GENERALIZATIONS OF ALDER'S CONJECTURE VIA A CONJECTURE OF KANG AND PARK

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Abstract. Integer partitions have long been of interest to number theorists, perhaps most notably Ramanujan, and are related to many areas of mathematics including combinatorics, modular forms, representation theory, analysis, and mathematical physics. Here, we focus on partitions with gap conditions and partitions with parts coming from fixed residue classes.

Let $\Delta_d^{(a,b)}(n) = q_d^{(a)}(n) - Q_d^{(b)}(n)$ where $q_d^{(a)}(n)$ counts the number of partitions of n into parts with difference at least d and size at least a, and $Q_d^{(b)}$ counts the number of partitions into parts $\equiv \pm b \pmod{d+3}$. In 1956, Alder conjectured that $\Delta_d^{(1,1)}(n) \geq 0$ for all positive n and d. This conjecture was proved partially by Andrews in 1971, by Yee in 2008, and was fully resolved by Alfes, Jameson and Lemke Oliver in 2011. Alder's conjecture generalizes several well-known partition identities, including Euler's theorem that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts, as well as the first of the famous Rogers-Ramanujan identities.

In 2020, Kang and Park constructed an extension of Alder's conjecture which relates to the second Rogers-Ramanujan identity by considering $\Delta_d^{(a,b,-)}(n) = q_d^{(a)}(n) - Q_d^{(b,-)}(n)$ where $Q_d^{(b,-)}(n)$ counts the number of partitions into parts $\equiv \pm b \pmod{d+3}$ excluding the $d+3-b$ part. Kang and Park conjectured that $\Delta_d^{(2,2,-)}(n) \geq 0$ for all $d \geq 1$ and $n \geq 0$, and proved this for $d = 2^r - 2$ and n even.

We prove Kang and Park's conjecture for all but finitely many d. Toward proving the remaining cases, we adapt work of Alfes, Jameson and Lemke Oliver to generate asymptotics for the related functions. Finally, we present a more generalized conjecture for higher $a = b$ and prove it for infinite classes of n and d.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, that sum to n. Let $p(n \mid \text{condition})$ be the number of partitions of n satisfying a certain condition. Euler famously proved that the number of partitions of a positive integer n into odd parts equals the number of partitions of n into distinct parts.

Two other celebrated partition identities are those of Rogers and Ramanujan. The first Rogers-Ramanujan identity states that the number of partitions of n into parts differing by 2 is equal to the number of partitions of n into parts that are congruent to $\pm 1 \pmod 5$ and the second Rogers-Ramanujan identity states that the number of partitions of n into parts differing by 2 and with parts at least 2 is equal to the number of partitions of n into parts that are congruent to $\pm 2 \pmod 5$. Motivated by these identities, Schur found that the number

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of partitions of n into parts differing by 3 or more among which no two consecutive multiples of 3 appear is equal to the number of partitions of n into parts congruent to $\pm 1 \pmod 6$.

After showing that no other such partition identities can exist, in 1956 Alder [2, 1] made a claim about a generalization of similar partition inequalities. Alder's conjecture states that the number of partitions of n into parts with difference of at least d is greater than or equal to than the number of partitions of n into parts congruent to $\pm 1 \pmod{d+3}$. Notice that the Euler, first Rogers-Ramanujan, and Schur identities are special cases of Alder's Conjecture. In 1971, Andrews [4] proved Alder's conjecture for $n > 0$ and $d = 2^r - 1$, $r \geq 4$. In 2004 and 2008, Yee [13, 12] proved the conjecture for $n > 0$, $d \geq 32$, and $d = 7$. In 2011, Alfes, Jameson, and Lemke Oliver [3] proved Alder's Conjecture for $n > 0$ and $4 \leq d \leq 30, d \neq 7, 15$, thus completely resolving the conjecture.

In 2020, Kang and Park [7] investigated how to generalize Alder's conjecture further by incorporating the second Rogers-Ramanujan identity. Kang and Park compared the partition functions

$$q_d^{(a)}(n) := p(n| \text{ parts} \geq a \text{ and parts differ by at least } d),$$
$$Q_d^{(b)}(n) := p(n| \text{ parts} \equiv \pm b \pmod{d+3}),$$

by defining the difference function

$$\Delta_d^{(a,b)}(n) := q_d^{(a)}(n) - Q_d^{(b)}(n),$$
$$\Delta_d^{(a)}(n) := \Delta_d^{(a,a)}(n).$$

Remark 1.1. *Notice that*

$$\begin{aligned} \textit{Euler's identity} &\iff \Delta_1^{(1)}(n) = 0 \textit{ for all } n > 0 \\ \textit{Rogers-Ramanujan (}1^{st}\textit{ identity)} &\iff \Delta_2^{(1)}(n) = 0 \textit{ for all } n > 0 \\ \textit{Rogers-Ramanujan (}2^{nd}\textit{ identity)} &\iff \Delta_2^{(2)}(n) = 0 \textit{ for all } n > 0 \\ \textit{Schur's identity} &\implies \Delta_3^{(1)}(n) \geq 0 \textit{ for all } n > 0. \end{aligned}$$

Using Kang and Park's notation, Alder's conjecture can be stated as

$$\Delta_d^{(1)}(n) = q_d^{(1)}(n) - Q_d^{(1)}(n) \geq 0 \tag{1}$$

for $d, n > 0$.

Kang and Park were interested in finding an analog of Alder's conjecture for the second Rogers-Ramanujan identity. However, by observing the data, they found that

$$\Delta_d^{(2)}(n) < 0 \text{ for some choices of } d, n > 0.$$

In order to find a suitable analog, Kang and Park modified $Q_d^{(2)}(n)$ by defining for $d, n > 0$,

$$Q_d^{(2,-)}(n) := p(n \,|\, \text{parts} \equiv \pm 2 \pmod{d+3}, \text{ excluding the part } d+1),$$
$$\Delta_d^{(2,-)}(n) := \Delta_d^{(2,2,-)}(n) := q_d^{(2)}(n) - Q_d^{(2,-)}(n),$$

and presented the following conjecture.

Conjecture 1.2 (Kang, Park [7], 2020)**.** *For all d, $n > 0$,*

$$\Delta_d^{(2,-)}(n) \geq 0.$$

Kang and Park [7] proved Conjecture 1.2 for $d = 2$ or $d = 2^s - 2$ for any positive integer $s \geq 5$ and for any positive even integer n.

By developing a new way of comparing these partition functions, we prove the remaining cases of Kang and Park's conjecture except for $d = 1$ and $3 \leq d \leq 61$ [1].

Theorem 1.3. *For $d \geq 62$ and $n > 0$,*

$$\Delta_d^{(2,-)}(n) \geq 0.$$

It is natural to ask whether Conjecture 1.2 can be generalized to consider higher $a = b$. We can think about a generalization of Kang and Park's definitions for $1 \leq b \leq d+2$, where

$$Q_d^{(b,-)}(n) := p(n \mid \text{parts} \equiv \pm b \pmod{d+3},\ \text{excluding the part } d+3-b),$$
$$\Delta_d^{(a,b,-)}(n) := q_d^{(a)}(n) - Q_d^{(b,-)}(n) \text{ and } \Delta_d^{(a,-)}(n) := \Delta_d^{(a,a,-)}(n).$$

This allows us to present the following conjecture.

Conjecture 1.4. *For all $d, n > 0$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

But in general, $\Delta_d^{(a,-)}(n)$ is not always nonnegative for $a \geq 4$. Surprisingly, removing just one more possible part from the parts that can build partitions of n counted by $Q_d^{(a,-)}(n)$ allows us to completely generalize Conjecture 1.2. Define, where $1 \leq b \leq d+2$,

$$Q_d^{(b,-,-)}(n) := p(n \mid \text{parts} \equiv \pm b \pmod{d+3}, \text{excluding the parts } b \text{ and } d+3-b),$$
$$\Delta_d^{(a,b,-,-)}(n) := q_d^{(a)}(n) - Q_d^{(b,-,-)}(n) \text{ and } \Delta_d^{(a,-,-)}(n) := \Delta_d^{(a,a,-,-)}(n).$$

We posit the following conjecture[2].

Conjecture 1.5. *For all $a, d, n > 0$,*

$$\Delta_d^{(a,-,-)}(n) \geq 0.$$

Note that (1) implies Conjecture 1.5 for $a = 1$ and all $d, n > 0$ and Theorem 1.3 implies Conjecture 1.5 for $a = 2$ where $d \geq 62$ and $n > 0$.

Our methods used to prove Theorem 1.3 can be applied to prove the following partial result toward Conjecture 1.5 for $a \geq 3$.

Theorem 1.6. *For $a > 0$, $d \geq 31a - 3$, where $d + 3$ is divisible by a, and any $n > 0$,*

$$\Delta_d^{(a,-,-)}(n) \geq 0.$$

Furthermore, by generalizing the methods of Andrews [4] and Yee [13], we prove the following additional results toward Conjecture 1.5. First, given $d, a > 0$, define r (dependent on d and a) to be the largest integer such that

$$2^r - 2^{a-1} \leq d. \tag{2}$$

Theorem 1.7. *For any $a, d, n > 0$ where $d = 2^s - 2^t$, with $s \geq t + 4$ and $t \geq 0$,*

$$\Delta_d^{(a,-,-)}\left(2^{a-1}n\right) \geq 0.$$

[1] The $d = 2$ case is simply the second Rogers-Ramanujan identity.

[2] For a discussion on why the exclusion of the b and $d + 3 - b$ parts are necessary see Section 7.

Theorem 1.8. *Let $a \geq 3$, and $d = 2^{a-1}m$ where $m \geq 31$ and $m \neq 2^r - 1$. Then for all $n \geq 2^{3-a}d + 2^{r+1-a}$,*

$$\Delta_d^{(a,-,-)}\left(2^{a-1}n\right) \geq 0.$$

Finally, we investigate and generalize the asymptotic results of Andrews [4], and Alfes, Jameson, and Lemke Oliver [3]. We conclude the following.

Theorem 1.9. *Let $a, n > 0$ and $d > 3$ such that $a < \frac{d+3}{2}$ and a is relatively prime to $d+3$. Then*

$$\lim_{n\to\infty} \Delta_d^{(a)}(n) = +\infty.$$

Remark 1.10. *Notice that for all $a, d, n > 0$,*

$$\Delta_d^{(a,-,-)}(n) \geq \Delta_d^{(a,-)}(n) \geq \Delta_d^{(a)}(n).$$

In order to prove these theorems, we begin, in Section 2, by generalizing some theorems from [4] along with establishing other useful lemmas. Then, in Section 3 we will prove two different modified versions of Alder's conjecture that we will use to prove Theorem 1.3, Theorem 1.6, and Proposition 7.3. In Section 4 we prove Theorem 1.3 by considering four cases based on the parity of n and d. Then, in Section 5, we modify the methods used to prove Theorem 1.3 to prove Theorem 1.6 as well as adapt the methods of [4] and [13] to prove Theorem 1.7 and Theorem 1.8, respectively. In Section 6, we prove Theorem 1.9 as well as analog the method of [3] to prove an explicit error term for the asymptotic of $\Delta_d^{(a)}(n)$ and remark on how asymptotics could be used to prove the remaining finite cases of Theorem 1.3. Finally, in Section 7, we make remarks on the removal of parts to make a suitable generalization of Alder's conjecture and present a partial proof of a Conjecture 1.4.

2. Preliminaries

In this section, we develop generating functions for our partition functions that can be used for computation and we also establish some important lemmas that we will employ in later sections.

As in Yee [13], we denote the coefficient of q^n in an infinite series $s(q)$ as $[q^n](s(q))$.

We now introduce q-Pochhammer notation which we will use for convenience.

$$(a;q)_0 := 1$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$$
$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$$

For example, the generating function for the unrestricted partition function, $p(n)$, is,

$$p(n) = \frac{1}{(q;q)_\infty}.$$

As another example, the Rogers-Ramanujan identities can also be stated in this notation. The first Rogers-Ramanujan identity is

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

and the second Rogers-Ramanujan identity is

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Now we construct generating functions for $q_d^{(a)}(n)$, $Q_d^{(b)}(n)$, $Q_d^{(b,-)}(n)$, and $Q_d^{(b,-,-)}(n)$.

Lemma 2.1. *For $a, d > 0$, the generating function for $q_d^{(a)}(n)$ is:*

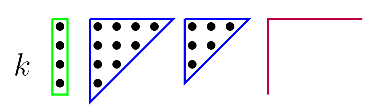
$$\sum_{n=0}^{\infty} q_d^{(a)}(n) q^n = \sum_{k=0}^{\infty} \frac{q^{d\binom{k}{2}+ka}}{(q;q)_k}.$$

This generating function appears in work of Alder [2], and the case where $a = 1$ is described graphically in [8]. We provide the full details of the combinatorial proof for arbitrary a here.

Proof. First, we consider triangles of dots having an equal height and width. We take a triangle of dots with height k and add $d-1$ triangles to the right with height $k-1$. Then we shift all of the dots to be left justified and we see this gives us a Ferrers' diagram of a partition with parts having a difference of at least d.

Next, we add $a-1$ columns of height k before the triangles and now we have a Ferrers' diagram of a partition with parts that have a difference of at least d and the smallest part is at least a.

From this base Ferrers' diagram, we can add any Ferrers' diagram of a partition with at most k parts and left justify to construct infinitely many partitions satisfying the partition condition counted by $q_d^{(a)}(n)$. To illustrate this, take the example where $k = 4$, $d = 2$, and $a = 2$. The d triangles are highlighted in blue, the addition of the $a-1$ columns of height k are highlighted in green, and the remaining possibilities of partitions with at most k parts in represented by the red.



When the dots in the diagram are shifted to the left to fill in the gaps we will be left with a Ferrer's diagram that represents a partition with a gap of at least d and parts at least a.

Now we construct our generating function using each of the three parts of our argument. The 'blue' part is the d triangles and to represent the choices of the number of dots we use $q^{d\binom{k}{2}+k}$. The 'green' part is the $a-1$ columns of height k and to represent these choices we use $q^{k(a-1)}$. The 'red' part is used for the remaining possibilities of partitions with at most k parts is represented by $\frac{1}{(q;q)_k}$.

Putting it all together we get

$$\sum_{n=0}^{\infty} q_d^{(a)}(n)q^n = \sum_{k=0}^{\infty} \frac{q^{d\binom{k}{2}+k}q^{k(a-1)}}{(q;q)_k}$$
$$= \sum_{k=0}^{\infty} \frac{q^{d\binom{k}{2}+ka}}{(q;q)_k}.$$

□

As stated in Alfes, Jameson, and Lemke Oliver [3], the generating function for $Q_d^{(b)}(n)$ is

$$\sum_{n=0}^{\infty} Q_d^{(b)}(n)q^n = \frac{1}{(q^{d+3-b};q^{d+3})_\infty (q^b;q^{d+3})_\infty}.$$

We can generalize this to obtain the generating function for $Q_d^{(b,-)}(n)$.

Lemma 2.2. *For $b, d > 0$, the generating function for $Q_d^{(b,-)}(n)$ is given by:*

$$\sum_{n=0}^{\infty} Q_d^{(b,-)}(n)q^n = \begin{cases} \frac{1}{(q^{2d+6-b};q^{d+3})_\infty} & \text{for } b = \frac{d+3}{2}, \\ \frac{1}{(q^{2d+6-b};q^{d+3})_\infty (q^b;q^{d+3})_\infty} & \text{otherwise.} \end{cases}$$

Proof. Begin by considering the generating function for $Q_d^{(b)}(n)$ which counts the number of partitions of n into parts congruent to $\pm b \pmod{d+3}$. To get the generating function for $Q_d^{(b,-)}(n)$, we need to remove the $(1 - q^{d+3-b})$ term from the expansion of $(q^{d+3-b};q^{d+3})_\infty$ in the denominator to get:

$$\sum_{n=0}^{\infty} Q_d^{(b,-)}(n)q^n = \frac{1}{(q^{2d+6-b};q^{d+3})_\infty (q^b;q^{d+3})_\infty}.$$

However, in the case that $2b = d + 3$, the above generating function double counts the use of each part, thus we eliminate the duplicate Pochhammer symbol to obtain

$$\sum_{n=0}^{\infty} Q_d^{(b,-)}(n)q^n = \frac{1}{(q^{2d+6-b};q^{d+3})_\infty}.$$

□

We employ a similar method to create the generating function for $Q_d^{(b,-,-)}(n)$.

Lemma 2.3. *For $b, d > 0$, the generating function for $Q_d^{(b,-,-)}(n)$ is given by:*

$$\sum_{n=0}^{\infty} Q_d^{(b,-,-)}(n)q^n = \begin{cases} \frac{1}{(q^{2d+6-b};q^{d+3})_\infty} & \text{for } b = \frac{d+3}{2}, \\ \frac{1}{(q^{2d+6-b};q^{d+3})_\infty (q^{d+3+b};q^{d+3})_\infty} & \text{otherwise.} \end{cases}$$

Proof. Modifying the generating function for $Q_d^{(b,-)}(n)$, we need to remove the $(1 - q^b)$ term from the denominator. In the case where $2b = d + 3$, this is already trivially taken care of. For all other cases we get

$$\sum_{n=0}^{\infty} Q_d^{(b,-,-)}(n)q^n = \frac{1}{(q^{2d+6-b};q^{d+3})_\infty (q^{d+3+b};q^{d+3})_\infty}.$$

□

In the proof of Theorem 1.8 we will need to use a theorem of Andrews [4]. Using notation of Andrews [4], let $\beta_d(x)$ be the least positive residue of x modulo d, let $b(x)$ be the number of terms appearing in the binary representation of x, and let $\nu(x)$ be the least 2^i in this representation.

Let $A = \{a(1), \dots, a(s)\}$ be a set of s distinct integers which satisfy $\sum_{i=1}^{k-1} a(i) < a(k)$ for $1 \le k \le s$ and denote the set of sums of elements from A by A' and the elements of A' by $\alpha(i)$. Let N be a positive integer such that $N \ge a(1) + a(2) + \cdots + a(s)$. Finally, let A_N be the set of all integers congruent to some $a(i) \pmod N$ and let A'_N be the set of all integers congruent to some $\alpha(i) \pmod N$. The following is a theorem from Andrews [5]:

Theorem 2.4 (Andrews [5], 1969)**.** *Let A_N, A'_N be defined as above, and let $D(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N and let $E(A'_N; n)$ be the number of partitions of n into parts taken from A'_N of the form $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ such that*

$$\lambda_{i+1} - \lambda_i \ge d \cdot b(\beta_d(\lambda_i)) + \nu(\beta_d(\lambda_i)) - \beta_d(\lambda_i). \tag{3}$$

Then $D(A_N; n) = E(A'_N; n)$.

Andrews [4] also uses the following theorem in his proof of certain cases of Alder's conjecture. First, define

$$\rho(R; n) := p(n \mid \text{parts from the set } R).$$

Theorem 2.5 (Andrews [4], 1971)**.** *Let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $y_1 = 1$ and $x_i \ge y_i$ for all i. Then for all $n > 0$,*

$$\rho(T; n) \ge \rho(S; n).$$

Yee [13] employs this theorem, for which she gives a combinatorial proof, to obtain her results on Alder's conjecture. Kang and Park [7] present a generalization of Theorem 2.5 which we generalize further as follows.

Lemma 2.6. *Let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be increasing sequences of positive integers with each y_i divisible by $m > 0$, such that $y_1 = m$ and $x_i \ge y_i$ for all i. Then for all $n > 0$,*

$$\rho(T; mn) \ge \rho(S; mn).$$

Proof. Let $\widetilde{S}_{mn}$ be the set of partitions of mn with parts coming from S, and $\widetilde{T}_{mn}$ be the set of partitions of mn with parts coming from T. We construct an injection, similar to that in Yee [13], where

$$\varphi : \widetilde{S}_{mn} \to \widetilde{T}_{mn}.$$

Let λ be a partition in $\widetilde{S}_{mn}$. Define e_i as the number of times x_i occurs in the partition λ. We note that $\sum_{i=1}^{\infty} x_i \cdot e_i = mn$ is clearly divisible by m. Also note that by assumption, all y_i are divisible by m. Thus,

$$\sum_{i=1}^{\infty} (x_i - y_i) e_i$$

is divisible by m.

We will define the sequence $\{f_i\}_{i=1}^{\infty}$ as follows

$$f_i = \begin{cases} e_1 + \frac{1}{m}\sum_{i=1}^{\infty}(x_i - y_i)e_i, & i = 1 \\ e_i, & i > 1, \end{cases}$$

We then define $\varphi(\lambda)$ to be the partition consisting of each y_i occurring f_i times. We will now verify that the desired properties are held by $\varphi(\lambda)$.

First, we see $\sum_{i=1}^{\infty} f_i y_i = e_1 y_1 + \sum_{i=1}^{\infty}(x_i - y_i)e_i + \sum_{i=2}^{\infty} y_i e_i = \sum_{i=1}^{\infty} x_i = mn$, which verifies that $\varphi(\gamma)$ is a partition of mn.

We now show φ is injective. Suppose λ and λ' are partitions of mn such that x_i appears e_i times in λ and x_i appears g_i times in λ'.

Suppose $\varphi(\lambda) = \varphi(\lambda')$. Then $g_i = e_i$ $\forall i \geq 2$ and

$$e_1 + \frac{1}{m}\sum_{i=1}^{\infty}(x_i - y_i)e_i = g_1 + \frac{1}{m}\sum_{i=1}^{\infty}(x_i - y_i)g_i.$$

From this, clearly $e_1 = g_1$. Thus $\lambda = \lambda'$.

Since φ is an injection, the result follows.

□

Lemma 2.6 will be used to prove Theorems 1.7 and 1.8.

We now present two lemmas that are useful in proving Theorem 1.3, Theorem 1.6, and later Proposition 7.3.

Lemma 2.7. *For all $a, d, n > 0$,*

$$q_d^{(a)}(n) \geq q_{\lceil \frac{d}{a} \rceil}^{(1)}\left(\left\lceil \frac{n}{a} \right\rceil\right).$$

Proof. Let $\overline{d}$ be the residue of $d \pmod a$, and let $\overline{n}$ be the residue of $n \pmod a$. Let $q_d^{(a)}(n)^*$ count the number of partitions with parts $\geq a$ and parts differing by at least d, with the added restriction that one part, the largest part, is $\equiv \overline{n} \pmod a$ and every other part is divisible by a. Also define

$$\hat{n} := \begin{cases} a - \overline{n} & \text{if } a \nmid n \\ 0 & \text{if } a \mid n \end{cases}$$

and

$$\hat{d} := \begin{cases} a - \overline{d} & \text{if } a \nmid d \\ 0 & \text{if } a \mid d \end{cases}.$$

We define a map between a partition, $\lambda = (\lambda_1, \ldots, \lambda_l)$, counted by $q_d^{(a)}(n)^*$ and a partition counted by $q_{\lceil \frac{d}{a} \rceil}^{(1)}\left(\left\lceil \frac{n}{a} \right\rceil\right)$. Let λ_1 be the largest part in the partition λ and define

$$f(\lambda) = \begin{cases} \frac{\lambda_i + \hat{n}}{a} & \text{if } i = 1 \\ \frac{\lambda_i}{a} & \text{if } i \geq 2 \end{cases}$$

We see that $f(\lambda)$ is a partition of $\frac{n+\hat{n}}{a} = \left\lceil \frac{n}{a} \right\rceil$. We also have that the difference between parts in the partition $f(\lambda)$ is at least $\frac{d+\hat{d}}{a} = \left\lceil \frac{d}{a} \right\rceil$, thus $f(\lambda)$ is indeed a partition counted by $q_{\lceil \frac{d}{a} \rceil}^{(1)}\left(\left\lceil \frac{n}{a} \right\rceil\right)$. Furthermore we see that this map is bijective since we can recover the preimage of $f(\lambda)$ by subtracting $\frac{\hat{n}}{a}$ from the largest part and multiplying every part by a. Since, clearly, $q_d^{(a)}(n) \geq q_d^{(a)}(n)^*$, we have our desired result. □

Lemma 2.8. *For $a, d, n > 0$ and $d+3$ and n both divisible by a,*

$$Q_d^{(a,-)}(n) = Q_{\frac{d+3}{a}-3}^{(1,-)}\left(\frac{n}{a}\right),$$
$$Q_d^{(a,-,-)}(n) = Q_{\frac{d+3}{a}-3}^{(1,-,-)}\left(\frac{n}{a}\right).$$

Proof. First, we have

$$\begin{aligned} Q_d^{(a,-)}(n) &= p(n|\text{ parts } \equiv \pm a \pmod{d+3},\ \text{parts } \neq d+3-a) \\ &= p\left(\frac{n}{a}\middle| \text{ parts } \equiv \pm 1 \left(\text{mod } \frac{d+3}{a}\right),\ \text{parts } \neq \frac{d+3}{a}-1\right) \\ &= Q_{\frac{d+3}{a}-3}^{(1,-)}\left(\frac{n}{a}\right). \end{aligned}$$

Similarly we have

$$\begin{aligned} Q_d^{(a,-,-)}(n) &= p(n|\text{ parts } \equiv \pm a \pmod{d+3},\ \text{parts } \neq a, d+3-a) \\ &= p\left(\frac{n}{a}\middle| \text{ parts } \equiv \pm 1 \left(\text{mod } \frac{d+3}{a}\right),\ \text{parts } \neq 1, \frac{d+3}{a}-1\right) \\ &= Q_{\frac{d+3}{a}-3}^{(1,-,-)}\left(\frac{n}{a}\right). \end{aligned}$$

□

3. A modification of Alder's conjecture

To prove Theorem 1.3, and later Proposition 7.3, we will need a slightly modified version of Alder's conjecture.

Proposition 3.1. *For any $k \geq 31$, $m > 0$,*

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m).$$

This proposition will be proved in the following three lemmas[3], which we will prove by adapting the methods of Andrews [4] and Yee [13]. We will write λ^x if the part λ appears x times as a part in a partition. Now we begin by showing Proposition 3.1 for small m.

Lemma 3.2. *For any $k \geq 20$ and $0 < m \leq 4k + 2^r$, where r is defined in (2) by letting $a = 1$ and $d = k$,*

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m).$$

Proof. Since r is the largest positive integer, such that $2^r \leq k+1$, we have that n is no greater than $5k+1$. We will assume that $q_k^{(1)}(m)$ is an increasing function. Following Yee [13], we let

$$U = \{1, k+2, 2k+1, 2k+3, 3k+2, 3k+4, 4k+3, 4k+5, 5k+4, \cdots\}$$

and note that $Q_{k-2}^{(1,-)}(m)$ counts the number of partitions of m into parts from U.

[3]See Section 6 for a comment on the cases when $1 \leq k \leq 31$ for Proposition 3.1

The lemma is trivially true for $n < k+2$, for $Q^{(1,-)}_{k-2}(m)$ is at most 1 and $q^{(1)}_k(m) \geq 1$.

For $n = k+2$, note that $Q^{(1,-)}_{k-2}(k+2) = 2$, with the partitions of $k+2$ being (1^{k+2}) and $(k+2)$. Also $q^{(1)}_k(k+2) \geq 2$, as it will have the partitions $(k+2)$ and $(1,(k+1))$.

We now move to the case $k+3 \leq m \leq 2k+2$. We have that $q^{(1)}_k(k+3) \geq 3$ with the partitions $(k+3)$, $(2,(k+1))$, and $(1,(k+2))$. Note that $Q^{(1,-)}_{k-2}(m)$ is at most 3 with partitions (1^m), $(1^{m-k-2},(k+2))$, and $(1^{m-2k-1},(2k-1))$. Since $q^{(1)}_k(m)$ is increasing, we have this case.

For $Q^{(1,-)}_{k-2}(2k+3)$ we can have the partitions $(2k+3)$, $(1^2,(2k+1))$, $(1^{k+1},(k+2))$ and (1^{2k+3}). We also have that $q^{(1)}_k(2k+3)$ counts at least the partitions $(k,(k+3))$, $(1,(2k+2))$, $(2,(2k+1))$, $(3,2k)$, and $(4,2k-1)$, so the lemma holds for $(2k+3)$.

We now consider $2k+4 \leq m \leq 4k+3$. Using the floor function, we analyze the number of partitions of $2k+4$ with only two parts, that is analyzing the partition

$$(y,(2k+4-y))$$

with $0 \leq y \leq \left\lfloor \frac{k}{2} \right\rfloor + 2$. Since $k \geq 20$, $q^{(1)}_k(m) \geq 12$. For $Q^{(1,-)}_{k-2}(m)$ we directly calculate its maximum possible value by considering the contribution of the different parts to possible partitions of m. For $2k+4 \leq m \leq 4k+3$ we have that the part $4k+3$ can contribute to at most one partition of m, which is of the form

$$(4k+3).$$

The part $3k+4$ can contribute to at most one partition of m, which is of the form

$$((3k+4),1^{m-3k-4}).$$

The part $3k+2$ can contribute to at most one partition of m, which is of the form

$$((3k+2),1^{m-3k-2}).$$

The part $2k+3$ can contribute to at most two partitions of m, which are of the form

$$((2k+3),(k+2),1^{m-3k-5}).$$
$$((2k+3),1^{m-2k-3}),$$

The part $2k+1$ can contribute to at most two partitions of m, which are of the form

$$((2k+1),(k+2),1^{m-3k-3}),$$
$$((2k+1),1^{m-2k-1}).$$

The part $k+2$ can contribute to at most three additional partitions of m, which are of the form

$$((k+2)^3,1^{m-3k-6}),$$
$$((k+2)^2,1^{m-2k-4}),$$
$$((k+2),1^{m-k-2}).$$

The part 1 can contribute to at most one additional partition of m, which is of the form

$$(1^m).$$

Noting that in some cases a part will not contribute to a partition of m at all, we have at most that $Q_{k-2}^{(1,-)}(m) \leq 11 \leq 12 \leq q_k^{(1)}(m)$.

We now proceed for $4k+3 < n < 4k+2^r$. We have from above that at $n \leq 5k+1$, thus it's sufficient to verify only in the range $4k+3 < n < 5k+1$. We have that $k + \lfloor \frac{k}{2} \rfloor + 2 \geq 32$. This implies that we have 32 values to choose for y to form the two part partitions of $4k+4$ of the form $(y, (4k+4-y))$, implying $q_k^{(1)}(m) \geq 32$. We now calculate the maximum of $Q_{k-2}^{(1,-)}(m)$. We have that the part $4k+5$ can contribute to at most one partition of m, which is of the form

$$((4k+5), 1^{m-4k-5}).$$

The part $4k+3$ can contribute to at most one partition of m, which is of the form

$$((4k+3), 1^{m-4k-3}).$$

The part $3k+4$ can contribute to at most two partitions of m, which are of the form

$$((3k+4), 1^{m-3k-4}),$$
$$((3k+4), (k+2), 1^{m-4k-6}).$$

The part $3k+2$ can contribute to at most two partitions of m, which are of the form

$$((3k+2), (k+2), 1^{m-4k-4}),$$
$$((3k+2), 1^{m-3k-2}).$$

The part $2k+3$ can contribute to at most five partitions of m, which are of the form

$$((2k+3)^2, 1^{m-2k-6}),$$
$$((2k+3), (k+2)^2, 1^{m-4k-7}),$$
$$((2k+3), (k+2), 1^{m-3k-5}),$$
$$((2k+3), (2k+1), 1^{m-4k-4}),$$
$$((2k+3), 1^{m-2k-3}).$$

The part $2k+1$ can contribute to at most four additional partitions of m, which are of the form

$$((2k+1)^2, 1^{m-4k-4}),$$
$$((2k+1), 1^{m-2k-1}),$$
$$((2k+1), (k+2)^2, 1^{m-4k-5}),$$
$$((2k+1), (k+2), 1^{m-3k-3}).$$

The part $k+2$ can contribute to at most four additional partitions of m, which are of the form

$$((k+2)^4, 1^{m-4k-8}),$$
$$((k+2)^3, 1^{m-3k-6})),$$
$$((k+2)^2, 1^{m-2k-4}),$$
$$((k+2), 1^{m-k-2})).$$

The part 1 can contribute to only one additional partition of m, which is of the form

$$(1^m).$$

Thus, we have that $Q_{k-2}^{(1,-)}(m) \leq 20 \leq q_k^{(1)}(m)$ which finishes our proof. □

To prove the remaining case of Proposition 3.1 we see from the work of [13] that we have

$$q_k^{(1)}(m) \geq [q^m](g_k^{(1)}(q)) \tag{4}$$

for $m \geq 4d + 2^r$, where r is defined by (2) and $k \neq 2^r - 1$, where

$$g_k^{(1)}(q) := \frac{(-q^{k+2^{r-1}}; q^{2k})_\infty}{(q^1; q^{2k})_\infty (q^{k+2}; q^{2k})_\infty \cdots (q^{k+2^{r-2}}; q^{2k})_\infty}.$$

Similarly, from the work of [4], we have that if $k = 2^s - 1$ and $s \geq 5$,

$$q_k^{(1)}(m) \geq \mathcal{L}_k^{(1)}(m), \tag{5}$$

where $\mathcal{L}_k^{(1)}(m) = \rho(T_s; m)$ and $\rho(T_s; m)$ counts the number of partitions with parts from the set

$$T_s = \{y | y \equiv 1, k + 2^1, ..., k + 2^{s-1} \pmod{2k}\}.$$

Now we can prove the following two lemmas.

Lemma 3.3. *If $k \geq 31$, $k \neq 2^r - 1$, where r is defined in (2) and $m \geq 4k + 2^r$, then we have*

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1)}(m).$$

Proof. Let $r \geq 5$, since $k \geq 31$. We define the sets S and T_r as follows:

$$S = \{x | x \equiv 1, -1 \pmod{k+1}\} \setminus \{k\}$$
$$T_r = \{y | y \equiv 1, k + 2^1, ..., k + 2^{r-2} \pmod{2k}\}.$$

We want to use Theorem 2.5 so we aim to show that $x_i \geq y_i$ for all i, where x_i and y_i are the i^{th} elements of S and T_r, respectively, when the elements of each set are arranged in increasing order.

Remark 3.4. *Notice that we only have to consider the set T_5, where $r = 5$, since, if we increase r, the S set remains unchanged but the T_r set will be given more residue classes* $\pmod{2k}$. *Adding another residue class to the elements of T_r will imply that the i^{th} element of T_r, will be less than the i^{th} element of T_5.*

Let

$$T_5 = \{y | y \equiv 1, k + 2, k + 4, k + 8 \pmod{2k}\}.$$

We derive the forms of each component, x_i and y_i, of S and T_5.

i	x_i	y_i
1	1	1
2	$(k+1)+1$	$k+2$
3	$2(k+1)-1$	$k+4$
4	$2(k+1)+1$	$k+8$
5	$3(k+1)-1$	$2k+1$
6	$3(k+1)+1$	$3k+2$
7	$4(k+1)-1$	$3k+4$
8	$4(k+1)+1$	$3k+8$
$\vdots$	$\vdots$	$\vdots$

Noting that $x_i = 1 = y_i$, we can consider the general forms of each x_i and y_i by considering the congruence classes of $i \pmod 4$:

i	x_i	y_i
$i = 4\alpha$ and $\alpha \geq 1$	$(2\alpha)(k+1)+1$	$k(2\alpha)+8-k$
$i = 4\alpha+1$ and $\alpha \geq 1$	$(2\alpha+1)(k+1)-1$	$k(2\alpha)+1$
$i = 4\alpha+2$	$(2\alpha+1)(k+1)+1$	$k(2\alpha)+2+k$
$i = 4\alpha+3$	$(2\alpha+2)(k+1)-1$	$k(2\alpha)+4+k$

We now show that we have our desired inequalities for each i given that $k \geq 31$.

(i) If $i = 4\alpha$, $x_i - y_i = (2\alpha)(k+1)+1-(k(2\alpha)+8-k) = 2\alpha+1+k-8 \geq 0$.
(ii) If $i = 4\alpha+1$, $x_i - y_i = (2\alpha+1)(k+1)-1-(k(2\alpha)+1) = 2\alpha+k-1 \geq 0$.
(iii) If $i = 4\alpha+2$, $x_i - y_i = (2\alpha+1)(k+1)+1-(k(2\alpha)+2+k) = 2\alpha \geq 0$.
(iv) If $i = 4\alpha+3$, $x_i - y_i = (2\alpha+2)(k+1)-1-(k(2\alpha)+2^2+k) = 2\alpha+k-3 \geq 0$.

Thus, by Theorem 2.5 and (4), we have

$$q_k^{(1)}(m) \geq [q^m](g_k^{(1)}(q)) \geq \rho(T_r; m) \geq \rho(S; m) = Q_{k-2}^{(1,-)}(m)$$

for $k \geq 31$, $k \neq 2^r - 1$ and $m \geq 4k + 2^r$. □

To complete the proof of Proposition 3.1 we show the following.

Lemma 3.5. *If $k = 2^s - 1$ for $s \geq 5$, and $m > 0$, we have*

$$q_k^{(1)}(m) \geq Q_{k-2}^{(1,-)}(m).$$

Proof. We aim to prove that $\rho(T_s; m) \geq Q_{k-2}^{(1,-)}(m)$ where $Q_{k-2}^{(1,-)}(m) = \rho(S; m)$ with T_s and S is defined as

$$S = \{x | x \equiv 1, -1 \pmod{k+1}\} \setminus \{k\}$$
$$T_s = \{y | y \equiv 1, k+2^1, ..., k+2^{s-1} \pmod{2k}\}.$$

Using the argument from Remark 3.4, we only need to look at the case where $s = 5$.

$$T_5 = \{y | y \equiv 1, k+2, k+4, k+8 \pmod{2k}\}$$

We see that these sets S and T_5 match those from the proof of Lemma 3.3, thus we have already shown that Theorem 2.5 and (4) implies our desired result. □

Lemmas 3.2, 3.3, and 3.5 prove Proposition 3.1.

We will also need another modification of Alder's conjecture to prove Theorem 1.6.

Proposition 3.6. *For any $k \geq 31$, $m > 0$,*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m).$$

Again we will adapt [13] and [4] to prove the following three lemmas which will prove Proposition 3.6. The proofs of these lemmas will follow similarly as above.

Lemma 3.7. *For any $k \geq 26$ and $0 < m \leq 4k + 2^r$ with r defined by (2),*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m).$$

Proof. Again following Yee [13], we have the set

$$U = \{k+1, 2k-1, 2k+1, 3k-1, 3k+1, 4k-1, 4k+1, 5k-1, \cdots\}$$

where $Q^{(1,-,-)}_{k-3}(m)$ counts the number of partitions of m into parts from U.

We note that $m < k+1$ that the result is trivial, since $Q^{(1,-,-)}_{k-3}(m) = 0$.

For $k+1 \leq m < 2k-1$, note that the maximum size of $Q^{(1,-,-)}_{k-3}(m)$ is 1, with the only partition being $(k+1)$. We have that $q^{(1)}_{k}(k+1) \geq 2$ with the partitions (1^{k+1}) and $(k+1)$. Th increasing property of $q^{(1)}_{k}(k+1)$ finishes this case.

For $2k-1 \leq m < 2k+4$ we note that for $2k-1$, that the following partitions $(2k-1), (1^{2k-1})$, and $(2k-2, 1)$ are always counted, since $k \geq 26$. Now we examine the maximum possible value of $Q^{(1,-,-)}_{k-3}(m)$. The part $2k+1$ can contribute to at most one partition of m, which is of the form

$$(2k+1).$$

The part $2k-1$ can contribute to at most one partition of m, which is of the form

$$(2k-1).$$

The part $k+1$ can contribute to at most one partition of m, which is of the form

$$(k+1)^2.$$

Thus the maximum that $Q^{(1,-,-)}_{k-3}(m)$ can be is 3, so this case is settled.

For $2k+4 \leq m \leq 4k+3$, we have that $q^{(1)}_{k}(2k+4) \geq 15$, by examining two part partitions of the form $(y, (2k+4-y))$, where $0 \leq y \leq \left\lfloor \frac{k}{2} \right\rfloor + 2$ and $k \geq 26$. We now calculate the maximum size of $Q^{(1,-,-)}_{k-3}(m)$. We have that The part $4k+1$ can contribute to at most one partition of m, which is of the form

$$(4k+1).$$

The part $4k-1$ can contribute to at most one partition of m, which is of the form

$$(4k-1).$$

The part $3k+1$ can contribute to at most two partitions of m to which are of the form

$$(3k+1),$$
$$((3k+1), (k+1)).$$

The part $3k-1$ can contribute to at most two partitions of m, which are of the form

$$((3k-1), (k+1))$$
$$(3k-1).$$

The part $2k+1$ can contribute to at most four partitions of m, which are of the form

$$((2k+1), (k+1)^2),$$
$$((2k+1)^2,$$
$$(2k-1), (2k+1)),$$
$$((2k+1), (k+1)).$$

The part $2k-1$ can contribute to at most four additional partitions of m, which are of the form

$$((2k-1)^2),$$
$$((2k-1),(k+1)^2),$$
$$((2k-1),(k+1)),$$
$$((2k-1)^2).$$

The part $k+1$ can contribute to only one additional partition of m, which is of the form

$$((k+1)^3).$$

So, we see that $Q^{(1,-,-)}_{k-3}(m)$ can be at most 15 and thus this case is also settled.

We now verify that the inequality holds for $4k+4 \le m \le 4k+2^r$. Since $2^r \le k+1$, we have that we simply need to verify $4k+4 \le m \le 5k+1$. We have that $q^{(1)}_k(4k+4) \ge 12$ by again considering only the two part partitions of the form $(y,(4k+k-y))$ where $0 \ge y \ge \lfloor \frac{k}{2} \rfloor + 2$ and $k \ge 26$. Now we calculate the maximum possible value of $Q^{(1,-,-)}_{k-3}(m)$. The part $5k+1$ can contribute to at most one partition of m, which is of the form

$$(5k+1).$$

The part $5k-1$ can contribute to at most one partition of m, which is of the form

$$(5k-1).$$

The part $4k+1$ can contribute to at most one partition of m which is of the form

$$(4k+1).$$

The part $4k-1$ can contribute to at most one partition of m, which is of the form

$$((4k-1),(k+1)).$$

The part $3k+1$ contributes to at most one partition of m, which is of the form

$$((3k+1),(2k-1)).$$

The part $3k-1$ contributes to at most three partitions of m, which are of the form

$$((3k-1),(k+1)^2),$$
$$((3k-1),(2k+1)),$$
$$((3k-1),(2k-1)).$$

The part $2k+1$ contributes to at most two additional partitions of m, which are of the form

$$((2k+1),(k+1)^2),$$
$$((2k+1),(2k-1),(k+1)).$$

The part $2k-1$ contributes to at most one additional partition of m, which is of the form

$$((2k-1)^2,(k+1)).$$

The part $k+1$ contributes to at most one additional partition of m, which is of the form

$$((k+1)^4).$$

This implies that $Q^{(1,-,-)}_k(m) \le 12 \le q^{(1)}_k(m)$. Thus, we have completed our proof.

$\square$

We can use (4) to also prove the following lemma.

Lemma 3.8. *If $k \geq 31$, $k \neq 2^r - 1$ and $m \geq 4k + 2^r$ with r defined in (2), then we have*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m).$$

Proof. The proof here follows similarly as above. Let $r \geq 5$, since $k \geq 31$. We define the sets S and T_r as follows:

$$S = \{x | x \equiv 1, -1 \pmod{k}\} \setminus \{k-1, 1\}$$
$$T_r = \{y | y \equiv 1, k+2^1, ..., k+2^{r-2} \pmod{2k}\}.$$

By Remark 3.4, we need only consider the case when $r = 5$ so we define

$$T_5 = \{y | y \equiv 1, k+2, k+4, k+8 \pmod{2k}\}.$$

We again derive the forms of each component, x_i and y_i, of S and T_5.

i	x_i	y_i
1	$k+1$	1
2	$2k-1$	$k+2$
3	$2k+1$	$k+4$
4	$3k-1$	$k+8$
5	$3k+1$	$2k+1$
6	$4k-1$	$3k+2$
7	$4k+1$	$3k+4$
8	$5k-1$	$3k+8$
$\vdots$	$\vdots$	$\vdots$

Noting that $y_i = 1$, we can consider the general forms of each x_i and y_i by considering the congruence classes of $i \pmod 4$:

i	x_i	y_i
$i = 4\alpha$ and $\alpha \geq 1$	$(2\alpha+1)(k) - 1$	$k(2\alpha) + 8 - k$
$i = 4\alpha + 1$ and $\alpha \geq 1$	$(2\alpha+1)(k) + 1$	$k(2\alpha) + 1$
$i = 4\alpha + 2$	$(2\alpha+2)(k) - 1$	$k(2\alpha) + 2 + k$
$i = 4\alpha + 3$	$(2\alpha+2)(k) + 1$	$k(2\alpha) + 4 + k$

We now show that we have our desired inequalities for each i given that $k \geq 31$.

(i) If $i = 4\alpha$, $x_i - y_i = (2\alpha+1)(k) - 1 - (k(2\alpha) + 8 - k) = 2k - 9 \geq 0$.
(ii) If $i = 4\alpha + 1$, $x_i - y_i = (2\alpha+1)(k) + 1 - (k(2\alpha) + 1) = k \geq 0$.
(iii) If $i = 4\alpha + 2$, $x_i - y_i = (2\alpha+2)(k) - 1 - (k(2\alpha) + 2 + k) = k - 3 \geq 0$.
(iv) If $i = 4\alpha + 3$, $x_i - y_i = (2\alpha+2)(k) + 1 - (k(2\alpha) + 4 + k) = k - 3 \geq 0$.

By Theorem 2.5 and (4), we have

$$q_k^{(1)}(m) \geq [q^m](g_k^{(1)}(q)) \geq \rho(T_r; m) \geq \rho(S; m) = Q_{k-3}^{(1,-,-)}(m)$$

for $k \geq 31$, where $k \neq 2^r - 1$ and $m \geq 4k + 2^r$. □

Again, (5) can be used in the proof of the following.

Lemma 3.9. *If $k = 2^s - 1$ for $s \geq 5$, and $m > 0$, we have*

$$q_k^{(1)}(m) \geq Q_{k-3}^{(1,-,-)}(m).$$

Proof. Similarly as above, we aim to prove that $\rho(T_s; m) \geq Q_{k-3}^{(1,-,-)}(m)$ where $Q_{k-3}^{(1,-,-)}(m) = \rho(S; m)$ with T_s and S is defined as

$$S = \{x | x \equiv 1, -1 \pmod{k}\} \setminus \{k-1, 1\}$$
$$T_s = \{y | y \equiv 1, k+2^1, ..., k+2^{s-1} \pmod{2k}\}.$$

From Remark 3.4, we only need to look at the case where $s = 5$.

$$T_5 = \{y | y \equiv 1, k+2, k+4, k+8 \pmod{2k}\}$$

We see that these sets S and T_5 also match those from the proof of Lemma 3.8, thus Theorem 2.5 and (5) implies our result. □

We see that Lemmas 3.7, 3.8, and 3.9 prove Proposition 3.6.

4. The proof of Kang and Park's conjecture

We prove Theorem 1.3 in cases, by considering the parities of d and n.

Remark 4.1. *The first case of Theorem 1.3, when both $n > 0$ and $d > 0$ are odd, is trivial. This is because in these cases $Q_d^{(2,-)}(n) = 0$, as explained by Kang and Park* [7].

Now we consider the cases not covered by Remark 4.1.

Lemma 4.2. *For $d \geq 61$, where $d > 0$ is odd and $n > 0$ is even,*

$$q_d^{(2)}(n) \geq Q_d^{(2,-)}(n).$$

Proof. We prove this by establishing the following string of inequalities,

$$q_d^{(2)}(n) \geq q_{\frac{d+1}{2}}^{(1)}\left(\frac{n}{2}\right) \geq Q_{\frac{d-3}{2}}^{(1,-)}\left(\frac{n}{2}\right) = Q_d^{(2,-)}(n).$$

The first inequality is a simple consequence of Lemma 2.7. The second inequality is a consequence of Proposition 3.1, as $\frac{d+1}{2} - \frac{d-3}{2} = 2$. Finally, the third inequality is a consequence of Lemma 2.8. This completes the proof. □

Lemma 4.3. *For $d \geq 62$, where $d > 0$ is even and $n > 0$ is even,*

$$q_d^{(2)}(n) \geq Q_d^{(2,-)}(n).$$

Proof. We prove this by establishing the following string of inequalities,

$$q_d^{(2)}(n) \geq q_{\frac{d}{2}}^{(1)}\left(\frac{n}{2}\right) \geq Q_{\frac{d}{2}-2}^{(1,-)}\left(\frac{n}{2}\right) = Q_{d-1}^{(2,-)}(n) \geq Q_d^{(2,-)}(n).$$

The first inequality is again a simple consequence of Lemma 2.7. The second inequality is a consequence of Proposition 3.1, as $\frac{d}{2} - \left(\frac{d}{2} - 2\right) = 2$. The equality follows from Lemma 2.8 if we replace d with $d-1$ in the lemma since $d - 1 + 3$ is even. This leaves the fourth inequality which we now show.

Let x_i and y_i be the i^{th} terms of the sets

$$S = \{x \mid x \equiv \pm 2 \pmod{d+3}, \text{ parts } \neq d+1\}$$

and

$$T = \{y \mid y \equiv \pm 2 \pmod{d+2},\ \text{parts } \neq d\}$$

where clearly $Q_d^{(2,-)}(n)$ counts the partitions of n with parts from S and $Q_{d-1}^{(2)}(n)$ counts the partitions of n with parts from T. The general forms of x_i and y_i are

i	x_i	y_i
$i = 4\alpha$ and $\alpha \geq 1$	$(2\alpha)(d+3)+2$	$(2\alpha)(d+2)+2$
$i = 4\alpha+1$ and $\alpha \geq 1$	$(2\alpha+1)(d+3)-2$	$(2\alpha+1)(d+2)-2$
$i = 4\alpha+2$	$(2\alpha+1)(d+3)+2$	$(2\alpha+1)(d+2)+2$
$i = 4\alpha+3$	$(2\alpha+2)(d+3)-2$	$(2\alpha+2)(d+2)-2.$

It is clear for all i that $x_i \geq y_i$. Since d is even, y_i is even for all i. Finally, given that $x_i = 2 = y_i$, applying Lemma 2.6 gives our desired conclusion. □

Lemma 4.4. *For $d \geq 62$, where $d > 0$ is even and $n > 0$ is odd,*

$$q_d^{(2)}(n) \geq Q_d^{(2,-)}(n).$$

Proof. Our ultimate goal is to show the following chain of inequalities

$$q_d^{(2)}(n) \geq q_{\frac{d}{2}}^{(1)}\left(\frac{n+1}{2}\right) \geq Q_{\frac{d}{2}-2}^{(1,-)}\left(\frac{n+1}{2}\right) = Q_{d-1}^{(2,-)}(n+1) \geq Q_d^{(2)}(n).$$

The first inequality is a simple consequence of Lemma 2.7. The second inequality follows from Proposition 3.1. Finally, the third equality can be shown using Lemma 2.8 if in the lemma we replace d with $d-1$, i.e. $d-1+3$ is even, and replace n with $n+1$ since $n+1$ is even. It remains to show that for odd n and even d, $Q_{d-1}^{(2,-)}(n+1) \geq Q_d^{(2,-)}(n)$. We prove this below.

Let $Q_d^{(2,-)}(n)$ count the number of partitions of n into parts from the set

$$V = \{2, 2d+4, 2d+8, 3d+7, 3d+11, ...\}$$

and let $Q_{d-1}^{(2,-)}(n+1)$ count the number of partitions of n into parts from the set

$$V' = \{2, 2d+2, 2d+6, 3d+4, 3d+8, ...\}.$$

Let a_i, b_i be the i^{th} terms of V and V' respectively where the elements are arranged in increasing order. Let $\gamma = (\gamma_1, ..., \gamma_r)$ be a partition counted by $Q_d^{(2,-)}(n)$ where γ_j is the j^{th} part in the partition when the parts are arranged in non-increasing order. We note that since n is odd, there must be an odd number of odd parts in γ.

We will construct a injection $\varphi : V \to V'$ to derive our inequality. We create a new partition λ that is constructed from γ as follows:

$$\begin{cases} \text{If } \gamma_j = 2, \text{ then replace it with one part of size 2 i.e. the part remains unchanged,} \\ \text{if } \gamma_j = a_i \text{ is even but } \neq 2, \text{ replace it with one part of size } b_i \text{ and } a_i - b_i \text{ parts of size 2,} \\ \text{if } \gamma_j = a_i \text{ is odd, replace it with one part of size } b_i, \\ \text{and add } \beta \text{ parts of size 2 where } 2\beta = \left(\sum(a_i - b_i)\right) + 1. \end{cases}$$

Then we rearrange the new parts of λ so that they are written in non-increasing order and note that the new partition, λ is a partition of $n+1$ and all parts in λ are congruent to elements of V'.

We prove that the mapping is injective. Let γ, γ' be two partitions counted by V. Assume $\gamma \neq \gamma'$. If γ and γ' have a different number of 2's, then they must have at least one $\gamma_j \neq \gamma'_j$ where γ_j and γ'_j are not 2. This means that at least one b_i in the image will be different so the images of γ and γ' are different.

If γ and γ' the same number of 2's then there must be at least one $\gamma_j \neq \gamma'_j$, since we assumed $\gamma \neq \gamma'$. Thus again there will be at least one b_i in the image that differs so their images are different and our map is injective. □

Now we see that Remark 4.1 and Lemmas 4.2, 4.3, and 4.4 prove Theorem 1.3.

5. The generalization of Kang and Park's conjecture

Recall Conjecture 1.5. We will use three different methods to prove partial results for this conjecture. We will begin by adapting some of the methods used to prove Theorem 1.3, then we will use the methods of Andrews [4] and Yee [13].

5.1. An adaptation of the methods in Section 4.

Remark 5.1. *We remark that in the case where $d+3$ is divisible by a and n is not divisible by a, Conjecture 1.5 is trivially true, since $Q_d^{(a,-,-)}(n) = 0$ for all a.*

Proof of Theorem 1.6. We aim to prove

$$q_d^{(a)}(n) \geq q^{(1)}_{\frac{d+3}{a}}\left(\frac{n}{a}\right) \geq Q^{(1,-,-)}_{\frac{d+3}{a}-3}\left(\frac{n}{a}\right) = Q_d^{(a,-,-)}(n).$$

The first inequality is a simple consequence of Lemma 2.7. The second inequality is a consequence of Proposition 3.6, as $\frac{d+3}{a} - \left(\frac{d+3}{a} - 3\right) = 3$. Finally, the third equality is a consequence of Lemma 2.8. This, along with the Remark 5.1, proves the theorem. □

5.2. A generalization of the method of Andrews.

We will utilize a method of Andrews [4] to prove Theorem 1.7.

Lemma 5.2. *Let s be the largest integer such that $2^s - 2^t \leq d$ for $s \geq t+4 \geq 4$ and let $\mathcal{L}_d^{(t)}(n)$ count the number of partitions with distinct parts that are congruent to $2^t, 2^{t+1}, \cdots, 2^{s-1} \pmod{d}$. Then*

$$q_d^{(a)}(n) \geq \mathcal{L}_d^{(t)}(n)$$

for $n > 0$.

Proof. In the notation of Theorem 2.4 set $N = d$ and $a(1) = 2^t$, $a(2) = 2^{t+1}$, …, $a(s-t) = 2^{s-1}$. We see that $\mathcal{L}_d^{(t)}(n) = D(A_N; n)$. By Theorem 2.4, $D(A_N; n) = E(A'_N; n)$ where $E(A'_N; n)$ has the following properties:

$$\text{(i) } n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$$
$$\text{(ii) } \lambda_{i+1} - \lambda_i \geq d \cdot b(\beta_d(\lambda_i)) + \nu(\beta_d(\lambda_i)) = \beta_d(\lambda_i)$$

We now consider the case where λ_i is congruent to $2^j \pmod{d}$ with $t \leq j \leq s-1$. If $\lambda_i \equiv 2^j \pmod{d}$, then we have

$$d \cdot b(\beta_d(\lambda_i)) + \nu(\beta_d(\lambda_i)) - \beta_d(\lambda_i) = d + 2^j - 2^j = d.$$

If $\lambda_i \not\equiv 2^j \pmod{d}$, then we have

$$d \cdot b(\beta_d(\lambda_i)) + \nu(\beta_d(\lambda_i)) - \beta_d(\lambda_i) \geq 2d + 2^t - (2^s - 2^t) \geq 2d + 2^t - d = d + 2^t.$$

This implies that the minimum differences of parts of partitions in $E(A'_N;n)$ is greater than d, thus $q_d^{(a)}(n) \geq \mathcal{L}_d^{(t)}(n)$. □

Lemma 5.3. *Let $d = 2^s - 2^t$ for some $s \geq t+4 \geq 4$. Let $\rho(T_s;n)$ be the number of partitions of $n > 0$ whose parts from the set T where*

$$T_s := \{2^t, d+2^{t+1}, d+2^{t+2}, ..., d+2^{s-1} \pmod{2d}\}.$$

Then $\rho(T_s;n) = \mathcal{L}_d^{(t)}(n)$ for all n divisible by 2^t.

Proof. We have that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{L}_d^{(t)}(n)q^n &= (-q^{2^t};q^d)_\infty(-q^{2^{t+1}};q^d)_\infty \cdots (-q^{2^{s-1}};q^d)_\infty \\
&= \frac{(-q^{2^t};q^d)_\infty(q^{2^t};q^d)_\infty}{(q^{2^t};q^d)_\infty} \cdots \frac{(-q^{2^{s-1}};q^d)_\infty(q^{2^{s-1}};q^d)_\infty}{(q^{2^{s-1}};q^d)_\infty} \\
&= \frac{(q^{2^{t+1}};q^d)_\infty}{(q^{2^t};q^{2d})_\infty(q^{d+2^t};q^{2d})_\infty} \cdots \frac{(q^{2^s};q^d)_\infty}{(q^{2^{s-1}};q^{2d})_\infty(q^{d+2^{s-1}};q^{2d})_\infty} \\
&= \frac{1}{(q^{2^t};q^{2d})_\infty(q^{d+2^{t+1}};q^{2d})_\infty \cdots (q^{d+2^{s-1}};q^{2d})_\infty} \\
&= \sum_{n=0}^{\infty} \rho(T_s;n)q^n.
\end{aligned}$$

□

Proof of Theorem 1.7. Let $d = 2^s - 2^t$ for some $s \geq t+4$ and let n be divisible by 2^t. From Lemma 5.2 and Lemma 5.3 we know that

$$q_d^{(a)}(n) \geq \mathcal{L}_d^{(t)}(n) = \rho(T_s;n).$$

Thus to prove our theorem we can use Theorem 2.6 to show that

$$\mathcal{L}_d^{(t)}(n) \geq Q_d^{(a,-,-)}(n).$$

We define

$$\begin{aligned}
S &= \{x | x \equiv a, d+3-a \pmod{d+3}\} \setminus \{a, d+3-a\} \\
T_s &= \{y | y \equiv 2^t, 2^{t+1}+d, 2^{t+2}+d, ..., 2^{s-1}+d \pmod{2d}\}
\end{aligned}$$

First note that $\rho(S;n)$, which counts the number of partitions of n with parts from S, is equivalent to $Q_d^{(a,-,-)}(n)$. Observe that all $y_i \in T_s$ are divisible by 2^t, since d is a multiple of 2^t, and that the first element in T_s is 2^t. To show $\rho(T_s;n) \geq \rho(S;n)$ it is thus sufficient to show that $x_i \geq y_i$ for all i.

Using the same argument from Remark 3.4 we need only examine the case when $s = t+4$ case, where

$$T_{t+4} = \{y | y \equiv 2^t, 2^{t+1}+d, 2^{t+2}+d, 2^{t+3}+d \pmod{2d}\}.$$

Below are the first few x_i, y_i terms

i	x_i	y_i
1	$(d+3)+a$	2^t
2	$2(d+3)-a$	$d+2^{t+1}$
3	$2(d+3)+a$	$d+2^{t+2}$
4	$3(d+3)-a$	$d+2^{t+3}$
5	$3(d+3)+a$	$2d+2^t$
6	$4(d+3)-a$	$3d+2^{t+1}$
7	$4(d+3)+a$	$3d+2^{t+2}$
8	$5(d+3)-a$	$3d+2^{t+3}$
$\vdots$	$\vdots$	$\vdots$

We now generalize the x_i and y_i terms by considering $i \pmod 4$.

i	x_i	y_i
$i=4\alpha$	$(2\alpha+1)(d+3)-a$	$\alpha(2d)+2^{t+3}-d$
$i=4\alpha+1$	$(2\alpha+1)(d+3)+a$	$\alpha(2d)+2^t$
$i=4\alpha+2$	$(2\alpha+2)(d+3)-a$	$\alpha(2d)+2^{t+1}+d$
$i=4\alpha+3$	$(2\alpha+2)(d+3)+a$	$\alpha(2d)+2^{t+2}+d$

We now show $x_i \geq y_i$, for all i given that $d \geq 2^{t+4}-2^t$.

(i) If $i=4\alpha$, $x_i-y_i=(2\alpha+1)(d+3)-a-(\alpha(2d)+2^{t+3}-d)=6\alpha+2d+3-2^{t+3}\geq 6\alpha+3\geq 0$.

(ii) If $i=4\alpha+1$, $x_i-y_i=(2\alpha+1)(d+3)+a-(\alpha(2d)+2^t)=6\alpha+d+3-2^t\geq 6\alpha+3\geq 0$.

(iii) If $i=4\alpha+2$, $x_i-y_i=((2\alpha+2)(d+3)-a-(\alpha(2d)+2^{t+1}+d)=d+6\alpha+6-2^{t+1}\geq 6\alpha+6\geq 0$.

(iv) If $i=4\alpha+3$, $x_i-y_i=(2\alpha+2)(d+3)+a-(\alpha(2d)+2^{t+2}+d)=d+6\alpha+6-2^{t+1}\geq 6\alpha+6\geq 0$.

Hence, we have

$$q_d^{(a)}(n)\geq \mathcal{L}_d^{(t)}(n)=\rho(T_s;n)\geq\rho(S;n)=Q_d^{(a,-,-)}(n)$$

which proves our theorem. □

5.3. **A generalization of the method of Yee.** Now we use a method of Yee [13] to prove Theorem 1.8. Using (2) we will generalize the following generating functions of Yee [13] to incorporate dependence on a, where $a,d,n>0$.

$$f_d^{(a)}(q):=\sum_{n=0}^{\infty}\mathcal{L}_d^{(a)}(n)q^n:=(-q^{2^{a-1}};q^d)_\infty(-q^{2^a};q^d)_\infty\cdots(-q^{2^{r-1}};q^d)_\infty.$$

$$k_d^{(a)}(q):=\sum_{n=0}^{\infty}\mathcal{K}_d^{(a)}(n)q^n:=\frac{1-q^{d+2^{a-1}}}{1-q^{2^{a-1}}}(-q^{d+2^{a-1}};q^d)_\infty(-q^{d+2^a};q^d)_\infty\cdots(-q^{d+2^{r-1}};q^d)_\infty.$$

$$g_d^{(a)}(q):=\sum_{n=0}^{\infty}\mathcal{G}_d^{(a)}(n)q^n:=\frac{(-q^{d+2^{r-1}};q^{2d})_\infty}{(q^{2^{a-1}};q^{2d})_\infty(q^{d+2^a};q^{2d})_\infty\cdots(q^{d+2^{r-2}};q^{2d})_\infty}.$$

We prove the following lemmas which relate these generating functions.

Lemma 5.4. *By the definitions above, for $a, d > 0$, $n > 0$ divisible by 2^{a-1}, and $d \neq 2^r - 2^{a-1}$ where r is defined by* (2),

$$\mathcal{L}_d^{(a)}(n) + \mathcal{L}_d^{(a)}(n - 2^r) \geq \mathcal{K}_d^{(a)}(n).$$

Proof. Let n be a multiple of 2^{a-1} and $d \neq 2^r - 2^{a-1}$. We see that

$$\mathcal{L}_d^{(a)}(n) + \mathcal{L}_d^{(a)}(n - 2^r) = [q^n]((1 + q^{2^r})f_d(q)).$$

Thus, it is sufficient to show

$$[q^n]((1 + q^{2^r})f_d(q)) \geq [q^n](k_d(q)) = \mathcal{K}_d^{(a)}(n).$$

So, we manipulate as follows:

$$\begin{aligned}
&(1 + q^{2^r})(f_d^{(a)}(q)) - k_d^{(a)}(q) \\
&= (1 + q^{2^r})(-q^{2^{a-1}}; q^d)_\infty \cdots (-q^{2^{r-1}}; q^d)_\infty - \frac{1 - q^{d+2^{a-1}}}{1 - q^{2^{a-1}}}(-q^{d+2^{a-1}}; q^d)_\infty \cdots (-q^{d+2^{r-1}}; q^d)_\infty \\
&= (1 + q^{2^{a-1}}) \cdots (1 + q^{2^{r-1}})(1 + q^{2^r}) - (1 + q^{2^{a-1}} + q^{(2^{a-1})2} + q^{(2^{a-1})3} + \cdots + q^d) \geq 0.
\end{aligned}$$

□

Lemma 5.5. *For any $a, d > 0$ and $n \geq 4d + 2^r$, where r is defined by* (2)*, we have*

$$\mathcal{K}_d^{(a)}(n) \geq \mathcal{G}_d^{(a)}(n).$$

Proof. With some algebraic manipulations, we obtain

$$\begin{aligned}
k_d^{(a)}(q) &= \frac{1 - q^{d+2^{a-1}}}{1 - q^{2^{a-1}}}(-q^{d+2^{a-1}}; q^d)_\infty(-q^{d+2^a}; q^d)_\infty \cdots (-q^{d+2^{r-1}}; q^d)_\infty \\
&= \frac{1 - q^{d+2^{a-1}}}{1 - q^{2^{a-1}}} \cdot \frac{(q^{2d+2^a}; q^{2d})_\infty}{(q^{d+2^{a-1}}; q^d)_\infty} \cdot \frac{(q^{2d+2^{a+1}}; q^{2d})_\infty}{(q^{d+2^a}; q^d)_\infty} \cdot \cdots \cdot \frac{(q^{2d+2^r}; q^{2d})_\infty}{(q^{d+2^{r-1}}; q^d)_\infty} \\
&= \frac{(q^{4d+2^r}; q^{4d})_\infty(-q^{d+2^{r-1}}; q^{2d})_\infty}{(q^{2^{a-1}}; q^{2d})_\infty(q^{3d+2^{a-1}}; q^{2d})_\infty(q^{d+2^a}; q^{2d})_\infty \cdots (q^{d+2^{r-2}}; q^{2d})_\infty}.
\end{aligned}$$

With this new form, we can interpret $k_d^{(a)}(q)$ and $\mathcal{K}_d^{(a)}(n)$ combinatorially. We define two sets:

$$S^+(n) := \left[\begin{array}{c}\text{The set of partitions of n into:}\\ \text{distinct parts} \equiv 4d + 2^r \pmod{4d} \text{ where there is an \textit{even} number of such parts,}\\ \text{and distinct parts} \equiv d + 2^{r-1} \pmod{2d},\\ \text{and (possibly repeated) parts} \equiv 2^{a-1}, 3d + 2^{a-1}, d + 2^a, ..., d + 2^{r-2} \pmod{2d}.\end{array}\right]$$

$$S^-(n) := \left[\begin{array}{c}\text{The set of partitions of n into:}\\ \text{distinct parts} \equiv 4d + 2^r \pmod{4d} \text{ where there is an \textit{odd} number of such parts,}\\ \text{and distinct parts} \equiv d + 2^{r-1} \pmod{2d},\\ \text{and (possibly repeated) parts} \equiv 2^{a-1}, 3d + 2^{a-1}, d + 2^a, ..., d + 2^{r-2} \pmod{2d}.\end{array}\right]$$

This definition implies $\mathcal{K}_d^{(a)}(n) = |S^+| - |S^-|$. Equivalently, $[q^n](k_d(n)) = |S^+(n)| - |S^-(n)|$. Additionally, denote $S(n) = S^+ \cup S^-$. Now, we define the sign of a partition π in $S(n)$ as

$$\text{sgn}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ has an even number of parts} \equiv 2^r \pmod{4d} \text{ (i.e. } \pi \in S^+(n)) \\ -1 & \text{if } \pi \text{ has an odd number of parts} \equiv 2^r \pmod{4d} \text{(i.e. } \pi \in S^-(n)). \end{cases}$$

We see that if $\mathrm{sgn}(\pi) = 1$ then $\pi \in S^+(n)$ and if $\mathrm{sgn}(\pi) = -1$ then $\pi \in S^-(n)$. Next, let $T(n)$ denote the set of partitions of n generated by $g_d^{(a)}(q)$. Observe that no partition in $T(n)$ has parts congruent to $4d + 2^r \pmod{4d}$ so we can say that $T(n) \subset S^+(n)$. In the following work, we create a sign reversing function $\varphi(\pi) : S(n) \to S(n)$ that fixes $T(n)$.

Following the notation of [13], we write $\pi_i \in \pi$ if π_i is a part of π and write π_i^x if π_i occurs x times as a part in a partition. Also, let m_{π_i} be the number of times π_i occurs as a part of π. Finally, let $\alpha = d + 2^r - 2^{a-1}$ with r defined by (2). We define the following:

$$\begin{aligned} x &:= \text{smallest integer } i \text{ such that } 2^{r-(a-1)}id + 2^r \in \pi \\ y &:= \text{smallest integer } j \text{ such that } (2^{r-(a-1)}j - 1)d + 2^{a-1} \in \pi \\ z &:= \text{smallest } l > y \text{ such that } m_{ld+2^{a-1}} \geq 2^{r-(a-1)} \end{aligned}$$

If there are no such x, y, z, we set $x = \infty$, $y = \infty$, or $z = \infty$. Let φ_{a-1} be defined by

$$\begin{cases} 2^{r-(a-1)}xd + 2^r \to (2^{r-(a-1)}x - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} & \text{if } x \leq y \text{ and } x < \infty, \\ (2^{r-(a+1)}y - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} \to 2^{r-(a+1)}yd + 2^r & \text{if } x > y, y < \infty, \text{ and } m_{2^{a-1}} \geq \frac{\alpha}{2^{a-1}}, \\ 2^{r-(a+1)}xd + 2^r \to (xd + 2^{a-1})^{2^{r-(a-1)}}, & \text{if } x > y,\ x < \infty,\ m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}, \text{ and } x \leq z, \\ (zd + 2^{a-1})^{2^{r-(a-1)}} \to 2^{r-(a-1)}zd + 2^r, & \text{if } x > y,\ m_{2^{a-1}} < \frac{\alpha}{2^{a-1}},\ x > z, \text{ and } z < \infty \end{cases}$$

and $\varphi_{a-1}(\pi) = \pi$ otherwise.

We notice that $\varphi_{a-1}^2(\pi) = \pi,$ and $\mathrm{sgn}(\pi) \cdot \mathrm{sgn}(\varphi_{a-1}(\pi)) = -1$ if $\varphi_{a-1}(\pi) \neq \pi$. Define

$$\begin{aligned} S_{a-1}(n) &= \{\pi \in S(n) | \varphi_{a-1}(\pi) \neq \pi\} \\ T_{a-1}(n) &= S(n) \setminus S_{a-1}(n) \end{aligned}$$

and notice that for any $\pi \in T_a(n)$, π has no part $\equiv 2^r \pmod{2^{r-(a-1)}d}$. Also $m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}$ and $z = \infty$ if $y < \infty$, in other words π has a part $\equiv (2^{r-2} - 2^{a-1})d + 2^{a-1} \pmod{2^{r-(a-1)}d}$.

We now establish an involution on $T_{a-1}(n)$. Let $\pi \in T_{a-1}(n)$, and define

$$\begin{aligned} u &:= \text{smallest integer } p \text{ such that } (2^{r-a}p - 1)d + 2^{a-1} \in \pi \\ x &:= \text{smallest odd integer } i \text{ such that } 2^{r-a}id + 2^{r-a} \in \pi \\ y &:= \text{smallest odd integer } j \text{ such that } (2^{r-a}j - 1)d + 2^{a-1} \in \pi \\ w &:= \text{smallest odd integer } l \text{ such that } m_{ld+2^a} \geq 2^{r-a} \\ z &:= \text{smallest odd integer } l > y \text{ such that } m_{ld+2^a} \geq 2^{r-a}, \end{aligned}$$

If no such u, x, y, w, and z exist, denote $u = \infty, x = \infty, y = \infty, w = \infty$, or $z = \infty$. Let $\varphi_a(\pi)$ be defined by replacing

$$\begin{cases} 2^{r-a}xd + 2^r \to (xd + 2^a)^{2^{r-a}} & \text{if } u < \infty, x < \infty, x \leq w \\ (wd + 2^a)^{2^{r-a}} \to 2^{r-a}wd + 2^r & \text{if } u < \infty, x > w \\ 2^{r-a}xd + 2^r \to (2^{r-a}x - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} & \text{if } u = \infty, x \leq y, x < \infty \\ (2^{r-a}y - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} \to 2^{r-a}yd + 2^r & \text{if } u = \infty, x > y, y < \infty, m_{2^{a-1}} \geq \frac{\alpha}{2^{a-1}} \\ 2^{r-a}xd + 2^r \to (xd + 2^a)^{2^{r-a}} & \text{if } u = \infty, x > y, x < \infty, m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}, x \leq z \\ (zd + 2^a)^{2^{r-a}} \to 2^{r-a}zd + 2^r & \text{if } u = \infty, x > y, m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}, x > z,\ z < \infty \end{cases}$$

Again we see, $\varphi_a^2(\pi) = \pi$ and $\operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\varphi_a(\pi)) = -1$ if $\varphi_a(\pi) \neq \pi$. Now, define

$$S_a(n) = \{\pi \in T_{a-1}(n) | \varphi_a(\pi) \neq \pi\}$$
$$T_a(n) = T_{a-1}(n) \setminus S_a(n).$$

For any $\pi \in T_a(n)$, π has no part $\equiv 2^r \pmod{2^{r-a}d}$ with r defined by (2) and $m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}$ if π has a part $\equiv (2^{r-a} - 2^{a-1})d + 2^{a-1} \pmod{2^{r-a}d}$.

Now define φ_t, S_t, and T_t for $a+1 \leq t \leq r-2$. For any $\pi \in T_{t-1}(n)$, define

$$u := \text{smallest integer } p \text{ such that } (2^{r-t+1}p - 1)d + 2^{a-1} \in \pi$$
$$x := \text{smallest odd integer } i \text{ such that } 2^{r-t}id + 2^r \in \pi$$
$$y := \text{smallest odd integer } j \text{ such that } (2^{r-t}j - 1)d + 2^{a-1} \in \pi$$
$$w := \text{smallest odd integer } l \text{ such that } m_{ld+2^t} \geq 2^{r-t}$$
$$z := \text{smallest odd integer } l > y \text{ such that } m_{ld+2^t} \geq 2^{r-t}$$

We define $\varphi_t(\pi)$ by

$$\begin{cases} 2^{r-t}xd + 2^r \to (xd + 2^t)^{2^{r-t}} & \text{if } u < \infty, x < \infty, x \leq w \\ (wd + 2^t)^{2^{r-t}} \to 2^{r-t}wd + 2^r & \text{if } u < \infty, x > w \\ 2^{r-t}xd + 2^r \to (2^{r-t}x - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} & \text{if } u = \infty, x \leq y, x < \infty \\ (2^{r-t}y - 1)d + 2^{a-1}, (2^{a-1})^{\frac{\alpha}{2^{a-1}}} \to 2^{r-t}yd + 2^r & \text{if } u = \infty, x > y, y < \infty, m_{2^{a-1}} \geq \frac{\alpha}{2^{a-1}} \\ 2^{r-t}xd + 2^r \to (xd + 2^t)^{2^{r-t}} & \text{if } u = \infty, x > y, x < \infty, m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}, x \leq z \\ (zd + 2^t)^{2^{r-t}} \to 2^{r-t}zd + 2^r & \text{if } u = \infty, x > y, m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}, x > z,\ z < \infty \end{cases}$$

and define $\varphi_t(\pi) = \pi$ otherwise.

From the definition of φ_t, we again have $\varphi_t^2(\pi) = \pi$, and $\operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\varphi_t(\pi)) = -1$ if $\varphi_t(\pi) \neq \pi$. Let,

$$S_t(n) = \{\pi \in T_{t-1}(n) | \varphi_t(\pi) \neq \pi\}$$
$$T_t(n) = T_{t-1}(n) \setminus S_t(n)$$

Thus, using φ_t, we see that any partitions in $S_t(n)$ have parts congruent to 2^r with r defined as (2) or $(2^{r-t} - 2^{a-1})d + 2^{a-1} \pmod{2^{r-t}d}$. Thus $T(n) \subset T_t(n)$. Also, note that for any $\pi \in T_{r-2}(n)$, π has no part $\equiv 2^r \pmod{4d}$, and $m_{2^{a-1}} < \frac{\alpha}{2^{a-1}}$ if π has a part $\equiv 3d + 2^{a-1} \pmod{2d}$. Since any partition in $T_{r-2}(n)$ has no parts $\equiv 2^r \pmod{4d}$, we see that the sign of partitions in $T_{r-2}(n)$ is 1. Thus $T_{r-2}(n) \subset S^+(n)$. Note that

$$S(n) = S_{a-1}(n) \cup S_a(n) \cup \cdots \cup S_{r-2}(n) \cup T_{r-2}(n)$$

with $S_i(n) \cap S_j(n) = \varnothing$ for $a-1 \leq i < j \leq r-2$ and $S_i \cap T_{r-2}(n) = \varnothing$ for $a-1 \leq i \leq r-2$.

Finally, we define φ on S(n) by

$$\varphi(\pi) = \begin{cases} \varphi_t(\pi) & \text{if } \pi \in S_t(n) \text{ for some } t, a-1 \leq t \leq r-2, \\ \pi & \text{otherwise.} \end{cases}$$

From the definition of φ_t, we see that φ is a sign reversing involution. We provide a brief overview on how φ_{a-1} affects $\operatorname{sgn}(\pi)$. In each case, we have that a part congruent to $2^r \pmod{4d}$ is being decomposed into parts not congruent to $2^r \pmod{4d}$ or that some parts that are not congruent to $2^r \pmod{4d}$ are being combined to create a part congruent to $2^r \pmod{4d}$. This changes $\operatorname{sgn}(\pi)$, but leaves the total sum of the parts the same.

Furthermore, when there is no part in π that is congruent to $2^r \pmod{4d}$, then φ_{a-1} is the identity map. Since partitions in $T(n)$ have no parts congruent to 2^r or $3d+2^{a-1} \pmod{2d}$, we see that $T(n) \subset T_{r-2}(n)$. Thus all partitions in $T(n)$ are fixed under φ. This shows that $|T(n)| \leq |S^+(n)| - |S^-(n)|$ for any $n \geq 0$, which completes the proof. □

Now we will prove some inequalities that relate $q_d^{(a)}(n)$ with $\mathcal{L}_d^{(a)}(n)$ and $Q_d^{(a,-,-)}(n)$ with $\mathcal{G}_d^{(a)}(n)$.

Lemma 5.6. *For $a > 0$, $n > 0$ divisible by 2^{a-1}, and $d \geq 2^{a-1} \cdot 31$ also divisible by 2^{a-1},*

$$\mathcal{G}_d^{(a)}(n) \geq Q_d^{(a,-,-)}(n) .$$

Proof. Let n and d be divisible by 2^{a-1} where $d \geq 2^{a-1} \cdot 31$. Note that this implies that $r \geq a+4$. Define

$$S := \{x | x \equiv a, d+3-a \pmod{d+3}\} \setminus \{a, d+3-a\}$$
$$T_r := \{y | y \equiv 2^{a-1}, d+2^a, ..., d+2^{r-2} \pmod{2d}\}$$

Once again, by the same argument in Remark 3.4, we need only consider the case $r = a+4$, where

$$T_{a+4} = \{y | y \equiv 2^{a-1}, d+2^a, d+2^{a+1}, d+2^{a+2} \pmod{2d}\}$$

Let x_i and y_i be the i^{th} smallest elements of S and T, respectively. Then notice that

i	x_i	y_i
$i = 4\alpha$ and $\alpha \geq 1$	$(2\alpha+1)(d+3) - a$	$\alpha(2d) + 2^{a+2} - d$
$i = 4\alpha + 1$	$(2\alpha+1)(d+3) + a$	$\alpha(2d) + 2^{a-1}$
$i = 4\alpha + 2$	$(2\alpha+2)(d+3) - a$	$\alpha(2d) + 2^a + d$
$i = 4\alpha + 3$	$(2\alpha+1)(d+3) + a$	$\alpha(2d) + 2^{a+2} - d$

Note that all y_i are divisible by 2^{a-1} since d is divisible by 2^{a-1} and the first element of T_5 is 2^{a-1}. We now show that $x_i \geq y_i$ for each i given that $d \geq 2^{a-1} \cdot 31$.

(i) If $i = 4\alpha$, $(2\alpha+1)(d+3) - a - (\alpha(2d) + 2^{a+2} - d) = 6\alpha + 3 + 2d - a - 2^{a+1} \geq 6\alpha + 3 \geq 0$.
(ii) If $i = 4\alpha+1$, $x_i - y_i = (2\alpha+1)(d+3) + a - (\alpha(2d) + 2^{a-1}) = 6\alpha + 3 + a + d - 2^{a-1} \geq 6\alpha + 3 + a \geq 0$.
(iii) If $i = 4\alpha+2$, $x_i - y_i = (2\alpha+2)(d+3) - a - (\alpha(2d) + 2^a + d) = 6\alpha + 6 + d - a - 2^{a-1} \geq 6\alpha + 6 \geq 0$.
(iv) If $i = 4\alpha+3$, $x_i - y_i = (2\alpha+1)(d+3) + a - (\alpha(2d) + 2^{a+2} - d) = 6\alpha + 3 + a + 2d - 2^{a+2} \geq 6\alpha + 3 + a \geq 0$.

Thus, by Theorem 2.6, we obtain

$$\mathcal{G}_d^{(a)}(n) \geq \rho(T; n) \geq \rho(S; n) = Q_d^{(a,-,-)}(n).$$

□

We need to introduce some more terminology which we modify from [13]. Let

$$A_d^{(a)} = \{x \mid x \equiv 2^i \pmod{d}, a-1 \leq i \leq r-1\}$$

and

$$A_d^{(a)'} = \{y \mid y \equiv i \pmod{d}, 2^{a-1} \leq i \leq 2^r - 2^{a-1}\}.$$

We see that these sets satisfy Theorem 2.4, so we see that

$$\mathcal{L}_d^{(a)}(n) = D(A_d^{(a)}; n) = E(A_d^{(a)'}; n).$$

Now we proceed to prove the following lemma.

Lemma 5.7. *For $a, d, n > 0$, where $d \neq 2^r - 2^{a-1}$ with r defined by (2),*

$$q_d^{(a)}(n) \geq \mathcal{L}_d^{(a)}(n) + \mathcal{L}_d^{(a)}(n - 2^r).$$

Proof. We wish to construct an injective map and we can do so by using the injection from the proof of Lemma 2.7 in [13], with the caveat that we must check that the additional restriction on $q_d^{(a)}(n)$, that parts are at least a, is also satisfied and modify one part of the injection slightly.

By setting $t = a$ in Lemma 5.2 we see that $q_d^{(a)}(n) \geq \mathcal{L}_d^{(a)}(n)$ and from Theorem 2.4 we see that this lemma holds if

$$q_d^{(a)}(n) - E(A_d^{(a)'}; n) \geq E(A_d^{(a)'}; n - 2^r).$$

The rest of the proof follows as in the proof Lemma 2.7 in [13] so we omit most of the details from [13], that the injections do indeed satisfy the necessary requirements, and check that the smallest part is $\geq a$ in each case below.

Let $X_n = q_d^{(a)}(n)$ and $Y_n = E(A_d^{(a)'}; n)$. Then we construct an injection $\psi : Y_{n-2^r} \to X_n \backslash Y_n$ for any $n > 4d + 2^r$. Let λ be a partition in Y_n, let λ_i be the parts of λ written in increasing order, and let $l(\lambda)$ be the number of parts of λ and define $\lambda_{l(\lambda)+1} = \infty$. Let

$$r_i = \begin{cases} \beta(\lambda_i), & \text{if } i \leq l(\lambda), \\ \infty, & \text{if } i = l(\lambda) + 1, \end{cases}$$

and

$$u_i = \begin{cases} \lambda_i = r_i, & \text{if } i \leq l(\lambda), \\ \infty, & \text{if } i = l(\lambda) + 1. \end{cases}$$

Let Z_{n-2^r} be a subset of Y_{n-2^r} whose partitions satisfy

$$\lambda_{i+1} - \lambda_i \geq 2d \text{ and } r_i + 2^r \leq d \tag{6}$$

for some i. For $\lambda \in Z_{n-2^r}$ let s be the smallest i such that (6) is satisfied. Let $\mu := \psi(\lambda)$ by

$$\mu_j = \begin{cases} \lambda_j + 2^r, & \text{if } j = s, \\ \lambda_j, & \text{if } j \neq s. \end{cases}$$

Then by the definition of $A_d^{(a)'}$, $\mu_j \geq a$ for all j.

Now subtract Z_{n-2^r} and its image under ψ to redefine ψ from $Y_{n-2^r} \setminus Z_{n-2^r}$ to $(X_n \setminus Y_n) \setminus \psi(Z_{n-2^r})$ and split into three cases.

Case I: $\lambda_1 \geq 2d$. We define $\mu := \psi_1(\lambda)$ by

$$\mu_j = \begin{cases} 2^r, & \text{if } j = 1, \\ \lambda_{j-1}, & \text{if } j \geq 2. \end{cases}$$

Where clearly, by the definition of $A_d^{(a)'}$, $\mu_j \geq a$ for all j.

Case II: $d \leq \lambda_1 < 2d$. Let s now be the smallest integer i such that $u_i + 2d \leq u_{i+1}$ and define $\mu := \psi_2(\lambda)$ by

$$\mu_j = \begin{cases} r_1, & \text{if } j = 1, \\ u_{j-1} + r_j, & \text{if } 2 \leq j \leq s, \\ u_{j-1} + 2^r, & \text{if } j = s+1, \\ \lambda_{j-1}, & \text{if } j \geq s+2. \end{cases}$$

Once again, by the definition of $A_d^{(a)'}$, all $\mu_j \geq a$.

Case III: $\lambda_1 < d$. This case is partitioned into five sub-cases.

Subcase (i): $u_2 \geq 6d$. We define $\mu := \psi_{3,1}(\lambda)$ by

$$\mu_j = \begin{cases} 2^r - 1, & \text{if } j = 1, \\ 2d + r_1, & \text{if } j = 2, \\ \lambda_{j-1} - 2d + 1, & \text{if } j = 3, \\ \lambda_{j-1}, & \text{if } j \geq 4. \end{cases}$$

Since $r \geq a + 4$ so $\mu_j \geq a$ for all j.

Subcase (ii): $u_2 = 4d$ or $5d$. We define $\mu := \psi_{3,2}(\lambda)$ by

$$\mu_j = \begin{cases} r_1, & \text{if } j = 1, \\ d + 2^r - 1, & \text{if } j = 2, \\ \lambda_{j-1} - d + 1, & \text{if } j = 3, \\ \lambda_{j-1}, & \text{if } j \geq 4. \end{cases}$$

By the definition of $A_d^{(a)'}$, $\mu_j \geq a$ for all j.

Subcase (iii): $u_2 = 3d$. Here we make slight modification's to the injection in [13] and show that the modification still satisfies all the necessary requirements. We define $\mu := \psi_{3,3}(\lambda)$ by

$$\mu_j = \begin{cases} 2^{a-1} - 1, & \text{if } j = 1, \\ d + r_1 + 1, & \text{if } j = 2, \\ \lambda_{j-1} - d + 2^r - 2^{a-1}, & \text{if } j = 3, \\ \lambda_{j-1}, & \text{if } j = 4, \\ \lambda_{j-1}, & \text{if } j \geq 5. \end{cases}$$

Clearly, $\mu_j \geq a$ for all j so we check that μ satisfies the difference condition. First we know that since $u_2 = 3d$, $b(r_1) \leq 3$ so $2^{a-1} \leq r_1 \leq 2^{r-a} + 2^{r-a-1} + 2^{r-a-2}$ with r defined by (2) and $2^{a-1} \leq r_i$ for any i. We see

$$\mu_1 + d = 2^{a-1} - 1 + d < d + 2^{a-1} + 1 \leq d + r_1 + 1 = \mu_2.$$

Suppose $r_1 = 2^{r-a} + 2^{r-a-1} + 2^t$ for some $t \le r-a-2$, then $r_2 \le 2^t$. We also have $2^{r-a} + 2^{r-a-1} = 2^{r-a+1} - 2^{r-a-1}$ and $2^{r-a+1} - 2^{r-a-1} + 2^t + 1 \le 2^r$ so

$$\begin{aligned}\mu_2 + d &= 2d + r_1 + 1\\ &= 2d + 2^{r-1} + 2^{r-a-1} + 2^t + 1\\ &= 2d + 2^{r-a+1} - 2^{r-a-1} + 2^t + 1\\ &\le 2d + 2^r\\ &\le 2d + r_2 + 2^r - 2^{a-1}\\ &= \mu_3.\end{aligned}$$

Now suppose $r_1 \le 2^{r-a} + 2^{r-a-1}$ then we have

$$\begin{aligned}\mu_2 + d &= 2d + r_1 + 1\\ &\le 2d + r^{r-a} + 2^{r-a-1} + 1\\ &= 2d + 2^{r-a+1} - 2^{r-a-1} + 1\\ &\le 2d + 2^r\\ &\le 2d + r_2 + 2^r - 2^{a-1}\\ &= \mu_3.\end{aligned}$$

Also, by definition, we know $\lambda_2 + d \le \lambda_3$ so,

$$\begin{aligned}\mu_3 + d &= 2d + r_2 + 2^r - 2^{a-1}\\ &\le 3d + r_2 + 2^r - 2^{a-1}\\ &\le 3d + r_2 + d\\ &= \lambda_2 + d\\ &\le \lambda_3 = \mu_4.\end{aligned}$$

Clearly, by definition, $\mu_i + d \le \mu_{i+1}$ for all $i \ge 4$. Since $2^{a-1} - 1 \notin A_d^{(a)\prime}$ we see that $\mu \notin Y_n \cup \overline{Z}_n$ and, by definition, $\psi_{3,3}$ is injective. Also it is obvious that if W_k is the set of μ mapped to by ψ_k then $W_{3,3} \cap W_l = \emptyset$ for $l = 1$, 2, $(3,1)$, $(3,2)$, $(3,5)$, and $(3,6)$. Since $r_1 \ne 2^r - 2^{a-1}$ we see that $W_{3,3} \cap W_{3,4} = \emptyset$ as well.

Subcase (iv): $u_2 = 2d$. We define $\mu := \psi_{3,4}(\lambda)$ by

$$\mu_j = \begin{cases} 2^r - r_1 - 1, & \text{if } j = 1,\\ \lambda_j, & \text{if } 2 \le j < l(\lambda),\\ \lambda_j + 2r_1 + 1, & \text{if } j = l(\lambda).\end{cases}$$

Since $u_2 = 2d$ and the fact that λ satisfies (3), $b(r_1) \le 2$ so $\mu_j \ge a$ for all j.

Subcase (v): $u_2 = d$. Let s be the smallest integer i such that $u_i + 2d \le u_{i+1}$. Let

$$x = \begin{cases} 5, & \text{if } r_{s-1} \ne 1 \text{ and } 4,\\ 10, & \text{if } r_{s-1} = 1 \text{ and } 4.\end{cases}$$

We define $\mu := \psi_{3,5}(\lambda)$ by

$$\mu_j = \begin{cases} \lambda_j, & \text{if } j < s-1, \\ \lambda_j + x, & \text{if } j = s-1, s, \\ \lambda_j, & \text{if } s < j < l(\lambda), \\ \lambda_j + 2^r - 2x, & \text{if } j = l(\lambda). \end{cases}$$

Finally, by the definition of $A_d^{(a)'}$, $\mu_j \geq a$ for all j.

□

Using Lemma 5.4, 5.5, 5.6, and 5.7 we have

$$q_d^{(a)}(n) \geq \mathcal{L}_d^{(a)}(n) + \mathcal{L}_d^{(a)}(n-2^r) \geq \mathcal{K}_d^{(a)}(n) \geq \mathcal{G}_d^{(a)}(n) \geq Q_d^{(a,-,-)}(n) \tag{7}$$

for d and n divisible by 2^{a-1} where $d \geq 2^{a-1} \cdot 31$, $d \neq 2^r - 2^{a-1}$ and $n > 4d + 2^r$. This proves Theorem 1.8.

Remark 5.8. *It is also possible to use similar generalizations of work of Yee* [13] *and Andrews* [4] *to prove certain cases of Theorem 1.3, however, such a generalization wouldn't be able to prove all of the remaining cases so we omit it.*

6. Asymptotic results

In this section we provide asymptotic bounds for the partition functions $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$, which in theory could be used to computationally prove the remaining finite cases of Theorem 1.3. First, we prove the asymptotic result in Theorem 1.9 for $\Delta_d^{(a)}(n)$ analogous to Andrews' [4] result on Alder's Conjecture.

Proof of Theorem 1.9. We begin by considering the asymptotic formulas for $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$ separately.

By work of Meinardus [10, Theorems 2 and 3], making the substitutions $k = 1, l = d, \alpha_l = \alpha$, $A_d = A$, $m = a$, and $C(1, l, m) = C(d, a)$, we have

$$q_d^{(a)}(n) \sim C(d,a) n^{-\frac{3}{4}} e^{2\sqrt{An}},$$

where

$$C(d,a) := \frac{1}{2\sqrt{\pi}} A^{\frac{1}{4}} \left(\alpha^{d+1-2a} \cdot (d(\alpha)^{d-1} + 1)\right)^{-\frac{1}{2}},$$

$$A := \frac{d}{2} \log^2 \alpha + \sum_{r=1}^{\infty} \frac{(\alpha)^{rd}}{r^2},$$

and $\alpha \in [0,1]$ is the positive real number such that $\alpha^d + \alpha - 1 = 0$. We can say

$$\log q_d^{(a)}(n) \sim \log\left(C(d,a) n^{-\frac{3}{4}}\right) + 2\sqrt{An} \sim 2\sqrt{An}.$$

Using Andrews [6, Ch. 6, Example 1, pg. 97] we have

$$Q_d^{(a)}(n) \sim \frac{\csc(\frac{\pi a}{d+3})}{(4\pi) 3^{\frac{1}{4}} (d+3)^{\frac{1}{4}}} n^{-\frac{3}{4}} e^{2\pi\sqrt{\frac{n}{3d+9}}}$$

for $a < \frac{d+3}{2}$ and a relatively prime to $d+3$. This gives us

$$\log Q_d^{(a)}(n) \sim \log\left(\frac{\csc(\frac{\pi a}{d+3})}{(4\pi)3^{\frac{1}{4}}(d+3)^{\frac{1}{4}}} n^{-\frac{3}{4}}\right) + 2\pi\sqrt{\frac{n}{3d+9}} \sim 2\pi\sqrt{\frac{n}{3d+9}}.$$

From Andrews [4, proof of Theorem 2] we have

$$A > \frac{\pi^2}{3d+9}$$

which implies that

$$\lim_{n\to\infty}\left(\log q_d^{(a)}(n) - \log Q_d^{(a)}(n)\right) = +\infty.$$

Thus,

$$\lim_{n\to\infty}\Delta_d^{(a)}(n) = \lim_{n\to\infty} q_d^{(a)}(n)\left(1 - \frac{Q_d^{(a)}(n)}{q_d^{(a)}(n)}\right) = +\infty.$$

□

Remark 1.10, when n is large enough, directly implies the following corollaries.

Corollary 6.1. *Let $a > 0$ and $d \geq 4$ such that $a < \frac{d+3}{2}$ and a is relatively prime to $d+3$ then*

$$\lim_{n\to\infty}\Delta_d^{(a,-)}(n) = +\infty.$$

Proof. Notice that for all d and n, $Q_d^{(b)}(n) \geq Q_d^{(b,-)}(n)$. Thus, $\Delta_d^{(a,-)}(n)$ will only grow faster than $\Delta_d^{(a)}(n)$ as n goes to ∞. □

Corollary 6.2. *Let $a > 0$, and $d \geq 4$ such that $a < \frac{d+3}{2}$ and a is relatively prime to $d+3$ then*

$$\lim_{n\to\infty}\Delta_d^{(a,-,-)}(n) = +\infty.$$

Proof. Similarly, for all d and n, $Q_d^{(b)}(n) \geq Q_d^{(b,-,-)}(n)$. Thus, $\Delta_d^{(a,-,-)}(n)$ will grow faster than $\Delta_d^{(a)}(n)$ as n goes to ∞. □

The asymptotic formulas for $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$ given by Meinardus [9, 10] and Andrews [6], while allowing us to examine the asymptotic behavior of $\Delta_d^{(a)}(n)$, can be made more explicit by closely following methods of Alfes, Jameson, and Lemke Oliver [3]. In the following, we draw heavily on their work, only making minor modifications to achieve explicit asymptotic results for arbitrary $a > 0$.

6.1. **The asymptotics of $Q_d^{(a)}(n)$.** First we state a theorem due to Xia [11].

Theorem 6.3 (Xia [11], 2011). *If $d > 0$, $0 < a < \frac{d+3}{2}$, and a is coprime to $d+3$ then for $n > 0$,*

$$Q_d^{(1)}(n) \geq Q_d^{(a)}(n).$$

Given Theorem 6.3, we see that under the given hypotheses, $Q_d^{(1)}(n)$ bounds $Q_d^{(a)}(n)$ from above, thus the asymptotic upper bounds from Alfes, Jameson, and Lemke Oliver [3, Theorem 2.1] for $Q_d^{(1)}(n)$ also bound $Q_d^{(a)}(n)$ for any $a \geq 1$ with the given hypotheses. Thus we obtain immediately the following result.

Theorem 6.4. *If $d \geq 4$, $n > 0$, $0 < a < \frac{d+3}{2}$, and a is coprime to $d+3$, then*

$$Q_d^{(a)}(n) \leq \frac{(3d+9)^{-\frac{1}{4}}}{4\sin\left(\frac{\pi}{d+3}\right)} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}} \frac{2\pi}{\sqrt{3(d+3)}}\right) + R(n)$$

where $R(n)$ is an explicitly bounded function given in [3, equation (2.10)].

6.2. **The asymptotics of $q_d^{(a)}(n)$.** In this section we will analog the asymptotics given by Alfes, Jameson and Lemke Oliver [3, Theorem 3.1] for $q_d^{(1)}(n)$.

Let α be the unique real number in $[0, 1]$ satisfying $\alpha^d + \alpha - 1 = 0$, and let $A := \frac{d}{2}\log^2\alpha + \sum_{r=1}^{\infty} \frac{\alpha^{rd}}{r^2}$.

Theorem 6.5. *If $n > 0$, then*

$$q_d^{(a)}(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d+1-2a}(d\alpha^{d-1}+1)}} n^{-3/4} e^{2\sqrt{An}} + r_d(n)$$

where $|r_d(n)|$ can be bounded explicitly.

Remark 6.6. *We get the main term from a direct application of* [10, Theorem 2] *using $k = 1, m = a, l = d, \mu = n$. We will derive $r_d(n)$ by following the method of Alfes, Jameson and Lemke Oliver* [3].

We first establish some preliminary facts. For a fixed $d \geq 4$, we have

$$f(z) := \sum_{n=0}^{\infty} q_d^{(a)}(n) e^{-nz} \tag{8}$$

with $z = x + iy$. Hence, through the Fourier expansion and other manipulations, we have that

$$q_d^{(a)}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z) e^{nz} dy. \tag{9}$$

Thus we see that to establish asymptotics for $q_d^{(a)}(n)$ we need strong estimates for $f(z)$.

Letting $\rho := 1 - \alpha$, we define the following two functions, modified from Meinardus [10],

$$H(w, z) := \prod_{n=0}^{\infty} (1 - we^{-(n+a)z})^{-1} \tag{10}$$

$$\Theta(w, z) := \sum_{n=-\infty}^{\infty} e^{-\frac{d}{2}n(n-1)z} w^{-n}, \tag{11}$$

where $\mathcal{R}(z) > 0, 0 < |w| < e^{aR(z)}$. Then, by the Cauchy Integral Theorem

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} H(w, z)\Theta(w, z) \frac{dw}{w}, \tag{12}$$

where $\mathcal{C}$ is a circle centered at the origin of radius ρ.

In the following lemma we aim to estimate $H(w, z)$ and $\Theta(w, z)$.

Lemma 6.7. *Let $\rho = \alpha^d = 1 - \alpha$, and let that $w = \rho e^{i\varphi}$ with $-\pi \le \varphi < \pi$. Then for $|y| \le x^{1+\epsilon}$ and $x < \beta$ where*

$$\beta := min \ \left(\frac{-\pi}{\log p}\xi, \frac{2\alpha^{2-d}}{\pi d}, \frac{1}{2d} + p\left(\frac{1}{2} - \frac{\pi^2}{24}\right)\right)^{\frac{1}{\epsilon}},$$

we have

$$H(w,z) = \exp\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2} + \left(a - \frac{1}{2}\right)\log(1-w) + f_1(w,z)\right) \tag{13}$$

and

$$\Theta(w,z) = \sqrt{\frac{2\pi}{dz}}\exp\left(\frac{\log^2 w}{2dz} - \frac{1}{2}\log w\right)(1 + f_2(w,z)), \tag{14}$$

where, as $x \to 0$, $f_1(w,z) = O(x^{1/2})$ and $f_2(w,z) = O\left(x + \exp\left(-\frac{c_0}{x}(\pi - |\varphi|) + c_1 x^{\epsilon-1}\right)\right)$ are explicitly bounded functions.

Proof. By Meinardus [10, equation (18)] and the inverse Mellin transform,

$$\log H(w,z) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} z^{-s}\Gamma(s)\zeta(s,a)D(s+1,w)ds, \tag{15}$$

where $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$ is the Hurtwitz zeta function, $s = \sigma + it$, $\Gamma(s)$ is the Gamma function, and $D(s,w) := \sum_{r\ge1}\frac{w^r}{r^s}$, which is defined as a function of s for all fixed w with $|w| < 1$. We remark that $\zeta(s,a)$ converges for $\sigma > 1$.

Note that if $\theta_0 := \arctan x^{\epsilon}$, then

$$|z^{1/2} - it| \le |z|^{1/2}e^{\theta_0|t|} \le (1 + x^{2\epsilon})^{1/4}x^{1/2}e^{\theta_0|t|}.$$

Also, note that $\zeta(0,a) = \frac{1}{2} - a$. We change the curve of integration and account for the poles at $s = 0$ and $s = 1$, to get

$$\log H(w,z) = \frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2} + \left(a - \frac{1}{2}\right)\log(1-w) + f_1(w,z)$$

where

$$\begin{aligned} |f_1(w,z)| &= \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} z^{-s}\Gamma(s)\zeta(s,a)D(s+1,w)ds \\ &\le (1 + x^{2\epsilon})^{\frac{1}{4}}2^{-\frac{5}{2}}\pi^{-\frac{3}{2}}\zeta\left(\frac{3}{2}, a\right)\frac{\rho}{1-\rho}\frac{4}{\frac{\pi}{2} - \theta_0}x^{\frac{1}{2}} := f_1(x). \end{aligned}$$

Since $x^{2\epsilon}$ and θ_0 both tend toward 0 as $x \to 0$, we have proved the first part of our lemma.

The remainder of the proof, which covers the bound of $\Theta(x,z)$ follows directly from Alfes, Jameson and Lemke Oliver [3, Lemma 3.5] where

$$f_2(\varphi,z) := f_2^0(z) + f_2^{\varphi}(z)\exp\left(\frac{2\pi}{dx(1+x^{2\epsilon})}\right) \tag{16}$$
$$:= e^{\frac{d|z|}{8}}\left[e^{\frac{dx\sqrt{1+x^{2\epsilon}}}{8}} - 1 + 2\frac{\exp\left(-\frac{4\pi^2(1-\xi)}{dx(1+x^{2\epsilon}}\right)}{1-\exp\left(-\frac{2\pi^2(1-\xi)}{dx(1+x^{2\epsilon}}\right)}\right]$$
$$+ 2\exp\left(\frac{2\pi}{dx(1+x^{2\epsilon})} - \frac{2\pi\log\rho}{d}x^{\epsilon-1} + \frac{d|z|}{8}\right).$$

□

Lemma 6.8. *Assuming the notation above, for $|y| \le x^{1+\epsilon}$ and $x < \beta$ where $\rho := 1-\alpha$ and $0 < \xi < 1$ is a constant, we have that*

$$f(z) = (\alpha^{d+1-2a}(d\alpha^{d-1}+1))^{-\frac{1}{2}} e^{\frac{A}{z}}(1 + f_{err}(z)),$$

where $f_{err}(z) = o(1)$ is an explicitly bounded function.

Proof. Recall equation (12). We decompose this integral into two components of a circle of radius ρ. Let $\varphi_0 = x^c$ with $\frac{3}{8} < c < \frac{1}{2}$. We have

$$f(z) = \frac{1}{2\pi i}\int_{-\rho e^{i\varphi_0}}^{\rho e^{i\varphi_0}} H(w,z)\Theta(w,z)\frac{dw}{w} + \frac{1}{2\pi i}\int_{\mathcal{B}} H(w,z)\Theta(w,z)\frac{dw}{w} \tag{17}$$

with $\mathcal{B}$ being the circle without the arc of $-\rho e^{i\varphi_0}$ to $\rho e^{i\varphi_0}$. We start with bounding the second integral involving $\mathcal{B}$ by using explicit bounds on $|H(w,z)\Theta(w,z)|$. We now provide a brief explanation of bounds for $|\Theta(w,z)|$ and $|H(w,z)|$. First,

$$|\Theta(w,z)| \le \sqrt{\frac{2\pi}{d|z|}}\rho^{-\frac{1}{2}}\exp\left(\frac{x\log^2\rho - \varphi^2 x}{2d(x^2+y^2)} + \frac{y\varphi\log\rho}{d(x^2+y^2)}\right)(1 + |f_2(w,z)|). \tag{18}$$

This quantity is derived from an application of the triangle inequality, $\frac{1}{\rho} \ge 1$, and application of the norm to z. Alfes, Jameson and Lemke Oliver [3] note Meinardus' [10] error of not considering the contribution of $\frac{y\varphi\log p}{d(x^2+y^2)}$. Since $\Theta(w,z)$ is not dependent on a we can simply employ the result of Alfes, Jameson and Lemke Oliver [3] for the bound of this function, including the correction of Meinardus' [10] error.

The bound provided for $|H(w,z)|$ is a simple application of $|1-w| \le (1+\rho)$ and reducing the logarithm and exponential. Thus we have

$$|H(w,z)| \le \exp(|f_1(w,z)|)(1+\rho)^{a-\frac{1}{2}}\exp\left(\Re\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2}\right)\right). \tag{19}$$

Noting that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, and evaluating the geometric series we have

$$\Re\left(\frac{1}{z}\sum_{r=1}^{\infty}\frac{w^r}{r^2}\right) = \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\cos(r\varphi)-1\right) - \frac{y\varphi\log(1-\rho)}{x^2+y^2} + \frac{y}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\sin(r\varphi)-r\varphi\right). \tag{20}$$

Further noting that $p = \alpha^d$, $(1-\rho) = \alpha$, and multiplying out everything from our bounds on $|\Theta(w,z)|$ and $|H(w,z)|$ yields

$$\begin{aligned} |H(w,z)\Theta(w,z)| \leq & \sqrt{\frac{2\pi}{d|z|}}\frac{(1+\rho)^{a-\frac{1}{2}}}{\rho^{1/2}} \\ & \cdot \exp\Bigg(|f_1(w,z)| + \frac{Ax}{x^2+y^2} - \frac{\varphi^2}{x^2+y^2} \\ & \cdot\left[\frac{1}{2d} - \frac{y}{\varphi^2 x}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\sin(r\varphi)-r\varphi\right) + \frac{1}{\varphi^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\cos(r\varphi)-1\right)\right]\Bigg) \\ & \cdot (1+|f_2(w,z)|). \end{aligned} \tag{21}$$

Using this notation, we have

$$\begin{aligned} \left|\int_{\mathcal{B}} H(w,z)\Theta(w,z)\frac{dw}{w}\right| \leq & \sqrt{\frac{2\pi}{d|z|}}\frac{(1+\rho)^{a-1/2}}{\rho^{1/2}}\exp\left(f_1(x) + \frac{Ax}{x^2+y^2}\right) \\ & \cdot\Bigg[(1+f_2^0(z))\int_{\mathcal{B}} e^{-\psi(\varphi,z)}d\varphi \\ & + f_2^{\varphi}(z)\int_{\mathcal{B}}\exp\left(-\psi(\varphi,z) + \frac{2\pi|\varphi|}{dx(1+x^{2\epsilon})}\right)\Bigg] \end{aligned} \tag{22}$$

with

$$\psi(w,z) := \frac{\varphi^2 x}{2d(x^2+y^2)} + \frac{x}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\cos(r\varphi)-1\right) - \frac{y}{x^2+y^2}\sum_{r=1}^{\infty}\frac{\rho^r}{r^2}\left(\sin(r\varphi)-r\varphi\right). \tag{23}$$

The two integrals in (22) do not depend on a so we refer the reader to Alfes, Jameson, and Lemke Oliver [3] for those bounds and define $E_{\mathcal{B}}(z)$ to be the bound of the second integral obtained from (22) and [3, equations (3.15), (3.17), and (3.18)].

Following [10, Equations (29) and (30)], the first integral of (17) is given by

$$I := \frac{1}{2\pi i}\int_{\rho e^{-i\varphi_0}}^{\rho e^{i\varphi_0}} H(w,z)\Theta(w,z)\frac{dw}{w} = \frac{\exp\left(\frac{A}{z} + \left(a - \frac{1+d}{2}\right)\log\alpha\right)}{\sqrt{2\pi dz}}(I_{main} + I_{error}), \tag{24}$$

where

$$I_{main} := \int_{-\varphi_0}^{\varphi_0}\exp\left(-\frac{\varphi^2}{2dz}(d\alpha^{d-1}+1)\right)d\varphi, \tag{25}$$

and

$$I_{error} := \int_{-\varphi_0}^{\varphi_0} \left(\exp\left(\log\left(\frac{1-\rho e^{i\varphi}}{1-\rho}\right) + f_3(w,z) + f_1(w,z)\right)(1+f_2(w,z)) - 1\right) \cdot \exp\left(-\frac{\varphi^2}{2dz}(d\alpha^{d-1}+1)\right) d\varphi \tag{26}$$

with $|f_3(w,z)| \leq \frac{\rho e}{6(1-\rho e)^2}\varphi^3$.

Since I_{main} and I_{error} do not depend on a, we refer to [3] for their bounds and conclude that

$$I = \frac{\alpha^{\left(a - \frac{1+d}{2}\right)}}{d\alpha^{d-1}+1} \exp\left(\frac{A}{z}\right) + \hat{E}_{\varphi_0}(z) \tag{27}$$

where

$$|\hat{E}_{\varphi_0}(z)| \leq E_{\varphi_0}(z)$$

and $E_{\varphi_0}(z)$ is defined as

$$E_{\varphi_0}(z) := \frac{\alpha^{\left(a - \frac{1+d}{2}\right)}}{\sqrt{2\pi d|z|}} e^{\frac{Ax}{x^2+y^2}} \cdot \left[\frac{(2\pi d|z|)^{\frac{1}{2}}}{\varphi_0(d\alpha^{d-1}+1)^{\frac{1}{2}}} \exp\left(-\frac{(\varphi_0^2)x(d\alpha^{d-1}+1)}{2d(x^2+y^2)}\right) + 2\varphi_0\left(\frac{1-\rho\cos\varphi_0}{1-\rho} \exp\left(f_1(x) + \frac{\rho e}{6(1-\rho e)^2}\varphi_0^3\right)(1+f_2(\varphi_0,z)) - 1\right)\right]. \tag{28}$$

Thus we see that

$$|f_{err}(z)| \leq (E_{\varphi_0}(z) + E_{\mathcal{B}}(z))(\alpha^{d+1-2a}(d\alpha^{d-1}))^{\frac{1}{2}} \exp\left(\frac{-Ax}{x^2+y^2}\right). \tag{29}$$

□

Lemma 6.9. *For $x < \beta$ and $x^{1+\epsilon} < |y| \leq \pi$ where f_1, f_2 are functions defined in Lemma 6.7 and η is an explicitly derived constant, we have*

$$|f(x+iy)| \leq \sqrt{\frac{2\pi}{dx}} e^{-\eta\rho x^{2\epsilon-1}} (1+f_2(\rho,x)) \exp\left(\frac{A}{x} + \left(a - \left(\frac{1}{2}+d\right)\right)\log(\alpha) + f_1(\rho,x)\right).$$

Proof. From the definition of $\Theta(w,z)$ we see that $|\Theta(w,z)| \leq \Theta(\rho,x)$. From how $H(w,z)$ is defined, we have that

$$\log|H(w,z)| \leq \log(H(\rho,x)) + \Re\left\{w\sum_{n\geq 1}^{\infty} e^{-(n+a)x}\right\} - \rho\sum_{n\geq 1}^{\infty} e^{-(n+a)x}.$$

To show this consider the bounds on the norm of $\log(H(w,z)$. We know that $\log|H(w,z)| = \Re\{\log(H(w,z)\}$, since $\log(z) = \log(|z|) + i\mathrm{Arg}(z)$ and by applying $\Re$ to both sides.

We now analyze the expression $\Re\{\log H(w,z)\}$. We have from the definition

$$H(w,z) = \prod_{n=1}^{\infty}(1 - we^{-(n+a)z})^{-1}.$$

Note that $\log(1-z)$ has the following power series

$$\log(1-z) = \sum_{m=1}^{\infty} \frac{z^m}{m}.$$

Applying the complex logarithm and the above power series results in the following chain of equalities:

$$\begin{aligned}\Re\{\log(H(w,z))\} &= \Re\Big\{\sum_{n=0}^{\infty}\log\left(1-we^{-(n+a)z}\right)^{-1}\Big\} = \Re\Big\{\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(we^{-(n+a)z})^m}{m}\Big\}\\ &= \Re\Big\{\sum_{n=0}^{\infty}\sum_{m=2}^{\infty}\frac{(we^{-(n+a)z})^m}{m}\Big\} + \Re\Big\{\sum_{n\geq 0}^{\infty} we^{-(n+a)z}\Big\}\\ &= \Re\Big\{\sum_{n=0}^{\infty}\sum_{m=2}^{\infty}\frac{(\rho e^{i\varphi}e^{-(n+a)x}e^{-(n+a)iy})^m}{m}\Big\} + \Re\Big\{\sum_{n\geq 1}^{\infty} we^{-(n+a)z}\Big\}.\end{aligned}$$

We will denote $\Re\big\{\sum_{n\geq 1}^{\infty} we^{-(n+a)z}\big\}$ as $\Re\{z_1\}$. Employing Euler's formula $e^{i\varphi} = \cos(\varphi) + \sin(\varphi)$ yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=2}^{\infty}\frac{(\rho\cos(\varphi)e^{-(n+a)x}\cos(-y(n+a))^m}{m} + \Re(z_1) \quad \leq \sum_{n=0}^{\infty}\sum_{m=2}^{\infty}\frac{(\rho e^{-(n+a)x})^m}{m} + \Re(z_1)\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(\rho e^{-(n+a)x})^m}{m} + \Re\{z_1\} - \rho\sum_{n=0}^{\infty} e^{-(n+a)z}\\ &= \log(H(\rho,x)) + \Re\{z_1\} - \rho\sum_{n=0}^{\infty} e^{-(n+a)z}\end{aligned}$$

as desired.

We now derive η. We do this by deriving an upper bound on the expression

$$\Re\Big\{\sum_{n=0}^{\infty} we^{-(n+a)z}\Big\} - \rho\sum_{n=0}^{\infty} e^{-(n+a)x}.$$

We first derive a closed form $\Re\sum_{n-0}^{\infty} we^{-(n+a)z}$. With some index manipulation, we have

$$\begin{aligned}&\Re\Big\{w\sum_{n=0}^{\infty} e^{-(n+a)z}\Big\}\\ &= \Re\Big\{e^{-(a-1)z}w\sum_{n=1}^{\infty} e^{-nz}\Big\}\\ &= e^{-(a-1)x}\cos((a-1)y)\Re\Big\{w\sum_{n=1}^{\infty} e^{-nz}\Big\}\\ &\leq e^{-(a-1)x}\Re\Big\{w\sum_{n=1}^{\infty} e^{-nz}\Big\}.\end{aligned}$$

In a similar fashion, we have that

$$-\rho\sum_{n=0}^{\infty} e^{-(n+a)x} = -\rho e^{-(a-1)x}\sum_{n=1}^{\infty} e^{-nx}.$$

This results in

$$\Re\left\{\sum_{n=0}^{\infty} we^{-(n+a)z}\right\} - \rho\sum_{n=0}^{\infty} e^{-(n+a)x} \leq e^{-(a-1)x}\left(\Re\left\{w\sum_{n=1}^{\infty} e^{-nz}\right\} + -\rho\sum_{n=1}^{\infty} e^{-nx}\right)$$

Using the estimation in [3] for the part in parentheses we get

$$\begin{aligned}&\frac{e^{-(a-1)x}(\Re\{w\sum_{n=1}^{\infty} e^{-nz}\} + -\rho\sum_{n=1}^{\infty} e^{-nx})}{-\rho x^{2\epsilon-1}}\\ &\geq e^{-(a-1)x}x^{1-2\epsilon}e^{-ax}\left(\frac{1}{1-e^{x}} - \frac{1}{\sqrt{1-2e^{-x}\cos(x^{1+\epsilon})+e^{-2x}}}\right)\\ &\geq e^{-(a-1)x}\beta^{1-2\epsilon}e^{-a\beta}\left(\frac{1}{1-e^{\beta}} - \frac{1}{\sqrt{1-2e^{-\beta}\cos(x^{1+\epsilon})+e^{-2\beta}}}\right) := \eta.\end{aligned}$$

Using these estimations we can derive the explicit bound on $|f(x+iy)|$ which proves the lemma. ☐

We now we can use Lemmas 6.8, 6.9 to prove Theorem 6.5.

Proof of Theorem 6.5. Recall that in preliminary facts, we deduced

$$q_d^{(a)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)e^{nz}dy.$$

Now, let $q_d^{(a)}(n) = I_1 + I_2$ where

$$I_1 := \frac{1}{2\pi}\int_{-x^{1+\epsilon}}^{x^{1+\epsilon}} f(z)e^{nz}dy \text{ and } I_2 := \frac{1}{2\pi}\left(\int_{-\pi}^{-x^{1+\epsilon}} + \int_{x^{1+\epsilon}}^{\pi}\right) f(z)e^{nz}dy.$$

In this proof, we let $x = \sqrt{\frac{A}{n}}$. Next, we split I_1 into the following integrals

$$I_1 = E_1 + E_2 + E_3$$

such that

$$\begin{aligned}\gamma &:= \frac{1}{2\pi\sqrt{\alpha^{d+1-2a}(d\alpha^{d-1}+1)}}\\ E_1 &:= \gamma e^{\frac{2A}{x}}\int_{-x^{1+\epsilon}}^{x^{1+\epsilon}} e^{\frac{-y^2A}{x^3}}dy\\ E_2 &:= \gamma e^{\frac{2A}{x}}\int_{-x^{1+\epsilon}}^{x^{1+\epsilon}} e^{\frac{-y^2A}{x^3}}\left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1\right)dy\\ E_3 &:= \gamma e^{\frac{2A}{x}}\int_{-x^{1+\epsilon}}^{x^{1+\epsilon}} e^{A\left(\frac{-xy^2+iy^3}{x^2(x^2+y^2)}\right)} f_{\text{err}}(z)dy.\end{aligned}$$

We can rewrite E_1 as follows:

$$E_1 = \gamma e^{\frac{2A}{x}}\int_{-x^{1+\epsilon}}^{x^{1+\epsilon}} e^{\frac{-y^2A}{x^3}}dy = \gamma e^{\frac{2A}{x}}\sqrt{\frac{\pi x^3}{A}} + E_1' \tag{30}$$

where

$$|E_1'| \leq \frac{\gamma}{A\sqrt{2}} x^{2-\epsilon} e^{\frac{2A}{x} - Ax^{2\epsilon-1}}. \tag{31}$$

For E_2 we split futher to get

$$\begin{aligned} E_2 =& \gamma e^{\frac{2A}{x}} \int_{|y|\leq x^{1+\epsilon_2}} e^{\frac{-y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1 \right) dy \\ &+ \gamma e^{\frac{2A}{x}} \int_{x^{1+\epsilon_2} \leq |y| \leq x^{1+\epsilon}} e^{\frac{-y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1 \right) dy \end{aligned}$$

where $\epsilon_2 > \epsilon$ and $\epsilon_2 > \frac{1}{3}$. Then, as in Alfes, Jameson, and Lemke Oliver [3],

$$\left| \gamma e^{\frac{2A}{x}} \int_{|y|\leq x^{1+\epsilon_2}} e^{\frac{-y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1 \right) dy \right| \leq \gamma e^{\frac{2A}{x}} (\exp\left(Ax^{3\epsilon_2-1}\right) - 1) \sqrt{\frac{\pi x^3}{A}}$$

and

$$\begin{aligned} &\left| \gamma e^{\frac{2A}{x}} \int_{x^{1+\epsilon_2} \leq |y| \leq x^{1+\epsilon}} e^{\frac{-y^2 A}{x^3}} \left(e^{A\left(\frac{y^4+ixy^3}{x^3(x^2+y^2)}\right)} - 1 \right) dy \right| \\ &\leq \gamma \exp\left(\frac{2A}{x} - \frac{Ax^{\epsilon_2-2}}{1+x^{2\epsilon}} \right) x^3(1+x^{2\epsilon}) + \frac{\gamma x^3}{A} \exp\left(\frac{2A}{x} - Ax^{\epsilon_2-2} \right). \end{aligned}$$

For E_3, using f_{err} as defined in Lemma 6.8,

$$|E_3| \leq \gamma e^{\frac{2A}{x}} |f_{\text{err}}^{\max}| (\pi x^3(1+x^{2\epsilon}))^{\frac{1}{2}}. \tag{32}$$

To bound I_2, we apply Lemma 6.9 to find that

$$|I_2| \leq \sqrt{\frac{2\pi}{dx}} e^{-\eta\rho x^{2\epsilon-1}} (1+f_2(\rho,x)) \exp\left(nx + \frac{A}{x} + \left(a - (\frac{1}{2}+d) \right) \log(\alpha) + f_1(\rho,x) \right). \tag{33}$$

Finally we obtain

$$q_d^{(a)}(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d+1-2a}(d\alpha^{d-1}+1)}} n^{-3/4} e^{2\sqrt{An}} + E_1' + E_2 + E_3 + I_2$$

where $|E_1' + E_2 + E_3 + I_2|$ is bounded using the expressions from above. The result follows with $|r_d(n)| \leq |E_1'| + |E_2| + |E_3| + |I_2|$. □

Now we can show the asymptotic result for $\Delta_d^{(a)}(n)$ which we analog from Alfes, Jameson, and Lemke Oliver [3, Theorem 1.2].

Theorem 6.10. *For $a, d, n > 0$ with $a < \frac{d+3}{2}$ and a coprime to $d+3$, we have*

$$\Delta_d^{(a)}(n) = q_d^{(a)}(n) - Q_d^{(a)}(n) = \frac{A^{1/4}}{2\sqrt{\pi\alpha^{d+1-2a}(d\alpha^{d-1}+1)}} n^{-3/4} e^{2\sqrt{An}} + \mathcal{E}_d(n)$$

where $\mathcal{E}_d(n) = r_d(n) - Q_d^{(a)}(n)$.

Proof. From Theorem 6.4 we get that $Q_d^{(a)}(n) = O\left(\frac{2\pi}{\sqrt{3d+9}}n^{1/2} + c_0 n^{1/6}\right)$ where c_0 is a positive constant. From Theorem 6.5, we examine the error term to see that

$$\begin{aligned} E_1' &= O(n^{-\frac{5}{6}} e^{2\sqrt{An}}) \\ E_2 &= O(n^{-\frac{3}{2}\epsilon_2 - \frac{1}{4}} e^{2\sqrt{An}}) \\ E_3 &= O(n^{-\frac{15}{16}} e^{2\sqrt{An}}) \\ I_2 &= O(n^{\frac{1}{4}} e^{2\sqrt{An} - \eta\rho x^{2\epsilon - 1}}). \end{aligned}$$

Choosing $\epsilon_2 \geq \frac{7}{18}$ gives us our result.

□

6.3. **A potential method of proving the remaining cases of Conjecture 1.2.** Given the asymptotic bounds for $\Delta_d^{(a)}(n)$ it appears possible to use the method of Alfes, Jameson, and Lemke Oliver [3] to prove the remaining finite cases of d by computation. One way this could be done is by taking the asymptotic bounds for $q_d^{(2)}(n)$ and $Q_d^{(2)}(n)$ and calculating the smallest n_0, for a fixed d, for which these bounds imply $\Delta_d^{(2)}(n) \geq 0$ for all $n > n_0$. By Remark 1.10, this would also imply $\Delta_d^{(2,-)}(n) \geq 0$ for all $n > n_0$. Then one could check that $\Delta_d^{(2,-)}(n) \geq 0$ for all smaller n and each $d < 62$ where $d+3$ is coprime to 2. However, this would still not resolve the odd $d < 61$ cases. Alternatively, one could take the asymptotic bounds for $q_d^{(1)}(n)$ and $Q_d^{(1)}(n)$ given by Alfes, Jameson, and Lemke Oliver [3], and using the same procedure outlined above, but also substituting $d-2$ for d in the asymptotic of $Q_d^{(1)}(n)$, one could prove the finite cases of Proposition 3.1 which would consequently prove the remaining cases of Conjecture 1.2. It is worth noting that this procedure, while in theory would produce the result, has its own computational challenges.

7. Discussion on the exclusion of parts

When considering the values of $\Delta_d^{(a)}(n)$ for $a > 1$ it is not hard to find that there are many examples where $\Delta_d^{(a)}(n) < 0$. Kang and Park introduce the notion of the 'dash,' which indicates a removal of a part from the set of allowed parts, in an attempt to generalize Alder's conjecture to include the second Rogers-Ramanujan identity. We introduce a second 'dash' to indicate a removal of a second part in order to generalize Conjecture 1.2 for arbitrary a. While at first the removal of the $d+3-a$ and a term indicated by the dashes may appear to be an artificial modification meant to clean up the results, we argue that its use is in fact quite natural.

One way to consider how the removal of the $d+3-a$ and the a term is natural is to look specifically at the values of $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$ when $\Delta_d^{(a)}(n)$ is negative. In the following we prove some examples of when $\Delta_d^{(a)}(n) < 0$.

Proposition 7.1. *For $b \geq 4$ and $r \geq 0$, we have*

$$\Delta_{b-2+r}^{(b,b)}(2b+1+r) < 0.$$

Proof. First let's look at $q_{b-2+r}^{(b)}(2b-1+r)$. Since $b < 2b+1+r$ we have at least one partition, $2b+1+r$ itself. Suppose there is a second partition, which obviously must have at least two parts. The smallest option we have is $b + (b+b-2+r) = 3b-2+r$, however, since $b \geq 4$, $3b - 2 + r > 2b + 1 + r$ so this can't be a partition of $2b + 1 + r$. Thus $q^{(b)}_{b-2+r}(2b + 1 + r) = 1$ for all $b \geq 4$ and $r \geq 0$.

Now, examining $Q^{(b)}_{b-2+r}(2b + 1 + r)$ we see that the possible choices for parts are $(1 + r), b, (b+2+2r), (2b+1+r), (2b+3+3r), (3b+2+2r), \dots)$. We can obviously construct the partitions $(2b+1+r)$ and $b+b+(1+r)$ from this list. Thus we know $Q^{(b)}_{b-2+r}(2b + 1 + r) \geq 2$ for all $b \geq 4$ and $r \geq 0$.

Finally we can conclude that $\Delta^{(b,b)}_{b-2+r}(2b + 1 + r) < 0$ for all $b \geq 4$ and $r \geq 0$. □

Proposition 7.2. *We have the following:*

(1) $\Delta^{(2k,2k,-)}_{3k-3}(4k) = -1$ *for* $k \geq 2$*,*
(2) $\Delta^{(2k,2k,-)}_{3k-3}(6k) = -1$ *for* $k \geq 4$*,*
(3) $\Delta^{(2k,2k,-)}_{5k-3}(8k) = -1$ *for* $k \geq 4$*,*
(4) $\Delta^{(3k,3k,-)}_{4k-3}(9k) = -1$ *for* $k \geq 4$*,*
(5) $\Delta^{(4k,4k,-)}_{5k-3}(12k) = -1$ *for* $k \geq 4$*.*

Proof of (1). First we will prove that $q^{(2k)}_{3k-3}(4k) = 1$ for $k \geq 2$. Since $4k > 2k$ for $k \geq 2$, we will have the partition $4k$ of $4k$. Suppose we wanted to construct a partition of $4k$ with more than one part we could, using the smallest available parts, take $2k + (2k + (3k - 3))$. However, $2k + (2k + (3k - 3)) = 7k - 3$ and, since $k \geq 2$, $7k - 3 > 4k$, this cannot be a partition of $4k$. Thus $q^{(2k)}_{3k-3}(4k) = 1$ for all $k \geq 2$.

Second we will show that $Q^{(2k,-)}_{3k-3}(4k) = 2$ for all $k \geq 2$. We are looking for partitions of $4k$ which come from parts $\equiv \pm 2k \pmod{3k}$ and not the part $-2k + 3k$. Enumerated, this is the set of parts $2k, 4k, 5k, 7k, 8k, \dots$. We see clearly that the only ways to construct a partition of $4k$ into parts from this list are $4k$ and $2k + 2k$. Thus $Q^{(2k,-)}_{3k-3}(4k) = 2$.

Therefore we have shown that $\Delta^{(2k,2k,-)}_{3k-3}(4k) = -1$ for $k \geq 2$.

□

Proof of (2). First we will prove that $q^{(2k)}_{3k-3}(6k) = 1$ for $k \geq 4$. Since $6k > 2k$ for $k \geq 4$, we will have the partition $6k$ of $6k$. Suppose we wanted to construct a partition of $6k$ with more than one part we could, using the smallest available parts, take $2k + (2k + (3k - 3))$. However, $2k + (2k + (3k - 3)) = 7k - 3$ and, since $k \geq 4$, $7k - 3 > 6k$, this cannot be a partition of $6k$. Thus $q^{(2k)}_{3k-3}(6k) = 1$ for all $k \geq 4$.

Second we will show that $Q^{(2k,-)}_{3k-3}(6k) = 2$ for all $k \geq 4$. We are looking for partitions of $6k$ which come from parts $\equiv \pm 2k \pmod{3k}$ and not the part $-2k + 3k$. Enumerated, this is the set of parts $2k, 4k, 5k, 7k, 8k, \dots$. We see that the only ways to construct a partition of $6k$ into parts from this list are $4k + 2k$ and $2k + 2k + 2k$. Thus $Q^{(2k,-)}_{3k-3}(6k) = 2$.

Therefore we have shown that $\Delta^{(2k,2k,-)}_{3k-3}(6k) = -1$ for $k \geq 4$.

□

Proof of (3). First we will prove that $q^{(2k)}_{5k-3}(8k) = 1$ for $k \geq 4$. Since $8k > 2k$ for $k \geq 4$, we will have the partition $8k$ of $8k$. Suppose we wanted to construct a partition of $8k$ with more than one part, we could using the smallest available parts, take $2k + (2k + (5k - 3))$. However, $2k + (2k + (5k - 3)) = 9k - 3$ and, since $k \geq 4$, we have $9k - 3 > 8k$, this cannot be a partition of $8k$. Thus $q^{(2k)}_{5k-3}(8k) = 1$ for all $k \geq 4$.

Second we will show that $Q_{5k-3}^{(2k,-)}(8k) = 2$ for all $k \geq 4$. We are looking for partitions of $8k$ which come from parts $\equiv \pm 2k \pmod{5k}$ and not the part $-2k+5k$. Enumerated, this is the set of parts $2k, 7k, 8k, 12k, 13k, \ldots$. We see that the only ways to construct a partition of $8k$ into parts from this list are $8k$ and $2k+2k+2k+2k$. Thus $Q_{5k-3}^{(2k,-)}(8k) = 2$.

Therefore we have shown that $\Delta_{5k-3}^{(2k,2k,-)}(8k) = -1$ for $k \geq 4$. □

Proof of (4). First we will prove that $q_{4k-3}^{(3k)}(9k) = 1$ for $k \geq 4$. Since $9k > 3k$ for $k \geq 4$, we will have the partition $9k$ of $9k$. Suppose we wanted to construct a partition of $9k$ with more than one part we could, using the smallest available parts, take $3k + (3k + (4k-3))$. However, $3k + (3k + (4k-3)) = 10k - 3$ and, since $k \geq 4$, $10k - 3 > 9k$ so this cannot be a partition of $9k$. Thus $q_{4k-3}^{(3k)}(9k) = 1$ for all $k \geq 4$.

Second we will show that $Q_{4k-3}^{(3k,-)}(9k) = 2$ for all $k \geq 4$. We are looking for partitions of $9k$ which come from parts $\equiv \pm 3k \pmod{4k}$ and not the part $-3k+4k$. Enumerated, this is the set of parts $3k, 5k, 7k, 9k, 11k, \ldots$. We see that the only ways to construct a partition of $9k$ into parts from this list are $9k$ and $3k+3k+3k$. Thus $Q_{4k-3}^{(3k,-)}(9k) = 2$.

Therefore we have shown that $\Delta_{4k-3}^{(3k,3k,-)}(9k) = -1$ for $k \geq 4$. □

Proof of (5). First we will prove that $q_{5k-3}^{(4k)}(12k) = 1$ for $k \geq 4$. Since $12k > 4k$ for $k \geq 4$, we will have the partition $12k$ of $12k$. Suppose we wanted to construct a partition of $12k$ with more than one part we could, using the smallest available parts, take $4k + (4k + (5k-3))$. However, $4k + (4k + (5k-3)) = 13k - 3$ and, since $k \geq 4$, $13k - 3 > 12k$ so this cannot be a partition of $12k$. Thus $q_{5k-3}^{(4k)}(12k) = 1$ for all $k \geq 4$.

Second we will show that $Q_{5k-3}^{(4k,-)}(12k) = 2$ for all $k \geq 4$. We are looking for partitions of $12k$ which come from parts $\equiv \pm 4k \pmod{5k}$ and not the part k. Enumerated, this is the set of parts $4k, 6k, 9k, 11k, 14k, \ldots$. We see clearly that the only ways to construct a partition of $12k$ into parts from this list are $6k+6k$ and $4k+4k+4k$. Thus $Q_{5k-3}^{(4k,-)}(12k) = 2$.

Therefore we have shown that $\Delta_{5k-3}^{(4k,4k,-)}(12k) = -1$ for $k \geq 4$. □

By examining the cases in Propositions 7.1 and 7.2, we find that the offending parts which yield more partitions for $Q_d^{(a)}(n)$ than $q_d^{(a)}(n)$ are precisely the parts equal to $d+3-a$ and a. Roughly speaking, when n is less than $2a+d$, $q_d^{(a)}(n)$ is bounded at 1. However the a and $d+3-a$ parts are small enough that they can contribute to partitions of n counted by $Q_d^{(a)}(n)$, especially when n is a multiple of a. This contribution of the parts $d+3-a$ and a to the partitions of small n is problematic since a is simply too large to allow any possibilities of nontrivial partitions of n that are counted by $q_d^{(a)}(n)$. By removing the 'small' parts $d+3-a$ and a from the possibilities of parts of partitions counted by $Q_d^{(a)}(n)$, we are rebalancing the scales to give 'equal opportunity' for nontrivial partitions of n. In other words, the removal of these parts for large n is unnecessary to maintain the nonnegativity of $\Delta_d^{(a)}(n)$, which is further evidenced by the proof of Theorem 1.9. Furthermore, for arbitrary $a \geq 3$, we only need to remove these two parts and no others to computationally observe that $\Delta_d^{(a,-,-)}(n) \geq 0$.

7.1. **A generalization of Kang and Park's conjecture for $a = 3$.**

We now will use the methods of Section 4 to prove a partial result regarding Conjecture 1.4.

We note that if d is divisible by 3, but n is not divisible by 3, then $Q_d^{(3,-)}(n) = 0$ so $\Delta_d^{(3,-)}(n)$ is trivially nonnegative.

We prove a nontrivial case of Conjecture 1.4 below.

Proposition 7.3. *For $d \geq 31$ and $n > 0$,*

$$q_{3d}^{(3)}(3n) \geq Q_{3d}^{(3,-)}(3n).$$

Proof. We aim to prove the following string of inequalities

$$q_{3d}^{(3)}(3n) \geq q_d^{(1)}(n) \geq Q_{d-2}^{(1,-)}(n) = Q_{3d}^{(3,-)}(3n).$$

The first inequality follows from Lemma 2.7. The second inequality is a consequence of Proposition 3.1. Finally, the third inequality is a consequence of Lemma 2.8. □

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