EXPRESSING THE k-RANK GENERATING FUNCTION AND RANK DIFFERENCE FUNCTIONS FOR MULTIPARTITIONS AS MODULAR FORMS

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ABSTRACT. Partition theory, an area of combinatorics and number theory, is deeply linked to the theory of modular forms. In 2008, Ahlgren and Treneer, amongst others, showed that rank difference functions for single partitions can be classified as either identically zero functions, weakly holomorphic modular forms, or "mock" objects. Using the arsenal of tools provided by complex analysis, we attempt to classify k-coefficient multipartitions in a similar manner. Ultimately, we relate restricted k-rank generating functions and k-rank difference functions to weakly holomorphic modular forms, as well as to identically zero functions.

1. INTRODUCTION

In 1918, the Indian mathematician Ramanujan collaborated with the English mathematician G. H. Hardy to study many areas of mathematics, including partition theory. Their work on partitions, along with Ramanujan's theta and mock theta functions, has inspired a great deal of mathematics in the fields of partition theory and modular forms.

A **partition** π of a positive integer n is a finite nonincreasing sequence of positive integers $\pi = \{\lambda_1, \lambda_2, ..., \lambda_r\}$ with $\lambda_i \geq \lambda_{i+1}$ such that their sum is equal to n. We call λ_i the ith **part** of the partition π, and we call n the **size** of π, denoted $|\pi|$.

Example 1.1. *The following is a partition of 11:*

$$\pi = 5+3+2+1.$$

We define $p(n)$, the partition function, to be the number of partitions of n, where n is a positive integer. The value of $p(n)$ is defined to be 1 for $n=0$ as we consider the empty set to be the sole partition of 0. In addition, $p(n)$ is defined to be 0 for $n<0$. The generating function for $p(n)$ has the following form, due to Euler

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}. \tag{1}$$

One of the most celebrated results in partition theory are Ramanujan's surprisingly simple yet remarkable congruences for $p(n)$:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \end{aligned}$$

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$$p(11n+6) \equiv 0 \pmod{11}.$$

Much work has been done on single partitions. In our research, we focused on generalizing some of this work to multipartitions.

Definition 1.2. *A **k-component multipartition**, or **k-partition** of a nonnegative integer n is a k-tuple*

$$\pi = (\pi_1, \pi_2, ..., \pi_k)$$

*where each π_i is a partition, called a **sub-partition**, and the sum of the sizes of the sub-partitions is equal to n, i.e.*

$$\sum_{i=1}^{k} |\pi_i| = n.$$

We have a k-partition function $p_k(n)$ which represents the number of k-partitions of a nonnegative integer n. The generating function for $p_k(n)$ is the k-th power of the generating function for $p(n)$ [And08] [Atk68], i.e.

$$\sum_{n\geq 0} p_k(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^k}.$$

Example 1.3. *The following are all the* 3*-partitions of 2:*

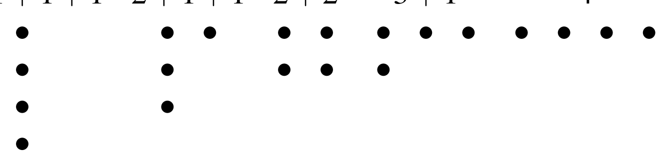
$$\begin{array}{ccc} (2,\emptyset,\emptyset) & (1+1,\emptyset,\emptyset) & (\emptyset,1,1) \\ (\emptyset,2,\emptyset) & (\emptyset,1+1,\emptyset) & (1,\emptyset,1) \\ (\emptyset,\emptyset,2) & (\emptyset,\emptyset,1+1) & (1,1,\emptyset). \end{array}$$

Thus, we can see that $p_3(2) = 9$.

We can depict partitions graphically through the use of Ferrers diagrams. A **Ferrers diagram** for $\pi = (\lambda_1, ..., \lambda_r)$ is a left-justified array of dots where the ith row has λ_i dots, in descending order.

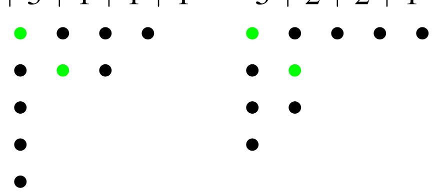
Example 1.4. *The partitions of 4 have the following Ferrers diagrams:*

$1+1+1+1 \quad 2+1+1 \quad 2+2 \quad 3+1 \quad 4$



We can use this graphical interpretation to help define the **conjugate** of a partition π, denoted π'. The conjugate π' of π has the Ferrers diagram obtained by exchanging the rows and columns of the Ferrers diagram for π. For example, the following two Ferrers diagrams represent partitions that are conjugates of each other:

$4+3+1+1+1 \quad 5+2+2+1$



We define a conjugate for a k-partition $\pi = (\pi_1, \pi_2, ..., \pi_3)$ as $\pi' = (\pi'_1, \pi'_2, ..., \pi'_3)$.

1.1. **Dyson's Rank.** There are several statistics used to categorize partitions and k-partitions. The primary statistic we look at is rank. For a partition $\pi = (\lambda_1, ... \lambda_r)$ of n, Dyson [Dys44] defined the **rank** of π as the largest part of π minus the number of parts of π, or

$$\text{rank}(\pi) = \lambda_1 - r.$$

In [Gar10], Garvan extends the definition of Dyson's rank to k-partitions. We call this extension **Dyson's multirank**, or **k-rank**. For a k-partition $\pi = (\pi_1, ..., \pi_k)$,

$$\text{rank}_k(\pi) = \text{rank}_k(\pi_1, ..., \pi_k) = \sum_{a=1}^{k} a \cdot \text{rank}(\pi_a).$$

It is useful to notice that the rank of a partition's conjugate is its negative, i.e.

$$\text{rank}(\pi) = -\text{rank}(\pi').$$

We use this idea in the following lemma.

Lemma 1.5. *For any k-partition π, we have $rank_k(\pi) = -rank_k(\pi')$.*

Proof. First, note that for a single partition π, we know $\text{rank}(\pi) = -\text{rank}(\pi')$. Let $\pi = (\pi_1, \pi_2, ..., \pi_k)$ be a k-partition. Then we have

$$\begin{aligned}
\text{rank}_k(\pi) &= \text{rank}(\pi_1) + 2\,\text{rank}(\pi_2) + \cdots + k\,\text{rank}(\pi_k) \\
&= -\text{rank}(\pi_1') - 2\,\text{rank}(\pi_2') - \cdots - k\,\text{rank}(\pi_k') \\
&= -(\text{rank}(\pi_1') + 2\,\text{rank}(\pi_2') + \cdots + k\,\text{rank}(\pi_k')) \\
&= -\text{rank}_k(\pi').
\end{aligned}$$

□

We let $N(m,n)$ count the number of partitions of n with rank m. From the previous relation between conjugates (or from [AG03][ASD54]) we observe that

$$N(m,n) = N(-m,n).$$

We extend this to the k case by defining $N_k(m,n)$ to be the number of k-partitions π of n with $\text{rank}_k(\pi)$ equal to m, and observe that similarly

$$N_k(m,n) = N_k(-m,n).$$

Example 1.6. *Consider the partitions given in Example 1.3. We can see that*

$$\begin{array}{lll}
\text{rank}_3(2,\emptyset,\emptyset) = 1 & \text{rank}_3(1+1,\emptyset,\emptyset) = -1 & \text{rank}_3(\emptyset,1,1) = 0 \\
\text{rank}_3(\emptyset,2,\emptyset) = 2 & \text{rank}_3(\emptyset,1+1,\emptyset) = -2 & \text{rank}_3(1,\emptyset,1) = 0 \\
\text{rank}_3(\emptyset,\emptyset,2) = 3 & \text{rank}_3(\emptyset,\emptyset,1+1) = -3 & \text{rank}_3(1,1,\emptyset) = 0.
\end{array}$$

So we have,

$$N_3(0,2) = 3.$$

The generating function for $N(m,n)$ is given in [Gar10] by

$$\mathcal{R}(z,q)=\sum_{\pi} z^{\mathrm{rank}(\pi)}q^{|\pi|}=\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N(m,n)z^m q^n \tag{2}$$

$$=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{\prod_{k=1}^{n}((1-zq^k)(1-z^{-1}q^k))}.$$

Garvan showed that you can extend this to generate the values of $N_k(m,n)$ by, $\mathcal{R}_2(z,q)=\mathcal{R}(z,q)\mathcal{R}(z^2,q)$. We generalize this generating function to $N_k(m,n)$ in the following way:

$$\mathcal{R}_k(z,q)=\sum_{\pi=(\pi_1,\dots,\pi_k)} z^{\mathrm{rank}_k(\pi)}q^{|\pi|}=\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N_k(m,n)z^m q^n.$$

Theorem 1.7. *The k-rank generating function can be written in terms of $\mathcal{R}(z,q)$ as*

$$\mathcal{R}_k(z,q)=\mathcal{R}(z,q)\mathcal{R}(z^2,q)\dots\mathcal{R}(z^k,q).$$

We let $N(s,m,n)$ be the number of partitions of n with rank congruent to s modulo m. We can also extend this definition to k-partitions by defining $N_k(s,m,n)$ to be the number of k-partitions of n with k-rank congruent to s modulo m.

We are particularly interested in **Rank difference functions**, which are functions of the form

$$\sum_{n\geq 0}(N_k(s,\ell,\ell n+b)-N_k(t,\ell,\ell n+b))q^n,$$

where ℓ is a prime, and b, s and t are integers reduced modulo ℓ. One reason we are so interested in them is because Atkin and Swinnerton-Dyer proved the first two Ramanujan congruences using rank difference functions [ASD54]. The main reason we are concerned with them though is because we know from the work of Ahlgren, Treneer [AT08] and others that in the $k=1$ case, we can categorize the rank difference functions as zero, modular forms, or "mock" objects. One of our goals is to be able to recognize when a rank difference function falls into one of these categories in the $k>1$ cases. This is further explored in Sections 4 and 5. The classifications for the rank difference functions when $k>1$ are of particular interest, as not much is currently known about them.

2. Introduction to Modular Forms

Modular forms are deeply entrenched in the theory of partitions. In fact, Ken Ono [Ono09] cites the theory of partitions as "a delightful 'testing ground' for some of the deepest developments in the theory of modular forms." As a demonstration of the intrinsic relationship between partition theory and modular forms, consider Euler's partition generating function in (1), which is closely related to the Dedekind eta-function, a modular form. The historical affinity between partition theory and modular forms is our primary motivation in relating the k-rank generating function and k-rank difference functions to weakly holomorphic modular forms (a relationship that has been observed for usual partitions), which we do in Sections 3 and 4.

2.1. *q-series.* A **q-series** is defined to be any formal power series of the form $\sum_{n\in\mathbb{Z}} a(n)q^n$. We can modify q-series by restricting across specific arithmetic progressions. For example,

Example 2.1. *Say we have a q-series from which we would like to remove all n that are congruent to p modulo m. We could write*

$$\left(\sum_{n=-\infty}^{\infty} a(n)q^n\right)\Bigg|_{n\not\equiv p \pmod m} = \sum_{n\not\equiv p \pmod m} a(n)q^n.$$

Restriction is a tool that will be very useful in our work, particularly with respect to relating our generating functions to modular forms. But first, we will discuss modular forms.

2.2. **Modular Group.** Before going into the full definition of modular forms, it is important to define the groups that dictates the symmetries modular forms exhibit.

Definition 2.2. *The **modular group**,* $\mathrm{SL}_2(\mathbb{Z})$ *is the group of all matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, *where* $a,b,c,d\in\mathbb{Z}$ *and* $ad-bc=1$. *Sometimes* $\mathrm{SL}_2(\mathbb{Z})$ *is denoted as* $\Gamma(1)$.

We are also interested in specific subgroups of the modular group called congruence subgroups. Some of the key subgroups include

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\mathrm{N}}\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\mathrm{N}}\right\},$$

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathrm{N}}\right\}.$$

Observe that these subgroups are ordered by containment

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

2.3. **Modular Forms.** The modular group acts on the upper half plane

$$\mathcal{H} = \{x+iy \in \mathbb{C} \mid y>0\}$$

by linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Definition 2.3. *For a congruence subgroup* Γ, *the **cusps** of* Γ *are defined to be the equivalence classes of* $\mathbb{Q}\cup\{i\infty\}$ *under the natural extension of the action of* Γ *on* $\mathcal{H}$.

Definition 2.4. *A function* $f:\mathcal{H}\to\mathbb{C}$ *is said to be a **modular form of weight** k **for a congruence subgroup** Γ if*

(a) *f is holomorphic on $\mathcal{H}$,*
(b) *f is "modular" that is to say for* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma$ *and* $z\in\mathcal{H}$, *then*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

(c) *The function f is holomorphic at the cusps of Γ. In particular f has a Fourier series of the form*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

where $q = e^{2\pi iz}$. If f is merely meromorphic at the cusps, we call is a weakly holomorphic modular form.

For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ we can classify modular forms in the following two ways:

- $M_k(\Gamma) =$ the set of holomorphic modular forms for Γ of weight k.
- $S_k(\Gamma) =$ The set of elements of $M_k(\Gamma)$ that vanish at the cusps.
- $M_k^!(\Gamma)$ = the set of weakly holomorphic forms for Γ of weight k.

One great feature is that the sets M_k, S_k, and $M_K^!$ are all complex vector spaces [Ono04]; and M_K and S_k are finite dimensional. Elements of $M_k(\Gamma)$ are called holomorphic modular forms, and elements of $S_k(\Gamma)$ are called cusp forms.

An important example of a modular form is the Dedekind eta-function

$$\eta(z) = q^{1/24} \prod_{n>0}(1-q^n). \tag{3}$$

However, not all q-series of interest are modular forms.

2.4. **Maass Forms.** Before we can define a harmonic weak Maass form we need a couple definitions to build up to it.

We define the weight k **hyperbolic Laplacian** to be

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

And for odd integers d, we define ε_d to be

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases}.$$

Definition 2.5. [Ono09] *If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then a **weight k harmonic weak Maass form** on a subgroup $\Gamma \subset \{\Gamma_1(N), \Gamma_0(N)\}$ is any smooth function $f : \mathcal{H} \to \mathbb{C}$ satisfying:*

(a) *For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathcal{H}$, we have*

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k f(z) & \text{if } k \in \mathbb{Z} \\ (\frac{c}{d})^{2k}\varepsilon_d^{-2k}(cz+d)^k f(z) & \text{if } \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \end{cases}$$

Where $(\frac{c}{d})$ denotes the extended Legendre symbol, and $\sqrt{z}$ is the principal branch of the holomorphic square root.

(b) *We have that $\Delta_k f = 0$.*

(c) *There is a polynomial $P_M = \sum_{n \le 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that*

$$f(z) - P_f(z) = O(e^{-\varepsilon y})$$

as $y \to +\infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Harmonic weak Maass forms can be decomposed into a holomorphic part and a nonholomprhic part. A **mock modular form** is the holomorphic part of that decomposition such that [Ono09]

Harmonic Weak Maass Form = Mock Modular Form + Non-Holomorphic part.

The set of harmonic weak Maass forms with non-holomorphic part equal to zero is the set of weakly holomorphic modular forms. This can be seen because when the non-holomorphic part is zero we have a harmonic weak Maass form, which is modular, equal to a mock modular form, which is holomorphic. This satisfies all of the reqirements for being a weakly holomorphic modular form.

The following are important slightly altered [Kan09] identities from Zwegers thesis [Zwe02]:

$$\Theta(z;\tau) = \sum_{n=-\infty}^{\infty} (-1)^{n+1} e^{\pi i(2n+1)z} q^{n(n+1)/2+1/8}, \tag{4}$$

and also

$$\mu(u,v;\tau) = \frac{e^{\pi i u}}{\Theta(v)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{2\pi i v} q^{(n^2+n)/2}}{1-e^{2\pi i u} q^n}. \tag{5}$$

The function $\mu(u,v;\tau)$ is the mock modular part of a Maass form call it $\hat{\mu}(u,v;\tau)$. There exist a function $R(u-v;\tau)$ such that the relation between the three is [Kan09][Zwe02]

$$\hat{\mu}(u,t;\tau) = \mu(u,v;\tau) + \frac{1}{2}R(u-v;\tau).$$

The **shadow** of a Maass form is a modular form that is associated with the non-holomorphic part of the Maass form, in this case $R(u-v;\tau)$. Note that this implies that the shadow is independent of the holomorphic part of a Maass form. In our example we can rewrite $R(u-v;\tau)$ to get the following relation:

$$\hat{\mu} = \mu + \int_{\bar{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz,$$

where $g(\tau)$ is the shadow. We have restated Proposition 1 from [GKST12] in the following theorem:

Theorem 2.6. *Using the definitions above, whenever $u-v = a\tau - b$, then*

$$-\sqrt{2} e^{2\pi i a(b+\frac{1}{2})} q^{-\frac{a^2}{2}} \mu(u,v;\tau)$$

is a weight $\frac{1}{2}$ mock theta function with shadow $g_{a+\frac{1}{2},b+\frac{1}{2}}(\tau)$.

By restricting across progressions that cause the shadow to be zero, we cause the non-holomorphic part to be zero as well.

This restriction can be done by a process called twisting.

Definition 2.7. *The **twist** of a q-series $f(z) = \sum_{n\geq 0} a(n)q^n$ by a Dirichlet Character, χ, is defined by*

$$f(z) \otimes \chi = \sum_{n\geq 0} \chi(n)a(n)q^n.$$

From [Ono04] [AT08] we have that twisting is an operation which maintains modularity, which is why we are able to twist modular forms and Maass forms. We note that twisting is also distributive over addition, i.e.

$$(A+B)\otimes\chi = (A\otimes\chi)+(B\otimes\chi).$$

Due to the properties of Dirichlet characters, one can show that for $f(z)=\sum_{n\geq 0} a(n)q^n$, a fixed positive integer N, and $d\in\mathbb{Z}/N\mathbb{Z}$,

$$\sum_{n\equiv d \pmod N} a(n)q^n = \frac{1}{\varphi(N)}\sum_{\chi \pmod N} \overline{\chi}(d) f(z)\otimes\chi. \tag{6}$$

We use this tool in the following lemma.

Lemma 2.8. *Suppose we have a Maass form of the form $\hat{\mu}=\mu+\int_{\overline{\tau}}^{i\infty}\frac{g(\tau)}{f(z)}dz$, where μ is a mock modular form with shadow $g(\tau)$. Furthermore, suppose that we can express $g(\tau)$ as*

$$g(\tau)=\sum_{n\equiv d_1,\ldots,d_k \pmod N} a(n)q^n.$$

Then, when we restrict both $\hat{\mu}$ and μ to not include $n\equiv d_1,\ldots,d_k \pmod N$, we get that

$$\hat{\mu}|_{n\not\equiv d_1,\ldots,d_k \pmod N} = \mu|_{n\not\equiv d_1,\ldots,d_k \pmod N}.$$

Proof. Suppose that $\hat{\mu}=\mu+\int_{\overline{\tau}}^{i\infty}\frac{g(\tau)}{f(z)}dz$ is a Maass form; and suppose that we can write

$$g(\tau)=\sum_{n\equiv d_1,\ldots,d_k \pmod N} a(n)q^n.$$

We know from (6) that for $h(z)=\sum_{n\geq 0} a(n)q^n$

$$\sum_{n\equiv d \pmod N} a(n)q^n = \frac{1}{\varphi(N)}\sum_{\chi \pmod N} \overline{\chi}(d) h(z)\otimes\chi.$$

Then,

$$
\begin{aligned}
\hat{\mu}|_{n\not\equiv d_1,\dots,d_k \pmod N} =& \hat{\mu} - \sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i) \hat{\mu} \otimes \chi \right) \\
=& \mu + \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz - \sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i) \left(\mu + \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \right) \otimes \chi \right) \\
=& \mu + \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \\
& - \sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i)(\mu \otimes \chi) + \sum_{\chi \pmod N} \overline{\chi}(d_i) \left(\int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \otimes \chi \right) \right) \\
=& \mu - \sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i) \mu \otimes \chi \right) + \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \\
& - \sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i) \left(\int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \right) \otimes \chi \right).
\end{aligned}
$$

We can clearly see that

$$
\sum_{i=1}^{k} \frac{1}{\varphi(N)} \left(\sum_{\chi \pmod N} \overline{\chi}(d_i) \left(\int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \right) \otimes \chi \right) = \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz
$$

by (6), and because all the q exponents of $\int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)}$ are equivalent to d_1, ..., d_k modulo N. Therefore,

$$
\begin{aligned}
\hat{\mu}|_{n\not\equiv d_1,\dots,d_k \pmod N} =& \mu - \sum_{i=1}^{k} \left(\frac{1}{\varphi(N)} \sum_{\chi \pmod N} \overline{\chi}(d_i) \mu \otimes \chi \right) + \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz - \int_{\overline{\tau}}^{i\infty} \frac{g(\tau)}{f(z)} dz \\
=& \mu - \sum_{i=1}^{k} \left(\frac{1}{\varphi(N)} \sum_{\chi \pmod N} \overline{\chi}(d_i) \mu \otimes \chi \right) \\
=& \mu|_{n\not\equiv d_1,\dots,d_k \pmod N}
\end{aligned}
$$

Therefore, we get that

$$
\hat{\mu}|_{n\not\equiv d_1,\dots,d_k \pmod N} = \mu|_{n\not\equiv d_1,\dots,d_k \pmod N}
$$

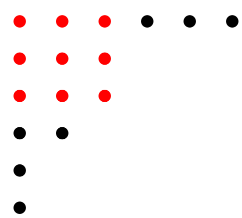
for any Maass form $\hat{\mu}$. □

3. Rank Generating Function as a Modular Form

In this section we will show that $\mathcal{R}_k(z,q)$ is a weakly holomorphic modular form under certain progressions. We begin by proving Theorem 1.7. A combinatorial proof of equation 2 is presented in Andrews' "Partitions:Yesterday and Today" [And79]. His proof makes use of an object called the Durfee square, which we will utilize in our proof of Theorem 1.7.

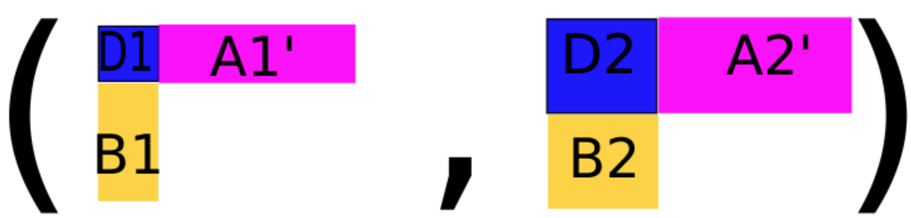
For a partition π, the Durfee square of π is a dxd square, where d is the largest integer such that π contains at least d parts of size greater than or equal to d.

Example 3.1. *The Durfee square of the partition* $\pi = 6+3+3+2+1+1$ *has dimension* $d = 3$. *Below is the Ferrer's diagram for* π*; the Durfee square is highlighted in red.*



Note that, in the Ferrer's diagram of π, the complement of the Durfee square defines two partitions, one to right of the Durfee square, and one below the Durfee square. Also note that the rank of π is equal to the number of parts in the conjugate of the partition to the right of the Durfee square minus the number of parts in the partition below the Durfee square.

Example 3.2. *The following graphic represents a Ferrer's diagram of a* 2*-partition* π *of* n*, with each Ferrer's diagram divided into sub-diagrams:*



We have that D_1*, with side length* d_1*, and* D_2*, with side length* d_2*, are the Durfee squares of* π_1 *and* π_2*, respectively, and* A_1'*,* A_2'*,* B_1*, and* B_2 *are the unique partitions on either side of the Durfee square.*
Note that

$$n = d_1^2 + d_2^2 + |A_1| + |A_2| + |B_1| + |B_2|.$$

We are now able to prove Theorem 1.7.

Proof. First, we recall the $k = 1$ case, as presented in [And79]. For each partition π of n, we can consider n as $d^2 + |A| + |B|$, where d is the side length of the unique Durfee square of π, and A, B are the two "leftover" partitions as denoted in Example 3.2. Note that A and B belong to the set of partitions whose parts are all at most d. So the generating function for all partitions in terms of Durfee square side length d is

$$\sum_{d=0}^{\infty} q^{d^2}((1+q^1+q^{1+1}+\ldots)\ldots(1+q^d+q^{d+d}+\ldots))((1+q^1+q^{1+1}+\ldots)\ldots(1+q^d+q^{d+d}+\ldots)) = \sum_{d=0}^{\infty} \frac{q^{d^2}}{(q;q)_d(q;q)_d}.,$$

where

$$(z;q)_n = \begin{cases} \prod_{j=0}^{n-1}(1-zq^j) & n>0 \\ 1 & n=0 \end{cases}$$

Now, in order to keep track of rank, we introduce a second variable, z. Note that $\frac{1}{(zq;q)_d}$ generates partitions with no part greater than d, with z keeping track of rank. So, since $\text{rank}(\pi) = \#(A) - \#(B)$, where $\#(A)$ denotes the number of parts of A, we have

$$\mathcal{R}(z,q) = \sum_{d=0}^{\infty} \frac{q^{d^2}}{(q;zq)_d(q;z^{-1}q)_d}.$$

Now, we look at the general k case. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_k)$ be a k-partition of n, and define $D_1, D_2, ..., D_k, A_1, ..., A_k, B_1, ..., B_k$ as in Example 3.2.

Note that

$$n = d_1^2 + d_2^2 + \cdots + d_k^2 + |A_1| + |A_2| + \cdots + |A_k| + |B_1| + |B_2| + \cdots + |B_k|.$$

For a k-partition π, we define the size of the Durfee square of π to equal the sum of the sizes of the Durfee squares of the k sub-partitions of π.
Thus, the size of the Durfee square of our k-partition π is $d_1^2 + d_2^2 + \cdots + d_k^2$, so the generating function for $p_k(n)$ can be written as follows:

$$\sum_{n\geq 0} p_k(n)q^n = \left(\sum_{d_1\geq 1} \frac{q^{d_1^2}}{(q;q)_{d_1}(q;q)_{d_1}}\right)\left(\sum_{d_2\geq 1} \frac{q^{d_2^2}}{(q;q)_{d_2}(q;q)_{d_2}}\right)\cdots\left(\sum_{d_k\geq 1} \frac{q^{d_k^2}}{(q;q)_{d_k}(q;q)_{d_k}}\right),$$

where $p_k(n)$ is the number of k-partitions of m.
We also want to keep track of rank; in order to do this, we introduce a second variable, z.
Note that:

$$rank_k(\pi) = \#(A_1) + 2\#(A_2) + \cdots + k\#(A_k) - \#(B_1) - 2\#(B_2) - \cdots - k\#(B_k),$$

where $\#(A)$ denotes the number of parts in the partition A. So we have

(7)
$$\sum_{n\geq 0}\sum_{m\geq 0} N_k(m,n)q^n z^m = \left(\sum_{D_1\geq 1} \frac{q^{D_1^2}}{(zq;q)_{D_1}(z^{-1}q;q)_{D_1}}\right)\left(\sum_{D_2\geq 1} \frac{q^{D_2^2}}{(z^2q;q)_{D_2}(z^{-2}q;q)_{D_2}}\right)\cdots\left(\sum_{D_k\geq 1} \frac{q^{D_k^2}}{(z^kq;q)_{D_k}(z^{-k}q;q)_{D_k}}\right).$$

The first sum in (7) represents the sub-partition π_1, while the i^{th} sum in (7) represents the sub-partition π_i. For π_1, we can treat the rank the same way that we did in the k=1 case. In calculating $rank_k(\pi)$, however, the rank of π_i counts i times as much. So we account for the unequal weighting by taking the variable we use for rank to the i^{th} power in the i^{th} sum.

□

Since we have a nice expression for $\mathcal{R}_k(z,q)$ in terms of $\mathcal{R}(z,q)$ from Theorem 1.7, we can expand upon information we know about $\mathcal{R}(z,q)$. D. Zagier [Zag09] derived the following formula [Kan09] using an identity of Gordon and McIntosh [GM03] that is comprised of functions detailed in Zwegers [Zwe02]:

$$\frac{q^{-1/24}\mathcal{R}(e^{2\pi i\alpha};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}=\frac{\eta(3\tau)^3/\eta(\tau)}{\Theta(3\alpha;3\tau)}-q^{-1/6}e^{-2\pi i\alpha}\mu(3\alpha,-\tau;3\tau)+q^{-1/6}e^{2\pi i\alpha}\mu(3\alpha,\tau;3\tau) \tag{8}$$

where $e^{2\pi i\alpha}\neq 1$ is a t-th root of unity, $q=e^{2\pi i\tau}$, and the remaining functions refer to Equations (3), (5), and (4).

From [Zwe02] and [Zag09], we have that for rational values a and b, $q^{\lambda}\mu(a\tau,b;\tau)$ is a mock modular form of weight 1/2 for a certain rational number λ with shadow a weight 3/2 unary theta series. With this knowledge the formula writes the rank generating function as three terms; the first term on the right is a weakly holomorphic modular form of weight $1/2$, and the other two terms are mock modular forms of weight $1/2$. Zagier [Zag09] (see Theorem 7.1) proved that, for any root of unity $\zeta=e^{2\pi i\alpha}\neq 1$, $q^{-1/24}f(\zeta,q)$ is a mock modular form of weight $1/2$ with shadow proportional to

$$(\zeta^{-1/2}-\zeta^{1/2})\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)n\sin(\pi n\alpha)q^{n^2/24}.$$

We can expand Zagier's formula to work for $\mathcal{R}_k(z,q)$ to find progressions for which we can claim modularity. We have by equation (8) that

$$\frac{q^{-k/24}\mathcal{R}_k(e^{2\pi i\alpha};q)}{\prod_{n=1}^{k}(e^{-n\pi i\alpha}-e^{n\pi i\alpha})}=\prod_{n=1}^{k}\left(\frac{\eta(3\tau)^3}{\eta(\tau)\Theta(3n\alpha;3\tau)}-q^{-1/6}e^{-2n\pi i\alpha}\mu(3n\alpha,-\tau;3\tau)+q^{-1/6}e^{2n\pi i\alpha}\mu(3n\alpha,\tau;3\tau)\right)$$

where α, q, and τ are defined as in equation (8).

Since μ is a mock theta function, there exists a correcting term R such that [Zwe02]

$$\hat{\mu}(u,v,\tau)=\mu(u,v,\tau)+\tfrac{1}{2}R(u-v;\tau), \tag{9}$$

where when you plug in values for u and v, $\hat{\mu}(u,v;\tau)$ has nice modularity properties and is a Maass form. We know that $\mu(u,v;\tau)$ has a shadow that is related to the value of $R(u,\tau)$; the shadow for $\mu(u,v;\tau)$ is [Zag09] [Kan09][GKST12]

$$g_{a,b}(\tau):=\sum_{v\in a+\mathbb{Z}}ve^{\pi iv^2\tau+2\pi ivb}=\sum_{v\in\mathbb{Z}}(v+a)e^{\pi i(v+a)^2\tau+2\pi i(v+a)b} \tag{10}$$

and its relation to $R(u,\tau)$ is

$$\int_{-\bar{\tau}}^{i\infty}\frac{g_{a+1/2,b+1/2}(z)}{\sqrt{-i(z+\tau)}}dz=-e^{-\pi ia^2+2\pi ia(b+1/2)}R(a\tau-b,\tau).$$

Theorem 3.3. *For any root of unity, $\zeta = e^{2\pi i\alpha} \neq 1$, and let $\alpha' = \gcd(\frac{1}{\alpha}, 3)$, then we have that*

$$\prod_{m=1}^{k}\left[\left(\frac{q^{-1/24}\mathcal{R}(\zeta^m;q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\bigg|V(24)\right)\right|_{n\not\equiv 1 \pmod{24}}$$

is a weakly holomorphic modular form of weight $\frac{k}{2}$ on $\Gamma_1\left(\frac{13824}{\alpha^5(\alpha')^2}\right)$, where in the restriction n represents the powers of q when we expand $\mathcal{R}$ to a q-series.

Proof. We give a proof by induction on k. When $k=1$, we start with equation (8) and apply $V(24)$ to get

$$\text{(11)}\qquad \frac{q^{-1/24}\mathcal{R}(\zeta;q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\bigg|V(24) = \frac{\eta(3\tau)^3/\eta(\tau)}{\Theta(3\alpha;3\tau)}\bigg|V(24) - q^{-1/6}e^{-2\pi i\alpha}\mu(3\alpha,-\tau;3\tau)\Big|V(24)$$
$$+q^{-1/6}e^{2\pi i\alpha}\mu(3\alpha,\tau;3\tau)\Big|V(24).$$

Since the V operator maintains modularity, the first term on the right hand side is still a modular form. We want to focus on the last two terms on the right hand side. We specifically want to look at the following expressions

$$-q^{-1/6}e^{-2\pi i\alpha}\hat{\mu}(3\alpha,-\tau;3\tau)|V(24), \quad \text{and} \quad q^{-1/6}e^{2\pi i\alpha}\hat{\mu}(3\alpha,\tau;3\tau)|V(24),$$

where $\hat{\mu}$ is as defined above in equation (9).

These $\hat{\mu}$ are Maass forms and we would like to find progressions for which their shadows are zero. Generally speaking, the shadows can be calculated from equation (10) and Theorem 2.6. Finding the shadow for either one can be achieved by looking at the shadow for $\hat{\mu}(3\alpha;k\tau;3\tau)$, where $k\in\{\pm 1\}$.

By Theorem 2.6, this means that $q^{-1/6}e^{2k\pi i3\alpha}\mu(3\alpha;k\tau;3\tau)$ is a weight $\frac{1}{2}$ mock theta function with shadow

$$\frac{kq^{-1/6}e^{2k\pi i3\alpha}}{-i\sqrt{2}e^{2k\pi i\frac{1}{3}(\frac{1}{2}-3\alpha)}q^{-\frac{3}{2}(\frac{k}{3})^2}}\cdot g_{\frac{k}{3}+\frac{1}{2},\frac{1}{2}-3\alpha}(3\tau) = \frac{ke^{2k\pi i3\alpha}}{-i\sqrt{2}e^{2k\pi i\frac{1}{3}(\frac{1}{2}-3\alpha)}}\sum_{n\in\mathbb{Z}}\left(n\pm\frac{1}{6}\right)e^{2\pi i(\frac{1}{2}-3\alpha)(n\pm\frac{1}{6})}q^{\frac{(6n\pm1)^2}{24}}.$$

We note that this equation holds, since $g_{a,b}(\tau) = g_{a+1,b}(\tau)$ from Proposition 1.15 in [Zwe02]. By observation we can see that the exponents of q of the shadow do not depend on 3α, and all the exponents are of the form $\frac{m}{24}$ for some integer m. By applying the $V(24)$ operator, we get that all of these exponents become $(6n\pm1)^2$. When we look at this modulo 24, we get that

$$\begin{aligned}(6n\pm1)^2 &= 36n^2 \pm 12n + 1\\ &\equiv 12n^2 \pm 12n + 1 \pmod{24}\\ &\equiv 12n(n\pm 1)+1 \pmod{24}.\end{aligned}$$

Note that either n or $n\pm1$ must be even while the other is is odd. So, we define $2a$ to be the even term, and b to be the odd term. Thus

$$36n^2 \pm 12n + 1 \equiv 24ab + 1 \pmod{24} \equiv 1 \pmod{24}.$$

Therefore, for both $-q^{-1/6}e^{-2\pi i\alpha}\hat{\mu}(3\alpha,-\tau;3\tau)|V(24)$, and $q^{-1/6}e^{2\pi i\alpha}\hat{\mu}(3\alpha,\tau;3\tau)|V(24)$ the shadows have exponents that are congruent to 1 modulo 24. Therefore, by restricting across this progression, Lemma 2.8 implies that

$$-q^{-1/6}e^{-2\pi i\alpha}\hat{\mu}(3\alpha,-\tau;3\tau)|V(24)|_{n\not\equiv 1 \pmod{24}} = -q^{-1/6}e^{-2\pi i\alpha}\mu(3\alpha,-\tau;3\tau)|V(24)|_{n\not\equiv 1 \pmod{24}}$$

and

$$q^{-1/6}e^{2\pi i\alpha}\hat{\mu}(3\alpha,\tau;3\tau)|V(24)|_{n\not\equiv 1 \pmod{24}} = q^{-1/6}e^{2\pi i\alpha}\mu(3\alpha,\tau;3\tau)|V(24)|_{n\not\equiv 1 \pmod{24}}.$$

This means that when we restrict both sides of equation (11) to these progressions, we get that

$$\left.\frac{q^{-1/24}\mathcal{R}(\zeta;q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\middle|V(24)\right|_{n\not\equiv 1 \pmod{24}}$$

is a weakly holomorphic modular form of weight $\frac{1}{2}$ for $\Gamma_1\left(\frac{13824}{\alpha^5(\alpha')^2}\right)$ as desired. This concludes the base case.

We now suppose that our theorem holds up through $k=n$. We want to show that it will hold for $k=n+1$. Using a similar argument to the base case, it follows that

$$\left.\frac{q^{-1/24}\mathcal{R}(\zeta^{n+1};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\middle|V(24)\right|_{n\not\equiv 1 \pmod{24}}$$

is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1\left(\frac{13824}{\alpha^5(\alpha')^2}\right)$ as desired. We also know that when we multiply two modular forms for the same group together, their product is a modular form with weight that is equal to the sum of the weights of the two modular forms. This means that

$$\prod_{m=1}^{n+1}\left[\left.\left(\frac{q^{-1/24}\mathcal{R}(\zeta^{m};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\middle|V(24)\right)\right|_{n\not\equiv 1 \pmod{24}}\right.$$

is a weakly holomorphic modular form of weight $\frac{n+1}{2}$ for $\Gamma_1\left(\frac{13824}{\alpha^5(\alpha')^2}\right)$.

Therefore,

$$\prod_{m=1}^{k}\left[\left.\left(\frac{q^{-1/24}\mathcal{R}(\zeta^{m};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}}\middle|V(24)\right)\right|_{n\not\equiv 1 \pmod{24}}\right.$$

where n represents the powers of q when we expand $\mathcal{R}$ to a q-series, is a weakly holomorphic modular form of weight $\frac{k}{2}$ on $\Gamma_1\left(\frac{13824}{\alpha^5(\alpha')^2}\right)$. □

4. Rank Difference Function as a Modular Form

As one of our primary goals was to be able to identify rank difference functions that are modular forms, we connect $\mathcal{R}_k(z,q)$ to rank difference functions. The following theorem generalizes Equation (3.7) from [BOR08].

Theorem 4.1. *Let $r\in\mathbb{Z}$ and t be a positive integer. Then, for $\zeta_t\neq 1$ a t-th root of unity, we have*

$$\sum_{n\geq 0}N_k(r,t,n)q^n = \frac{1}{t}\sum_{n\geq 0}p_k(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}\mathcal{R}_k(\zeta_t^j,q)\right).$$

Proof. Suppose that $r,t \in \mathbb{Z}$ where $t > 0$, and $\zeta_t \neq 1$ is a t-th root of unity. To show that both sides of the above equation are equal, we show that the coefficients of q^n are equal for all n. The left hand side clearly has its q^n coefficient equal to $N_k(r,t,n)$.

Looking at the right hand side, we see that by Theorem 1.7

$$\mathcal{R}_k(\zeta_t^j, q) = \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N_k(m,n)\zeta_t^{mj}q^n.$$

Also, note that

$$\frac{1}{t}\zeta_t^0\mathcal{R}_k(1,q) = \frac{1}{t}\sum_{n\geq 0} p_k(n)q^n.$$

Thus, we can write the right hand side as

$$\begin{aligned}\frac{1}{t}\sum_{j=0}^{t-1}\left(\zeta_t^{-rj}\mathcal{R}_k(\zeta_t^j,q)\right) &= \frac{1}{t}\sum_{j=0}^{t-1}\left(\zeta_t^{-rj}\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N_k(m,n)\zeta_t^{mj}q^n\right)\\ &= \sum_{n\geq 0}\sum_{m\in\mathbb{Z}}\sum_{j=0}^{t-1}\frac{1}{t}\zeta_t^{-rj}N_k(m,n)\zeta_t^{mj}q^n.\end{aligned}$$

So, the q^n coefficient of the right hand side is

$$\sum_{m\in\mathbb{Z}}\sum_{j=0}^{t-1}\frac{1}{t}\zeta_t^{-rj}N_k(m,n)\zeta_t^{mj} = \sum_{m\in\mathbb{Z}}\frac{1}{t}N_k(m,n)\sum_{j=0}^{t-1}\zeta_t^{(m-r)j}.$$

We note that ζ_t^{m-r} is still a t-th root of unity. Furthermore, for any ε_t a t-th root of unity,

$$\sum_{j=0}^{t-1}\varepsilon_t^j = \begin{cases} 0 & \text{if } \varepsilon_t \neq 1 \\ t & \text{if } \varepsilon_t = 1\end{cases}.$$

Therefore, the only times when we have something non-zero is when $m-r$ is a multiple of t, or $m \equiv r \pmod t$. So, we have that

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{j=0}^{t-1}\frac{1}{t}\zeta_t^{-rj}N_k(m,n)\zeta_t^{mj} &= \sum_{m\in\mathbb{Z}}\frac{1}{t}N_k(m,n)\sum_{j=0}^{t-1}\zeta_t^{(m-r)j}\\ &= \sum_{m\equiv r \pmod t} N_k(m,n)\\ &= N_k(r,t,n).\end{aligned}$$

This shows that both sides have the same coefficient for q^n for all n. Therefore,

$$\begin{aligned}\sum_{n\geq 0} N_k(r,t,n)q^n &= \frac{1}{t}\sum_{j=0}^{t-1}\left(\zeta_t^{-rj}\mathcal{R}_k(\zeta_t^j,q)\right)\\ &= \frac{1}{t}\sum_{n\geq 0}p_k(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}\mathcal{R}_k(\zeta_t^j,q)\right).\end{aligned}$$

□

We can use this connection to show the following theorem.

Theorem 4.2. *Let $t > 2$ be a prime, $t' = \gcd(t,3)$, $\zeta_t \neq 1$ be a t-th root of unity, $k,s,t \in \mathbb{Z}^+$ with $s,t \neq 0$. Suppose we can write the rank generating function acted upon by $V(24)$ as $\mathcal{R}(\zeta_t^c;q^{24}) = \sum_{n\geq 0} b_c(n)q^{24n}$. Then, for integers $a_1,\ldots a_k$ such that $a_i \not\equiv 1 \pmod{24}$,*

$$\frac{1}{t}\sum_{j=1}^{t-1}(\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{h=1}^{k} q\mathcal{R}(\zeta_t^{jh};q^{24})|_{n\equiv a_h \pmod{24}}$$
$$= \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}} \left(N_k\left(r,t,\frac{n-k}{24}\right) - N_k\left(s,t,\frac{n-k}{24}\right)\right)q^n$$

is a modular form of weight $\frac{k}{2}$ on $\Gamma_0\left(\frac{13824t^5}{(t')^2}\right)$.

Before showing how this is true, we would like to take a moment to point out the following lemma.

Lemma 4.3. *For a product of k q-series with coefficients $c_i(n)$, for $i = 1,\ldots,k$,*

$$\prod_{i=1}^{k}\left(\sum_{n\equiv a_i \pmod{N}} c_i(n)q^n\right)$$
$$= \left(\prod_{i=1}^{k}\left(\sum_{n\geq 0} c_i(n)q^n\right)\right)\Bigg|_{\substack{n\equiv a_1+\cdots+a_k \pmod{N}\\ n\geq a_1+\cdots+a_k}} - \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{N}\\ n\geq a_1+\cdots+a_k\\ m_i\not\equiv a_i\\ m_1+\cdots+m_k=n}} \left(\prod_{i=1}^{k} c_i(m_i)\right)$$

Proof. Suppose we have k q-series that are of the form $\sum_{n\geq 0} c_i(n)q^n$, and consider

$$\prod_{i=1}^{k}\left(\sum_{n\equiv a_i \pmod{N}} c_i(n)q^n\right). \tag{12}$$

We will construct the right hand side in Lemma 4.3 by starting with the product of the original series

$$\prod_{i=1}^{k}\left(\sum_{n\geq 0} c_i(n)q^n\right). \tag{13}$$

We first note that after expanding the product in equation (12). The only possible values for powers of q are $n \geq a_1+\cdots+a_k$ with $n \equiv a_1+\cdots+a_k \pmod{N}$. Therefore, we need to restrict (13) as follows

$$\left(\prod_{i=1}^{k}\left(\sum_{n\geq 0} c_i(n)q^n\right)\right)\Bigg|_{\substack{n\equiv a_1+\cdots+a_k \pmod{N}\\ n\geq a_1+\cdots+a_k}}. \tag{14}$$

However, we now have contributions to any single coefficient that come from $c_i(n)$ where $n \not\equiv a_i$. We would like to remove these all at once. We can express any coefficient of q^n in (13) as

$$\sum_{\substack{m_i\geq 0\\ m_1+\cdots+m_k=n}} \prod_{i=1}^{k} c_i(m_i).$$

The undesired coefficients occur when $m_i \not\equiv a_i$, so by restricting to those undesired parts, we get

$$\sum_{\substack{n\equiv a_1+\cdots+a_k \pmod N\\ n\geq a_1+\cdots+a_k\\ m_i\not\equiv a_i\\ m_1+\cdots+m_k=n}} \left(\prod_{i=1}^{k} c_i(m_i)\right).$$

We can then remove the undesired parts from (14), and we get

$$\begin{aligned}&\prod_{i=1}^{k}\left(\sum_{n\not\equiv a_i \pmod N} c_i(n)q^n\right)\\ &\quad=\left(\prod_{i=1}^{k}\left(\sum_{n\geq 0} c_i(n)q^n\right)\right)\Bigg|_{\substack{n\equiv a_1+\cdots+a_k \pmod N\\ n\geq a_1+\cdots+a_k}} - \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod N\\ n\geq a_1+\cdots+a_k\\ m_i\not\equiv a_i\\ m_1+\cdots+m_k=n}} \left(\prod_{i=1}^{k} c_i(m_i)\right)\end{aligned}$$

as desired. □

We are now ready to prove Theorem 4.2

Proof. By Theorem 3.3 that for any $m\in\mathbb{Z}$ such that $\zeta_t^m \neq 1$,

$$\left(\frac{q^{-1/24}\mathcal{R}(\zeta_t^m;q)}{e^{\frac{-\pi i}{t}}-e^{\frac{\pi i}{t}}}\Bigg| V(24)\right)\Bigg|_{n\not\equiv 1 \pmod{24}}$$

is a weakly holomorphic modular form of weight $\frac{1}{2}$ for $\Gamma_0\left(\frac{13824t^5}{(t')^2}\right)$. Via twisting, we can maintain modularity, and say that

$$\prod_{h=1}^{k} q\cdot \mathcal{R}(\zeta_t^{jh};q^{24})|_{n\equiv a_h \pmod{24}}$$

is a modular form of weight $\frac{k}{2}$ for $\Gamma_0\left(\frac{13824t^5}{(t')^2}\right)$, when $a_h \not\equiv 1 \pmod{24}$. Therefore, the left hand side of the equation from the statement of Theorem 4.2 is clearly a modular form. Now we need to show that the two sides are equal.

By Theorem 4.1, we have that

$$\frac{1}{t}\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q)\right)=\sum_{n\geq 0}(N_k(r,t,n)-\frac{1}{t}p_k(n))q^n.$$

By Lemma 4.3,

$$\begin{aligned}&\frac{1}{t}\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q)|_{n\equiv a_h \pmod{24}}\right)\\ &=\sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\left(N_k(r,t,n)-\frac{1}{t}p_k(n)-\frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\sum_{\substack{m_i\not\equiv a_i \pmod{24}\\ m_1+\cdots+m_k=n}}\prod_{b=1}^{k}a_{bj}(m_b)\right)q^n.\end{aligned}$$

We can then subtract by the similar equation, obtained by replacing r with s, to get

$$\frac{1}{t}\sum_{j=1}^{t-1}\left((\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q)|_{n\equiv a_h \pmod{24}}\right)=\sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\Bigg(N_k(r,t,n)-N_k(s,t,n) - \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\sum_{\substack{m_i\not\equiv a_i \pmod{24}\\ m_1+\cdots+m_k=n}}\prod_{b=1}^{k}a_{bj}(m_b) + \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-sj}\sum_{\substack{m_i\not\equiv a_i \pmod{24}\\ m_1+\cdots+m_k=n}}\prod_{b=1}^{k}a_{bj}(m_b)\Bigg)q^n.$$

The powers of t-th roots of unity will loop through all possible t-th roots of unity since t is a prime number. Since the roots of unity loop, the last two terms on the right hand side cancel each other out. Which gives us

$$\frac{1}{t}\sum_{j=1}^{t-1}\left((\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q)|_{n\equiv a_h \pmod{24}}\right) = \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\left(N_k(r,t,n)-N_k(s,t,n)\right)q^n.$$

We now continue our transformation to make the left hand side match what we desire:

$$\frac{1}{t}\sum_{j=1}^{t-1}\left((\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q)|_{n\equiv a_h \pmod{24}}\right) = \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\left(N_k(r,t,n)-N_k(s,t,n)\right)q^n.$$

Thus replacing q by q^{24}, we get

$$\frac{1}{t}\sum_{j=1}^{t-1}\left((\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{m=1}^{k}\mathcal{R}(\zeta_t^{mj},q^{24})|_{n\equiv a_h \pmod{24}}\right) = \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\left(N_k(r,t,n)-N_k(s,t,n)\right)q^{24n}$$

So, multiplying by q^k gives

$$\frac{1}{t}\sum_{j=1}^{t-1}\left((\zeta_t^{-rj}-\zeta_t^{-sj})\prod_{m=1}^{k}q\cdot\mathcal{R}(\zeta_t^{mj},q^{24})|_{n\equiv a_h \pmod{24}}\right) = \sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24}\\ n\geq a_1+\cdots+a_k}}\left(N_k(r,t,n)-N_k(s,t,n)\right)q^{24n+k}.$$

This gives us our desired left hand side, and by change of variables, we can write the right hand side as

$$\sum_{\substack{n\equiv a_1+\cdots+a_k \pmod{24} \\ n\geq a_1+\cdots+a_k}} \left(N_k\left(r,t,\frac{n-k}{24}\right)-N_k\left(s,t,\frac{n-k}{24}\right)\right)q^n$$

as desired. □

5. ZEROS

Recall that in addition wanting to identify when rank difference functions were weakly holomorphic modular forms, we also wanted to identify them as identically zero. Also recall that rank difference functions of a given modulus ℓ are of the form

$$R_kD_{st}(b)=\sum_{n\geq 0}(N_k(s,\ell,ln+b)-N_k(t,\ell,ln+b))q^n.$$

It is difficult to identify when these functions are equal to zero looking at them analytically. Instead, we used a MAPLE program, the code for which is provided in Appendix A. We did extensive calculations for the first thirty or so terms of the rank-difference funtions for $k=2,3,4,5,6$ and $\ell=3,5,7$. We observed that a few of these rank difference functions appeared to be identically zero. We then were able to prove some of these observations.

The first case arises from the symmetry of residue classes with respect to conjugacy.

Theorem 5.1. *For $k,\ell\in\mathbb{N}$ and $s,b\in\mathbb{Z}_\ell$, we have $R_kD_{s,-s}(b)=0$.*

Proof. For $k,\ell\in\mathbb{Z}$, $b\in\mathbb{Z}_\ell$, define $S_{b,\ell,k}=\{\pi:\text{rank}_k(\pi)\equiv b \pmod{\ell}\}$
Claim: There exists a bijection between $S_{s,\ell,k}$ and $S_{-s,\ell,k}$.

Let $\pi\in S_{s,\ell,k}$. Then define a mapping $g:S_{r(m)}\to S_{-s,\ell,k}$ by $g(\pi)=\pi'$.
By Lemma 1.5 $\text{rank}_k(\pi)=-\text{rank}_k(\pi')$, and therefore the mapping g is a bijection.

Since a bijection does exist between $S_{s,\ell,k}$ and $S_{-s,\ell,k}$, we can conclude that $RD_{s,-s}(b)=0$. □

Theorem 5.2. *Let $p\geq 3$ be a prime, let k be of the form mp, $(m+1)(p-1)$, or $mp+\frac{p-1}{2}$, where $m\in\mathbb{Z}_{\geq 0}$. Furthermore, let $s,t\neq 0$. Then, for all $b\in\mathbb{Z}_p$, we have that $R_kD_{st}(b)=0$.*

Proof. We can break this proof up into six distinct cases. Let $p\geq 3$ be a prime, define $S_{b,\ell,k}$ as in the proof of theorem 5.1.

- Case 1: $k=\frac{p-1}{2}$.
  We want to show that, for some $a,b\in\mathbb{Z}_p$ there exists an isomorphism between $S_{a,p,k}$ and $S_{b,p,k}$. Suppose first that k is even. Let

$$\phi_{1a}:(\pi_1,\ldots,\pi_k)\to(\pi'_k,\pi_1,\pi'_{k-1},pi_2,\ldots,\pi'_{\frac{k}{2}+1},\pi_{\frac{k}{2}}).$$

We must show that $\phi_{1a} : S_{R,p,k} \to S_{2R,p,k}$ is an isomoprhism. Let $\pi = (\pi_1,...,\pi_k) \in S_{R,p,k}$. Then,

$$\begin{aligned} \operatorname{rank}(\pi_1) + 2\operatorname{rank}(\pi_2) + \cdots + k\operatorname{rank}(\pi_k) &\equiv R \pmod p \\ 2\operatorname{rank}(\pi_1) + 4\operatorname{rank}(\pi_2) + \cdots + (p-1)\operatorname{rank}(\pi_k) &\equiv 2R \pmod p \\ -(p-1)\operatorname{rank}(\pi'_k) + 2\operatorname{rank}(\pi_1) + \cdots - (p-k+1)\operatorname{rank}(\pi_{\frac{k}{2}+1}) + k\operatorname{rank}(\pi_{\frac{k}{2}}) &\equiv 2R \pmod p \\ \operatorname{rank}(\pi'_k) + 2\operatorname{rank}(\pi_1) + \cdots + (k-1)\operatorname{rank}(\pi_{\frac{k}{2}+1}) + k\operatorname{rank}(\pi_{\frac{k}{2}}) &\equiv 2R \pmod p. \end{aligned}$$

This tells us that the k-rank of $(\pi'_k, \pi_1, \pi'_{k-1}, \pi_2, ..., \pi'_{\frac{k}{2}+1}, \pi_{\frac{k}{2}})$ is congruent to $2R$ modulo p. So ϕ_{1a} is indeed a mapping between $S_{R,p,k} \to S_{2R,p,k}$. Furthermore, it is clear that ϕ_{1a} is injective. By similar logic, we can see that ϕ_{1a} injectively maps

$$S_{2R,p,k} \to S_{4R,p,k}, ..., S_{2^{p-2}R,p,k} \to S_{R,p,k}.$$

(Note that $2^{p-1} \equiv 1 \pmod p$). Since ϕ_{1a} is injective, we have

$$|S_{R,p,k}| \leq |S_{2R,p,k}| \leq \cdots \leq |S_{2^{p-2}R,p,k}| \leq |S_{R,p,k}|.,$$

and

$$|S_{R,p,k}| = |S_{2R,p,k}| = \cdots = |S_{2^{p-2}R,p,k}| = |S_{R,p,k}|$$

easily follows. So if k is even, the theorem holds in this case.
Suppose now that k is odd. Let

$$\phi_{1b} : (\pi_1, ..., \pi_k) \to (\pi'_k, \pi_1, \pi'_{k-1}, \pi_2, ..., \pi'_{\frac{k+1}{2}+1}).$$

I will show that ϕ_{1b} maps $S_{R,p,k} \to S_{2R,p,k}$. Let $(\pi_1, ..., \pi_k) \in S_{R,p,k}$. Then,

$$\begin{aligned} \operatorname{rank}(\pi_1) + 2\operatorname{rank}(\pi_2) + \cdots + k\operatorname{rank}(\pi_k) &\equiv R \pmod p \\ 2\operatorname{rank}(\pi_1) + 4\operatorname{rank}(\pi_2) + \cdots + (p-k)\operatorname{rank}(\pi_{\frac{k+1}{2}}) &\equiv 2R \pmod p \\ -(p-1)\operatorname{rank}(\pi'_k) + 2\operatorname{rank}(\pi_1) + \cdots - (p-k)\operatorname{rank}(\pi'_{\frac{k+1}{2}}) &\equiv 2R \pmod p \\ \operatorname{rank}(\pi'_k) + 2\operatorname{rank}(\pi_1) + \cdots + k\operatorname{rank}(\pi_{\frac{k+1}{2}}) &\equiv 2R \pmod p. \end{aligned}$$

This tells us that the k-rank of $(\pi'_k, \pi_1, \pi'_{k-1}, \pi_2, ..., \pi'_{\frac{k+1}{2}})$ is congruent to $2R$ modulo p. So ϕ_{1b} is an injective mapping between $S_{R,p,k} \to S_{2R,p,k}$. We can also see that ϕ_{1b} injectively maps

$$S_{2R,p,k} \to S_{4R,p,k}, ..., S_{2^{p-2}R,p,k} \to S_{R,p,k}.$$

Which means that

$$|S_{R,p,k}| \leq |S_{2R,p,k}| \leq \cdots \leq |S_{2^{p-2}R,p,k}| \leq |S_{R,p,k}|.$$

This means that

$$|S_{R,p,k}| = |S_{2R,p,k}| = \cdots = |S_{2^{p-2}R,p,k}| = |S_{R,p,k}|.$$

So, the theorem holds.

- Case 2: $k = p-1$. Consider the mapping

$$\phi_2 : (\pi_1, ..., \pi_{p-1}) \to (\pi_{(p+1)/2}, \pi_1, ..., \pi_{p-1}, \pi_{(p-1)/2}).$$

We want to show that ϕ_2 maps $S_{R,p,k}$ to $S_{2R,p,k}$. Let $(\pi_1, ..., \pi_k) \in S_{R,p,k}$. Then,

$$\operatorname{rank}(\pi_1) + 2\operatorname{rank}(\pi_2) + \cdots + (p-1)\operatorname{rank}(\pi_k) \equiv R \pmod{p}$$
$$2\operatorname{rank}(\pi_1) + 4\operatorname{rank}(\pi_2) + \cdots + 2(p-1)\operatorname{rank}(\pi_k) \equiv 2R \pmod{p}$$
$$\operatorname{rank}(\pi_{(k+2)/2}) + 2\operatorname{rank}(\pi_1) + \cdots + (p-1)\operatorname{rank}(\pi_{k/2}) \equiv 2R \pmod{p}.$$

These equations clearly show that $(\pi_{(p+1)/2}, \pi_1, ..., \pi_{p-1}, \pi_{(p-1)/2}) \in S_{2R,p,k}$. So ϕ_2 is injective. We can also see that ϕ_2 maps injectively

$$S_{2R,p,k} \to S_{4R,p,k}, ..., S_{2^{p-2}R,p,k} \to S_{R,p,k}.$$

So, as in the first case, we can see that

$$|S_{R,p,k}| \leq |S_{2R,p,k}| \leq \cdots \leq |S_{2^{p-2}R,p,k}| \leq |S_{R,p,k}|,$$

and thus

$$|S_{R,p,k}| = |S_{2R,p,k}| = \cdots = |S_{2^{p-2}R,p,k}| = |S_{R,p,k}|.$$

So, the theorem holds for this case.

- Case 3: $k = p$ We look at the map

$$\phi_3 : (\pi_1, ..., \pi_{p-1}, \pi_p) \to (\pi_{(p+1)/2}, \pi_1, ..., \pi_{p-1}, \pi_{(p-1)/2}, \pi_p).$$

We want to show that ϕ_3 injectively maps $S_{R,p,k}$ to $S_{2R,p,k}$. Let $(\pi_1, ..., \pi_k) \in S_{R,p,k}$. Then,

$$\operatorname{rank}(\pi_1) + 2\operatorname{rank}(\pi_2) + \cdots + (p-1)\operatorname{rank}(\pi_k) + p\operatorname{rank}(\pi_p) \equiv R \pmod{p}$$
$$2\operatorname{rank}(\pi_1) + 4\operatorname{rank}(\pi_2) + \cdots + 2(p-1)\operatorname{rank}(\pi_k) + 2p\operatorname{rank}(\pi_p) \equiv 2R \pmod{p}$$
$$\operatorname{rank}(\pi_{(k+2)/2}) + 2\operatorname{rank}(\pi_1) + \cdots + (p-1)\operatorname{rank}(\pi_{k/2}) + p\operatorname{rank}(\pi_p) \equiv 2R \pmod{p}.$$

These equations clearly show that $(\pi_{(p+1)/2}, \pi_1, ..., \pi_{p-1}, \pi_{(p-1)/2}, \pi_p) \in S_{2R,p,k}$. So, our injective mapping holds. We can also see that ϕ_3 maps

$$S_{2R,p,k} \to S_{4R,p,k}, ..., S_{2^{p-2}R,p,k} \to S_{R,p,k}.$$

Therefore,

$$|S_{R,p,k}| \leq |S_{2R,p,k}| \leq \cdots \leq |S_{2^{p-2}R,p,k}| \leq |S_{R,p,k}|.$$

This means that

$$|S_{R,p,k}| = |S_{2R,p,k}| = \cdots = |S_{2^{p-2}R,p,k}| = |S_{R,p,k}|.$$

So, the theorem holds for this case.

- Case 4: $k = mp$ for some $m \in \mathbb{N}$. Define a map ϕ_4 as follows. First, we divide the partition into m groups of p subpartitions. Then we apply the third case's ϕ_3 mapping to each of these m groups. Now we can see that ϕ_4 injectively maps $S_{R,p,k} \to S_{2R,p,k}$. As a result, we have $|S_{R,p,k}| = |S_{Q,p,k}|$, provided $R, Q \neq 0$.

- Case 5: $k = mp + \frac{p-1}{2}$. As in the fourth case, we define a mapping ϕ_5 by splitting each partition into m groups of p subpartitions and one group of $\frac{p-1}{2}$ subpartitions. Then we can apply the mapping ϕ_3 from Case 3 on the groups of p, and one of the mappings from Case 1 on the group of $\frac{p-1}{2}$ depending on whether $\frac{p-1}{2}$ is odd or even. Then ϕ_5 injectively maps $S_{R,p,k} \to S_{2R,p,k}$. As a result, we have $|S_{R,p,k}| = |S_{Q,p,k}|$, provided $R, Q \neq 0$.

- Case 6: $k = mp + p - 1$. As in the fourth case, we define a mapping ϕ_6 by splitting each partition into m groups of p subpartitions and one group of $p-1$ subpartitions. Then we can use the mapping ϕ_3 from Case 3 on each group of p subpartitions, and the mapping ϕ_2 from Case 2 on the group of $p-1$ subpartitions. This ϕ_6 injectively maps $S_{R,p,k} \to S_{2R,p,k}$. As a result, we have $|S_{R,p,k}| = |S_{Q,p,k}|$, so long as $R, Q \neq 0$.

Since we've shown all possible cases, the theorem holds. □

6. Open ends of discovery

Given that we spent just two months investigating partition theory and modular forms, we naturally have a lot of unanswered questions and ideas of paths that could be interesting to follow. In this section, we will share what we started, but did not finish.

First of all, in addition to our theorems about identically zero rank difference functions, we observed some rank-difference functions that appeared to be identically zero.

Conjecture 6.1. *For $\ell = 5$, $k = 2$, and for all $s, t \in \mathbb{Z}_5$,*

$$RD_{st}(2) = 0$$

and

$$RD_{st}(4) = 0.$$

There exist multitudes of statistics when it comes to partitions. Though we primarily looked at the Dyson k-rank, we also briefly explored a couple of other statistics, namely, the k-crank and the Hammond-Lewis Birank.

Definition 6.2. *For a partition π, let $\omega(\pi)$ denote the number of ones in π, let $\ell(\pi)$ denote the largest part of π, and let $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The **crank** of π is defined to be*

$$\operatorname{crank}(\pi) = \begin{cases} \ell(\pi) & \text{if } \omega(\pi) = 0 \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

Example 6.3. *Consider the partition $\pi = 5+3+2+2+1+1$. Then $\omega(\pi) = 2$, and $\mu(\pi) = 2$, so $\operatorname{crank}(\pi) = 2 - 2 = 0$. For a k-partition $\pi = (\pi_1, \pi_2, \ldots \pi_k)$, we look at the unweighted **k-crank**,*

defined by

$$\operatorname{crank}_k(\pi) = \sum_{i=1}^{k} \operatorname{crank}(\pi_i).$$

Define $M_1(m,n)$ to be the number of partitions π of n with crank$(\pi) = m$. We can extend this definition to k-partitions by letting $M_k(m,n)$ denote the number of k-partitions of n with unweighted k-crank equal to m. We can sort partitions into residue classes via crank with the function $M_k(s,m,n)$, which equals the number of k-partitions of n with k-crank congruent to s, modulo m. It is shown in [AG88], [Gar94], [AG03] that $M_1(m,n) = M_1(-m,n)$. In [Gar10], Garvan also gives the following generating function for $M_1(m,n)$:

$$F(z,q) = \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M_1(m,n)z^m q^n = (1-z)q + \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Conjecture 6.4. *The generating function for $M_k(m,n)$ can be written as*

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M_k(m,n)z^m q^n = \prod_{i=1}^{k} F(z^i,q).$$

Definition 6.5. *Define the crank difference function for a k-partition, modulo ℓ, as*

$$C_kD_{st}(b) = \sum_{n=0}^{\infty} \left(M_k(s,\ell,\ell n+b) - M_k(t,\ell,\ell n+b)\right) q^n.$$

Using Maple and the generating function in Conjecture 6.4, we calculated some coefficients of the crank-difference functions for $k = 2,3,4,5,6$ modulo 3, 5, and 7 (refer to appendix for code and data).

Conjecture 6.6. *For prime $\ell > 2$, $k \equiv \ell-1, 0 \pmod{\ell}$, $s,t \neq 0$, and for all b,*

$$C_kD_{st}(b) = 0.$$

Conjecture 6.7. *For $k = 3$ and $\ell = 5$,*

$$C_kD_{23}(1) = 0.$$

Conjecture 6.8. *For $k = 6$, $\ell = 5$, and for all b,*

$$C_kD_{23}(b) = 0.$$

We also looked at the Hammond-Lewis birank.

Definition 6.9. *For a 2-partition $\pi = (\pi_1,\pi_2)$, we define the* ***Hammond-Lewis Birank*** *as follows:*

$$\text{HL-birank}(\Pi) = \#\pi_1 - \#\pi_2.$$

where # denotes the number of parts in the partition π_i. The generating function for the Hammond-Lewis birank [Gar10] *is given by*

$$\sum_{\pi=(\pi_1,\pi_2)} z^{\text{HL-birank}(\pi)} q^{|\pi|} = \frac{1}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Define

$$H(m,n) = |\{|\pi| = n \mid \text{HL-birank}(\pi) = m\}|,$$

and

$$H(s,m,n) = |\{|\pi| = n \mid \text{HL-birank}(\pi) = s \pmod{m}\}|.$$

For a 2-partition, define the ***Hammond-Lewis birank difference function****, modulo ℓ, as*

$$HD_{st}(b) = \sum_{n=0}^{\infty} (H(s,\ell,ln+b) - H(t,\ell,\ell n+b))\, q^n.$$

Using Maple and the above generating function, we calculated some coefficients of Hammond-Lewis birank difference functions modulo 3, 5 and 7. Please see the appendix for Maple code and data. Garvan [Gar10] proved that, for $\ell = 5$, $b = 2,3,4$, and for all s,t, $HD_{st}(b) = 0$. We were able to observe some other zeros.

Theorem 6.10. *For all b,ℓ,s,*

$$HD_{s,-s}(b) = 0.$$

Proof. Consider the mapping $g : (\pi_1,\pi_2) \to (\pi_2,\pi_1)$. We can easily see that

$$\text{HL-birank}(\pi_1,\pi_2) = -\,\text{HL-birank}(\pi_2,\pi_1).$$

The mapping g is also clearly bijective. Therefore, there exists a bijection between the set of 2-partitions whose HL-birank is congruent to $s \pmod{\ell}$ and the set of 2-partitions whose Hammond-Lewis birank is congruent to $-s \pmod{\ell}$. □

Conjecture 6.11. *For $\ell = 5$ and $s,t \neq 0$,*

$$HD_{st}(0) = 0.$$

Conjecture 6.12. *For $\ell = 5$ and $s,t \in \{0,2,3\}$,*

$$HD_{st}(1) = 0.$$

Beyond proving these conjectures, we could look for identically zero difference functions for other values of k and l. Furthermore, attempting to classify multipartition difference functions with respect to any of the other multitudes of existing partition statistics could yield interesting results.

After identifying the rank difference functions that were identically zero, our next task was to see if we could identify rank difference functions that are modular forms. Two ways we thought we might accomplish this were by

(a) rewriting the rank difference functions as a quotient in terms of Dedekind's eta function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

and

(b) rewriting the rank difference functions as a quotient of products.

The Maple programs that we employed to go down these routes are in the appendix. They were created with the help of Frank Garvan's [Gar99] guide, "A q-product tutorial for a q-series MAPLE package."

Regretfully, although both programs worked excellently, we were unable to identify any nonzero modular forms through this method. This is something that could further be explored. Finally, we noticed some patterns for congruences for the multipartition function which hark back to Ramanujan's congruences.

Conjecture 6.13.

(a) *For* $k \equiv 0,1,2,4 \pmod 5$, $p_k(5n+4) \equiv 0 \pmod 5$.
(b) *For* $k \equiv 0,1,4 \pmod 7$, $p_k(7n+5) \equiv 0 \pmod 7$.
(c) *For* $k \equiv 1,-1 \pmod{11}$, $p_k(11n+6) \equiv 0 \pmod{11}$.

When $k = 1$, these congruences are Ramanujan's congruences, proved combinatorially by Atkin and Swinnerton-Dyer [ASD54]. Furthermore, for all three congruences, the case where $k \equiv 1 \pmod{1,5,7}$, respectively, was shown by Lazarev, Mizuhara, Reid, and Swisher in 2010 [LMRS10].

Appendix A. MAPLE Code

```
>cau:=(a,q,n)-> product((1-a*q^(k-1)),k=1..n);
>f:=(z,q,m)->series(1+sum(q^(n^2)/(cau(z*q,q,n)*cau(q/z,q,n)),n=1..3*m),
                    q=0,3*m);
>rank2:=m->series(coeff((series(f(z,q,m)*f(z^2,q,m),q=0,2*m)),q,m),z=0,m^2);
>getmods:=proc(m,n)          #n is the number to be partitioned, m is the
                              number which is modulus
 local mo,co,x,curmo,rankgf;
 mo:=array(0..m-1);          #this array will hold all the mods
 co:=array(0..4*n-4);        #this array will hold the coefficients of z for a
                              given n
 for x from 0 to m-1 do:
 mo[x]:=0;                   #sets the mod counters to 0
 od:
 rankgf:=rank2(n);           #generates the z's, so that the function isn't
                              called repeatedly
 for x from 0 to 4*n-4 do:
    co[x]:=coeff(rankgf,z,x-(2*n-2));  #stores the coefficients
    curmo:=modp(x-(2*n-2),m);   #finds the mod we should be at
    mo[curmo]:=mo[curmo]+co[x]; #adds the coefficient to the current number
                                 with ranks congruent to each mod m.
    od:
 print(mo);                  #prints the mods.
 end;
>getmods2:=proc(k,l,n)         #k = the number of sub-partitions; l=the
                                modulus; n=the number being partitioned
 local mo,x,d,curmo,curmo2,rankgf;
 mo:=array(0..l-1);
 for x from 0 to l-1 do:
    mo[x]:=0;
    od:
 rankgf:=series(coeff((series(product(f(z^d,q,n),d=1..k),q=0,2*n)),q,n),
               z=0,k*(n-1)+1);
 for x from 0 to k*(n-1) do:
    curmo:=modp(x-k*(n-1),l);
```

```
    curmo2:=modp(-1*curmo,l);
    mo[curmo]:=mo[curmo]+coeff(rankgf,z,x-k*(n-1));
    if (k*(n-1)-x)>0 then:
       mo[curmo2]:=mo[curmo2]+coeff(rankgf,z,x-k*(n-1));
       end if;
    od:
 print(mo);
 end;
>rankdiff:=proc(k,l,n)
 local x,y,z,mods,rd,fd;
 mods:=getmods2(k,l,n);
 z:=0;
 rd:=array(0..(l^2-1)/8-1);
 fd:=fopen(\datafile,APPEND,TEXT);
 fprintf(fd,"This is for k=%d, l=%d, and n=%d \n",k,l,n);
 for x from 0 to (l-3)/2 do:
    for y from x+1 to (l-1)/2 do:
       rd[z]:=mods[x]-mods[y];
       z:=z+1;
       fprintf(fd, "RD%d%d \t\t",x,y);
       od:
    od:
 fprintf(fd,"\n");
 for x from 0 to (l^2-1)/8-1 do:
    fprintf(fd,convert(rd[x],string));
    fprintf(fd,"\t\t");
    od:
 fprintf(fd,"\n");
 fclose(fd);
 end;
>crank:=(z,q,n)->cau(q,q,n)/(cau(z*q,q,n)*cau(q/z,q,n));
>cgetmods:=proc(k,l,n)          #n is the number to be partitioned, l is the
                                 number which is modulus, k is the number of
                                 subpartitions
 local mo,x,curmo,crankgf, fd,s;
 mo:=array(0..l-1);          #this array will hold all the mods
 for x from 0 to l-1 do:
    mo[x]:=0;                   #sets the mod counters to 0
    od:
 print("Generating for", k, " and ", l, " and ", n);
 crankgf:=series(coeff(series(product(crank(z^m,q,n+1),m=1..k),q,10*n),q,n),z,
                10*n^2);
 for x from -k*n to k*n do:
    curmo:=modp(x,l);   #finds the mod we should be at
```

```
   mo[curmo]:=mo[curmo]+coeff(crankgf,z,x); #adds the coefficient to the
                                             current  number with ranks
                                             congruent to each mod m.
   od:
fd:=fopen(\datafilecrank,APPEND,TEXT);
fprintf(fd,"This is for k=%d, l=%d, and n=%d \n",k,l,n);
for x from 0 to l-1 do:
   fprintf(fd, "mod%d \t\t",x);
   od:
fprintf(fd,"\n");
for x from 0 to l-1 do:
   fprintf(fd,convert(mo[x],string));
   fprintf(fd,"\t\t");
   od:
fprintf(fd,"\n");
fclose(fd);
return(mo);                  #prints the mods.
end;
>HL:=(z,q,m)->series(1/(cau(z*q,q,2*m)*cau(q/z,q,2*m)),q=0,3*m);
>hlgetmods:=proc(m,n)          #n is the number to be partitioned, m is the
                                number which is modulus
local mo,x,curmo,rankgf,fd;
mo:=array(0..m-1);           #this array will hold all the mods
for x from 0 to m-1 do:
   mo[x]:=0;                    #sets the mod counters to 0
   od:
print("Generating for",m," and ",n);
rankgf:=series(coeff(HL(z,q,n),q,n),z,3*n);         #generates the z's, so
                                                     that the function isn't
                                                     called repeatedly, also
                                                     saves time.
for x from -n to n do:
   curmo:=modp(x,m);   #finds the mod we should be at
   mo[curmo]:=mo[curmo]+coeff(rankgf,z,x); #adds the coefficient to the
                                            current number with ranks
                                            congruent to each mod m.
   od:
fd:=fopen(\hldatafile,APPEND,TEXT);
fprintf(fd,"This is for k=2, l=%d, and n=%d \n",m,n);
for x from 0 to m-1 do:
   fprintf(fd, "mod%d \t\t",x);
   od:
fprintf(fd,"\n");
for x from 0 to m-1 do:
```

```
    fprintf(fd,convert(mo[x],string));
    fprintf(fd,"\t\t");
    od:
 fprintf(fd,"\n");
 fclose(fd);
 end;
>etamake := proc (f, q, last)
 local fp, tc, exq, g, aa, ld, h, hh, i, cf, etaprod, alast, sf;
 sf := series(f, q, last+10);
 fp := convert(sf, polynom);
 tc := tcoeff(fp, q); exq := ldegree(fp, q);
 g := normal(fp/(tc*q^exq));
 aa := tc;
 ld := 1;
 alast := last-exq;
 while 0 < ld do:
    h := series(g-1, q = 0, alast+1);
    hh := 0;
    for i to alast do:
       hh := hh+coeff(h, q, i)*q^i
       od:
    h := hh;
    if h = 0 then
       ld := 0
    else
       ld := ldegree(h, q)
    end if;
    cf := coeff(h, q, ld);
    if 0 < ld then
       exq := exq+(1/24)*ld*cf;
       aa := eta(ld*tau)^(-cf)*aa;
       g := g*etaq(q, ld, alast)^cf
    end if
    od:
 etaprod := q^exq*aa;
 RETURN(etaprod) ;
 end proc;
 >etaq := proc (q, i, trunk)
  local k, x, z1, z, w;
  z1 := (1/6)*(i+sqrt(i*i+24*trunk*i))/i;
  z := 1+trunc(evalf(z1));
  x := 0;
  for k from -z to z do:
     w := (1/2)*i*k*(3*k-1);
```

```
     if w <= trunk then
        x := x+q^w*(-1)^k
        end if
     end do;
 RETURN(x);
 end proc;
>aqprod := proc (a, q, n)
 local x, i;
 if type(n, nonnegint) then
    x := 1;
    for i to n do:
       x := x*(1-a*q^(i-1))
       od:
 else:
    x := ``(a, q)[n];
    end if;
 RETURN(x);
 end proc;
>prodmake := proc ()
 local ft, f0, _a, _b, i, n, j, d, _A, _B, sum1, sum2, divj, divjb, m, prd;
 if 4 < nargs then:
    ERROR(` number of arguments must be 3 or 4.`)
    end if;
 f := args[1];
 q := args[2];
 T := args[3];
 ft := series(f, q, T+5);
 if whattype(ft) = series then:
    f0 := coeff(ft, q, 0);
    if f0 = 1 then:
       _b := series(f, q, T);
       _b := convert(_b, polynom);
       for i to T-1 do:
          _B[i] := coeff(_b, q, i)
          od:
       _A[1] := _B[1];
       for n from 2 to T-1 do:
          sum2 := 0;
          for j to n-1 do:
             divj := numtheory[divisors](j);
             sum1 := 0;
             for d in divj do:
                sum1 := expand(sum1+d*_A[d])
                od:
```

```
            sum2 := expand(sum2+_B[n-j]*sum1)
            od:
         sum1 := 0;
         divjb := `minus`(numtheory[divisors](j), {n});
         for d in divjb do:
            sum1 := expand(sum1+d*_A[d])
            od:
         sum2 := expand(n*_B[n]-sum2-sum1);
         _A[n] := sum2/n
         od:
      if nargs = 3 then:
         prd := product((1-q^m)^(-_A[m]), m = 1 .. T-1);
         RETURN(prd);
      else:
         bseq := [seq(-_A[m], m = 1 .. T-1)];
         lst := convert(bseq, list);
         RETURN(lst)
         end if;
   else:
      ERROR(`coeff of q^0 must be 1`);
      end if;
else:
   ERROR(`f must be a series`);
   end if;
end proc
```

References

[AG88] George E. Andrews and F. G. Garvan. Dyson's crank of a partition. *Bull. Amer. Math. Soc. (N.S.)*, 18(2):167–171, 1988.

[AG03] A. O. L. Atkin and F. G. Garvan. Relations between the ranks and cranks of partitions. *Ramanujan J.*, 7(1-3):343–366, 2003. Rankin memorial issues.

[And79] George E. Andrews. *Partitions: yesterday and today*. New Zealand Mathematical Society, Wellington, 1979. With a foreword by J. C. Turner.

[And98] George E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[And08] George E. Andrews. A survey of multipartitions: congruences and identities. In *Surveys in number theory*, volume 17 of *Dev. Math.*, pages 1–19. Springer, New York, 2008.

[ASD54] A. O. L. Atkin and P. Swinnerton-Dyer. Some properties of partitions. *Proc. London Math. Soc. (3)*, 4:84–106, 1954.

[AT08] Scott Ahlgren and Stephanie Treneer. Rank generating functions as weakly holomorphic modular forms. *Acta Arith.*, 133(3):267–279, 2008.

[Atk68] A. O. L. Atkin. Ramanujan congruences for $p_{-k}(n)$. *Canad. J. Math. 20 (1968), 67-78; corrigendum, ibid.*, 21:256, 1968.

[BO10] Kathrin Bringmann and Ken Ono. Dyson's ranks and Maass forms. *Ann. of Math. (2)*, 171(1):419–449, 2010.

[BOR08] Kathrin Bringmann, Ken Ono, and Robert C. Rhoades. Eulerian series as modular forms. *J. Amer. Math. Soc.*, 21(4):1085–1104, 2008.

[Dys44] F. J. Dyson. Some guesses in the theory of partitions. *Eureka (Cambridge)*, 8:10–15, 1944.

[Gar94] Frank G. Garvan. Generalizations of Dyson's rank and non-Rogers-Ramanujan partitions. *Manuscripta Math.*, 84(3-4):343–359, 1994.

[Gar99] Frank Garvan. A q-product tutorial for a q-series MAPLE package. *Sém. Lothar. Combin.*, 42:Art. B42d, 27, 1999. The Andrews Festschrift (Maratea, 1998).

[Gar10] F. G. Garvan. Biranks for partitions into 2 colors. In *Ramanujan rediscovered*, volume 14 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 87–111. Ramanujan Math. Soc., Mysore, 2010.

[GKST12] Sharon Garthwait, Soon-Yi Kang, Holly Swisher, and Stephanie Treneer. Mock and false theta functions arising from eta-quotients. In Preparation., 2012.

[GM03] Basil Gordon and Richard J. McIntosh. Modular transformations of Ramanujan's fifth and seventh order mock theta functions. *Ramanujan J.*, 7(1-3):193–222, 2003. Rankin memorial issues.

[Kan09] Soon-Yi Kang. Mock Jacobi forms in basic hypergeometric series. *Compos. Math.*, 145(3):553–565, 2009.

[Laz11] Oleg Lazarev. Mock modular forms. pages 1–19, 2011.

[LMRS10] Oleg Lazarev, Matt Mizuhara, Ben Reid, and Holly Swisher. Extension of a proof of the ramanujan congruences for multipartitions. In Preparation., 2010.

[LO08] Jeremy Lovejoy and Robert Osburn. Rank differences for overpartitions. *Q. J. Math.*, 59(2):257–273, 2008.

[MO11] Maria Monks and Ken Ono. Modular forms arising from $Q(n)$ and Dyson's rank. *Adv. in Appl. Math.*, 46(1-4):457–466, 2011.

[Mon10] Maria Monks. Number theoretic properties of generating functions related to Dyson's rank for partitions into distinct parts. *Proc. Amer. Math. Soc.*, 138(2):481–494, 2010.

[Ono04] Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[Ono09] Ken Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. In *Current developments in mathematics, 2008*, pages 347–454. Int. Press, Somerville, MA, 2009.

[Zag09] Don Zagier. Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann). *Astérisque*, (326):Exp. No. 986, vii–viii, 143–164 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008.

[Zuc87] I. J. Zucker. A systematic way of converting infinite series into infinite products. *J. Phys. A*, 20(1):L13–L17, 1987.

[Zwe02] S.P. Zwegers. *Mock Theta Functions*. 2002.