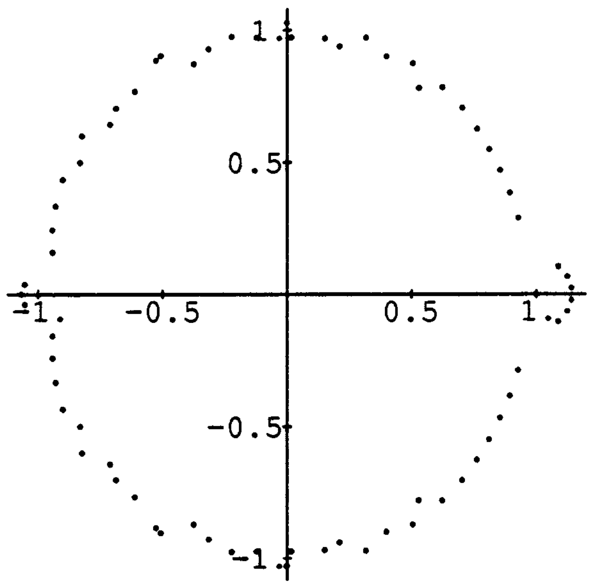
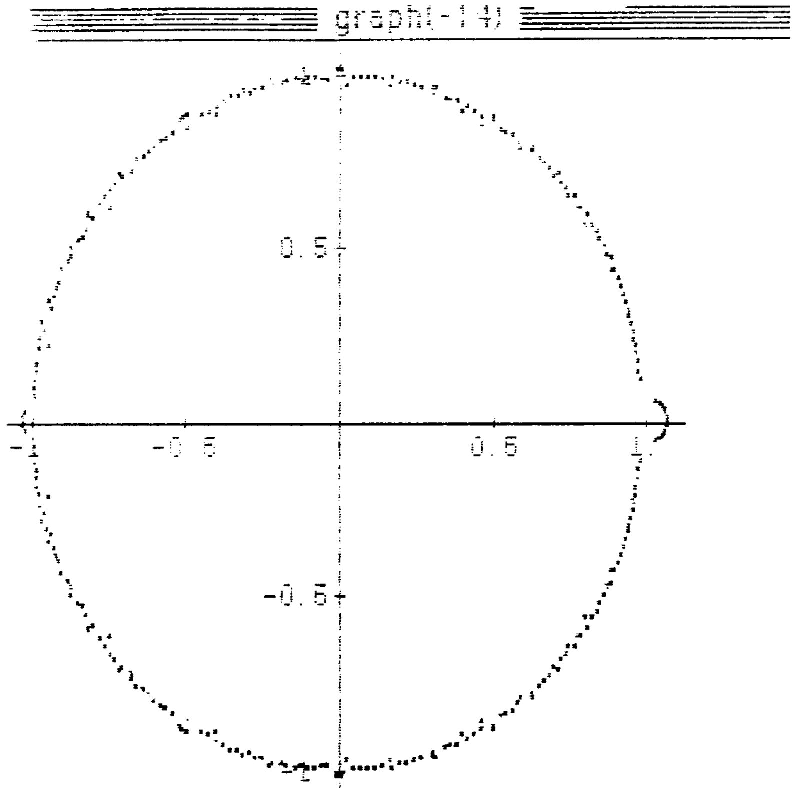


Roots of the polynomial corresponding to

$$N = \omega(7) = \frac{7(3\cdot 7-1)}{2} = 70:$$

$$x^{70} - x^{69} - x^{68} + x^{65} + x^{63} - x^{58} - x^{55}$$

$$+ x^{48} + x^{44} - x^{35} - x^{30} + x^{19} + x^{13} - 1.$$



Roots of the polynomial corresponding to $N = \omega(-14)$

$= -\frac{14}{2}(3\cdot -14 - 1) = 301$:

$x^{301} - x^{300} - x^{299} + \cdots + 1$ #

The apparent "bumps," which look much like circular arcs, occur primarily around roots of unity $e^{2\pi i/k}$ for small k. These plots prompt the questions:

1. Why are these polynomial roots distributed more heavily around such roots of unity?

2. Why do the "bumps" seem to form circular arcs, and what are the radii and angle sizes of these arcs?

3. Can we expect to see such nice plots for other recurrences in the class under consideration? For example, if we set $\alpha(k) = \omega(k) + 1$, will we get such predictable behavior in the spectra?

The first question is fairly easy to answer heuristically if we know of the generating function for the partitions. If we did not have this information, the question would be very difficult to answer; I know of no estimates for the partition function which are based soley on the recurrence relation and are independent of the generating function.

This idea, of using computers to crank out plots of spectra for various quadratics $\alpha(k)$ so that a person can try to guess asymptotic behavior of the spectra (and thus of the sequence), may seem awfully naive. However, the history of useless mathematics, a.k.a. number theory, has many examples of conjectures first noticed numerically and later proved (or at least remained unsettled). The law of quadratic reciprocity was first conjectured by Euler, who noticed the result via his numerical computations. Gauss something similar for the prime number theorem, and of course we cannot forget about the Riemann Hypothesis! In this setting, perhaps computer spectra plots for various recurrences similar to ours may lead to conjectures regarding the existence of canonical forms for generating functions. After giving a partial answer to question 1, I will give a couple examples of other recurrence relations which could fall prey to such simple computing-and-conjecturing; however, these examples have known generating functions, so their spectral behavior can be (at least partially) explained via estimates using the generating functions!

This type of haystack-searching does have a bright side– whenever the spectral behavior is apparently unpredictable, we have a clue as to the complexity of any canonical form for a generating function (e.g., rational functions in products similar to that for $p(n)$) that could possibly work. I will give spectral plots for a few quadratics $\alpha(k)$ which are just shifts of the pentagonal numbers by integer constants; this is meant to show that similar quadratics do not necessarily imply similar sequences or generating functions. In fact, the plots will seem to become less and less predictable as the shift integer gets larger! This will hopefully lend some support to the notion that the pentagonal numbers occupy an important place in additive number theory. This discussion is related to question 3, and there will be more on this in a few pages... for now, I will present a very non-rigorous argument to answer question 1.

The generating function for the partition function is, as given earlier,

$$\frac{1}{\prod_{n=1}^{\infty}(1-x^n)}.$$

If we denote by $F_N(x)$ the Nth partial product $\prod_{n=1}^{N}$ of the above, then it is easily seen (using Euler's pentagonal numbers theorem) that the characteristic polynomial of degree N, for N a pentagonal number, is just

$$x^N F_N(\frac{1}{x}) - E_N(x),$$

where $E_N(x)$ is a polynomial of degree $1 + 2 + \cdots + N = N/2(N+1)$, with all terms of degree greater than N. The roots of $F_N(x)$ are the same as the roots of $x^N F_N(\frac{1}{x})$, so it should at least be suspected that the roots of the characteristic polynomials may approach the roots of the generating function with error term. To make this precise, one probably needs facts about polynomials with all coefficients 0, 1, or -1; I have not been able to find a precise treatment which proves that the roots of the characteristic equations do indeed approach the unit circle as they so obviously do! Given more time to check the standard literature on polynomial root estimates, I'm sure that the argument could be filled in precisely. A particular reference I intend to search through is Marden's The Geometry of the Zeros (copy. 1949, American Mathematical Society).

Of course, this presumably weak statement (about the roots heading to the unit circle as the polynomials' degrees increase) says nothing specific about the manner in which the roots approach the unit circle, and this is the problem of question 2 above. I will mention here that knowledge of the generating function will allow integral estimates via Cauchy's integral not just around $z = 0$, but by substitution, also around $z = e^{2\pi i/k}$, so that we can find specific information regarding the radii and angular measurements of the apparent circular arcs straddling the roots of unity in the computer plots.

If we examine the circle method analysis of $p(n)$'s generating function, we see a strong analogy to our plots. For the circle method, we estimate contributions to an integral by the so-called major arcs around low order roots of unity (which constitute the "heavy" singularities of the function and thus make a quick truncated estimate of $p(n)$ surprisingly close to actual values!). In our spectrum plots, we see that the eigenvalues of maximum modulus (indeed, just about all of the ones which lie just outside the unit circle) do more than just approach low order roots of unity– they do so in a very patternistic manner. I do not know whether the integral estimates of the major arcs in Rademacher's analysis reflect this behavior; if they do not, but instead use a less complex estimate, then it is certainly worth looking into. I intend to continue searching for analogies between the circle method and the spectra plots, because at the very least the plots provide a high-tech yet simplistic and interesting motivation for the circle method analysis.

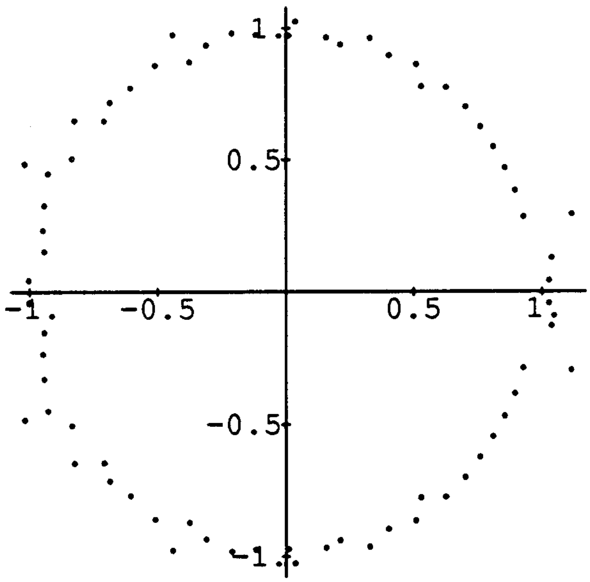
If we compare the known estimate for $p(n)$, we ought to be able to gain rough estimates for the radii and angle measurements of the circular arcs appearing in the plot. For example, if we want to know how fast the radii tend to zero, we can use the major arc estimates from the circle method individually for more precise results, or we can perhaps get a rough estimate as follows. $p(n)$, for n a pentagonal number, grows with the sum of the nth powers of the n eigenvalues straddling $1 + 0i$. If the maximum modulus is λ, then we'd have, very roughly, $p(n)$ grows with λ^n, so that the asymptotic $p(n) \sim \frac{e^{C\sqrt{n}}}{4n\sqrt{3}}$ gives $\lambda \sim \frac{e^{C/\sqrt{n}}}{4n\sqrt{3}}$. This estimate ignores contributions made by other eigenvalues with moduli relatively large. Another approach that might take these other eigenvalues around $1 + 0i$ into account is to assume that the eigenvalues are evenly spaced on the circular arc with supposed radius λ. Then we can estimate "by eye" the angle measure of the arc and get a sum of nth powers of eigenvalues which ought to be near $p(n)$. If any simplifications are possible, so much the better! One estimate I got in this manner is (supposing that the angle approaches $\pi/2$ rads as $n \to \infty$)

$$p(n) \sim \sum_{i=1}^{n} (n + \frac{\rho}{2}(r - 2\cos(\frac{\pi}{2}(1 + \frac{i}{n}))))^n .$$

Of course, this cannot be true in the sense of the limit of the ratio of the right and left sides being 1, as the analysis completely ignores the "bumps" around other low order roots of unity. I would like to continue this chain of ideas further, in the hopes that a "good" estimate of $p(n)$ can be made purely on the basis of estimating correctly enough of the "bumps" around roots of unity.

Returning now to the discussion of question 3, it should again be noted that the spectra plots may not give such as a nice clue as to the form of the generating function as for the partition recurrence. They may, however, give some information as to the possibility of a generating function of a certain form existing for the sequence at hand. On the following pages are more spectra plots– since the plots for $\alpha(k) = \omega(k)$ =(a pentagonal number) are so suggestive, I have chosen to "perturb" the quadratic $\omega(k)$ by shifting it up and down a number of units as given by the number l accompanying each plot. Thus the first plot for $l = 1$ corresponds to $\alpha(k) = \omega(k) + 1$. An immediate observation is the fact that as l increases in absolute value away from $\omega(k)$, the plots seem to become more scrambled, especially in the vicinity of $-1+0i$. Corresponding to this observation is the fact that when $\alpha(k) = \omega(k)$, we have the especially simple product generating function for $p(n)$. And as an off-the-record comment, the fact that $\omega(k)$ has minimum $-1/24$, the constant which is well-known to those who have studied Dedekind's eta function, which is related very closely to $p(n)$'s generating function. In fact, the circle method analysis that yields Hardy, Ramanujan, and Rademacher's results uses the transformation formula for the Dedekind eta function. The big question is: is this minimum value of $\omega(k)$ just a coincidence, or is it pertinent in some way?

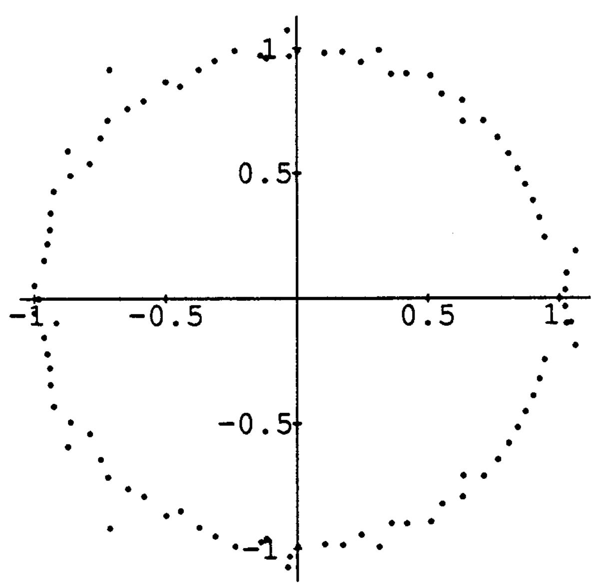
Plot corresponding to $l = 1$; polynomial of degree $\omega(7)+1 = 71$



poly. degree

$= \omega(8) - 1$

$= 91$

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Do[Print[pent[n]," " , gen[pent[n]]], {n,2,5}]
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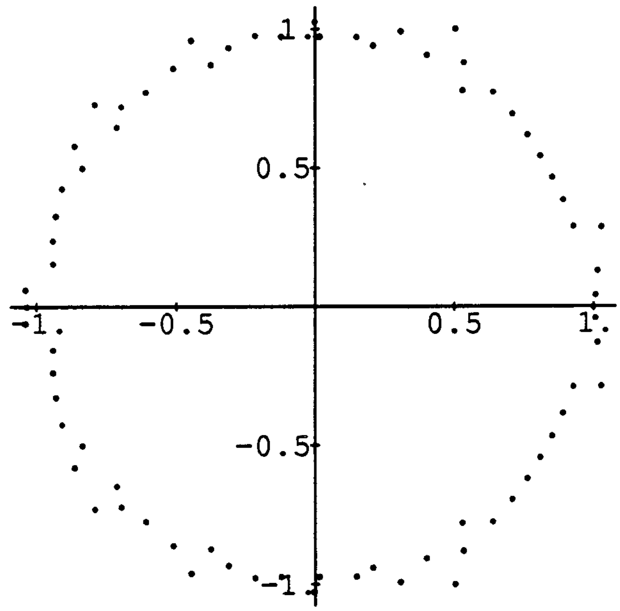
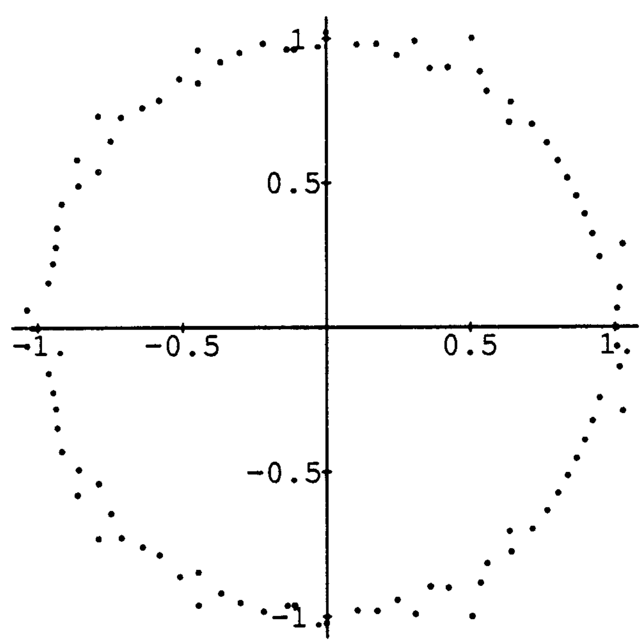
$l = 4$

degree = $w(7) + 4 = 74$



degree = $w(8) + 4 = 96$



13 $1 - x^{3} - x^{4} + x^{13}$

20 $-1 + x^{5} + x^{7} - x^{10} - x^{11} + x^{20}$

30 $1 - x^{7} - x^{10} + x^{15} + x^{17} - x^{20} - x^{21} + x^{30}$

43 $-1 + x^{9} + x^{13} - x^{20} - x^{23} + x^{28} + x^{30} - x^{33} - x^{34} + x^{43}$

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Do[ListPlot[roots[pent[n]], AspectRatio->1],{n,7,7}]
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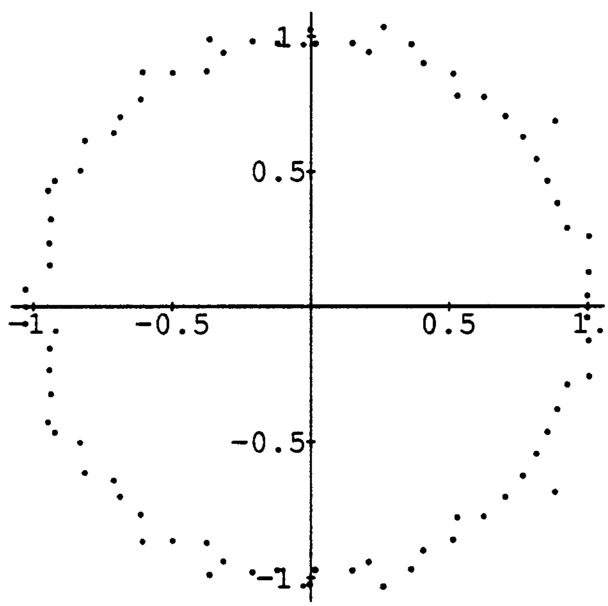
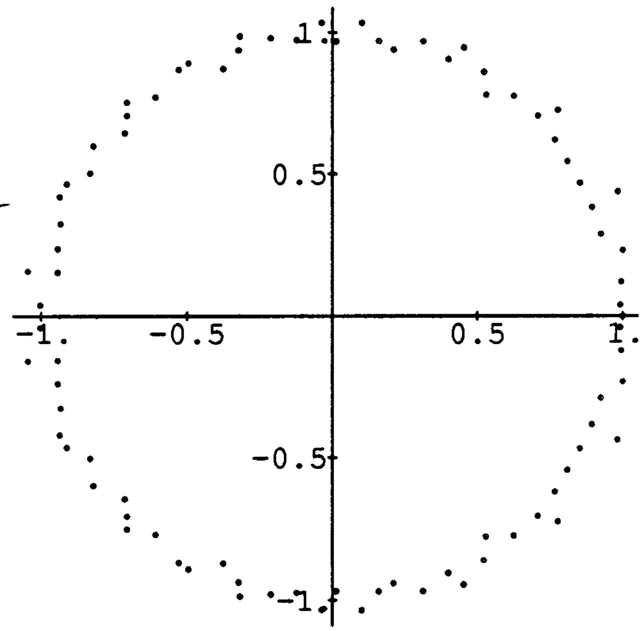
$\ell = 8$

degree $= \omega(7) + 8$
$= 78$

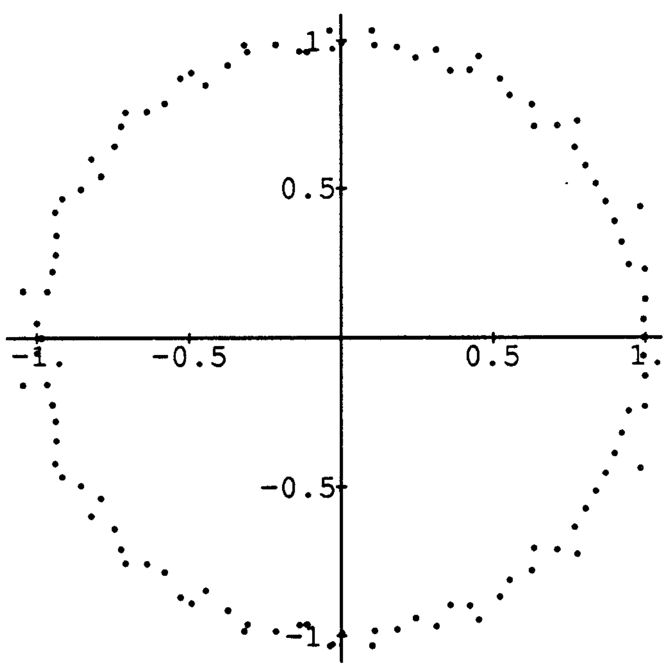


$l = 15$:

degree = $\omega(7) + 15 = 85$



degree = $\omega(8) + 15 = 107$



I mentioned earlier that I would give a couple of examples of sequences which satisfy the more general "growing"(pseudo-linear, as named in the title) recurrence relation where the coefficients in the relation for $s_{\alpha(k)}(n)$ are not restricted to 0, 1, or -1, but instead may be any integers, and $\alpha(k)$ may be any sequence, not necessarily quadratic (or even polynomial). Of course, the cases where a nice generating function exists similar to that for $p(n)$ are going to occur when the parameters in the recurrence follow some simple formulae of their own. For example, Jacobi's triple product identity yields the formal identity (which is analytically correct for $\|x\| < 1$):

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{(m^2+m)/2}.$$

If we denote the above product by $J(x)$, we can use the binomial theorem to expand the function $\frac{1}{J(x)}$ in a product of Taylor series. Then we can multiply this out formally to yield a sequence which may be thought of combinatorially as a weighted partition of n, where the weights are triangular numbers arising from the binomial expansion. In order to now find a recurrence relation for this sequence, we use the sum for $J(x)$: we multiply it by $J(x)$ and set the convolution sum for the coefficients equal to the coefficients of 1, i.e., all except the first are simply zero. The recurrence we obtain has coefficients which are just the ascending odd positive integers, and the index relation (corresponding to $\alpha(k)$ earlier) is just given by the triangular numbers. If we were to perform spectra plots for this recurrence, we ought to see behavior that is even stronger than that observed for $p(n)$'s recurrence. Afterall, we should find roughly three times as many eigenvalues near low-order rootss of unity. Given more time, this would have made an interesting computation!

Another example is found by using the product identity

$$\prod_{n=0}^{\infty}(1-x^{2^n}) = \sum_{m=0}^{\infty}(-1)^{\nu(m)}x^m,$$

where $\nu(m)$ is the number of 1's in the binary representation of m. Substituting x^2 for x and simplifying in the generating function

$$\frac{1}{\prod_{m=0}^{\infty}(1-x^{2^k})} = \sum_{n=0}^{\infty} g(n)x^n$$

yields the recurrence

$$g(2n) - g(2n-1) = g(n),$$

and obviously $g(0) = 1 = g(1)$. However, a slightly more intriguing recurrence relation may also be obtained using the method that gave the last example's as well as $p(n)$'s recurrences:

$$\sum_{k=0}^{n}(-1)^{\nu(k)}g(n-k) = 0.$$

As before, plotting the spectra corresponding to truncated forms of this recurrence ought to reveal familiar behavior near 2^n-roots-of-unity.

For the asymptotic methods discussed in this paper, I refer the reader to Tom Apostol's Modular Functions and Dirichlet Series in Number Theory, chapter 5 especially. A deeper circle method analysis of $p(n)$ may be found in Rademacher's Topics in Analytic Number Theory, chapter 13. For an elementary treatment of $p(n)$, see Apostol's undergraduate number theory book Introduction to Analytic Number Theory. All three of these books were published by Springer-Verlag.