INEQUALITIES FOR POSITIVE RANK AND CRANK MOMENTS OF OVERPARTITIONS

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Abstract. In recent work, Andrews, Chan and Kim extend a result of Garvan about even rank and crank moments of partitions. In a similar fashion we extend a result of Mao about even rank moments of overpartitions. We study positive Dyson-rank, M_2-rank, first residual crank, and second residual crank moments of overpartitions. We denote these positive k-th rank and crank moments by $\overline{N}_k^+(n)$, $\overline{N2}_k^+(n)$, $\overline{M}_k^+(n)$, and $\overline{M2}_k^+(n)$, respectively. We prove a number of inequalities involving the rank and residual crank moments. In particular, we prove a conjecture of Mao which states that: for $n \geq 2$ and k a positive integer,

$$\overline{N}_k^+(n) > \overline{N2}_k^+(n).$$

1. Introduction and Statement of Results

A *partition* of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. We call the elements of a partition *parts*, and denote by $p(n)$ the number of partitions of n. For example, the partitions of 4 are given by

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Hence, $p(4) = 5$. The generating function for $p(n)$, first given by Euler, can be written

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{1.1}$$

We will use the standard q-series notation

$$(a_1, \ldots, a_j)_n = (a_1, \ldots, a_j; q)_n = \prod_{i=0}^{n-1} (1-a_1q^i)\cdots(1-a_jq^i),$$

where n is a positive integer or ∞, and define $(a_1, \ldots, a_j)_0 = 1$. Ramanujan's celebrated congruences

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$

and

$$p(11n+6) \equiv 0 \pmod{11}$$

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have inspired much work in the theory of partitions. In particular, there are two partition statistics called the *rank* and the *crank*, which give a combinatorial explanation for these congruences. The *rank* or *Dyson-rank* was introduced by Dyson [9]. The rank of a partition λ is defined by

$$rank(\lambda) \coloneqq l(\lambda) - n(\lambda),$$

where $l(\lambda)$ is the largest part of λ and $n(\lambda)$ is the number of parts in λ. For example, the rank of the partition $3+1$ of 4 is

$$rank(3+1) = l(3+1) - n(3+1) = 3 - 2 = 1.$$

The *crank* was introduced by Andrews and Garvan [3]. The crank of a partition λ is defined by

$$crank(\lambda) \coloneqq \begin{cases} l(\lambda), & \text{if } r(\lambda) = 0, \\ \omega(\lambda) - r(\lambda), & \text{if } r(\lambda) \geq 1, \end{cases}$$

where $r(\lambda)$ is the number of 1's in λ and $\omega(\lambda)$ is the number of parts of λ strictly greater than $r(\lambda)$. For example, for the partition $3+1$ of 4, we have $r(\lambda) = 1$, so the crank is

$$crank(3+1) = \omega(3+1) - r(3+1) = 1 - 1 = 0.$$

An *overpartition* of a positive integer n is a partition of n where the first occurrence of a part may be overlined. We denote by $\overline{p}(n)$ the number of overpartitions of n. The generating function of $\overline{p}(n)$ can be written [8]

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q)_\infty}{(q)_\infty}. \tag{1.2}$$

For example, there are 14 overpartions of 4, which are given by

$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad \overline{2}+2,$$

$$2+1+1, \quad \overline{2}+1+1, \quad 2+\overline{1}+1, \quad \overline{2}+\overline{1}+1, \quad 1+1+1+1, \quad \overline{1}+1+1+1.$$

Thus, $\overline{p}(4) = 14$.

For $n \geq 1$, let $N(m,n)$ count the number of partitions of n with rank m. When $n = 0$ we define $N(0,0) = 1$ and $N(m,0) = 0$ for $m \neq 0$. For $n > 1$, we let $M(m,n)$ count the number of partitions of n with crank m. When $n \leq 1$ we define

$$M(m,n) \coloneqq \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), \text{ or } (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

We note that $N(m,n) = N(-m,n)$ [9] and $M(m,n) = M(-m,n)$ [3]. The rank generating function is defined by

$$R(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n.$$

Similarly, the crank generating function is defined by

$$C(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n.$$

We see in [2] that

$$R(z,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (q/z)_n},$$

and

$$C(z,q) = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty}.$$

In [11], Garvan studies the even moments of the rank and crank partition statistics. The *k-th rank moment* is defined by

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n),$$

and the *k-th crank moment* is defined by

$$M_k(n) := \sum_{m=-\infty}^{\infty} m^k M(m,n).$$

Since $N(m,n) = N(-m,n)$ and $M(m,n) = M(-m,n)$, we see that $N_k(n) = M_k(n) = 0$ for odd k.

Interestingly, the number of smallest parts in the partitions of n can be expresses by a difference of the second crank and rank moments. In fact,

$$spt(n) = \frac{1}{2}\left(M_2(n) - N_2(n)\right),$$

where $spt(n)$ is the total number of appearances of the smallest part in each partition of n [1]. For example, $spt(4) = 10$. We underline the smallest parts of the partitions of 4:

$$\underline{4}, \quad 3+\underline{1}, \quad \underline{2}+\underline{2}, \quad 2+\underline{1}+\underline{1}, \quad \underline{1}+\underline{1}+\underline{1}+\underline{1}.$$

By using symmetrized rank and crank moments as well as Bailey chains, Garvan [11] proves that for $n,k \geq 1$,

$$M_{2k}(n) > N_{2k}(n). \tag{1.3}$$

In [2], Andrews, Chan, and Kim define positive rank and crank moments for partitions. The *positive k-th rank moment* is given by

$$N_k^+(n) := \sum_{m=1}^{\infty} m^k N(m,n),$$

and the *positive k-th crank moment* is given by

$$M_k^+(n) := \sum_{m=1}^{\infty} m^k M(m,n).$$

We note that for every even k, $N_k(n) = 2N_k^+(n)$ and $M_k(n) = 2M_k^+(n)$. Hence,

$$spt(n) = M_2^+(n) - N_2^+(n).$$

In [2], the authors prove, using a more elementary approach, that for all $n,k \geq 1$,

$$M_k^+(n) > N_k^+(n),$$

which implies Garvan's result. In particular, they found a combinatorial interpretation of the difference $M_k^+(n) - N_k^+(n)$ for all k.

In [14], Mao finds an analogous result to equation (1.3) in the overpartition setting. Mao compares two distinct ranks for overpartitions, the *Dyson-rank* and the *M_2-rank*. The *Dyson-rank* or *D-rank* of an overpartition is defined by

$$D\text{-}rank(\lambda) := l(\lambda) - n(\lambda),$$

i.e., it is the same as the rank of λ. Furthermore, the *M_2-rank* of an overpartition was introduced by Lovejoy [13]. Let λ_o denote the subpartition of non-overlined odd parts of λ. The M_2-rank of an overpartition λ is given by

$$M_2\text{-}rank(\lambda) := \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_o) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if the largest part of λ is odd and not overlined and $\chi(\lambda) = 0$ otherwise. For $n \geq 1$, we let $\overline{N}(m,n)$ denote the number of overpartitions of n with D-rank m. When $n = 0$, we define $\overline{N}(0,0) = 1$ and $\overline{N}(m,0) = 0$ for $m \neq 0$. For $n \geq 1$, we let $\overline{N2}(m,n)$ denote the number of overpartitions of n with M_2-rank m. When $n = 0$, we define $\overline{N2}(0,0) = 1$ and $\overline{N2}(m,0) = 0$ for $m \neq 0$. We note that $\overline{N}(m,n) = \overline{N}(-m,n)$ [12] and $\overline{N2}(m,n) = \overline{N2}(-m,n)$ [13]. The D-rank generating function is defined by

$$\overline{R}(z,q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m,n) z^m q^n. \tag{1.4}$$

Similarly, the M_2-rank generating function is defined by

$$\overline{R2}(z,q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m,n) z^m q^n. \tag{1.5}$$

We see in [12] that

$$\overline{R}(z,q) = \sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n}, \tag{1.6}$$

and in [13] that

$$\overline{R2}(z,q) = \sum_{n=0}^{\infty} \frac{(-1)_{2n} q^n}{(zq^2, q^2/z; q^2)_n}. \tag{1.7}$$

In [14], Mao employs similar techniques as Garvan to study even moments of these two overpartition statistics. The *k-th D-rank moment* is defined by

$$\overline{N}_k(n) := \sum_{m=-\infty}^{\infty} m^k \overline{N}(m,n), \tag{1.8}$$

and the *k-th M_2-rank moment* is defined by

$$\overline{N2}_k(n) := \sum_{m=-\infty}^{\infty} m^k \overline{N2}(m,n). \tag{1.9}$$

As in the partition setting, we observe that $\overline{N}_k(n) = \overline{N2}_k(n) = 0$ for odd k. In [14], Mao proves that for $n \geq 2$ and $k \geq 1$,

$$N_{2k}(n) > N2_{2k}(n).$$

Similarly to Andrews, Chan, and Kim, Mao defines the *positive k-th D-rank moment* by

$$\overline{N}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{N}(m,n), \tag{1.10}$$

and the *positive k-th M_2-rank moment* by

$$\overline{N2}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{N2}(m,n). \tag{1.11}$$

We now state our first results.

Theorem 1.1. *For $n \geq 2$, we have*

$$\overline{N}_1^+(n) > \overline{N2}_1^+(n).$$

Theorem 1.2. *For $n \geq 2$ and k a positive integer, we have*

$$\overline{N}_k^+(n) > \overline{N2}_k^+(n).$$

Mao conjectured Theorem 1.2 in [14]. We prove Theorem 1.1 first to highlight the techniques used in the proof of the subsequent theorem. For proving our results we define the positive k-th D-rank generating function by

$$\overline{R}_k^+(q) := \sum_{n=1}^{\infty} \overline{N}_k^+(n) q^n, \tag{1.12}$$

and the positive k-th M_2-rank generating function by

$$\overline{R2}_k^+(q) := \sum_{n=1}^{\infty} \overline{N2}_k^+(n) q^n. \tag{1.13}$$

In Section 4, we will derive q-series representations for $\overline{R}_k^+(q)$ and $\overline{R2}_k^+(q)$.

Bringmann, Lovejoy, and Osburn introduced the first and second residual cranks of overpartitions [6]. Let λ_α be the subpartition of non-overlined parts of λ. The *first residual crank* of an overpartition λ is defined by

$$\textit{residual-crank}1(\lambda) := \textit{crank}(\lambda_\alpha).$$

Let λ_e be the subpartition of non-overlined even parts of λ. The *second residual crank* of an overpartition of λ is defined by

$$\textit{residual-crank}2(\lambda) := \frac{\textit{crank}(\lambda_e)}{2}.$$

For $n > 1$, we let $\overline{M}(m,n)$ denote the number of overpartitions of n with first residual crank m. When $n \leq 1$ we define

$$\overline{M}(m,n) := \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), \text{ or } (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

For $n > 1$, we let $\overline{M2}(m,n)$ denote the number of overpartitions of n with second residual crank m. When $n \leq 1$ we define

$$\overline{M2}(m,n) := \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), \text{ or } (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

We note that $\overline{M}(m,n) = \overline{M}(-m,n)$ and $\overline{M2}(m,n) = \overline{M2}(-m,n)$ [6]. The first residual crank generating function is defined by

$$\overline{C}(z,q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{M}(m,n) z^m q^n. \tag{1.14}$$

Similarly, the second residual crank generating function is defined by

$$\overline{C2}(z,q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{M2}(m,n) z^m q^n. \tag{1.15}$$

We see in [6] that

$$\overline{C}(z,q) = (-q;q)_\infty C(z,q) = \frac{(q^2;q^2)_\infty}{(zq;q)_\infty (q/z;q)_\infty}, \tag{1.16}$$

and

$$\overline{C2}(z,q) = \frac{(-q;q)_\infty}{(q;q^2)_\infty} C(z,q^2) = \frac{(-q;q)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (zq^2;q^2/z;q^2)_\infty}. \tag{1.17}$$

The *k-th first residual crank moment* is defined by

$$\overline{M}_k(n) := \sum_{m=-\infty}^{\infty} m^k \overline{M}(m,n), \tag{1.18}$$

and the *k-th second residual crank moment* is defined by

$$\overline{M2}_k(n) := \sum_{m=-\infty}^{\infty} m^k \overline{M2}(m,n). \tag{1.19}$$

In [6], Bringmann, Lovejoy, and Osburn prove the following relationships between second moments of ranks and cranks for overpartitions:

$$\overline{spt}(n) = \overline{M}_2(n) - \overline{N}_2(n),$$

and

$$\overline{spt2}(n) = \overline{M2}_2(n) - \overline{N2}_2(n),$$

where $\overline{spt}(n)$ is the total number of appearances of the smallest part in each overpartition of n and $\overline{spt2}(n)$ is the total number of appearances of the smallest part in each overpartition of n, provided the smallest part is even [6]. We define the *k-th positive fist residual crank moment* by

$$\overline{M}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{M}(m,n), \tag{1.20}$$

and the *k-th positive second residual crank moment* by

$$\overline{M2}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{M2}(m,n). \tag{1.21}$$

We find an interesting relationship between the residual crank positive moments.

Theorem 1.3. *For $n \geq 1$, we have*

$$\overline{M}_1^+(n) > \overline{M2}_1^+(n).$$

For proving our results we also define the k-th positive first residual crank moment generating function by

$$\overline{C}_k^+(q) := \sum_{n=1}^{\infty} \overline{M}_k^+(n) q^n, \tag{1.22}$$

and the k-th positive second residual crank moment generating function by

$$\overline{C2}_k^+(q) := \sum_{n=1}^{\infty} \overline{M2}_k^+(n) q^n. \tag{1.23}$$

In Section 4, we will derive q-series representations for $\overline{C}_k^+(q)$ and $\overline{C2}_k^+(q)$. We find further relationships between the overpartition rank and residual crank positive moments.

Theorem 1.4. *For $n \geq 2$, we have*

$$\overline{N}_1^+(n) > \overline{M2}_1^+(n),$$

in particular,

$$\overline{N}_1^+(n) = 2\overline{M2}_1^+(n).$$

Theorem 1.5. *For $n \geq 2$ and k a positive integer, we have*

$$\overline{N}_k^+(n) > \overline{M2}_k^+(n).$$

Theorem 1.6. *For $n \geq 2$ and $k \geq 2$, we have*

$$\overline{M2}_k^+(n) > \overline{N2}_k^+(n).$$

We observe that Theorem 1.2 for $k \geq 2$, follows from Theorem 1.5 and Theorem 1.6.

In the next section, we derive q-series representations for first moment generating functions. We discuss the relationship between the rank and residual crank generating functions and the first moment generating functions. Furthermore, we prove an important lemma for establishing the positivity of differences between the first moment generating functions. In Section 3, we prove our results about first moments by showing that the coefficients of the differences of the generating functions are positive. For example, since

$$\overline{R}_1^+(q) - \overline{R2}_1^+(q) = \sum_{n=1}^{\infty} \left(\overline{N}_1^+(n) - \overline{N2}_1^+(n) \right) q^n,$$

we observe that this method suffices. In Section 4, we generalize the techniques used in Section 2 and obtain the k-th moment generating functions. Additionally, we give alternative representations of these k-th moment generating functions. In Section 5, we prove our results for k-th moments using similar strategies as Section 3. In the Section, we suggest further inequalities for k-th moments.

2. Preliminaries I

In this section, we will derive q-series representations for the first moment generating functions $\overline{R}_1^+(q)$, $\overline{R2}_1^+(q)$, $\overline{C}_1^+(q)$ and $\overline{C2}_1^+(q)$. Lemma 2.1 provides a general way to obtain positive first moment generating functions from their 2-variable generating functions.

Lemma 2.1. *Let $a(m,n)$ be sequence defined for $n \geq 0$, $m \in \mathbb{Z}$ such that $a(m,0) = 0$ for all $m \geq 1$. The generating function for $a(m,n)$ is defined by*

$$G(z,q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a(m,n) z^m q^n,$$

the positive k-th moment is defined by

$$H_k^+(n) := \sum_{m=1}^{\infty} m^k a(m,n),$$

and the positive k-th moment generating function is defined by

$$G_k^+(q) := \sum_{n=1}^{\infty} H_k^+(n) q^n.$$

Then,

$$G_1^+(q) = \lim_{z\to 1} D_1^+(z,q),$$

where $D_1^+(z,q)$ is the series comprised of those terms of $z\frac{\partial}{\partial z}G(z,q)$ with positive powers of z.

Proof. By definition of $G(z,q)$,

$$\begin{aligned} z\frac{\partial}{\partial z}G(z,q) &= z\frac{\partial}{\partial z}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} a(m,n)z^m q^n \\ &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} ma(m,n)z^m q^n. \end{aligned}$$

Now consider the double sum were m ranges over only positive values. So we have

$$D_1^+(z,q) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} ma(m,n)z^m q^n.$$

Let $z \to 1$. Then

$$\begin{aligned} \lim_{z\to 1} D_1^+(z,q) &= \lim_{z\to 1}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} ma(m,n)z^m q^n \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} ma(m,n)q^n \\ &= \sum_{m=1}^{\infty} ma(m,0) + \sum_{n=1}^{\infty} H_1^+(n)q^n \\ &= G_1^+(q). \end{aligned}$$

□

We note that in the proof of Lemma 2.1, taking the non-negative powers of z then applying $z\frac{\partial}{z\partial}$ is the same as first applying $z\frac{\partial}{z\partial}$ and then taking the positive powers of z. We will use this fact in the proof of Lemma 2.3.

We will use the following lemma in our proof of Lemma 2.3.

Lemma 2.2. *We have*

$$\frac{(1-z)(1-1/z)q^n}{(1-zq^n)(1-q^n/z)} = 1 - \frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty} z^m q^{mn} - \frac{(1-q^n)}{(1+q^n)}\sum_{m=1}^{\infty} z^{-m} q^{mn}. \tag{2.1}$$

Proof. We see that

$$\begin{aligned}
\frac{(1-z)(1-1/z)q^n}{(1-zq^n)(1-q^n/z)} &= (1-z)(1-1/z)q^n\sum_{m=0}^{\infty}z^mq^{mn}\sum_{m=0}^{\infty}z^{-m}q^{mn}\\
&= (1-z)(1-1/z)q^n\sum_{m=0}^{\infty}q^{mn}\sum_{j=0}^{m}z^{m-2j}\\
&= (-z+2-1/z)\sum_{m=1}^{\infty}q^{mn}\sum_{j=0}^{m-1}z^{m-1-2j}\\
&= \sum_{m=1}^{\infty}q^{mn}\left(-z^m+2\sum_{j=0}^{m-1}(-1)^jz^{m-j-1}+2\sum_{j=0}^{m-2}(-1)^jz^{-(m-j-1)}-z^{-m}\right)\\
&= \sum_{m=1}^{\infty}q^{mn}\left(\sum_{j=0}^{m}(-1)^{j+1}z^{m-j}+\sum_{j=0}^{m-1}(-1)^{j+1}z^{-(m-j)}\right)\\
&+ q^n+\sum_{m=2}^{\infty}q^{mn}\left(\sum_{j=0}^{m-1}(-1)^jz^{m-j-1}+\sum_{j=0}^{m-2}(-1)^jz^{-(m-j-1)}\right)\\
&= 1-1-\sum_{m=1}^{\infty}q^{mn}\left(\sum_{j=0}^{m}(-1)^jz^{m-j}+\sum_{j=0}^{m-1}(-1)^jz^{-(m-j)}\right)\\
&+ q^n+\sum_{m=1}^{\infty}q^{(m+1)n}\left(\sum_{j=0}^{m}(-1)^jz^{m-j}+\sum_{j=0}^{m-1}(-1)^jz^{-(m-j)}\right)\\
&= 1-(1-q^n)\sum_{m=1}^{\infty}q^{mn}\left(\sum_{j=0}^{m}(-1)^jz^{m-j}+\sum_{j=0}^{m-1}(-1)^jz^{-(m-j)}\right)\\
&= 1-(1-q^n)\sum_{m=1}^{\infty}q^{mn}\sum_{j=0}^{m}(-1)^jz^{m-j}-(1-q^n)\sum_{m=1}^{\infty}q^{mn}\sum_{j=0}^{m-1}(-1)^jz^{-(m-j)}\\
&= 1-(1-q^n)\sum_{j=0}^{\infty}(-1)^jq^{jn}\sum_{m=0}^{\infty}z^mq^{mn}-(1-q^n)\sum_{j=0}^{\infty}(-1)^jq^{jn}\sum_{m=1}^{\infty}z^{-m}q^{mn}\\
&= 1-\frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty}z^mq^{mn}-\frac{(1-q^n)}{(1+q^n)}\sum_{m=1}^{\infty}z^{-m}q^{mn}.
\end{aligned}$$

□

In the next lemma, we express the generating function for the first D-rank moment, the first M_2-rank moment, and the first first residuals crank moments in terms of q-series.

Lemma 2.3. *The generating functions* $\overline{R}_1^+(q)$, $\overline{R2}_1^+(q)$, $\overline{C}_1^+(q)$, *and* $\overline{C2}_1^+(q)$ *are*

$$\overline{R}_1^+(q)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{1-q^{2n}}, \tag{2.2}$$

$$\overline{R2}_1^+(q) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}}{1-q^{4n}}, \tag{2.3}$$

$$\overline{C}_1^+(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n(n+1)/2}}{1-q^{n}}, \tag{2.4}$$

$$\overline{C2}_1^+(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{1-q^{2n}}. \tag{2.5}$$

Proof. By equation (1.6) we have

$$\overline{R}(z,q) = \sum_{n=0}^{\infty}\frac{(-1)_n q^{n(n+1)/2}}{(zq,q/z)_n}.$$

In [12], Watson's transformation is used to show that

$$\begin{aligned}\overline{R}(z,q) &= \sum_{n=0}^{\infty}\frac{(-1)_n q^{n(n+1)/2}}{(zq,q/z)_n}\\ &= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-1/z)(-1)^n q^{n^2+n}}{(1-zq^n)(1-q^n/z)}\right).\end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned}\overline{R}(z,q) &= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-1/z)(-1)^n q^{n^2+n}}{(1-zq^n)(1-q^n/z)}\right)\\ &= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty}z^m q^{mn}-\frac{(1-q^n)}{(1+q^n)}\sum_{m=1}^{\infty}z^{-m}q^{mn}\right)\right).\end{aligned}$$

Now we consider only the terms of the right hand side of the above equation that have non-negative powers of z. Then we have

$$\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty}z^m q^{mn}\right)\right). \tag{2.6}$$

Since we are considering (2.6) as a formal power series, we can rewrite it as

$$\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^n)}{(1+q^n)}\left(\frac{1}{1-q^n z}\right)\right)\right).$$

Now we apply the operator $z\frac{\partial}{\partial z}$ and obtain

$$2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^n)zq^{n^2+n}}{(1+q^n)(1-q^n z)^2}.$$

Now we let $z\to 1$, then

$$\begin{aligned}\lim_{z\to 1}2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^n)zq^{n^2+n}}{(1+q^n)(1-q^n z)^2} &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^n)q^{n^2+n}}{(1+q^n)(1-q^n)^2}\\ &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{(1+q^n)(1-q^n)}\end{aligned}$$

$$= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{1-q^{2n}}.$$

Therefore by Lemma 2.1, we have

$$\overline{R}_1^+(q) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{1-q^{2n}}.$$

In [13] by Theorem 1.2, we have that the coefficient of $a^r b^s c^t z^u q^N$ in

$$\sum_{n=0}^{\infty}\frac{(-1/a,-q/b;q^2)_n(abcq)^n}{(zq^2,cq^2/z;q^2)_n} \tag{2.7}$$

is equal to the number of overpartitions λ of N such that r is the sum of the number of overlined even parts of λ and the number of non-overlined odd parts of λ, s is the number of odd parts of λ, t is the number of parts of λ, and u is the M_2-rank of λ.

If we let $a=b=c=1$ in (2.7), then we see that the coefficient of $z^u q^N$ in

$$\sum_{n=0}^{\infty}\frac{(-1,-q;q^2)_n q^n}{(zq^2,q^2/z;q^2)_n}$$

is equal to the number of overpartitions λ of N such that u is the M_2-rank of λ. We notice that

$$\begin{aligned}
(-1,-q;q^2)_n &= \prod_{j=0}^{n-1}(1+q^{2j})(1+q^{2j+1})\\
&= \prod_{j=0}^{2n-1}(1+q^j)\\
&= (-1)_{2n}.
\end{aligned}$$

Hence, we have equation (1.7), namely that the q-series representation for $\overline{R2}(z,q)$ is

$$\overline{R2}(z,q) = \sum_{n=0}^{\infty}\frac{(-1)_{2n}q^n}{(zq^2,q^2/z;q^2)_n}.$$

In [13], Watson's transformation is used to show that

$$\begin{aligned}
\overline{R2}(z,q) &= \sum_{n=0}^{\infty}\frac{(-1)_{2n}q^n}{(zq^2,q^2/z;q^2)_n}\\
&= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-1/z)(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-q^{2n}/z)}\right).
\end{aligned}$$

By substituting $2n$ for n in equation (2.1), we have

$$\begin{aligned}
\overline{R2}(z,q) &= \sum_{n=0}^{\infty}\frac{(-1)_{2n}q^n}{(zq^2,q^2/z;q^2)_n}\\
&= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^{2n})}{(1+q^{2n})}\sum_{m=0}^{\infty}z^m q^{2mn}-\frac{(1-q^{2n})}{(1+q^{2n})}\sum_{m=1}^{\infty}z^{-m}q^{2mn}\right)\right).
\end{aligned}$$

Now we consider only the terms of the right hand side of the above equation that have non-negative powers of z. Then we have

$$\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^{2n})}{(1+q^{2n})}\sum_{m=0}^{\infty}z^m q^{2mn}\right)\right).$$

Which we can rewrite as

$$\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\left(1-\frac{(1-q^{2n})}{(1+q^{2n})}\left(\frac{1}{1-q^{2n}z}\right)\right)\right).$$

Now we apply the operator $z\frac{\partial}{\partial z}$ and obtain

$$2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^{2n})zq^{n^2+2n}}{(1+q^{2n})(1-q^{2n}z)^2}.$$

Now we let $z\to 1$. Then

$$\begin{aligned}
\lim_{z\to 1}2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^{2n})zq^{n^2+2n}}{(1+q^{2n})(1-q^{2n}z)^2} &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^{2n})q^{n^2+2n}}{(1+q^{2n})(1-q^{2n})^2}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}}{(1+q^{2n})(1-q^{2n})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}}{1-q^{4n}}.
\end{aligned}$$

Therefore by Lemma 2.1, we have

$$\overline{R2}_1^+(q)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}}{1-q^{4n}}.$$

By equation (1.16) we have

$$\overline{C}(z,q)=(-q;q)_\infty C(z,q).$$

Hence, by the proof of Theorem 1 in [2] we see that

$$\overline{C}_1^+(q)=(-q;q)_\infty C_1^+(q)=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{\frac{n^2+n}{2}}}{1-q^n}.$$

By equation (1.17) we have

$$\overline{C2}(z,q)=\frac{(-q;q)_\infty}{(q;q^2)_\infty}C(z,q^2).$$

Hence, by the proof of Theorem 1 in [2] we see that

$$\overline{C2}_1^+(q)=\frac{(-q;q)_\infty}{(q;q^2)_\infty}C_1^+(q^2)=\frac{(-q;q)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n(n+1)}}{1-q^{2n}}=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}}{1-q^{2n}}.$$

□

In the following lemma we make an observation about q-series. We will use this lemma throughout the main proofs in Sections 3 and 5. It is a crucial fact in observing that the differences of the moment generating functions have positive coefficients.

Lemma 2.4. *Let S be the set defined by*

$$S = \left\{ \frac{(1-q^{a_1})\cdots(1-q^{a_n})}{(1+q^{b_1})\cdots(1+q^{b_m})} \Big| a_1,\ldots,a_n,b_1,\ldots,b_m \in \mathbb{Z}^+, n,m \geq 0,\ a_i \neq a_j, b_i \neq b_j \ \forall\ i,j \right\},$$

where if $n=0$ or $m=0$ we assume that the empty product is 1. Let $f(q) \in S$. Then,

$$\frac{(-q)_\infty}{(q)_\infty} \cdot f(q)$$

has non-negative power series coefficients.

Proof. Let $f(q) \in S$, then by definition of S,

$$\frac{(-q)_\infty}{(q)_\infty} f(q) = \left(\prod_{\substack{i=1 \\ i \neq a_1,\ldots,a_n}}^{\infty} (1+q^i) \right) \left(\prod_{\substack{i=1 \\ i \neq b_1,\ldots,b_n}}^{\infty} \left(\frac{1}{1-q^i} \right) \right),$$

where $a_1,\ldots,a_n,b_1,\ldots,b_m \in \mathbb{Z}^+$ and $a_i \neq a_j, b_i \neq b_j$ for all i,j clearly has non-negative power series coefficients. □

3. Main Proofs I

In this section we prove Theorems 1.1, 1.3, and 1.4.

3.1. **Proof of Theorem 1.1.** We recall that Theorem 1.1 states that for $n \geq 2$,

$$\overline{N}_1^+(n) > \overline{N2}_1^+(n).$$

Proof. By equation (2.2) and equation (2.3) we have

$$\begin{aligned}
\overline{R}_1^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+n}(1+q^{2n})}{(1-q^{2n})(1+q^{2n})} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \left((-1)^{n+1} q^{n^2+n} \sum_{m=0}^{\infty} (q^{4mn} + q^{4mn+2n}) \right) \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+1} q^{n^2+n} (q^{4mn} + q^{4mn+2n}) \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} q^{(2j+1)^2+(2j+1)} (q^{4(2j+1)m} + q^{4(2j+1)m+2(2j+1)}) \\
&- 2\frac{(-q)_\infty}{(q)_\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} q^{(2j+2)^2+(2j+2)} (q^{4(2j+2)m} + q^{4(2j+2)m+2(2j+2)}),
\end{aligned}$$

and

$$\begin{aligned}
\overline{R2}_1^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+2n}}{1-q^{4n}} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \left((-1)^{n+1} q^{n^2+2n} \sum_{m=0}^{\infty} q^{4mn} \right)
\end{aligned}$$

$$\begin{aligned} &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+1}q^{n^2+2n+4mn} \\ &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+2(2j+1)+4(2j+1)m} \\ &- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+2)^2+2(2j+2)+4(2j+2)m}. \end{aligned}$$

Thus,

$$\begin{aligned} \overline{R}_1^+(q) - \overline{R2}_1^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+(2j+1)+4(2j+1)m} + q^{(2j+1)^2+4(2j+1)m+3(2j+1)} \\ &- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+2)^2+(2j+2)+4(2j+2)m} - q^{(2j+2)^2+4(2j+2)m+3(2j+2)} \\ &- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+2(2j+1)+4(2j+1)m} + q^{(2j+2)^2+2(2j+2)+4(2j+2)m} \\ &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+(2j+1)+4(2j+1)m} - q^{(2j+1)^2+2(2j+1)+4(2j+1)m} \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+2)^2+2(2j+2)+4(2j+2)m} - q^{(2j+2)^2+4(2j+2)m+3(2j+2)} \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+4(2j+1)m+3(2j+1)} - q^{(2j+2)^2+(2j+2)+4(2j+2)m} \\ &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+(2j+1)+4(2j+1)m}(1-q^{(2j+1)}) \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+2)^2+2(2j+2)+4(2j+2)m}(1-q^{(2j+2)}) \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+3(2j+1)+4(2j+1)m}(1-q^{(2+4m)}). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{R}_1^+(q) - \overline{R2}_1^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+(2j+1)+4(2j+1)m}(1-q^{(2j+1)}) \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+2)^2+2(2j+2)+4(2j+2)m}(1-q^{(2j+2)}) \\ &+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}q^{(2j+1)^2+3(2j+1)+4(2j+1)m}(1-q^{(2+4m)}). \end{aligned} \tag{3.1}$$

By Lemma 2.4 and equation (3.1) we see the difference $\overline{R}_1^+(q) - \overline{R2}_1^+(q)$ has positive coefficients. Therefore, for $n \geq 2$, $\overline{N}_1^+(n) > \overline{N2}_1^+(n)$. □

3.2. **Proof of Theorem 1.3.** We recall that Theorem 1.3 states that for $n \geq 1$,

$$\overline{M}_1^+(n) > \overline{M2}_1^+(n).$$

Proof. By equation (2.4) and equation (2.5) we have

$$\begin{aligned}
\overline{C}_1^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}}}{1-q^n}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}}(1+q^n)}{1-q^{2n}}
\end{aligned}$$

and

$$\overline{C2}_1^+(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}}{1-q^{2n}}.$$

Hence,

$$\begin{aligned}
\overline{C}_1^+(q) - \overline{C2}_1^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}}(1+q^n)}{1-q^{2n}} - \frac{(-1)^{n+1}q^{n^2+n}}{1-q^{2n}}\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}}(1+q^n-q^{\frac{n^2+n}{2}})}{1-q^{2n}}.
\end{aligned}$$

We reindex the sum by odd and even n and obtain

$$\begin{aligned}
\overline{C}_1^+(q) - \overline{C2}_1^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+1)^2+(2j+1)}{2}}(1+q^{(2j+1)}-q^{\frac{(2j+1)^2+(2j+1)}{2}})}{1-q^{2(2j+1)}}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+2)^2+(2j+2)}{2}}(1+q^{(2j+2)}-q^{\frac{(2j+2)^2+(2j+2)}{2}})}{1-q^{2(2j+2)}}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+1)^2+(2j+1)}{2}}(1+q^{(2j+1)}-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-q^{2(2j+2)})}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+2)^2+(2j+2)}{2}}(1+q^{(2j+2)}-q^{\frac{(2j+2)^2+(2j+2)}{2}})(1-q^{2(2j+1)})}{(1-q^{2(2j+2)})(1-q^{2(2j+1)})}
\end{aligned}$$

We observe that

$$\begin{aligned}
& q^{\frac{(2j+1)^2+(2j+1)}{2}}(1+q^{(2j+1)}-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-q^{2(2j+2)})\\
-\ & q^{\frac{(2j+1)^2+(2j+1)}{2}+(2j+2)}(1+q^{2j+2}-q^{\frac{(2j+1)^2+(2j+1)}{2}+(2j+2)})(1-q^{2(2j+1)})\\
=\ & q^{\frac{(2j+1)^2+(2j+1)}{2}}((1+q^{(2j+1)}-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-q^{2(2j+2)})\\
-\ & q^{2j+2}(1+q^{2j+2}-q^{\frac{(2j+1)^2+(2j+1)}{2}+(2j+2)})(1-q^{2(2j+1)}))
\end{aligned}$$

$$
\begin{aligned}
&= q^{\frac{(2j+1)^2+(2j+1)}{2}}(1+q^{(2j+1)}-q^{\frac{(2j+1)^2+(2j+1)}{2}}-q^{2(2j+2)}-q^{3(2j+1)+2}+q^{\frac{(2j+1)^2+(2j+1)}{2}+2(2j+2)}\\
&- q^{2j+2}-q^{2(2j+2)}+q^{\frac{(2j+1)^2+(2j+1)}{2}+2(2j+2)}+q^{2(2j+1)+(2j+2)}+q^{2(2j+1)+2(2j+2)}-q^{\frac{(2j+1)^2+(2j+1)}{2}+4(2j+2)})\\
&= q^{\frac{(2j+1)^2+(2j+1)}{2}}(q^{(2j+1)}(1-q)+q^{2(2j+1)+(2j+2)}(1-q)+1-2q^{2(2j+2)}+q^{2(2j+1)+2(2j+2)}\\
&- q^{\frac{(2j+1)^2+(2j+1)}{2}}(1-2q^{2(2j+2)}+q^{2(2j+1)+2(2j+2)}))\\
&= q^{\frac{(2j+1)^2+(2j+1)}{2}}((q^{(2j+1)}+q^{2(2j+1)+(2j+2)})(1-q)\\
&+ (1-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-2q^{2(2j+2)}+q^{2(2j+1)+2(2j+2)}))\\
&= q^{\frac{(2j+1)^2+(2j+1)}{2}}((q^{(2j+1)}+q^{2(2j+1)+(2j+2)})(1-q)\\
&+ (1-q^{\frac{(2j+1)^2+(2j+1)}{2}})((1-q^{2(2j+2)})^2+q^{2(2j+1)+2(2j+2)}(1-q^2))).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overline{C}_1^+(q)-\overline{C2}_1^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+1)^2+(2j+1)}{2}+(2j+1)}(1+q^{(2j+1)+(2j+2)})(1-q)}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+1)^2+(2j+1)}{2}}(1-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-q^{2(2j+2)})}{(1-q^{2(2j+1)})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{\frac{(2j+1)^2+(2j+1)}{2}+2(2j+1)+2(2j+2)}(1-q^{\frac{(2j+1)^2+(2j+1)}{2}})(1-q^2)}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}.
\end{aligned}
\tag{3.2}
$$

By Lemma 2.4 and equation (3.2) we see the difference $\overline{C}_1^+(q)-\overline{C2}_1^+(q)$ has positive coefficients. Therefore, for $n\geq 2$, $\overline{M}_1^+(n)>\overline{M2}_1^+(n)$. □

3.3. **Proof of Theorem 1.4.** We recall that Theorem 1.4 states that for $n\geq 2$,

$$\overline{N}_1^+(n)>\overline{M2}_1^+(n).$$

Proof. By equation (2.2) and equation (2.5) we notice that $\overline{R}_1^+(q)=2\cdot\overline{C2}_1^+(q)$. Again by equation (2.5) we have

$$\overline{C2}_1^+(q)=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}}{1-q^{2n}}.$$

Reindexing the sum by odd and even terms of n, we see that

$$\overline{C2}_1^+(q)=\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{(2j+1)^2+2j+1}}{1-q^{2(2j+1)}}-\frac{q^{(2j+2)^2+2j+2}}{1-q^{2(2j+2)}}.$$

We cross multiply to obtain

$$
\begin{aligned}
\frac{q^{(2j+1)^2+2j+1}}{1-q^{2(2j+1)}}-\frac{q^{(2j+2)^2+2j+2}}{1-q^{2(2j+2)}} &= \frac{q^{(2j+1)^2+2j+1}(1-q^{2(2j+2)})-q^{(2j+2)^2+2j+2}(1-q^{2(2j+1)})}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}\\
&= \frac{q^{(2j+1)^2+2j+1}(1-q^{2(2j+2)}-q^{2(2j+2)}(1-q^{2(2j+1)}))}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}
\end{aligned}
$$

$$= \frac{q^{(2j+1)^2+2j+1}(1-2q^{2(2j+2)}+q^{2(2j+2)+2(2j+1)})}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}.$$

We notice that

$$\begin{aligned}1-2q^{2(2j+2)}+q^{2(2j+2)+2(2j+1)} &= (1-q^{2(2j+2)})^2-q^{4(2j+2)}+q^{2(2j+2)+2(2j+1)}\\ &= (1-q^{2(2j+2)})^2+q^{2(2j+2)+2(2j+1)}(1-q^2).\end{aligned}$$

Therefore,

$$\overline{R}_1^+(q)-\overline{C2}_1^+=\overline{C2}_1^+(q) \tag{3.3}$$

$$\begin{aligned}&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{(2j+1)^2+2j+1}((1-q^{2(2j+2)})^2+q^{2(2j+2)+2(2j+1)}(1-q^2))}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}\\ &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{(2j+1)^2+2j+1}(1-q^{2(2j+2)})}{(1-q^{2(2j+1)})}+\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\frac{q^{(2j+1)^2+2(2j+2)+3(2j+1)}(1-q^2)}{(1-q^{2(2j+1)})(1-q^{2(2j+2)})}.\end{aligned}$$

By Lemma 2.4 and equation (3.3) we see the difference $R_1^+(q)-C2_1^+(q)$ has positive coefficients. Therefore, for $n\geq 2$, $\overline{N}_1^+(n)>\overline{M2}_1^+(n)$. □

4. Preliminaries II

In this section we will derive q-series representations for the k-th moment generating functions $\overline{R}_k^+(q)$, $\overline{R2}_k^+(q)$, $\overline{C}_k^+(q)$ and $\overline{C2}_k^+(q)$. In the following lemma we generalize Lemma 2.1 to connect the rank and residual crank generating functions to k-th moment generating functions .

Lemma 4.1. *Let $a(m,n)$ be sequence defined for $n\geq 0$, $m\in\mathbb{Z}$ such that $a(m,0)=0$ for all $m\geq 1$. The generating function for $a(m,n)$ is defined by*

$$G(z,q):=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}a(m,n)z^mq^n,$$

the positive k-th moment is defined by

$$H_k^+(n):=\sum_{m=1}^{\infty}m^ka(m,n),$$

and the positive k-th moment generating function is defined by

$$G_k^+(q):=\sum_{n=1}^{\infty}H_k^+(n)q^n.$$

Then,

$$G_k^+(q)=\lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1}D_1^+(z,q),$$

where $G_1^+(z,q)$ is the series comprised of those terms of $z\frac{\partial}{\partial z}G(z,q)$ with positive powers of z.

Proof. By the proof of Lemma 2.1,

$$G_1^+(z,q)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}ma(m,n)z^mq^n.$$

We observe that

$$\begin{aligned}\left(z\frac{\partial}{\partial z}\right)^{k-1} G_1^+(z,q) &= \left(z\frac{\partial}{\partial z}\right)^{k-1}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} ma(m,n)z^m q^n\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m^k a(m,n)z^m q^n.\end{aligned}$$

Let $z\to 1$. Then

$$\begin{aligned}\lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1} G_1^+(z,q) &= \lim_{z\to 1}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m^k a(m,n)z^m q^n\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m^k a(m,n)q^n\\ &= \sum_{m=1}^{\infty} m^k a(m,0)+\sum_{n=1}^{\infty} H_k^+(n)q^n\\ &= G_k^+(q).\end{aligned}$$

□

The next lemma connects the repeated application of the differential operator to the q-series representations of the k-th moment generating functions. We will use this lemma in the proof of Lemma 4.3. The content of the following lemma is found in Section 5 of [2].

Lemma 4.2. *Let $A_1(t)=1$. Then*

$$\left(z\frac{\partial}{\partial z}\right)^{k-1}\frac{z}{(1-q^n z)^2}=\frac{zA_k(zq^n)}{(1-zq^n)^{k+1}},$$

where

$$A_k(t)=A_{k,0}+A_{k,1}t+\cdots+A_{k,k}t^{k-1}$$

is a polynomial of degree $k-1$ with $A_{k,m}$ statisfying the recursive relation

$$A_{k,m}=(m+1)A_{k-1,m}+(k-m)A_{k-1,m-1}\quad (1\le m\le k-1).$$

We note that the polynomials $A_k(t)$ for $k\geq 1$, are Eulerian polynomials [2]. The coefficients $A_{k,m}$ for $1\le m\le k-1$, are Eulerian numbers and positive integers. The reader can find more details in [10].

In Lemma 4.3 we express the generating functions for the k-th D-rank, M_2-rank, first residual crank, and second residual crank moments in terms of q-series.

Lemma 4.3. *The generating functions $\overline{R}_k^+(q)$, $\overline{R2}_k^+(q)$, $\overline{C}_k^+(q)$, and $\overline{C2}_k^+(q)$ are*

$$\overline{R}_k^+(q)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}(1-q^n)A_k(q^n)}{(1+q^n)(1-q^n)^{k+1}},\tag{4.1}$$

$$\overline{R2}_k^+(q)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}(1-q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k+1}},\tag{4.2}$$

$$\overline{C}_k^+(q)=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{\frac{n^2+n}{2}}(1-q^n)A_k(q^n)}{(1-q^n)^{k+1}},\tag{4.3}$$

$$\overline{C2}_k^+(q) = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+n}(1-q^{2n})A_k(q^{2n})}{(1-q^{2n})^{k+1}}. \tag{4.4}$$

Proof. By Lemma 4.1 and the proof of equation (2.2),

$$\begin{aligned}
\overline{R}_k^+(q) &= \lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1} 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^n)zq^{n^2+n}}{(1+q^n)(1-q^nz)^2} \\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+n}\frac{(1-q^n)}{(1+q^n)}\left(\lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1}\frac{z}{(1-q^nz)^2}\right).
\end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}
\overline{R}_k^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+n}\frac{(1-q^n)}{(1+q^n)}\left(\lim_{z\to 1}\frac{zA_k(zq^n)}{(1-zq^n)^{k+1}}\right) \\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+n}(1-q^n)A_k(q^n)}{(1+q^n)(1-q^n)^{k+1}}.
\end{aligned}$$

By Lemma 4.1 and the proof of equation (2.3),

$$\begin{aligned}
\overline{R2}_k^+(q) &= \lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1} 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^{2n})zq^{n^2+2n}}{(1+q^{2n})(1-q^{2n}z)^2} \\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+2n}\frac{(1-q^{2n})}{(1+q^{2n})}\left(\lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1}\frac{z}{(1-q^{2n}z)^2}\right).
\end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}
\overline{R2}_k^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+2n}\frac{(1-q^{2n})}{(1+q^{2n})}\left(\lim_{z\to 1}\frac{zA_k(zq^{2n})}{(1-zq^{2n})^{k+1}}\right) \\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{n^2+2n}(1-q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k+1}}.
\end{aligned}$$

By equation (1.16) we have

$$\overline{C}(z,q) = (-q;q)_\infty C(z,q).$$

Hence, by work in [2] we have,

$$\begin{aligned}
\overline{C}_k^+(q) &= \lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1-q^n)zq^{\frac{n^2+n}{2}}}{(1-q^nz)^2} \\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{\frac{n^2+n}{2}}(1-q^n)\left(\lim_{z\to 1}\left(z\frac{\partial}{\partial z}\right)^{k-1}\frac{z}{(1-q^nz)^2}\right).
\end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}
\overline{C}_k^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}q^{\frac{n^2+n}{2}}(1-q^n)\left(\lim_{z\to 1}\frac{zA_k(zq^n)}{(1-zq^n)^{k+1}}\right) \\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{q^{\frac{n^2+n}{2}}(1-q^n)A_k(q^n)}{(1-q^n)^{k+1}}.
\end{aligned}$$

By equation (1.17) we have
$$\overline{C2}(z,q) = \frac{(-q;q)_\infty}{(q;q^2)_\infty} C(z,q^2).$$
Hence, by work in [2] we have,
$$\begin{aligned}
\overline{C2}^+_k(q) &= \lim_{z\to 1} \left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1-q^{2n})zq^{n^2+n}}{(1-q^{2n}z)^2} \\
&= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2+n}(1-q^{2n}) \left(\lim_{z\to 1} \left(z\frac{\partial}{\partial z}\right)^{k-1} \frac{z}{(1-q^{2n}z)^2} \right).
\end{aligned}$$
By Lemma 4.2 we have
$$\begin{aligned}
\overline{C}^+_k(q) &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2+n}(1-q^{2n}) \left(\lim_{z\to 1} \frac{zA_k(zq^{2n})}{(1-zq^{2n})^{k+1}} \right) \\
&= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+n}(1-q^{2n})A_k(q^{2n})}{(1-q^{2n})^{k+1}}.
\end{aligned}$$
□

We rewrite the results from Lemma 4.3 in a useful way. Let $\frac{1}{(1-q^{2n})^r} = \sum_{m=0}^{\infty} b_{m,r} q^{2mn}$ and define a_ℓ such that $A_k(t) = \sum_{\ell=0}^{k-1} a_\ell t^\ell$. By equation (4.1) we have
$$\begin{aligned}
\overline{R}^+_k(q) &= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+n}A_k(q^n)}{(1+q^n)(1-q^n)^k} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{q^{n^2+n}(1+q^n)^k A_k(q^n)}{(1+q^n)(1-q^{2n})^k} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \left((-1)^{n+1} \frac{q^{n^2+n}(1+q^n)^k A_k(q^n)}{(1+q^n)} \sum_{m=0}^{\infty} b_{m,k} q^{2mn} \right) \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+1} \frac{b_{m,k} q^{n^2+n+2mn}(1+q^n)^k A_k(q^n)}{(1+q^n)} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+1} \frac{b_{m,k} q^{n^2+n+2mn}(1+q^n)^k \left(\sum_{\ell=0}^{k-1} a_\ell q^{n\ell}\right)}{(1+q^n)} \\
&= 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{k-1} (-1)^{n+1} \frac{a_\ell b_{m,k} q^{n^2+n+n(2m+\ell)}(1+q^n)^k}{(1+q^n)}.
\end{aligned}$$
We recall that $(1+q^n)^k = \sum_{i=0}^{k} \binom{k}{i} q^{ni}$. Therefore,
$$\overline{R}^+_k(q) = 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{k-1} (-1)^{n+1} \frac{a_\ell b_{m,k} q^{n^2+n+n(2m+\ell)} \left(\sum_{i=0}^{k} \binom{k}{i} q^{ni}\right)}{(1+q^n)}$$

$$= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}\frac{a_\ell b_{m,k}\binom{k}{i}q^{n^2+n+n(2m+\ell+i)}}{(1+q^n)}$$

$$= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}\frac{a_\ell b_{m,k}\binom{k}{i}q^{n^2+n+n(2m+\ell+i)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}.$$

$\overline{R2}_k^+(q)$, $\overline{C}_k^+(q)$, and $\overline{C2}_k^+(q)$ can be rewritten in a similar manner to obtain the following equations:

$$\overline{R}_k^+(q) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}\frac{a_\ell b_{m,k}\binom{k}{i}q^{n^2+n+n(2m+\ell+i)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}, \tag{4.5}$$

$$\overline{R2}_k^+(q) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}(-1)^{n+1}\frac{a_\ell b_{m,k}q^{n^2+2n+n(2m+2\ell)}(1+q^n)}{(1+q^n)(1+q^{2n})}, \tag{4.6}$$

$$\overline{C}_k^+(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}\frac{a_\ell b_{m,k}\binom{k}{i}q^{\frac{n^2+n}{2}+n(2m+\ell+i)}(1+q^n)(1+q^{2n})}{(1+q^n)(1+q^{2n})}, \tag{4.7}$$

$$\overline{C2}_k^+(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}(-1)^{n+1}\frac{a_\ell b_{m,k}q^{n^2+n+n(2m+2\ell)}(1+q^n)(1+q^{2n})}{(1+q^n)(1+q^{2n})}. \tag{4.8}$$

5. Main Proofs II

In this section we prove Theorems 1.2, 1.5, and 1.6.

5.1. **Proof of Theorem 1.2.** We recall that Theorem 1.2 states that for $n \geq 2$,

$$\overline{N}_k^+(n) > \overline{N2}_k^+(n).$$

Proof. By equation (4.1) and equation (4.2),

$$\begin{aligned}
\overline{R}_k^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}a_\ell b_{m,k}\binom{k}{i}\frac{q^{n^2+n+n(2m+\ell+i)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}(-1)^{n+1}a_\ell b_{m,k}\frac{q^{n^2+n+n(2m+\ell)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k-1}(-1)^{n+1}a_\ell b_{m,k}\left(\binom{k}{i}-1\right)\frac{q^{n^2+n+n(2m+\ell+i)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k}(-1)^{n+1}a_\ell b_{m,k}\frac{q^{n^2+n+n(2m+\ell+i)}(1+q^{2n})}{(1+q^n)(1+q^{2n})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell)}(1+q^{2(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k-1}a_\ell b_{m,k}\left(\binom{k}{i}-1\right)\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell+i)}(1+q^{(2j+2)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}
\end{aligned}$$

$$
\begin{aligned}
&+\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k}a_\ell b_{m,k}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell+i)}(1+q^{(2j+2)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+k)}(1+q^{2(2j+2)})}{(1+q^{(2j+2)})(1+q^{2(2j+2)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k-1}a_\ell b_{m,k}\left(\binom{k}{i}-1\right)\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+i)}(1+q^{(2j+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+i)}(1+q^{(2j+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{R2}_k^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}(-1)^{n+1}a_\ell b_{m,k}\frac{q^{n^2+2n+n(2m+2\ell)}(1+q^n)}{(1+q^n)(1+q^{2n})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)^2+2(2j+1)+(2j+1)(2m+2\ell)}(1+q^{(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+2(2j+2)+(2j+2)(2m+2\ell)}(1+q^{(2j+2)})}{(1+q^{(2j+2)})(1+q^{2(2j+2)})}.
\end{aligned}
$$

By the above we have

$$\overline{R}_k^+(q)-\overline{R2}_k^+(q)=T_1+T_2+T_3+T_4.$$

Where T_1, T_2, T_3, and T_4 are defined as follows:

$$
\begin{aligned}
T_1 &:= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell)}(1+q^{2(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)^2+2(2j+1)+(2j+1)(2m+2\ell)}(1+q^{(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})},
\end{aligned}
$$

$$
\begin{aligned}
T_2 &:= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+2(2j+2)+(2j+2)(2m+2\ell)}(1+q^{(2j+2)})}{(1+q^{(2j+2)})(1+q^{2(2j+2)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+k)}(1+q^{2(2j+2)})}{(1+q^{(2j+2)})(1+q^{2(2j+2)})},
\end{aligned}
$$

$$
\begin{aligned}
T_3 &:= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k}a_\ell b_{m,k}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell+i)}(1+q^{(2j+2)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}\\
&-\ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+i)}(1+q^{(2j+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})},
\end{aligned}
$$

and

$$\begin{aligned}
T_4 &:= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k-1}a_\ell b_{m,k}\left(\binom{k}{i}-1\right)\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell+i)}(1+q^{(2j+2)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{k-1}a_\ell b_{m,k}\left(\binom{k}{i}-1\right)\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+i)}(1+q^{(2j+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}.
\end{aligned}$$

Now we manipulate T_1, T_2, T_3 and T_4.

$$\begin{aligned}
T_1 &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+1)(2j+1)}(1+q^{2(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+2\ell+2)(2j+1)}(1+q^{(2j+1)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+1)(2j+1)}(1-q^{(\ell+1)(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+3)(2j+1)}(1-q^{\ell(2j+1)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})}.
\end{aligned}$$

$$\begin{aligned}
T_2 &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+2)^2+(2m+2\ell+2)(2j+2)}(1+q^{(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+2)^2+(2m+\ell+k+1)(2j+2)}(1+q^{2(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+2)^2+(2m+2\ell+2)(2j+2)}(1-q^{(k-\ell-1)(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+2)^2+(2m+2\ell+3)(2j+2)}(1-q^{(k-\ell)(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}.
\end{aligned}$$

$$\begin{aligned}
T_3 &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)}(1+q^{(2j+2)})}{(1+q^{2j+2})(1+q^{(2j+1)})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+2)^2+(2m+\ell+i+1)(2j+2)}(1+q^{(2j+1)})}{(1+q^{2j+2})(1+q^{(2j+1)})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)}-q^{(2j+2)^2+(2m+\ell+i+1)(2j+2)}}{(1+q^{2j+2})(1+q^{(2j+1)})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1}a_\ell b_m\frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)+(2j+2)}-q^{(2j+2)^2+(2m+\ell+i+1)(2j+2)+(2j+1)}}{(1+q^{2j+2})(1+q^{(2j+1)})}
\end{aligned}$$

$$= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)}(1-q^{(2j+2)+(2m+\ell+i+1)})}{(1+q^{2j+2})(1+q^{(2j+1)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)+(2j+2)}(1-q^{(2j+1)+(2m+\ell+i+1)})}{(1+q^{2j+2})(1+q^{(2j+1)})}.$$

$$T_4 \quad = \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+1)^2+(2m+i+\ell+1)(2j+1)}(1+q^{(2j+2)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}$$

$$- \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+2)^2+(2m+i+\ell+1)(2j+2)}(1+q^{(2j+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}$$

$$= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+1)^2+(2m+i+\ell+1)(2j+1)}(1-q^{(2j+1)+(2j+2)+(2m+i+\ell+1)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+2)^2+(2m+i+\ell+1)(2j+2)}(1-q^{2(2j+1)+(2m+i+\ell+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}.$$

Therefore,

$$\overline{R}_k^+(q) - \overline{R2}_k^+(q) = T_1 + T_2 + T_3 + T_4 \tag{5.1}$$

$$= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+1)(2j+1)}(1-q^{(\ell+1)(2j+1)})}{(1+q^{(2j+1)})(1+q^{2(2j+1)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+3)(2j+1)}(1-q^{\ell(2j+1)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+2)^2+(2m+2\ell+2)(2j+2)}(1-q^{(k-\ell-1)(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+2)^2+(2m+2\ell+3)(2j+2)}(1-q^{(k-\ell)(2j+2)})}{(1+q^{2(2j+2)})(1+q^{(2j+2)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)}(1-q^{(2j+2)+(2m+\ell+i+1)})}{(1+q^{2j+2})(1+q^{(2j+1)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} a_\ell b_m \frac{q^{(2j+1)^2+(2m+\ell+i+2)(2j+1)+(2j+2)}(1-q^{(2j+1)+(2m+\ell+i+1)})}{(1+q^{2j+2})(1+q^{(2j+1)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+1)^2+(2m+i+\ell+1)(2j+1)}(1-q^{(2j+1)+(2j+2)+(2m+i+\ell+1)})}{(1+q^{(2j+1)})(1+q^{(2j+2)})}$$

$$+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{k-1}\sum_{\ell=0}^{k-1} a_\ell \left(\binom{k}{i}-1\right) b_m \frac{q^{(2j+2)^2+(2m+i+\ell+1)(2j+2)}(1-q^{2(2j+1)+(2m+i+\ell+1)})}{(1+q^{(2j+2)})(1+q^{(2j+1)})}.$$

By Lemma 2.4 and equation (5.1) we see the difference $\overline{R}_k^+(q) - \overline{R2}_k^+(q)$ has positive coefficients. Therefore, for $n \geq 2$, $\overline{N}_k^+(n) > \overline{N2}_k^+(n)$. □

5.2. **Proof of Theorem 1.5.** We recall that Theorem 1.5 states that for $n \geq 2$,

$$\overline{N}_k^+(n) > \overline{M2}_k^+(n).$$

First we prove a Lemma to be used in our proof of the theorem.

Lemma 5.1. *We have*

$$\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i (1 - q^{k-(2i-1)}).$$

Proof. Consider the sum

$$\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i.$$

Since $\binom{k}{i} = \binom{k}{k-i}$, we observe that

$$\binom{k}{i} - \binom{k}{i-1} = -\left(\binom{k}{k-(i-1)} - \binom{k}{k-i} \right). \tag{5.2}$$

When k is even $\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i$ has an even number of terms and we can pair the first and the last terms together, the second and second to last terms together, and so on and obtain

$$\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i = \sum_{i=1}^{\frac{k}{2}} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i - \left(\binom{k}{k-(i-1)} - \binom{k}{k-i} \right) q^{k-(i-1)}.$$

Therefore, by equation (5.2) we see that

$$\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i = \sum_{i=1}^{\frac{k}{2}} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i (1 - q^{k-(2i-1)}).$$

Since k is even, $\frac{k}{2} = \lfloor \frac{k}{2} \rfloor$. Therefore we have Lemma 5.1 for k even. When k is odd $\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i$ has an odd number of terms. The middle term of the sum is

$$\left(\binom{k}{\lfloor \frac{k}{2} \rfloor + 1} - \binom{k}{\lfloor \frac{k}{2} \rfloor} \right) q^{\lfloor \frac{k}{2} \rfloor + 1},$$

since k s odd. We observe that $\lfloor \frac{k}{2} \rfloor + 1 = k - \lfloor \frac{k}{2} \rfloor$ by the parity of k. Therefore

$$\binom{k}{\lfloor \frac{k}{2} \rfloor + 1} = \binom{k}{k - \lfloor \frac{k}{2} \rfloor} = \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

Hence, the middle term of $\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i$ is 0. So, we can pair the first and last terms together, the second and second to last terms together, and so on excluding the middle term. We obtain that

$$\sum_{i=1}^{k} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \left(\binom{k}{i} - \binom{k}{i-1} \right) q^i - \left(\binom{k}{k-(i-1)} - \binom{k}{k-i} \right) q^{k-(i-1)}.$$

Therefore, by equation (5.2) we see that

$$\sum_{i=1}^{k}\left(\binom{k}{i}-\binom{k}{i-1}\right)q^i=\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(\binom{k}{i}-\binom{k}{i-1}\right)q^i(1-q^{k-(2i-1)}),$$

for k odd. □

Now we prove Theorem 1.5.

Proof. By equation (4.5) and equation (4.8) we have

$$\begin{aligned}
\overline{R}_k^+(q) &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}(-1)^{n+1}a_\ell b_{m,k}\binom{k}{i}\frac{q^{n^2+n+n(2m+\ell+i)}}{(1+q^n)}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}a_\ell b_{m,k}\binom{k}{i}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+\ell+i)}}{(1+q^{(2j+1)})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}a_\ell b_{m,k}\binom{k}{i}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+\ell+i)}}{(1+q^{(2j+2)})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}a_\ell b_{m,k}\binom{k}{i}\frac{q^{(2j+1)(2j+2+2m+\ell+i)}(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k}a_\ell b_{m,k}\binom{k}{i}\frac{q^{(2j+2)(2j+3+2m+\ell+i)}(1+q^{2j+1})}{(1+q^{2j+1})(1+q^{2j+2})},
\end{aligned}$$

and

$$\begin{aligned}
\overline{C2}_k^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}(-1)^{n+1}a_\ell b_{m,k}\frac{q^{n^2+n+n(2m+2\ell)}(1+q^n)}{(1+q^n)}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)^2+(2j+1)+(2j+1)(2m+2\ell)}(1+q^{(2j+1)})}{(1+q^{(2j+1)})}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)^2+(2j+2)+(2j+2)(2m+2\ell)}(1+q^{(2j+2)})}{(1+q^{(2j+2)})}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+1)(2j+2+2m+2\ell)}(1+q^{2j+1})(1+2^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{(2j+2)(2j+3+2m+2\ell)}(1+q^{2j+1})(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}.
\end{aligned}$$

Let $\gamma=(2j+1)(2j+2+2m+\ell)$ and $\xi=(2j+2)(2j+3+2m+\ell)$. By the above we have

$$\overline{R}_k^+(q)-\overline{C2}_k^+(q)=U_1+U_2+U_3+U_4.$$

Where U_1, U_2, U_3, and U_4 are defined as follows:

$$U_1 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\left(2\frac{q^\gamma}{(1+q^{2j+1})} - \frac{q^{\gamma+(2j+1)\ell}(1+q^{2j+1})}{(1+q^{2j+1})}\right),$$

$$U_2 := 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\left(\frac{\sum_{i=1}^{k}\binom{k}{i-1}q^{\gamma+(2j+1)i}(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})} - \frac{\sum_{i=0}^{k-1}\binom{k}{i}q^{\xi+(2j+2)i}(1+q^{2j+1})}{(1+q^{2j+1})(1+q^{2j+2})}\right),$$

$$U_3 := 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{\sum_{i=1}^{k}\left(\binom{k}{i}-\binom{k}{i-1}\right)q^{\gamma+(2j+1)i}}{(1+q^{2j+1})},$$

and

$$U_4 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\left(\frac{q^{\xi+(2j+2)\ell}(1+q^{2j+2})}{(1+q^{2j+2})} - 2\frac{q^{\xi+(2j+2)k}}{(1+q^{2j+2})}\right).$$

Now we manipulate U_1, U_2, U_3, and U_4.

$$\begin{aligned}
U_1 &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\left(2\frac{q^\gamma}{(1+q^{2j+1})} - \frac{q^{\gamma+(2j+1)\ell}(1+q^{2j+1})}{(1+q^{2j+1})}\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma\left(2-q^{(2j+1)\ell}(1+q^{2j+1})\right)}{(1+q^{2j+1})}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma\left((1-q^{(2j+1)\ell})+(1-q^{(2j+1)(\ell+1)})\right)}{(1+q^{2j+1})}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma(1-q^{(2j+1)\ell})}{(1+q^{2j+1})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma(1-q^{(2j+1)(\ell+1)})}{(1+q^{2j+1})}.
\end{aligned}$$

$$\begin{aligned}
U_2 &= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{\sum_{i=1}^{k}\binom{k}{i-1}q^{\gamma+(2j+1)i}(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{\sum_{i=0}^{k-1}\binom{k}{i}q^{\xi+(2j+2)i}(1+q^{2j+1})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma\sum_{i=1}^{k}\binom{k}{i-1}q^{(2j+1)i}(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma\sum_{i=0}^{k-1}\binom{k}{i}q^{(2j+2)i+4j+4+2m+\ell}(1+q^{2j+1})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&= 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^\gamma\sum_{i=0}^{k-1}\binom{k}{i}q^{(2j+1)(i+1)}(1+q^{2j+2})}{(1+q^{2j+1})(1+q^{2j+2})}
\end{aligned}$$

$$\begin{aligned}
&- \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\gamma\sum_{i=0}^{k-1}\binom{k}{i}q^{(2j+2)i+4j+4+2m+\ell}(1+q^{2j+1})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\gamma\sum_{i=0}^{k-1}\binom{k}{i}q^{(2j+1)(i+1)}(1-q^{i+2j+3+2m+\ell}))}{(1+q^{2j+1})(1+q^{2j+2})}\\
&+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\gamma\sum_{i=0}^{k-1}\binom{k}{i}q^{(2j+1)(i+1)+2j+2}(1-q^{i+2j+2+2m+\ell})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1}a_\ell b_{m,k}\binom{k}{i}\frac{q^{\gamma+(2j+1)(i+1)}(1-q^{i+2j+3+2m+\ell})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&+ \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1}a_\ell b_{m,k}\binom{k}{i}\frac{q^{\gamma+(2j+1)(i+1)+2j+2}(1-q^{i+2j+2+2m+\ell})}{(1+q^{2j+1})(1+q^{2j+2})}.
\end{aligned}$$

$$U_3 = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{\sum_{i=1}^{k}\left(\binom{k}{i}-\binom{k}{i-1}\right)q^{\gamma+(2j+1)i}}{(1+q^{2j+1})}.$$

By Lemma 5.1,

$$\begin{aligned}
U_3 \quad &= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\gamma\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(\binom{k}{i}-\binom{k}{i-1}\right)q^{(2j+1)i}(1-q^{(2j+1)(k-(2i-1))})}{(1+q^{2j+1})}\\
&= \quad 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}a_\ell b_{m,k}\left(\binom{k}{i}-\binom{k}{i-1}\right)\frac{q^{\gamma+(2j+1)i}(1-q^{(2j+1)(k-(2i-1))})}{(1+q^{2j+1})}.
\end{aligned}$$

$$\begin{aligned}
U_4 \quad &= \quad \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\left(\frac{q^{\xi+(2j+2)\ell}(1+q^{2j+2})}{(1+q^{2j+2})}-2\frac{q^{\xi+(2j+2)k}}{(1+q^{2j+2})}\right)\\
&= \quad \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\xi\left(q^{(2j+2)\ell}(1+q^{2j+2})-2q^{(2j+2)k}\right)}{(1+q^{2j+2})}\\
&= \quad \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\xi\left(q^{(2j+2)\ell}(1-q^{(2j+2)(k-\ell)})+q^{(2j+2)(\ell+1)}(1-q^{(2j+2)(k-\ell-1)})\right)}{(1+q^{2j+2})}\\
&= \quad \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^\xi(1-q^{(2j+2)(k-\ell)})}{(1+q^{2j+2})}\\
&+ \quad \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k}\frac{q^{\xi+(2j+2)(\ell+1)}(1-q^{(2j+2)(k-\ell-1)})}{(1+q^{2j+2})}.
\end{aligned}$$

Therefore,

$$\overline{R}_k^+(q)-\overline{C2}_k^+(q)=U_1+U_2+U_3+U_4 \tag{5.3}$$

$$
\begin{aligned}
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^{\gamma}(1-q^{(2j+1)\ell})}{(1+q^{2j+1})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^{\gamma}(1-q^{(2j+1)(\ell+1)})}{(1+q^{2j+1})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1} a_\ell b_{m,k}\binom{k}{i}\frac{q^{\gamma+(2j+1)(i+1)}(1-q^{i+2j+3+2m+\ell})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&+ 2\frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=0}^{k-1} a_\ell b_{m,k}\binom{k}{i}\frac{q^{\gamma+(2j+1)(i+1)+2j+2}(1-q^{i+2j+2+2m+\ell})}{(1+q^{2j+1})(1+q^{2j+2})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}\sum_{i=1}^{\lfloor \frac{k}{2}\rfloor} a_\ell b_{m,k}\left(\binom{k}{i}-\binom{k}{i-1}\right)\frac{q^{\gamma+(2j+1)i}(1-q^{(2j+1)(k-(2i-1))})}{(1+q^{2j+1})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^{\xi}(1-q^{(2j+2)(k-\ell)})}{(1+q^{2j+2})}\\
&+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k}\frac{q^{\xi+(2j+2)(\ell+1)}(1-q^{(2j+2)(k-\ell-1)})}{(1+q^{2j+2})}.
\end{aligned}
$$

By Lemma 2.4 and equation (5.3) we see the difference $\overline{R}_k^+(q) - \overline{R2}_k^+(q)$ has positive coefficients. Therefore, for $n \geq 2$, $\overline{N}_k^+(n) > \overline{M2}_k^+(n)$. $\square$

5.3. **Proof of Theorem 1.6.** We recall that Theorem 1.6 states that for $n \geq 2$ and $k \geq 2$,

$$\overline{M2}_k^+(n) > \overline{N2}_k^+(n).$$

Proof. By equation (4.4) and equation (4.2),

$$
\begin{aligned}
\overline{C2}_k^+(q) - \overline{R2}_k^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}(1-q^{2n})(1+q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k+1}}\\
&- 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+2n}(1-q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k+1}}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}(1+q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k}}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2q^{n^2+2n}A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k}}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}(1-2q^{n}+q^{2n})A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k}}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}(1-q^{n})^2A_k(q^{2n})}{(1+q^{2n})(1-q^{2n})^{k}}
\end{aligned}
$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}A_k(q^{2n})}{(1+q^{2n})(1+q^n)^2(1-q^{2n})^{k-2}}.$$

We now recall that we let $\frac{1}{(1-q^{2n})^r} = \sum\limits_{m=0}^{\infty} b_{m,r}q^{2mn}$ and define a_ℓ such that $A_k(t) = \sum\limits_{\ell=0}^{k-1} a_\ell t^\ell$. Now we have,

$$\begin{aligned}
\overline{C2}_k^+(q) - \overline{R2}_k^+(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}q^{n^2+n}A_k(q^{2n})}{(1+q^{2n})(1+q^n)^2}\sum_{m=0}^{\infty} b_{m,k-2}q^{2mn}\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{m,k-2}\frac{(-1)^{n+1}q^{n^2+n+2mn}A_k(q^{2n})}{(1+q^{2n})(1+q^n)^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{m,k-2}\frac{(-1)^{n+1}q^{n^2+n+2mn}\sum_{\ell=0}^{k-1} a_\ell q^{2\ell n}}{(1+q^{2n})(1+q^n)^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{(-1)^{n+1}q^{n^2+n+n(2m+2\ell)}}{(1+q^{2n})(1+q^n)^2}.
\end{aligned}$$

Let $\gamma = (2j+1)^2 + (2j+1) + (2j+1)(2m+2\ell)$ and $\xi = (2j+2)^2 + (2j+2) + (2j+2)(2m+2\ell)$. We reindex the sum by odd and even n and obtain

$$\begin{aligned}
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\gamma}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\xi}{(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\gamma(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\xi(1+q^{2(2j+1)})(1+q^{(2j+1)})^2}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\gamma(1+2q^{(2j+2)}+2q^{2(2j+2)}+2q^{3(2j+2)}+q^{4(2j+2)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&- \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\xi(1+2q^{(2j+1)}+2q^{2(2j+1)}+2q^{3(2j+1)}+q^{4(2j+1)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= V_1 + V_2 + V_3 + V_4 + V_5.
\end{aligned}$$

Where V_1, V_2, V_3, V_4, and V_5 are defined as follows:

$$V_1 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1} a_\ell b_{m,k-2}\frac{q^\gamma - q^\xi}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2},$$

$$V_2 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+(2j+2)}-q^{\xi+(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2},$$

$$V_3 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}-q^{\xi+2(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2},$$

$$V_4 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+3(2j+2)}-q^{\xi+3(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2},$$

and

$$V_5 := \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma+4(2j+2)}-q^{\xi+4(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}.$$

Now we manipulate V_1, V_2, V_3, V_4, and V_5.

$$\begin{aligned}
V_1 &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma}-q^{\xi}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma}(1-q^{2(2j+2)+(2m+2\ell)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma}(1-q^{2(2j+2)+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)}))}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.
\end{aligned}$$

$$\begin{aligned}
V_2 &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+(2j+2)}-q^{\xi+(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+(2j+2)}(1-q^{4j+3+(2m+2\ell)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+(2j+2)}(1-q^{4j+3+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.
\end{aligned}$$

$$\begin{aligned}
V_3 &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}-q^{\xi+2(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}(1-q^{4j+2+(2m+2\ell)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}\\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}(1-q^{4j+2+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.
\end{aligned}$$

$$V_4 = \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+3(2j+2)}-q^{\xi+3(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+3(2j+2)}(1-q^{4j+1+(2m+2\ell)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+3(2j+2)}(1-q^{4j+1+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.$$

$$V_5 = \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma+4(2j+2)}-q^{\xi+4(2j+1)}}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma+4(2j+2)}(1-q^{4j+(2m+2\ell)})}{(1+q^{2(2j+1)})(1+q^{(2j+1)})^2(1+q^{2(2j+2)})(1+q^{(2j+2)})^2}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma+4(2j+2)}(1-q^{4j+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.$$

Therefore,

$$\overline{C2}_k^+(q) - \overline{R2}_k^+(q) = V_1+V_2+V_3+V_4+V_5 \tag{5.4}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma}(1-q^{2(2j+2)+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)}))}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}$$

$$+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}(1-q^{4j+2+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}$$

$$+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+2(2j+2)}(1-q^{4j+2+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}$$

$$+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}2a_\ell b_{m,k-2}\frac{q^{\gamma+3(2j+2)}(1-q^{4j+1+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}$$

$$+ \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{k-1}a_\ell b_{m,k-2}\frac{q^{\gamma+4(2j+2)}(1-q^{4j+(2m+2\ell)})(1-q^{(2j+1)})(1-q^{(2j+2)})}{(1-q^{4(2j+1)})(1+q^{(2j+1)})(1-q^{4(2j+2)})(1+q^{(2j+2)})}.$$

By Lemma 2.4 and equation 5.4 we see the difference $\overline{C2}_k^+(q) - \overline{R2}_k^+(q)$ has positive coefficients. Therefore, for $n \geq 2$ and $k \geq 2$, $\overline{M2}_k^+(n) > \overline{N2}_k^+(n)$. □

6. Concluding Remarks

We recall the result of Andrews, Chan, and Kim. For all positive integers n and k,

$$M_k^+(n) > N_k^+(n). \tag{6.1}$$

In addition they define

$$ospt_k(n) := R_k^+(q) - R2_k^+(q),$$

where $ospt_k(n)$ is a conbinatorial interpretation of the difference and $ospt_2(n) = spt(n)$ [2]. Furthermore, we recall that Bringmann, Lovejoy, and Osburn [6] give an $spt(n)$ analog for overpartitions:

$$\overline{spt}(n) = \overline{M}_2(n) - \overline{N}_2(n),$$

$$\overline{spt2}(n) = \overline{M2}_2(n) - \overline{N2}_2(n),$$

and

$$\overline{spt1}(n) = \overline{spt}(n) - \overline{spt2}(n).$$

Computational evidence suggests an analog to equation (6.1) for the overpartition setting. The conjectured inequality furthers Mao's conjecture by involving the residual cranks reminiscent of the crank-rank inequality given by Andrews, Chan, and Kim.

Conjecture 6.1. *For $n \geq 2$, we have*

$$\overline{M}_1^+(n) > \overline{N}_1^+(n) > \overline{M2}_1^+(n) > \overline{N2}_1^+(n).$$

By Theorem 1.4 we have for $n \geq 2$,

$$\overline{N}_1^+(n) > \overline{M2}_1^+(n).$$

So, it is left to be proved that for $n \geq 2$,

$$\overline{M}_1^+(n) > \overline{N}_1^+(n),$$

and

$$\overline{M2}_1^+(n) > \overline{N2}_1^+(n).$$

Further, we conjecture the inequality for all positive k.

Conjecture 6.2. *For $n \geq 2$ and k a positive integer, we have*

$$\overline{M}_k^+(n) > \overline{N}_k^+(n) > \overline{M2}_k^+(n) > \overline{N2}_k^+(n).$$

By Theorem 1.5 we have for $n \geq 2$ and k a positive integer,

$$\overline{N}_k^+(n) > \overline{M2}_k^+(n).$$

By Theorem 1.6 we have for $n \geq 2$ and $k \geq 2$,

$$\overline{M2}_k^+(n) > \overline{N2}_k^+(n).$$

Therefore, it is left to be proved that for $n \geq 2$ and k a positive integer,

$$\overline{M}_k^+(n) > \overline{N}_k^+(n),$$

and

$$\overline{M2}_1^+(n) > \overline{N2}_1^+(n).$$

Based on the computational evidence for Conjecture 6.2 and the existence of $\overline{spt}(n)$, $\overline{spt1}(n)$, and $\overline{spt2}(n)$ we suspect an $opst_k(n)$ like function may exist for overpartitions. In particular, we conjecture the existence of an $\overline{ospt}_k(n)$ that relates the difference of $\overline{M}_k^+(n)$ and $\overline{N}_k^+(n)$, $\overline{ospt2}_k(n)$ that relates the difference of $\overline{M2}_k^+(n)$ and $\overline{N2}_k^+(n)$, and $\overline{ospt1}_k(n)$ that relates the difference of $\overline{M}_k^+(n) - \overline{N}_k^+(n)$ and $\overline{M2}_k^+(n) - \overline{N2}_k^+(n)$.

Conjecture 6.3. *There exists $\overline{ospt}_k(n)$, $\overline{ospt1}_k(n)$, and $\overline{ospt2}_k(n)$ such that*

$$\overline{ospt}_k(n) = \overline{M}_k^+(n) - \overline{N}_k^+(n),$$

$$\overline{ospt2}_k(n) = \overline{M2}_k^+(n) - \overline{N2}_k^+(n),$$

and

$$\overline{ospt1}_k(n) = \left(\overline{M}_k^+(n) - \overline{N}_k^+(n)\right) - \left(\overline{M2}_k^+(n) - \overline{N2}_k^+(n)\right)$$

where $\overline{ospt}_k(n)$, $\overline{ospt1}_k(n)$, and $\overline{ospt2}_k(n)$ are combinatorial interpretations of the differences.

Computational evidence also suggests the following congruence relationships. We see a nice relationship between which values of n the differences are congruent to 1 modulo 2 and what $\overline{spt}(n)$, $\overline{spt1}(n)$, and $\overline{spt2}(n)$ count.

Conjecture 6.4. *If $n = 2^m k^2$ where $m \geq 0$ and k is an odd positive integer, then*

$$\overline{M}_1^+(n) - \overline{N}_1^+(n) \equiv 1 \pmod 2,$$

otherwise

$$\overline{M}_1^+(n) - \overline{N}_1^+(n) \equiv 0 \pmod 2.$$

If $n = 2^m$ where $m \geq 1$, then

$$\overline{M2}_1^+(n) - \overline{N2}_1^+(n) \equiv 1 \pmod 2,$$

otherwise

$$\overline{M2}_1^+(n) - \overline{N2}_1^+(n) \equiv 0 \pmod 2.$$

If $n = k^2$ where k is an odd positive integer, then

$$(\overline{M}_1^+(n) - \overline{N}_1^+(n)) - (\overline{M2}_1^+(n) - \overline{N2}_1^+(n)) \equiv 1 \pmod 2,$$

otherwise

$$(\overline{M}_1^+(n) - \overline{N}_1^+(n)) - (\overline{M2}_1^+(n) - \overline{N2}_1^+(n)) \equiv 0 \pmod 2.$$

We close with a nice observation about the overpartition function.

Theorem 6.5. *If $n = k^2$ where k is a positive integer, then*

$$\overline{p}(n) \equiv 2 \pmod 4,$$

otherwise

$$\overline{p}(n) \equiv 0 \pmod 4.$$

Proof. Let n be a positive integer. Consider the partitions of n with at least two distinct parts. The number of ways to overline a partition is equal to 2^m where mm is the number of distinct parts of the partition. 4 divides 2^m for $m \geq 2$. Therefore, the number of overpartitions of n with at least two distinct parts is a multiple of 4. Now, consider the partitions of n with exactly one distinct part. If n is not a square of a positive integer, then n has two partitions with exactly one distinct part, namely the partitions n and $1 + \cdots + 1$. Therefore, n has $2 + 2 = 4$ overpartitions with exactly one distinct part. So, the number of overpartitions of n is divisible by 4. Consider $n = k^2$ where k is a positive integer. If $k = 1$, then n has one partitions with exactly one distinct part, namely 1. Therefore, n has 2 overpartitions with exactly one distinct part. If $k > 1$, then n has three partitions with exactly one distinct part, namely the partitions n, $k + \cdots + k$, and $1 + \cdots + 1$. therefore, n has $2 + 2 + 2 = 6 = 4 + 2$ overpartitions with exactly one distinct part. So, the number of overpartitions of n has remainder 2 when divided by 4. □

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8. Appendix-Notation

q-series notation:

$$(a_1,\dots,a_j)_0=(a_1,\dots,a_j;q)_0=1$$

$$(a_1,\dots,a_j)_n=(a_1,\dots,a_j;q)_n=\prod_{i=0}^{n-1}(1-a_1q^i)\cdots(1-a_jq^i)$$

and

$$(a_1,\dots,a_j)_\infty=(a_1,\dots,a_j;q)_\infty=\prod_{i=0}^{\infty}(1-a_1q^i)\cdots(1-a_jq^i).$$

$N(m,n) \coloneqq \textit{the number of partitions of } n \textit{ with rank } m$
$M(m,n) \coloneqq \textit{the number of partitions of } n \textit{ with crank } m$
$\overline{N}(m,n) \coloneqq \textit{the number of overpartitions of } n \textit{ with D-rank } m$
$\overline{N2}(m,n) \coloneqq \textit{the number of overpartitions of } n \textit{ with } M_2\textit{-rank } m$
$\overline{M}(m,n) \coloneqq \textit{the number of overpartitions of } n \textit{ with first residual crank } m$
$\overline{M2}(m,n) \coloneqq \textit{the number of overpartitions of } n \textit{ with second residual crankm}$

rank generating function: $R(z,q) \coloneqq \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^mq^n$

crank generating function: $C(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n$

D-rank generating function: $\overline{R}(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m,n) z^m q^n$

M_2-rank generating function: $\overline{R2}(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m,n) z^m q^n$

first residual crank generating function: $\overline{C}(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M}(m,n) z^m q^n$

second residual crank generating function: $\overline{C2}(z,q) \coloneqq \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M2}(m,n) z^m q^n$

k-th rank moment: $N_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k N(m,n)$

k-th crank moment: $M_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k M(m,n)$

k-th D-rank moment: $\overline{N}_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k \overline{N}(m,n)$

k-th M_2-rank moment: $\overline{N2}_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k \overline{N2}(m,n)$

k-th first residual crank moment: $\overline{M}_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k \overline{M}(m,n)$

k-th second residual crank moment: $\overline{M2}_k(n) \coloneqq \sum_{m=-\infty}^{\infty} m^k \overline{M2}(m,n)$

Positive k-th rank moment: $N_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k N(m,n)$

Positive k-th crank moment: $M_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k M(m,n)$

Positive k-th D-rank moment: $\overline{N}_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k \overline{N}(m,n)$

Positive k-th M_2-rank moment: $\overline{N2}_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k \overline{N2}(m,n)$

Positive k-th first residual crank moment: $\overline{M}_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k \overline{M}(m,n)$

Positive k-th second residual crank moment: $\overline{M2}_k^+(n) \coloneqq \sum_{m=1}^{\infty} m^k \overline{M2}(m,n)$

k-th rank moment generating function: $R_k^+(q) \coloneqq \sum_{n=1}^{\infty} N_k^+(n) q^n$

k-th crank moment generating function: $C_k^+(q) \coloneqq \sum_{n=1}^{\infty} M_k^+(n) q^n$

k-th D-rank moment generating function: $\overline{R}_k^+(q) \coloneqq \sum_{n=1}^{\infty} \overline{N}_k^+(n) q^n$

k-th M_2-rank moment generating function: $\overline{R2}_k^+(q) \coloneqq \sum_{n=1}^{\infty} \overline{N2}_k^+(n) q^n$

k-th first residual crank moment generating function: $\overline{C}_k^+(q) \coloneqq \sum_{n=1}^{\infty} \overline{M}_k^+(n) q^n$

k-th second residual crank moment generating function: $\overline{C2}_k^+(q) \coloneqq \sum_{n=1}^{\infty} \overline{M2}_k^+(n) q^n$

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