# INHOMOGENEOUS QUANTUM WALKS

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ABSTRACT. We examined three different types of one-dimensional quantum walks, all exhibit some degree of inhomogeneity. The first is inhomogeneous only insofar as the coin operator differs at even and odd positions along the line. The second has a coin operator that is inhomogeneous and oscillatory in time. The third has a coin operator that is inhomogeneous and oscillatory in space, differing at every position along the line. Much progress is made towards attaining asymptotics for the first walk, although our analytical solution does not yet agree with the numerics. An asymptotic approach is attempted for the second walk, although it seems that this approach may not be well-suited to the walk. The third walk, it seems, requires more machinery, although progress is made using symmetry and a pentadiagonal transition operator is obtained. This general pentadiagonal operator gives hope for much progress in the near future.

## 1. INTRODUCTION

A fair amount of work has already been done on homogeneous quantum walks on the line, the most well-studied of these being the Hadamard walk. However, due to the difficult in obtaining these solutions, there is very little work on inhomogeneous quantum walks. As such, we attempt in the following to begin making real progress toward solving these walks.

1.1. **Brief Overview Of Quantum Walks.** The quantum walk is a kind of unitary stochastic process, the quantum analog to the classical random walk, familiar to probabilists and others. Some features of the quantum walk are:

(1) Separate coin and walker state spaces, full Hilbert space is: $\mathcal{H}_{\text{full}} = \mathcal{H}_c \otimes \mathcal{H}_w$.
(2) For our systems so far, $\mathcal{H}_c$ is a two-dimensional complex vector space over $\mathbb{Z}_2$ i.e.,

$$|\text{Coin}\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a|\uparrow\rangle + b|\downarrow\rangle.$$

(3) $\mathcal{H}_w$ is (for us) an infinite-dimensional complex vector space over $\mathbb{Z}$.
(4) The full wave function is given at an arbitrary time as:[1]

$$|\psi(n,t)\rangle = a_t(n)|\uparrow\rangle + b_t(n)|\downarrow\rangle \otimes |n\rangle$$

and, even more fully:

$$|\psi(t)\rangle = \sum_n |\psi(n,t)\rangle$$

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[1] Note where our time dependence is.

(5) Note that the probability for being in a given state is $|\langle\psi(\text{state})||\psi(\text{state})\rangle|^2$.

The evolution of the quantum walk is governed by:

(1) A unitary evolution operator.
(2) The full unitary operator on Hilbert space is given by:

$$U = S \cdot \left( C \cdot \begin{pmatrix} a \\ b \end{pmatrix} \otimes |n\rangle \right)$$

where $S$ is the shift operator, $C$ is the coin operator, $\begin{pmatrix} a \\ b \end{pmatrix}$ is the coin state, and $|n\rangle$ is the position state.

1.2. **The Hadamard Walk.** This is the most familiar type of quantum walk. A brief survey of its behavior is as follows:

(1) $S(M|c\rangle \otimes |v\rangle)$
(2) Coin operator is the 2 x 2 Hadamard operator:

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(3) Initial state:

$$|c\rangle \otimes |v\rangle = (a_0|\uparrow\rangle + b_0|\downarrow\rangle) \otimes |0\rangle = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \otimes |0\rangle$$

(4) Shift operator:

$$S = |\uparrow\rangle\langle\uparrow| \otimes |n+1\rangle\langle n| + |\downarrow\rangle\langle\downarrow| \otimes |n-1\rangle\langle n|$$

(1) Coefficients at time t:

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix}$$

(2) Yields these recurrence relations for time evolution:

$$a_t = \frac{1}{\sqrt{2}}(a_{t-1} + b_{t-1})$$

$$b_t = \frac{1}{\sqrt{2}}(a_{t-1} - b_{t-1})$$

## 2. Our Problems

We worked with the following types of inhomogeneous quantum walks:

(1) Slightly Generalized P and Q Even-Odd Operators, An Inhomogeneous Case. We solved with $p = 1 - q$.
   - Use $C_{even} = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{p} \end{pmatrix}$ and $C_{odd} = \begin{pmatrix} \sqrt{q} & \sqrt{1-q} \\ \sqrt{1-q} & -\sqrt{q} \end{pmatrix}$

(2) Spatially Inhomogeneous:

$$\begin{pmatrix} \cos(n) & \sin(n) \\ \sin(n) & -\cos(n) \end{pmatrix}$$

where $\forall n$, $0 < f(n) < 1$

(3) Temporally Inhomogeneous:

$$\begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix})$$

## 3. THE EVEN-ODD WALK

Here our coin operator is different for even and odd sites. In general:

$$C_{even} = \begin{pmatrix} \sqrt{q} & \sqrt{1-q} \\ \sqrt{1-q} & -\sqrt{q} \end{pmatrix}$$

$$C_{odd} = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{1-p} \end{pmatrix}$$

where $p \neq q$.

A quick analysis leading to a recursion relation is as follows:

(1) $|\psi(n,t+1)\rangle = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} |\psi(n-1,t)\rangle + \begin{pmatrix} 0 & 0 \\ \sqrt{1-p} & -\sqrt{1-p} \end{pmatrix} |\psi(n+1,t)\rangle$ **when** $n$ **is even**.

(2) (When $n$ is odd replace p with q)

Call the first matrix $M_{o+}$ and the second $M_{o-}$

Taking discrete Fourier transforms of both sides for this first (even) case yields[2]:

(1) $|\bar{\psi}(k,t+1)\rangle = \sum_n e^{ikn} M_{o+} |\psi(n-1,t)\rangle + \sum_n e^{ikn} M_{o-} |\psi(n+1,t)\rangle$[3]

(2) Which eventually yields to $|\bar{\psi}(k,t+1)\rangle = (e^{ik} M_{o+} + e^{-ik} M_{o-}) |\bar{\psi}(k,t)\rangle$ and, by analogy, $|\bar{\psi}(k,t)\rangle = (e^{ik} M_{e+} + e^{-ik} M_{e-}) |\bar{\psi}(k,t-1)\rangle$ where $t$ is odd and $t-1$ is even.

Next we package our operators together into:

$$M_e = M_{e+} + M_{e-} = \begin{pmatrix} e^{ik}\sqrt{p} & e^{ik}\sqrt{1-p} \\ e^{-ik}\sqrt{1-p} & -e^{-ik}\sqrt{p} \end{pmatrix}$$

$$M_o = M_{o+} + M_{o-} = \begin{pmatrix} e^{ik}\sqrt{q} & e^{ik}\sqrt{1-q} \\ e^{-ik}\sqrt{1-q} & -e^{-ik}\sqrt{q} \end{pmatrix}$$

The real crux is to realize that these operators commute, so that even though we have to apply them in alternating order, we may write this as:

$$M^t = (M_e M_o)^{t/2} \text{(t even)}$$

[2]Where k varies over the interval $[0, 2\pi]$

[3]Remember that n = 2m

$$M^t = M_o(M_eM_o)^{(t-1)/2}\text{(t odd)}$$

So, we are back to a familiar eigenvalue problem for this ugly matrix $M_eM_o$:

$$\lambda_{1,2} = \sqrt{pq}\cos 2k + \sqrt{(1-p)(1-q)} \pm \sqrt{(\sqrt{pq}\cos 2k + \sqrt{(1-p)(1-q)})^2 - 1}$$

$$\vec{e}_{1,2} = (1+\cos^2(2k) + A \pm \cos(2k)A^{1/2})^{-1} \left( \begin{array}{c} (\cos(2k) \pm A\left(\frac{\sqrt{pq}}{-e^{-2ik}\sqrt{p(1-p)}+\sqrt{q(1-p)}}\right) \\ 1 \end{array} \right)$$

where $A = \cos(2k) + 2\cos(2k)(1-p-q+pq) - \frac{p+q}{pq} + 1$.

Then we will have a slightly altered picture at odd times, but it is simple to calculate one extra application of the matrix $M_o$, which will yield the full k-space picture then needing to be inverse Fourier transformed:

$$|\tilde{\psi}(k,t)\rangle = \begin{cases} (M_eM_o)^{t/2}|\psi(k,0)\rangle & : \text{(t even)} \\ M_o(M_eM_o)^{(t-1)/2}|\psi(k,0)\rangle & : \text{(t odd)} \end{cases}$$

to explicitly calculate this, we use the diagonal matrix:

$$M_eM_o = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and re-express the initial state in the eigenbasis:

(1) In the original basis, $|\psi(k,0)\rangle = \sum_{n=-\infty}^{\infty} |\psi(n,0)\rangle e^{ikn} = |\psi(0,0)\rangle = |\uparrow\rangle \otimes |0\rangle$.

(2) To change bases we we do $A^{-1}(A|\psi(k,0)\rangle)$ where $A$ is specified only in the case we are examining.

3.1. **Case where p = 1 -q.** Here things simplify substantially, for instance

(1) $(1-p)(1-q) = pq = q(1-q)$ which we will call $B$.

(2) This, in turn, simplifies $A$, which is now just $\cos(2k) + 2B\cos(2k) - \frac{1}{B} + 1$.

(3) For $\lambda_{1,2}$ we have now $\sqrt{B}(\cos(2k)+1 \pm \sqrt{(\cos(2k)+1)^2 - \frac{1}{B}})$ or, slightly cleaner, $\lambda_{1,2} = \sqrt{B}(2\cos^2(k) \pm \sqrt{(2\cos^2(k))^2 - \frac{1}{B}})$

(4) And, finally, we have the normalized eigenvectors $\vec{e}_{1,2} = (1+A)^{-1} \left( \begin{array}{c} \cos(2k) \pm A(\frac{\sqrt{B}}{-e^{-2ik}\sqrt{B}+q} \\ 1 \end{array} \right)$

Our change of basis matrix (for the eigenbasis) is:

$$P = \frac{1}{1+A} \begin{pmatrix} \cos(2k)+F & \cos(2k)-F \\ 1 & 1 \end{pmatrix}$$

where $F = A\left(\frac{\sqrt{B}}{-e^{-2ik}\sqrt{B}+q}\right)$,
and its inverse is:

$$P^{-1} = \frac{1+A}{2F}\begin{pmatrix} 1 & -\cos(2k)+F \\ -1 & \cos(2k)+F \end{pmatrix}$$

Applying the change of basis to our initial state, which is purely up for now:

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to (1+A)^{-1}\begin{pmatrix} \cos(2k)+F \\ 1 \end{pmatrix}$$

This gives:

$$(M_e M_o)^l \vec{v} = P^{-1}(1+A)^{-1}\begin{pmatrix} \lambda_1^l(\cos(2k)+F) \\ \lambda_2^l \end{pmatrix}$$

which finally gives us $|\tilde{\psi}(k,t)\rangle$ **for even times** as:

$$\frac{1}{2F}\begin{pmatrix} \lambda_1^{t/2}(\cos(2k)+F)+\lambda_2^{t/2}(F-\cos(2k)) \\ -\lambda_1^{t/2}(\cos(2k)+F)+\lambda_2^{t/2}(F+\cos(2k)) \end{pmatrix}$$

using $\frac{t-1}{2}$ and applying an extra $M_o$ yields $|\tilde{\psi}(k,t)\rangle$ **for odd times** as:

$$\frac{1}{2F}((\lambda_1^{(t-1)/2}(\cos(2k)+F)+\lambda_2^{(t-1)/2}(F-\cos(2k)))\begin{pmatrix} e^{ik}\sqrt{q} \\ e^{-ik}\sqrt{1-q} \end{pmatrix}$$
$$+\frac{1}{2F}(-\lambda_1^{(t-1)/2}(\cos(2k)+F)+\lambda_2^{(t-1)/2}(F+\cos(2k)))\begin{pmatrix} e^{ik}\sqrt{1-q} \\ -e^{-ik}\sqrt{q} \end{pmatrix}$$

Concentrating on the even-time picture for now, we have a coefficient to inverse Fourier transform:

$$\tilde{a}_t(k) = \frac{\lambda_1^{t/2}(\cos(2k)+F)}{2F} - \lambda_2^{t/2}(\frac{F(k)-\cos(2k)}{2F(k)})$$

Given a linear speed of propagation, we substitute $n = \alpha t$, so the integral for $a_t(k)$ in its entirety is:

$$\frac{1}{2\pi}\int_0^{2\pi}\tilde{a}_t(k)e^{-ikn}dk = \frac{1}{2\pi}(\int_0^{2\pi}\lambda_1^t\frac{(\cos(2k)+F(k))e^{-ik\alpha t}}{2F(k)}dk - \int_0^{2\pi}\lambda_2^t e^{-ik\alpha t}\frac{F(k)-\cos(2k)}{2F(k)}dk) = \frac{1}{2\pi}(I_1+I_3)$$

To do this integral, we employ asymptotic methods- the details of which are relegated to the appendices.

3.2. **Expressing All The Coefficients.** Overall we have:

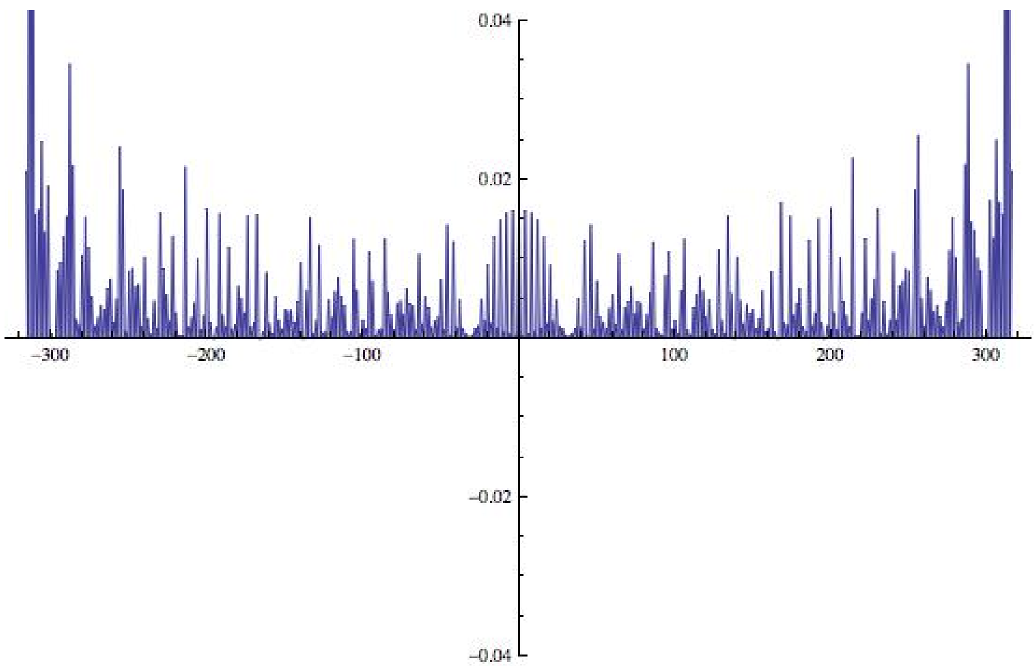
$$a_{t_{even}} = \frac{1}{2\pi}(I_1+I_3)$$

$$b_{t_{even}} = \frac{1}{2\pi}(-I_1+I_{3_b})$$

$$a_{t_{odd}} = (\frac{1}{2\pi})(\sqrt{q}(I_{1_{odd}}+I_{3_{odd}})+\sqrt{1-q}(-I_{1_{odd}}+I_{3_{odd}}))$$

$$b_{t_{odd}} = (\frac{1}{2\pi})(\sqrt{1-q}(I_{1_{odd,B}} + I_{3_{odd,B}}) + \sqrt{q}(I_{1_{odd,B}} - I_{3_{odd,B}}))$$

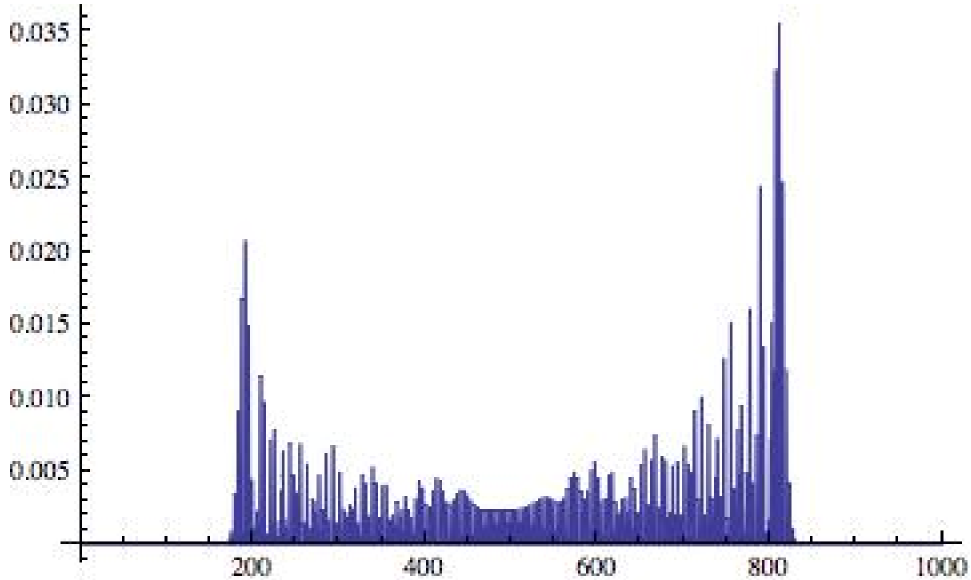
3.3. **Results From The Analytics And Numerics.** Here we have an asymptotic solution with $q = .1$ and $t = 1000$.



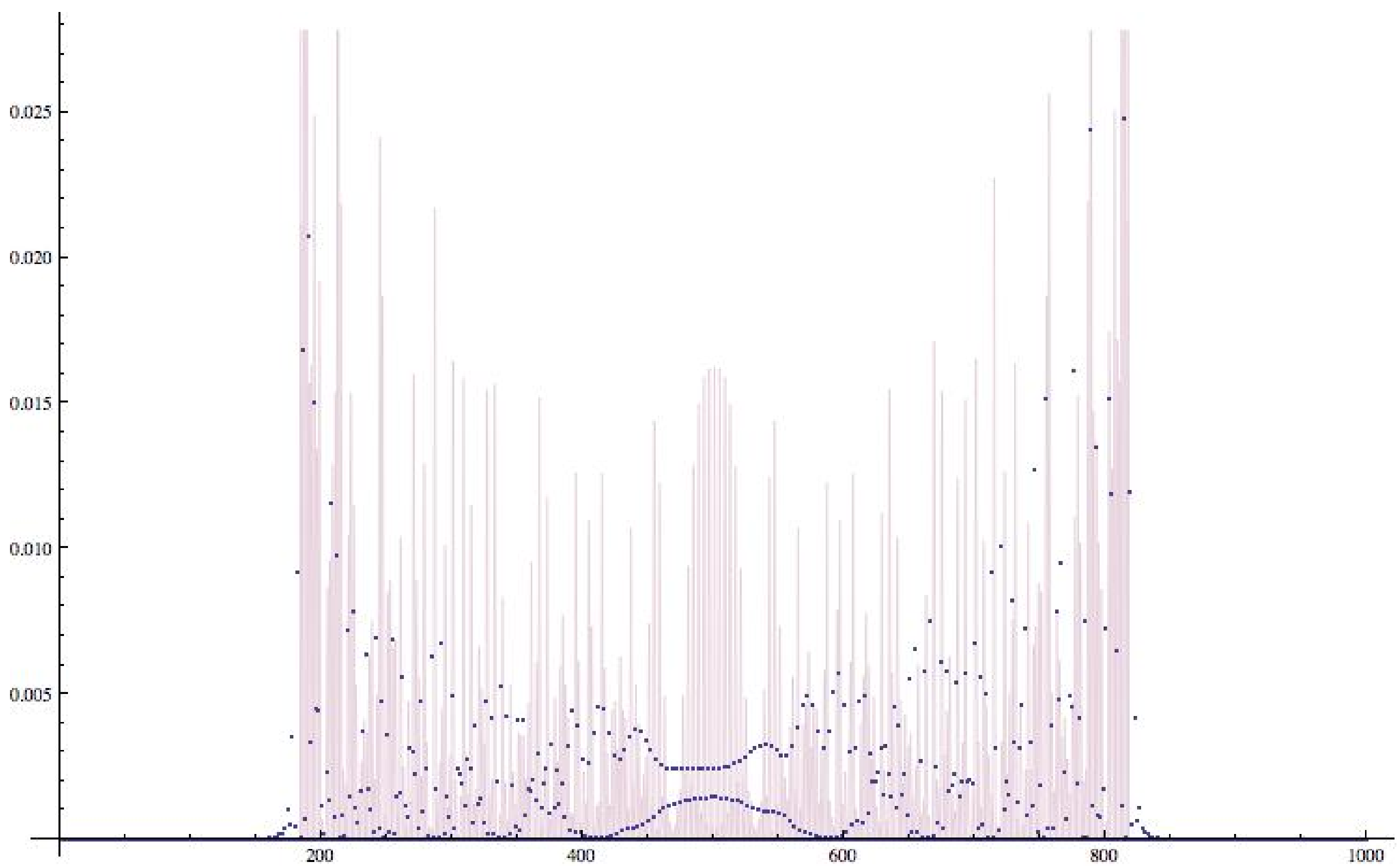

Our analytical (asymptotic) solution has the following features:

(1) Even-odd position alternation with even/odd times, like the Hadamard walk.
(2) Traveling peaks whose rate of propagation is ballistic and depends on the value of $q$, reaching a maximum at $q = .5$ and dying off symmetrically as $q$ approaches one.
(3) Unreasonably fast growth of the probability beyond the position of furthest propagation, instead of the expected immediate dying-off. Also, after the fast growth behavior begins, the even-odd alternation (a physical requirement due to wave interference) stops.

This last feature is undesired, and points to some mistake in our analysis.
Here is the corresponding numerical solution:




where the origin of the process is at position 500 on the graph, instead of zero. This solution also shows even-odd alternation, traveling peaks at the same rate, and dying-off at the same position that the analytical solution begins growing unreasonably fast. However, it gives rather different values across the entire space as seen here, with both systems plotted on the same graph:



Here the analytical solution is in red, with spikes, and the numerical solution is in blue, with dots.

## 4. More Analysis or For The Future

This is now giving fairly good results. However, note that the values of the numerics and analytics vary over the entire space. In the near future, we hope to determine the cause of this discrepancy, which is likely some simple and, as yet, undiscovered error in our analysis.

## 5. Time-Inhomogeneous Problem

For the case where the coin operator varies with respect to time, we consider two different coin operators:

$$C_t = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} \tag{1}$$

and

$$D_t = \begin{pmatrix} \cos(\frac{2\pi t}{N}) & \sin(\frac{2\pi t}{N}) \\ \sin(\frac{2\pi t}{N}) & -\cos(\frac{2\pi t}{N}) \end{pmatrix} \tag{2}$$

5.1. **First Time-Inhomogeneous Coin Operator.** First, we consider the coin $C_t$. For this coin, we get the following recurrence relations for the coefficients $a$ and $b$, where $|\psi(n,t)\rangle = \binom{a}{b}$:

$$a_{t+1}(n) = \cos(t)a_t(n-1) + \sin(t)b_t(n-1) \tag{3}$$

$$b_{t+1}(n) = \sin(t)a_t(n+1) - \cos(t)b_t(n+1) \tag{4}$$

Relatively straightforward Fourier analysis gives the following relations for $\tilde{a}$ and $\tilde{b}$, where $|\tilde{\psi}(k,t)\rangle = \binom{\tilde{a}}{\tilde{b}}$:

$$\tilde{a}_{t+1}(k) = e^{ik}\cos(t)\tilde{a}_t(k) + e^{ik}\sin(t)\tilde{b}_t(k) \tag{5}$$

$$\tilde{b}_{t+1}(k) = e^{-ik}\sin(t)\tilde{a}_t(k) - e^{-ik}\cos(t)\tilde{b}_t(k) \tag{6}$$

or equivalently,

$$|\tilde{\psi}(k,t+1)\rangle = M_t|\tilde{\psi}(k,t)\rangle \tag{7}$$

where

$$M(t) = \begin{pmatrix} e^{ik}\cos(t) & e^{ik}\sin(t) \\ e^{-ik}\sin(t) & -e^{-ik}\cos(t) \end{pmatrix} \tag{8}$$

We would like to find a matrix $M$ such that $|\tilde{\psi}(k,t)\rangle = M|\tilde{\psi}(k,0)\rangle$. Unfortunately, this matrix is:

(9) $$M = \prod_{j=0}^{t} M_{t-j}$$

which is somewhat hard to work with. To turn this product into a single matrix so we can solve for $\tilde{a}$ and $\tilde{b}$ separately, we can do the following:

$$M = \prod_{j=0}^{t} M_{t-j} = \exp\left(\ln\left(\prod_{j=0}^{t} M_{t-j}\right)\right) = \exp\left(\sum_{j=0}^{t} \ln M_j\right)$$

First, to find $\ln M_t$, we must diagonalize $M_t$. The eigenvalues of this matrix are:

(10) $$\lambda_{1,2}(t) = i\sin(k)\cos(t) \pm \sqrt{-\sin^2(k)\cos^2(t)+1}$$

and the corresponding normalized eigenvectors are:

(11) $$e_{1,2}(t) = \left(\cos(t)(A(t)^2 \pm 4e^{ik}\cos(k)B(t) + B(t)^2) + 1\right)^{-1/2} \begin{pmatrix} A(t) \mp B(t) \\ 2\sin(t) \end{pmatrix}$$

where $A(t) = \cos(t)(1+e^{2ik})$ and $B(t) = \sqrt{-\sin^2(k)\cos^2(t)+1}$.
This gives the change of basis matrix:

(12) $$P(t) = \begin{pmatrix} C(t)(A(t)-B(t)) & D(t)(A(t)+B(t)) \\ 2C(t)\sin(t) & 2D(t)\sin(t) \end{pmatrix}$$

where $C(t) = (\cos(t)(A(t)^2+4e^{ik}\cos(k)B(t)+B(t)^2)+1)^{-1/2}$ and $D(t) = (\cos(t)(A(t)^2-4e^{ik}\cos(k)B(t)+B(t)^2)+1)^{-1/2}$.
The inverse of the change of basis matrix is then

(13) $$P^{-1}(t) = -\frac{1}{4B(t)C(t)D(t)\sin(t)} \begin{pmatrix} 2D(t)\sin(t) & -D(t)(A(t)+B(t)) \\ -2C(t)\sin(t) & C(t)(A(t)-B(t)) \end{pmatrix}$$

So, we can compute:

$$\sum_{j=0}^{t} \ln M_j = \sum_{j=0}^{t} P(j) \begin{pmatrix} \ln(\lambda_1(j)) & 0 \\ 0 & \ln(\lambda_2(j)) \end{pmatrix} P^{-1}(j) = \begin{pmatrix} S_1 & S_2 \\ S_3 & i\pi(t+1) - S_1 \end{pmatrix}$$

where

$$S_1 = -\sum_{j=0}^{t} \frac{1}{2B(j)}[(A(j)-B(j))\ln(\lambda_1(j)) - (A(j)+B(j))\ln(\lambda_2(j))]$$
$$S_2 = \sum_{j=0}^{t} \frac{A^2(j)-B^2(j)}{4B(j)\sin(j)}[\ln(\lambda_1(j)) - \ln(\lambda_2(j))]$$
$$S_3 = -\sum_{j=0}^{t} \frac{\sin(j)}{B(j)}[\ln(\lambda_1(j)) - \ln(\lambda_2(j))]$$

Now, to find $M$, we must take the exponential of this matrix, which means we need to diagonalize it. This matrix has eigenvalues:

$$\mu_{1,2} = \pm\frac{1}{2}\left(\sqrt{-\pi^2(t+1)^2 + 4(S_2S_3 - i\pi(t+1)S_1 + S_1^2)} + i\pi(t+1)\right)$$

and corresponding eigenvectors:

(14) $$e_{1,2} = \begin{pmatrix} \frac{1}{L_\pm}(2S_3) \\ \frac{1}{L_\pm}(-2S_1 + i\pi(t+1) \pm K) \end{pmatrix}$$

where

$$K = \sqrt{-\pi^2(t+1)^2 + 4(S_2S_3 - i\pi(t+1)S_1 + S_1^2)}$$
$$L_\pm = \sqrt{4S_1(S_1 - i\pi(t+1) \mp K) + 4S_3^2 - \pi^2(t+1)^2 \pm 2i\pi K(t+1) + K^2}$$

This gives the change of basis matrix:

(15) $$P = \begin{pmatrix} \frac{2S_3}{L_+} & \frac{2S_3}{L_-} \\ \frac{1}{L_+}(-2S_1 + i\pi(t+1) + K) & \frac{1}{L_-}(-2S_1 + i\pi(t+1) - K) \end{pmatrix}$$

which has determinant:

(16) $$\det(P) = -\frac{4KS_3}{L_+L_-}$$

and inverse:

(17) $$\frac{1}{\det(P)}\begin{pmatrix} \frac{1}{L_-}(-2S_1 + i\pi(t+1) - K) & -\frac{2S_3}{L_-} \\ -\frac{1}{L_+}(-2S_1 + i\pi(t+1) + K) & \frac{2S_3}{L_+} \end{pmatrix}$$

So, we have:

$$\exp\left(\ln M_j\right) = \exp\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = P\exp\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} P^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

where

$$\begin{aligned}
p &= -\frac{1}{2K}(-2S_1 + i\pi(t+1))(e^{\mu_1} + e^{\mu_2}) - \frac{1}{2}(e^{\mu_1} - e^{\mu_2}) \\
q &= \frac{S_3}{K}(e^{\mu_1} + e^{\mu_2}) \\
r &= -\frac{1}{4KS_3}(-2S_1 + i\pi(t+1) + K)(-2S_1 + i\pi(t+1) - K)(e^{\mu_1} + e^{\mu_2}) \\
s &= p + (e^{\mu_1} - e^{\mu_2})
\end{aligned}$$

Now, we have:

$$|\tilde{\psi}(k,t)\rangle = \begin{pmatrix} p & q \\ r & s \end{pmatrix} |\tilde{\psi}(k,0)\rangle \tag{18}$$

If we take our initial state to be purely up, so $|\tilde{\psi}(k,0)\rangle = \binom{1}{0}$, this gives:

$$|\tilde{\psi}(k,t)\rangle = \begin{pmatrix} p \\ r \end{pmatrix} \tag{19}$$

Now, to find $|\psi(n,t)\rangle$, we take the inverse Fourier transform:

$$|\psi(n,t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} |\tilde{\psi}(k,t)\rangle e^{-ik\alpha t} dk = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} p \\ r \end{pmatrix} e^{-ik\alpha t} dk$$

which we can finally split up to find $a_t(n)$ and $b_t(n)$. First we deal with $a_t(n)$:

$$\begin{aligned}
a_t(n) &= \frac{1}{2\pi}\int_0^{2\pi} p e^{-ik\alpha t} dk \\
&= \int_0^{2\pi} G_1 e^{it(\frac{\pi}{2} + \frac{1}{t}K - \alpha k)} dk + \int_0^{2\pi} G_1 e^{it(\frac{\pi}{2} - \frac{1}{t}K - \alpha k)} dk + \int_0^{2\pi} G_2 e^{it(\frac{1}{2it}K + \frac{\pi}{2} - \alpha k)} dk + \int_0^{2\pi} G_3 e^{it(\frac{1}{2it}K - \frac{\pi}{2} - \alpha k)} dk
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= -\frac{1}{4\pi K}(-2S_1 + i\pi(t+1))e^{i\pi/2} \\
G_2 &= -\frac{1}{4\pi}e^{i\pi/2} \\
G_2 &= -\frac{1}{4\pi}e^{-i\pi/2}
\end{aligned}$$

To set up for the stationary phase approximation, let:

$$\phi_1 = \frac{\pi}{2} + \frac{1}{t}K - \alpha k$$
$$\phi_2 = \frac{\pi}{2} - \frac{1}{t}K - \alpha k$$
$$\phi_3 = \frac{1}{2it}K + \frac{\pi}{2} - \alpha k$$
$$\phi_4 = \frac{1}{2it}K - \frac{\pi}{2} - \alpha k$$

We attempt to solve the first integral using the stationary phase method. For large $t$, we can say that $K \approx \sqrt{(\pi t)^2 + i\pi t S_1}$, so:

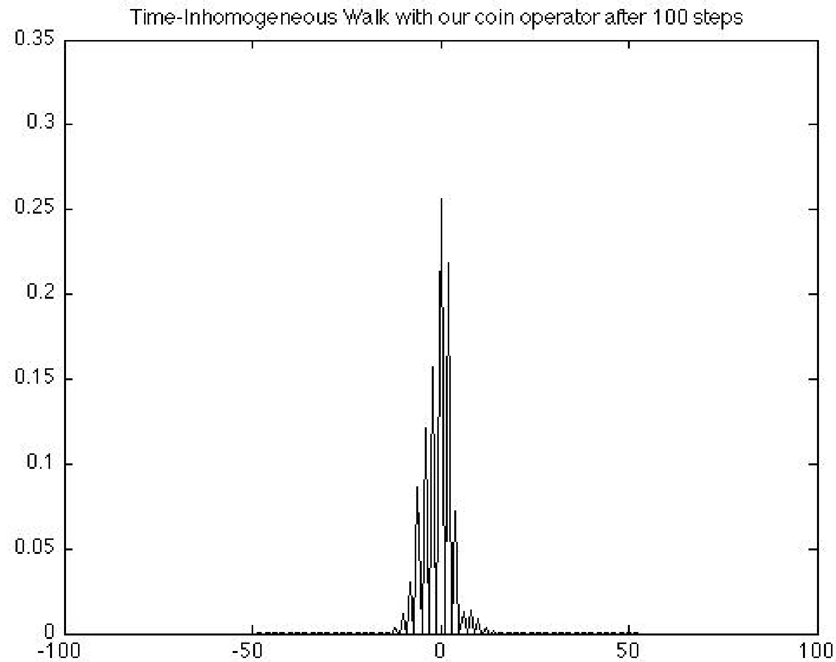
$$\phi_1 = \frac{\pi}{2} + \frac{i}{t}\sqrt{(\pi t)^2 + i\pi t S_1} - \alpha k$$

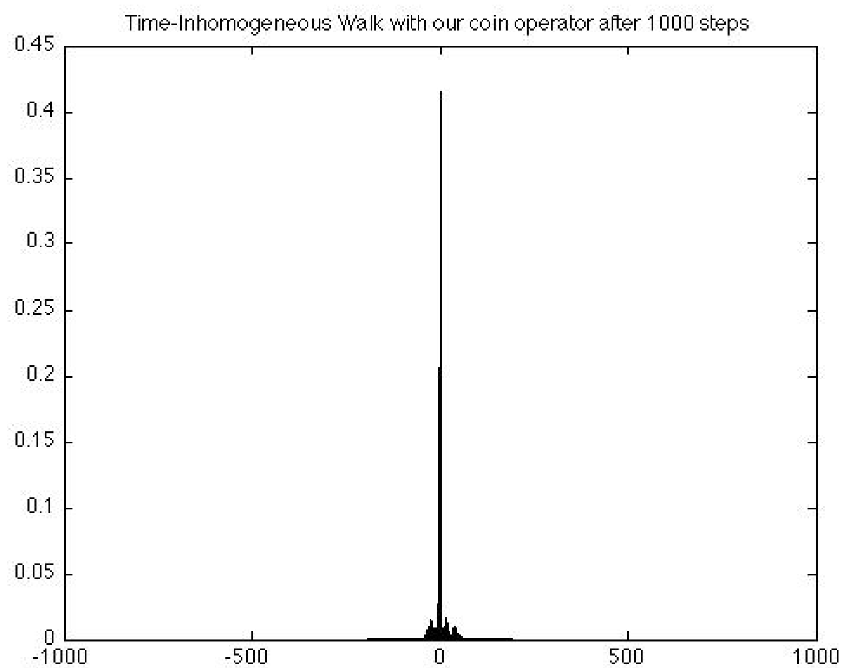
Then to find the stationary points, we must first take the derivative:

$$\begin{aligned}
\phi_1' &= -\frac{\pi}{2}((\pi t)^2 + i\pi t S_1)^{-1/2}\frac{dS_1}{dk} - \alpha \\
&= -\alpha + \left[\sum_{j=0}^{t}\left(\frac{\ln(\lambda_2(j))(2iB_j e^{2ik}\cos(j) - \cos^2(j)\cos(k)\sin(k))}{B_j^2}\right.\right. \\
&\quad + \frac{\ln(\lambda_1(j))(2iB_j e^{2ik}\cos(j) + \cos^2(j)\cos(k)\sin(k))}{B_j^2} \\
&\quad + \frac{\cos^2(j)\cos(k)\sin(k)\ln(\lambda_1(j))(A_j - B_j)}{2B_j^3} \\
&\quad + \frac{\cos^2(j)\cos(k)\sin(k)\ln(\lambda_2(j))(A_j + B_j)}{2B_j^3} \\
&\quad + \frac{(iB_j\cos(j)\cos(k) + \cos^2(j)\cos(k)\sin(k))(A_j + B_j)}{2B_j^2\lambda_2(j)} \\
&\quad \left.\left. + \frac{(iB_j\cos(j)\cos(k) - \cos^2(j)\cos(k)\sin(k))(A_j - B_j}{2B_j^2\lambda_1(j)}\right)\right] \Bigg/ \\
&\quad \left[2\left(t^2 - \frac{it}{\pi}\sum_{j=0}^{t}\left[\frac{\ln(\lambda_1(j))(A_j - B_j) + \ln(\lambda_2(j))(A_j + B_j)}{2B_j}\right]\right)^{1/2}\right]
\end{aligned}$$

We will have stationary points at values of $k$ for which this is zero. Thus far, we have been unable to find these values. Unfortunately, since this expression depends explicitly on time, finding the stationary points numerically in order to proceed with the stationary phase method is not an option. Finding the stationary points analytically will probably not be possible, so the stationary phase method is unlikely to provide a solution to this problem, and another approach is needed.

5.2. **Numerics for First Time-Inhomogeneous Problem.** Although we do not have a complete analytic solution, we do have numerical results:




5.3. **Second Time-Inhomogeneous Problem.** Since we are stuck on the first problem, we look instead at our other time-inhomogeneous coin operator:

$$D_t = \begin{pmatrix} \cos(\frac{2\pi t}{N}) & \sin(\frac{2\pi t}{N}) \\ \sin(\frac{2\pi t}{N}) & -\cos(\frac{2\pi t}{N}) \end{pmatrix}$$

The first few steps with this coin operator are very similar to those for the last coin operator; we get:

$$M_t = \begin{pmatrix} e^{ik}\cos(\frac{2\pi t}{N}) & e^{ik}\sin(\frac{2\pi t}{N}) \\ e^{-ik}\sin(\frac{2\pi t}{N}) & -e^{-ik}\cos(\frac{2\pi t}{N}) \end{pmatrix}$$

Then we have:

$$|\tilde{\Psi}(k,t)\rangle = \left(\prod_{j=0}^{t} M_{t-j}\right)|\tilde{\Psi}(k,0)\rangle$$

But with this coin operator, notice that we actually only have $N$ different matrices. For now, we consider only $N = 3$. Then we have three cases. For $t \equiv 0 \bmod 3$ we have:

$$|\tilde{\Psi}(k,t)\rangle = M_0(M_2M_1M_0)^{t/3}$$

For $t \equiv 1 \bmod 3$, we have:

$$|\tilde{\Psi}(k,t)\rangle = M_1M_0(M_2M_2M_0)^{(t-1)/3}$$

And for $t \equiv 2 \bmod 3$, we have:

$$|\tilde{\Psi}(k,t)\rangle = (M_2M_1M_0)^{(t+1)/3}$$

Note that:

$$M_2M_1M_0 = \begin{pmatrix} e^{ik}(-\frac{3}{4}+\frac{1}{4}e^{2ik}) & -e^{-ik}(-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}e^{2ik} \\ e^{ik}(\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}e^{-2ik}) & -e^{-ik}(-\frac{3}{4}+\frac{1}{4}e^{-2ik}) \end{pmatrix}$$

which has eigenvalues:

$$-\frac{1}{8}e^{-3ik}\left((e^{2ik}-1)^3 \pm \sqrt{64e^{6ik}+(e^{2ik}-1)^6}\right)$$

and corresponding eigenvectors:

$$\begin{pmatrix} A \pm B \\ 2\sqrt{2}(e^{2ik}+e^{4ik}) \end{pmatrix}$$

where:

$$A = (e^{2ik}-1)^3$$
$$B = 2\sqrt{2}\sqrt{e^{6ik}\cos^2(k)(15-8\cos(2k)+\cos(4k)}$$

Then the change of basis matrix is:

$$\begin{pmatrix} \frac{1}{L_+}(A+B) & \frac{1}{L_-}(A-B) \\ \frac{1}{L_+}(2\sqrt{2}(e^{2ik}+e^{4ik})) & \frac{1}{L_-}(2\sqrt{2}(e^{2ik}+e^{4ik})) \end{pmatrix}$$

which has determinant:

$$\det(P) = \frac{16}{L_+L_-}(e^{2ik}+e^{4ik})B$$

and inverse:

$$\frac{1}{\det(P)} \begin{pmatrix} \frac{1}{L_-}(2\sqrt{2}(e^{2ik}+e^{4ik})) & -\frac{1}{L_-}(A-B) \\ -\frac{1}{L_+}(2\sqrt{2}(e^{2ik}+e^{4ik})) & \frac{1}{L_+}(A+B) \end{pmatrix}$$

So, considering first the case where $t \equiv 0 \bmod 3$, and starting in a purely up state, we have:

$$|\tilde{\psi}(k,t)\rangle = M_0(M_2M_1M_0)^{t/3} = M_0P^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{a}_t(k) \\ \tilde{b}_t(k) \end{pmatrix}$$

where:

$$\begin{aligned} \tilde{a}_t(k) &= \frac{1}{2}\left(\left(\frac{A}{B}+1\right)\lambda_1^{t/3}e^{ik} + \left(\frac{A}{B}-1\right)\lambda_2^{t/3}e^{-ik}\right) \\ \tilde{b}_t(k) &= \frac{1}{2}\left(-\frac{L_-}{L_+}\left(\frac{A}{B}+1\right)\lambda_1^{t/3}e^{ik} - \frac{L_-}{L_+}\left(\frac{A}{B}+1\right)\lambda_2^{t/3}e^{-ik}\right) \end{aligned}$$

Now we can attempt to take the inverse Fourier transform to find $a_t(n)$ and $b_t(n)$. First we work with $a_t(n)$:

$$\begin{aligned} a_t(n) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{a}_t(k)e^{-ik\alpha t}dk \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left(\frac{A}{B}+1\right)\lambda_1^{t/3}e^{ik}e^{-ik\alpha t}dk + \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(\frac{A}{B}-1\right)\lambda_2^{t/3}e^{-ik}e^{-ik\alpha t}dk \\ &= \frac{1}{4\pi}\int_0^{2\pi}\left(\frac{A}{B}+1\right)e^{it((1/3i)\ln(\lambda_1)-k\alpha)}e^{ik}dk + \frac{1}{4\pi}\int_0^{2\pi}\left(\frac{A}{B}-1\right)e^{it((1/3i)\ln(\lambda_2)-k\alpha)}e^{ik}dk \end{aligned}$$

So, to set up for the stationary phase method, we take:

$$\begin{aligned} \phi_1 &= \frac{1}{3i}\ln(\lambda_1) - k\alpha \\ \phi_2 &= \frac{1}{3i}\ln(\lambda_2) - k\alpha \end{aligned}$$

Then

$$\phi_1' = \frac{1}{3i\lambda_1}(3i\sin^2(k)\cos(k) + \frac{1}{2}(1-\sin^6(k))^{-1/2}(-6\sin^5(k)\cos(k)) - k$$
$$\phi_2' = \frac{1}{3i\lambda_2}(3i\sin^2(k)\cos(k) - \frac{1}{2}(1-\sin^6(k))^{-1/2}(-6\sin^5(k)\cos(k)) - k$$

Although these expressions are much simpler than those for the other time-inhomogeneous coin, we have unfortunately still not been able to solve for these stationary points. However, even if we cannot obtain the stationary points analytically, these expressions do not depend on time, so they can be solved for numerically in order to continue using the stationary phase method.

## 6. Space-Inhomogenous Problem

Along with looking at coin operators that change with time, we explored coin operators that vary with position. For inhomogeneity of space, the generic coin operator is

$$C_n = \begin{pmatrix} \sqrt{f(n)} & \sqrt{1-f(n)} \\ \sqrt{1-f(n)} & -\sqrt{f(n)} \end{pmatrix}.$$

Specifically, we focused our research on the coin operator with strictly positive square roots where $f(n) = \cos^2(\omega n + \varphi)$ and $\omega = 1$, $\varphi = 0$. That is,

$$C_n = \begin{pmatrix} \cos(n) & \sin(n) \\ \sin(n) & -\cos(n) \end{pmatrix}.$$

The following recurrence relations define the coefficients $a$ and $b$ at a particular time and location:

$$a_t(n) = a_{t-1}(n-1)\cos(n-1) + b_{t-1}(n-1)\sin(n-1) \tag{20}$$

$$b_t(n) = a_{t-1}(n+1)\sin(n+1) - b_{t-1}(n+1)\cos(n+1). \tag{21}$$

Ideally, we would be able to calculate $\binom{a_t(n)}{b_t(n)}$ at any position $n$ and at any time $t$ based on $\binom{a_0(n)}{b_0(n)}$, the initial state of the walker at the location $n$. However, this is not so straightforward, because the coin operator is different at every location $n$ and multiplying different coin operators together leads to long products of sines and cosines that do not simplify or converge.

To get around this problem, we tried numerous techniques, some more promising than others.

### 6.1. Z Transform with Respect to Time.

We initially tried taking a Z transform of the recurrence relations with respect to time. This gave us the recurrence relations:

$$\hat{a}_z(n) = \sum_{t=-\infty}^{\infty} z^{-t}\, a_t(n) = z\left(\cos(n-1)\,\hat{a}_z(n-1) + \sin(n-1)\,\hat{b}_z(n-1)\right)$$

$$\hat{b}_z(n) = \sum_{t=-\infty}^{\infty} z^{-t}\, b_t(n) = z\left(\sin(n+1)\,\hat{a}_z(n-1) - \cos(n+1)\,\hat{b}_z(n-1)\right).$$

In matrix form, this is:

$$\begin{pmatrix} \hat{a}_z(n) \\ \hat{b}_z(n) \end{pmatrix} = z\begin{pmatrix} \cos(n-1) & \sin(n-1) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{a}_z(n-1) \\ \hat{b}_z(n-1) \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ \sin(n+1) & -\cos(n+1) \end{pmatrix}\begin{pmatrix} \hat{a}_z(n+1) \\ \hat{b}_z(n+1) \end{pmatrix}.$$

Then, we worked backward and replaced $\binom{\hat{a}_z(n-1)}{\hat{b}_z(n-1)}$ and $\binom{\hat{a}_z(n+1)}{\hat{b}_z(n+1)}$ with their respective recursions, and so on. However, we once again found ourselves dealing with matrix elements that consisted of products of sines and cosines that did not simplify.

6.2. **Fourier Transform with Respect to Position.** We returned to our initial recurrence and instead applied a Fourier transform with respect to position. This yielded the recurrence relations:

$$\tilde{a}_t(k) = \frac{1}{2}e^{ik}(\tilde{a}_{t-1}(k-1) + i\tilde{b}_{t-1}(k-1) + \tilde{a}_{t-1}(k+1) - i\tilde{b}_{t-1}(k+1))$$

$$\tilde{b}_t(k) = \frac{i}{2}e^{-ik}(\tilde{a}_{t-1}(k-1) + i\tilde{b}_{t-1}(k-1) - \tilde{a}_{t-1}(k+1) + i\tilde{b}_{t-1}(k+1)).$$

In matrix form, this is:

$$\begin{pmatrix}\tilde{a}_t(k)\\ \tilde{b}_t(k)\end{pmatrix} = \frac{1}{2}M_-\begin{pmatrix}\tilde{a}_{t-1}(k-1)\\ \tilde{b}_{t-1}(k-1)\end{pmatrix} + \frac{1}{2}M_+\begin{pmatrix}\tilde{a}_{t-1}(k+1)\\ \tilde{b}_{t-1}(k+1)\end{pmatrix} \tag{22}$$

where

$$M_- = \begin{pmatrix} e^{ik} & ie^{ik}\\ ie^{-ik} & -e^{-ik}\end{pmatrix} \text{and } M_+ = \begin{pmatrix} e^{ik} & -ie^{ik}\\ -ie^{-ik} & -e^{-ik}\end{pmatrix}.$$

Notice the similarities between $M_-$ and $M_+$. The matrix elements are exactly the same except that the entries not on the diagonal are positive for $M_-$ but are negative for $M_+$. Now, if we replace $\binom{\tilde{a}_{t-1}(k-1)}{\tilde{b}_{t-1}(k-1)}$ and $\binom{\tilde{a}_{t-1}(k+1)}{\tilde{b}_{t-1}(k+1)}$ with their respective recurrence relations, and so on, predictable patterns begin to appear.

6.2.1. *The Middle Term.* Given $\binom{\tilde{a}_{t-l}(k)}{\tilde{b}_{t-l}(k)}$ on the right side of equation (**??**), where $l$ is of course even, the 2x2 coefficient matrix will be diagonal. Now, we want to be able to determine the diagonal entries. The middle terms for 2, 4, 6, and 8 steps backward are:

$$\begin{aligned}
\vec{v}_t(k) &\to \frac{1}{2}(e^{ik}+e^{-ik})\begin{bmatrix} e^{ik} & 0\\ 0 & e^{-ik}\end{bmatrix}\vec{v}_{t-2}(k)\\
&\to \frac{1}{8}(e^{ik}+e^{-ik})\begin{bmatrix} e^{-ik}+3e^{3ik} & 0\\ 0 & e^{ik}+3e^{-3ik}\end{bmatrix}\vec{v}_{t-4}(k)\\
&\to \frac{1}{32}(e^{ik}+e^{-ik})\begin{bmatrix} 4e^{ik}+2e^{-3ik}+10e^{5ik} & 0\\ 0 & 4e^{-ik}+2e^{3ik}+10e^{-5ik}\end{bmatrix}\vec{v}_{t-6}(k)\\
&\to \frac{1}{128}(e^{ik}+e^{-ik})\begin{bmatrix} 9e^{-ik}+15e^{3ik}+5e^{-5ik}+35e^{7ik} & 0\\ 0 & 9e^{ik}+15e^{-3ik}+5e^{5ik}+35e^{-7ik}\end{bmatrix}\vec{v}_{t-8}(k)
\end{aligned}$$

Observe that the lower right entries are similar to the upper left entries except that the exponential is conjugate. Therefore, we need only to observe the pattern of the coefficients of the upper left entries. This table documents the coefficients for 2, 4, 6, 8, and 10 steps backward:

| | $e^{-7ik}$ | $e^{-5ik}$ | $e^{-3ik}$ | $e^{-ik}$ | $e^{ik}$ | $e^{3ik}$ | $e^{5ik}$ | $e^{7ik}$ | $e^{9ik}$ |
|---|---|---|---|---|---|---|---|---|---|
| $t-2$ | | | | | 1 | | | | |
| $t-4$ | | | | 1 | | 3 | | | |
| $t-6$ | | | 2 | | 4 | | 10 | | |
| $t-8$ | | 5 | | 9 | | 15 | | 35 | |
| $t-10$ | 14 | | 24 | | 36 | | 56 | | 126 |

These entries can also be represented like this:

| | $e^{-7ik}$ | $e^{-5ik}$ | $e^{-3ik}$ | $e^{-ik}$ | $e^{ik}$ | $e^{3ik}$ | $e^{5ik}$ | $e^{7ik}$ | $e^{9ik}$ |
|---|---|---|---|---|---|---|---|---|---|
| $t-2$ | | | | | 1 | | | | |
| $t-4$ | | | | 2-1 | | 2+1 | | | |
| $t-6$ | | | 6-4 | | 4 | | 6+4 | | |
| $t-8$ | | 20-15 | | 12-3 | | 12+3 | | 20+15 | |
| $t-10$ | 70-56 | | 40-16 | | 36 | | 40+16 | | 70+56 |

In this way, a pattern becomes evident:

| | $e^{-7ik}$ | $e^{-5ik}$ | $e^{-3ik}$ | $e^{-ik}$ | $e^{ik}$ | $e^{3ik}$ | $e^{5ik}$ | $e^{7ik}$ | $e^{9ik}$ |
|---|---|---|---|---|---|---|---|---|---|
| $t-2$ | | | | | $\binom{0}{0}$ | | | | |
| $t-4$ | | | | $\binom{2}{1}-\frac{1}{2}\binom{2}{1}$ | | $\binom{2}{1}+\frac{1}{2}\binom{2}{1}$ | | | |
| $t-6$ | | | $\binom{4}{2}-\frac{2}{3}\binom{4}{2}$ | | $2!\binom{2}{1}$ | | $\binom{4}{2}+\frac{2}{3}\binom{4}{2}$ | | |
| $t-8$ | | $\binom{6}{3}-\frac{3}{4}\binom{6}{3}$ | | $2!\left(\binom{4}{2}-\frac{1}{4}\binom{4}{2}\right)$ | | $2!\left(\binom{4}{2}+\frac{1}{4}\binom{4}{2}\right)$ | | $\binom{6}{3}+\frac{3}{4}\binom{6}{3}$ | |
| $t-10$ | $\binom{8}{4}-\frac{4}{5}\binom{8}{4}$ | | $2!\left(\binom{6}{3}-\frac{2}{5}\binom{6}{3}\right)$ | | $3!\binom{4}{2}$ | | $2!\left(\binom{6}{3}+\frac{2}{5}\binom{6}{3}\right)$ | | $\binom{8}{4}+\frac{4}{5}\binom{8}{4}$ |

This pattern for the upper left matrix elements goes beyond $l = 10$ and is applicable to the bottom right matrix entry as well. In particular, if $\frac{l}{4} = u$ where $u \in \mathbb{Z}^+ \setminus \{0\}$, then the upper left matrix entries are given by

$$\sum_{p=1}^{\frac{l}{4}} p!\left[\left(A - \frac{l-(4p-2)}{l}A\right)e^{((4p-1)-l)ik} + \left(A + \frac{l-(4p-2)}{l}A\right)e^{(l-(4p-3))ik}\right] \tag{23}$$

where $A = \binom{l-2p}{\frac{l-2p}{2}}$, and the bottom right entries are given by

$$\sum_{p=1}^{\frac{l}{4}} p!\left[\left(A - \frac{l-(4p-2)}{l}A\right)e^{-((4p-1)-l)ik} + \left(A + \frac{l-(4p-2)}{l}A\right)e^{-(l-(4p-3))ik}\right]. \tag{24}$$

If $\frac{l-2}{4} = v$ where $v \in \mathbb{Z}^+$, then the upper left matrix entries are given by

$$(25) \quad w!\binom{2w-2}{w-1}e^{ik} + \sum_{q=1}^{\frac{l-2}{4}} q!\left[\left(B + \frac{l-(4p-2)}{l}B\right)e^{((4p-1)-l)ik} + \left(B + \frac{l-(4p-2)}{l}B\right)e^{(l-(4p-3))ik}\right]$$

where $B = \binom{l-2q}{\frac{l-2q}{2}}$ and $w = \frac{l+2}{4}$, and the bottom right entries are given by

$$(26) \quad w!\binom{2w-2}{w-1}e^{ik} + \sum_{q=1}^{\frac{l-2}{4}} q!\left[\left(B + \frac{l-(4p-2)}{l}B\right)e^{-((4p-1)-l)ik} + \left(B + \frac{l-(4p-2)}{l}B\right)e^{-(l-(4p-3))ik}\right].$$

These take care of the coefficient matrices of $\binom{\tilde{a}_{t-l}(k)}{\tilde{b}_{t-l}(k)}$, but what about the coefficient matrices of $\binom{\tilde{a}_{t-l}(j)}{\tilde{b}_{t-l}(j)}$ where $j \neq k$? As it turns out, there is much symmetry involved here.

6.2.2. *The Periphery Terms.* For any $s \in \mathbb{Z}^+$, if

$$\begin{pmatrix} \tilde{a}_{t-l}(k-s) \\ \tilde{b}_{t-l}(k-s) \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

then

$$\begin{pmatrix} \tilde{a}_{t-l}(k+s) \\ \tilde{b}_{t-l}(k+s) \end{pmatrix} = \begin{pmatrix} E & -F \\ -G & H \end{pmatrix}$$

by symmetry. This suggests that we consider an initial wavefunction that is symmetric, meaning $\binom{\tilde{a}_0(k-s)}{\tilde{b}_0(k-s)} = \binom{\tilde{a}_0(k+s)}{\tilde{b}_0(k+s)}$ for all $s \in \mathbb{Z}^+$. This way, the coefficient operator matrix will be diagonal.

In order to make use of this, however, we need to be able to determine the diagonal entries of the coefficient matrix. We already know the central entries based on equations (**??**) through (**??**). The wavefront entries, that is the entries that correspond to the $\binom{\tilde{a}_0(k-s)}{\tilde{b}_0(k-s)}$ of largest $s$ such that $\binom{\tilde{a}_0(k-s)}{\tilde{b}_0(k-s)} \neq \binom{0}{0}$, follow the pattern:

$$\frac{1}{2^{l-1}}\lambda^{l-1}\begin{pmatrix} e^{ik} & 0 \\ 0 & -e^{-ik} \end{pmatrix}.$$

The second wavefront entries, that is the entries that correspond to the $\binom{\tilde{a}_0(k-d)}{\tilde{b}_0(k-d)}$ of largest $d$ such that $\binom{\tilde{a}_0(k-(d+1))}{\tilde{b}_0(k-(d+1))} \neq \binom{0}{0}$, follow the pattern:

$$\frac{1}{2^{l-1}}\lambda^{l-3}\gamma\begin{pmatrix} le^{2ik}+(l-4) & 0 \\ 0 & -\left((l-4)+le^{-2ik}\right) \end{pmatrix}.$$

We have not yet explored the other wavefornts, but we do believe that they, too, will have predicatable patterns. Then we would be able to solve the system of equations:

$$\begin{bmatrix} \tilde{a}_t(k) \\ \tilde{b}_t(k) \\ \tilde{a}_t(k-1)+\tilde{a}_t(k+1) \\ \tilde{b}_t(k-1)+\tilde{b}_t(k+1) \\ \tilde{a}_t(k-2)+\tilde{a}_t(k+2) \\ \tilde{b}_t(k-2)+\tilde{b}_t(k+2) \\ \vdots \end{bmatrix} = D^t \begin{bmatrix} \tilde{a}_0(k) \\ \tilde{b}_0(k) \\ \tilde{a}_0(k-1)+\tilde{a}_0(k+1) \\ \tilde{b}_0(k-1)+\tilde{b}_0(k+1) \\ \tilde{a}_0(k-2)+\tilde{a}_0(k+2) \\ \tilde{b}_0(k-2)+\tilde{b}_0(k+2) \\ \vdots \end{bmatrix}$$

where $D$ is the diagonal matrix. Finally, taking the inverse Fourier transform would allow us to find the coefficients in normal space, as desired.

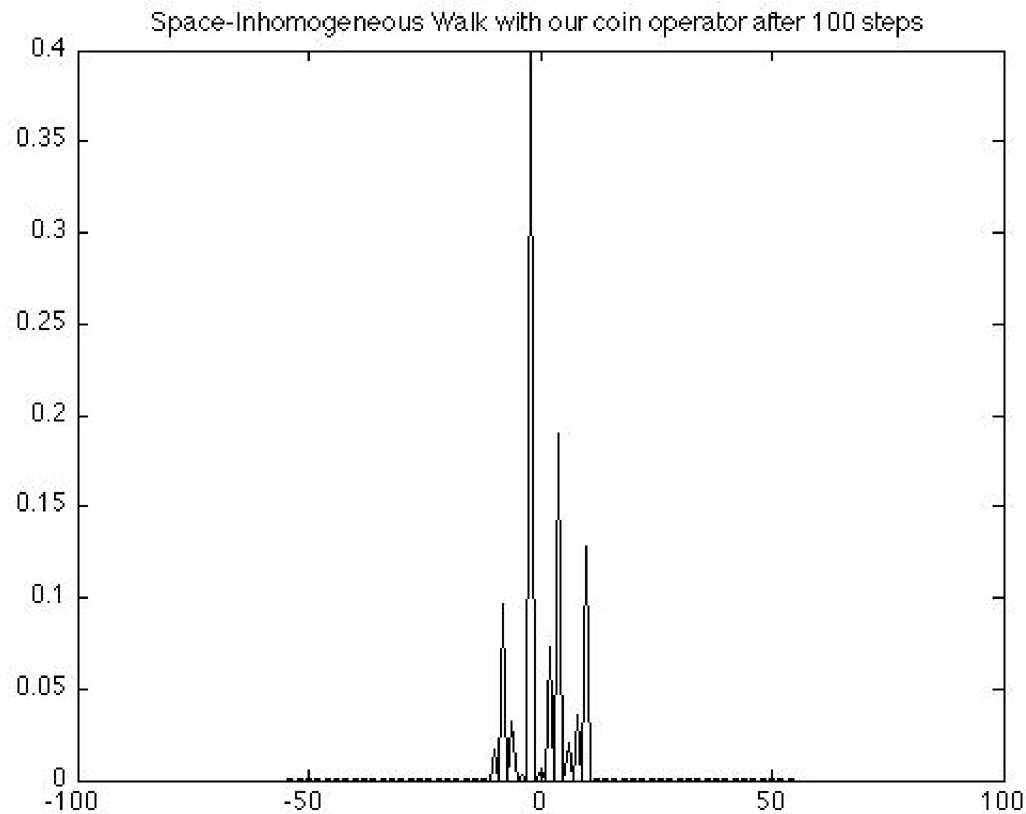
However, the requirement that the initial wavefunction be symmetric limits the usefulness of this method. Therefore, we continued to explore other options.

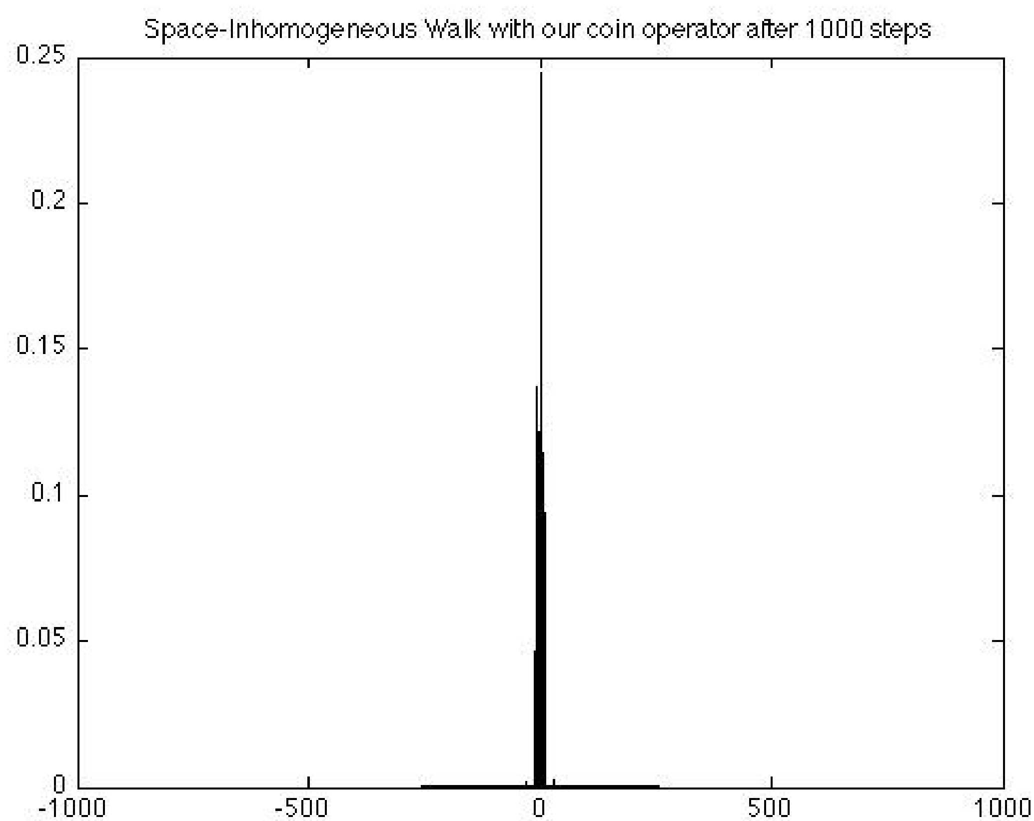
6.3. **Pentadiagonal Operator.** We once again returned to our initial recurrence relations, (**??**) and (**??**). This time, using the recurrence relations, we constructed an operator to be applied to the wavefunction at time $t-1$ that would produce the wavefunction at time $t$:

$$\begin{bmatrix} \vdots \\ a_t(n-2) \\ b_t(n-2) \\ a_t(n-1) \\ b_t(n-1) \\ a_t(n) \\ b_t(n) \\ a_t(n+1) \\ b_t(n+1) \\ a_t(n+2) \\ b_t(n+2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & & & & & \\ & 0 & 0 & 0 & & & & & & & & \\ & 0 & 0 & \sin(n-1) & -\cos(n-1) & & & & & & & \\ & \cos(n-2) & \sin(n-2) & 0 & 0 & 0 & & & & & & \\ & & 0 & 0 & 0 & \sin(n) & -\cos(n) & & & & & \\ & & & \cos(n-1) & \sin(n-1) & 0 & 0 & 0 & & & & \\ & & & & 0 & 0 & 0 & \sin(n+1) & -\cos(n+1) & & & \\ & & & & & \cos(n) & \sin(n) & 0 & 0 & 0 & & \\ & & & & & & 0 & 0 & 0 & \sin(n+2) & -\cos(n+2) & \\ & & & & & & & \cos(n+1) & \sin(n+1) & 0 & 0 & \\ & & & & & & & & 0 & 0 & 0 & \\ & & & & & & & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ a_{t-1}(n-2) \\ b_{t-1}(n-2) \\ a_{t-1}(n-1) \\ b_{t-1}(n-1) \\ a_{t-1}(n) \\ b_{t-1}(n) \\ a_{t-1}(n+1) \\ b_{t-1}(n+1) \\ a_{t-1}(n+2) \\ b_{t-1}(n+2) \\ \vdots \end{bmatrix}$$

Notice that there are four non-zero diagonals. The topmost diagonal has the same entries as the bottommost diagonal except the entries are negative. The elements of the middle two non-zero diagonals are similar to each other, as well. As of yet, we have not had time to explore the effect of taking $t$ iterates of the operator or to find other ways of representing the operator that would make solving $|\psi(t)\rangle = O^t|\psi(0)\rangle$ easier, but this method seems to have potential.

6.4. **Numerics.** Even though we have not yet found analytical solutions, numerical results give us some indication of the probability distribution of our quantum walk.


Space-Inhomogeneous Walk with our coin operator after 100 steps
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-100
-50
0
50
100


Space-Inhomogeneous Walk with our coin operator after 1000 steps
0.25
0.2
0.15
0.1
0.05
0
-1000
-500
0
500
1000

It appears that the probability distribution is sub-ballistic and disperses very slowly from the origin. This makes sense in the context of the our problem. Hopefully, in the future we will be able to find analytical solutions that match these numerics.

6.5. **Future Work.** Spatially inhomogenous coin operators have not extensively been explored. Our work focused on one particular spatially inhomogenous coin operator and its effect on the probability distribution of a quantum walk. Although the numerics suggest to us that the probability distribution is sub-ballistic, proving this analytically is difficult. The methods we tried that seem to produce the most promising results and warrant more attention in the future are the pentadiagonal operator and the diagonal operator that is applied to a symmetric wavefunction.

## Appendix A. Stationary Phase

Here is a useful procedure which shows up repeatedly in our work:
The idea here is that we want to evaluate an integral of the form:

$$I = \int_{-\infty}^{\infty} f(x) e^{it\phi(x)} dx, \tag{27}$$

where the phase $\phi(x)$ must tend to vary very quickly over the range of integration, and where $f(x)$ must be slow by comparison. Any area where $e^{it\phi(x)}$ is oscillating rapidly will contribute very little (almost nothing) to the whole integral, because positive and negative contributions will cancel each other out. However, from points of stationary phase where $\frac{d\phi}{dx} = 0$, we do accrue significant contributions to the whole integral. Call these points $x_i$. Note that since we need $f(x)$ to vary even slower than $\phi(x)$, its main contributions to the integral also come from the $f(x_i)$ terms.

Near one we have the following taylor series approximation for $\phi(x)$:

(1) $\phi(x) = \phi(x_i) + \frac{d\phi}{dx}|_{x=x_i}(x - x_i) + \frac{1}{2}\frac{d^2\phi}{dx^2}|_{x=x_i} + O(x^3)$
(2) But since $\frac{d\phi}{dx}|_{x=x_i} = 0$ this reduces to:
(3) $\phi(x) = \phi(x_i) + \frac{1}{2}\frac{d^2\phi}{dx^2}|_{x=x_i} + O(x^3)$.

Disregarding higher order terms, we have the following approximation for the integral:

$$I \approx \int_{-\infty}^{\infty} f(x_i) e^{it\phi(x_i)} e^{it\frac{1}{2}\phi''(x_i)(x-x_i)^2} dx \tag{28}$$

Cleaning up by pulling out constants, we have:

$$I \approx f(x_i) e^{it\phi(x_i)} \int_{-\infty}^{\infty} e^{it\frac{1}{2}\phi''(x_i)(x-x_i)^2} dx \tag{29}$$

which is a Gaussian integral contributing $\sqrt{\frac{2\pi}{t\phi''(x_x)}} e^{i\frac{\pi}{4}}$, so that our whole integral is now:

$$I \approx f(x_i)e^{it\phi(x_i)}\sqrt{\frac{2\pi}{t\phi''(x_x)}}e^{i\frac{\pi}{4}\mu} \tag{30}$$

where $\mu = sgn(\phi''(k_s))$

## APPENDIX B. THE INTEGRALS

Given a linear speed of propagation, we substitute $n = \alpha t$, so the integral for $a_t(k)$ in its entirety is:

$$a_t(k) = \frac{1}{2\pi}\left(\int_0^{2\pi} \lambda_1^t e^{-ik\alpha t}\frac{\cos(2k)+F(k)}{2F(k)}dk + \int_0^{2\pi} \lambda_2^t e^{-ik\alpha t}\frac{F(k)-\cos(2k)}{2F(k)}dk\right) = \frac{1}{2\pi}(I_1+I_3). \tag{31}$$

Having noticed that the $\lambda_{1,2}$ terms oscillate very quickly at large $t$ as compared to the other terms, we concentrate first on getting $\lambda_{1,2}^t$ in a form which is amenable to the method of stationary phase using first $\lambda_1$ as our example:

(1) $\lambda_1^t = B^{t/2}(2\cos^2(k) + \sqrt{(2\cos^2(k))^2 - \frac{1}{B}})^t$.
(2) For now, substitute in $x$ for $2\cos^2(k)$.
(3) Doing so, we have $\lambda_1^t = \exp(it\frac{\ln(B^{1/2}(x+\sqrt{x^2-\frac{1}{B}}))}{i})$, set $\phi(k) = \frac{\ln(B^{1/2}(x+\sqrt{x^2-\frac{1}{B}}))}{i} - k\alpha$, so that we have $\exp(it\phi(k))$,
(4) Similarly, we define $\xi_k = \frac{\ln(B^{1/2}(x-\sqrt{x^2-\frac{1}{B}})}{i} + k\alpha$ for $\lambda_2^t$.

Now, the method of stationary phase requires first that we solve for $\frac{d\phi}{dk} = 0$ and $\frac{d\xi}{dk} = 0$.
Here is the procedure for $\phi(k)$:

(1) $\phi(k) = \frac{1}{i}\ln(B^{1/2}(x+\sqrt{x^2-\frac{1}{B}})) - k\alpha$, set $\omega(k) = \frac{1}{i}\ln(B^{1/2}(x+\sqrt{x^2-\frac{1}{B}}))$.
(2) We want to solve $\omega'(k) - i\alpha = 0$, which after evaluating the derivative is $i\alpha = \frac{-4\sin(k)\cos(k)}{\sqrt{4\cos^4(k)-\frac{1}{B}}}$.
(3) Solving this gives us $k_s = \pm\arccos(\pm\sqrt{\frac{1}{2(\alpha^2-4)}(-4\pm\sqrt{\frac{\alpha^4}{B}-\frac{4\alpha^2}{B}+16})}$.
(4) Note that $x(k) = 2\cos^2(k)$, so we have $x(k_s) = \frac{1}{\alpha^2-4}(-4\pm\sqrt{\frac{\alpha^4}{B}-\frac{4\alpha^2}{B}+16})$. Note that we actually have multiple stationary points- four for each of the plus-minus terms that shows up in $x(k_s)$- most of which are invisible, but over which we will, eventually, be taking a sum.

A similar analysis for $\xi(k)$ shows that its stationary points are the same ones! (It differs only by a sign in front of the square root, which leaves the leading $\frac{dx}{dk}$ term unchanged) So, we are now nearly ready to take the integral, after a short preliminary analysis yielding bounds on $\alpha$.

(1) Disregarding the purely k-dependent terms in the integral, it can be seen that the exponential function's integral shrinks suddenly down to zero for $|\alpha|$ much larger than $\frac{1}{\sqrt{2}}$.
(2) So, we constrain will ourselves to $\frac{-1}{\sqrt{2}} \leq \alpha \leq \frac{1}{\sqrt{2}}$.

(3) In practice, our integral will turn out to grow rapidly outside of an even smaller interval in $\alpha$ that depends on the value of $q$.

Now, the integration:

(1) By the method of stationary phase, we have that for large $t$

$$\int_a^b \lambda_1^t G(k)dk \approx \sum_{k_s} G(k_s)e^{it\phi(k_s)}\sqrt{\frac{2\pi}{t\phi''(k_s)}}e^{i\frac{\pi}{4}\mu} \tag{32}$$

(2) We begin by analyzing $\phi(k_s) = \frac{1}{i}\ln(B^{1/2}(x(k_s)+\sqrt{x^2(k_s)-\frac{1}{B}})) - k_s\alpha$, where explicitly $x(k_s) = \frac{1}{\alpha^2-4}(-4\pm\sqrt{\frac{\alpha^4}{B}-\frac{4\alpha^2}{B}+16})$.

(3) We have $e^{it\phi(k_s)} = \lambda_1^t e^{-it\alpha k_s}$.

(4) We next calculate $\frac{d^2\phi}{dk^2}$ at the stationary points.

(5) Simplifying yields $\frac{d^2\phi}{dk^2} = \frac{i(2x(k)-2)(2+4B)+B(3+\frac{x^2(k)}{4}-\frac{3}{8}-\frac{1}{2}(x(k)-1))}{B(\frac{1}{B}+x^2(k))^{(3/2)}}$.

(6) And $\frac{d^2\phi}{dk^2}|_{k=k_s}$

(7) For the spatial portion, we have $G(k_s) = \frac{(\cos 2k_s+F(k_s))}{2F(k_s)} = (x(k_s)-1)(1+2B)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}} + (\frac{1}{B}+1)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}}+\frac{1}{2}$

(8) Now we are ready to approximate our integral $\sum_{k_s} G(k_s)e^{it\phi(k_s)}\sqrt{\frac{2\pi}{t\phi''(k_s)}}e^{i\frac{\pi}{4}\mu}$:

  (a) Are there any more simplifications?

We have for $I_1 = \sum_{k_s} G(k_s)e^{it\phi(k_s)}\sqrt{\frac{2\pi}{t\phi''(k_s)}}e^{i\frac{\pi}{4}\mu}$:

$$\begin{aligned} &e^{i\frac{\pi}{4}\mu}\sqrt{2\pi}\sum_{k_s}\frac{\lambda_1^t(k_s)}{\sqrt{t}}\left((x(k_s)-1)(1+2B)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}}+(\frac{1}{B}+1)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}}+\frac{1}{2}\right)\times \\ &\qquad e^{-i\alpha t k_s}\left(\frac{2i(2+4B(x(k_s)-1)+B(\frac{x^2(k_s)}{4}-\frac{x(k_s)}{2}+\frac{17}{8}))}{B(\frac{1}{B}+x^2(k_s))^{3/2}}\right)^{-1/2} \end{aligned} \tag{33}$$

Finally, we are led to examine $I_3$, whose analysis uses- for the most part- material already gathered for $I_1$, so refer back for some specific values as needed.

(1) $I_3 = \int_0^{2\pi} \lambda_2^t e^{-ik\alpha t}\frac{F(k)-\cos(2k)}{2F(k)}dk$.

(2) Using $x = 2\cos^2(k)$, we have $\lambda_2^t = B^{t/2}(x-\sqrt{x^2-\frac{1}{B}})^t$ and our important logarithm is $\xi(k) = \frac{\ln B^{1/2}(x-\sqrt{x^2-\frac{1}{B}})}{i} - k\alpha$, so that we can immediately see **the only difference is one sign**.

(3) Carrying through as before (refer to above), we have the same stationary points (they are unaffected by the sign change).

(4) We still wish to evaluate using stationary phase for large $t$, so

$$\int_0^{2\pi} \lambda_2^t M(k) \approx \sum_{k_s} M(k_s) e^{it\xi(k_s)} \sqrt{\frac{2\pi}{t\xi''(k_s)}} e^{i\frac{\pi}{4}} \tag{34}$$

(5) As before, we calculate $\xi''(k_s)$:

(a) Changing the appropriate sign yields $\frac{d^2\xi}{dk^2} = \frac{d^2x}{dk^2} \frac{1-\frac{x}{\sqrt{x^2-\frac{1}{B}}}}{i(x-\sqrt{x^2-\frac{1}{B}}} + \frac{dx}{dk}\frac{d}{dk}(\frac{1-\frac{x}{\sqrt{x^2-\frac{1}{B}}}}{i(x-\sqrt{x^2-\frac{1}{B}}})$.

(b) Cleaning yields eventually that $\xi''(k_s) = -\phi''(k_s)$.

(6) Then the purely spatial part (the difference between $\xi(k_s)$ and $\phi(k_s)$ themselves is simple enough that we can omit this step):

(a) We now have $\frac{1}{2} - \frac{\cos(2k)}{2F(k)}$ for the spatial portion, which is just $\frac{1}{2} - G(k_s)$, where $G(k)$ is the other spatial portion.

Putting this all together yields:

$I_3 = \sum_{k_s} (\frac{1}{2} - G(k_s)) e^{it\xi(k_s)} \sqrt{\frac{2\pi}{t\xi''(k_s)}} e^{i\frac{\pi}{4}\mu} =$

$$e^{i\frac{\pi}{4}\mu}\sqrt{2\pi}\sum_{k_s}\frac{\lambda_2^t(k_s)}{\sqrt{t}}\left(-(x(k_s)-1)(1+2B)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}} - (\frac{1}{B}+1)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}} + \frac{1}{2}\right) \times e^{-i\alpha t k_s}\left(\frac{-2i(2+4B(x(k_s)-1)+B(\frac{x^2(k_s)}{4}-\frac{x(k_s)}{2}+\frac{17}{8}))}{B(\frac{1}{B}+x^2(k_s))^{3/2}}^{-1/2}\right) \tag{35}$$

Now we can put both expressions together and get:

$a_t = \frac{1}{2\pi}(I_1 - I_3)$

Still concentrating on the even-time picture, we have some sign changes in the $b_t$ integral. The spatial term in $I_{3_b}$ is now $\frac{F+\cos(2k}{2F}$, and the whole integral is now $b_t = \frac{1}{2\pi}(-I_1 + I_{3_b})$. Calculating these changes gives us:

$I_{3_b} = \sum_{k_s} (\frac{1}{2} + G(k_s)) e^{it\xi(k_s)} \sqrt{\frac{2\pi}{t\xi''(k_s)}} e^{i\frac{\pi}{4}\mu} =$

$$e^{i\frac{\pi}{4}\mu}\sqrt{2\pi}\sum_{k_s}\frac{\lambda_2^t(k_s)}{\sqrt{t}}\left((x(k_s)-1)(1+2B)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}} + (\frac{1}{B}+1)\frac{q-e^{-2ik_s}\sqrt{B}}{2\sqrt{B}} + \frac{1}{2}\right) \times e^{-i\alpha t k_s}\left(\frac{-2i(2+4B(x(k_s)-1)+B(\frac{x^2(k_s)}{4}-\frac{x(k_s)}{2}+\frac{17}{8}))}{B(\frac{1}{B}+x^2(k_s))^{3/2}}^{-1/2}\right) \tag{36}$$

and:

$b_t = \frac{1}{2\pi}(-I_1 + I_{3_b})$

Now we can set our sights on the odd-time picture, where we have a few additional changes. Letting $J_i$'s be the integrands corresponding to the integrals used above $(I_1, I_3, I_{3_b})$, we examine the odd-time case:

We have two main groups of terms for each coefficient:

(37) $$a_{t_{odd}} = \int (J_1 + J_3) e^{ik} \sqrt{1-q} dk + \int (-J_1 + J_3) e^{ik} \sqrt{1-q} dk$$

(38) $$b_{t_{odd}} = \int (J_1 + J_{3_b}) e^{-ik} \sqrt{1-q} dk + \int (J_1 - J_{3_b}) e^{-ik} \sqrt{q} dk$$

where we need only slightly adjust our approach to integration, keeping track of some subtle sign changes.

Let us explicitly write out only one such integral, the very first one:

(39) $$\frac{\sqrt{1-q}}{2\pi} \int_0^{2\pi} \lambda_1^t(k) e^{-ik\alpha t} e^{ik} G(k) dk$$

We account for the new term $e^{ik}$ by adjusting our spatial function, and still proceed with the same stationary phase analysis. This gives us two terms with positive exponential function $e^{ik}$ $I_{1_{odd}}$ and $I_{3_{odd}}$, and two with negative exponential function $I_{1_{odd,B}}$ and $I_{3_{odd,B}}$.

## Appendix C. Corrections For The Future Pertaining To The Even-Odd Problem

It turns out that our eigenvalues and eigenvectors for the even-odd problem may, in fact, be incorrect. Recalculating these yields:

(40) $$\lambda_{1,2} = \frac{1}{2} e^{-2ik} (\sqrt{-B} + 2e^{2ik}\sqrt{-B} + e^{4ik}\sqrt{-B} \pm \sqrt{-4e^{-4ik} - B(1+e^{2ik})^4})$$

and

(41) $$\vec{e}_{1,2} = \begin{pmatrix} \frac{1}{2(e^{2ik}(q-1)+q)} (\sqrt{-B} - e^{4ik}\sqrt{-B} \mp [\sqrt{2}\sqrt{-e^{4ik}(2+3B+B(4\cos(2k)+\cos(4k))} \\ 1 \end{pmatrix}$$

Using these slightly different functions and recalculating the diagonalized matrix, etc. will hopefully yield results which match the numerics.

## Appendix D. Approaches For The Future Pertaining To The Time Inhomogeneous Problem

By adjusting the pentadiagonal operator obtained in the spatially inhomogeneous problem, we have obtained a similar operator for the time inhomogeneous quantum walk. This operator is identical, but with all variables depending on space, the $n-1$, $n$, $n+1$'s etc., replaced with one variable- the time $t$ in question. Iterating this operator on the initial state and incrementing the time variable should yield the wave function at an arbitrary time. This approach hints at an explanation for the peculiar fast shrinking behavior of this walk, as large products of circular functions are clearly fast shrinking, because all circular functions are bounded by one.

## REFERENCES

[1] [Brun, 2003] Brun, Todd A., Hilary A. Carteret, and Andris Ambainis. *Quantum Walks driven by many coins.* Phys. Rev. A 67, 052317 (2003).
[2] [Dimcovic, 2012] Zlatko Dimcovic. Discrete-time Quantum Walks via Interchange Framework and Memory in Quantum Evolution, Oregon State University Doctoral Thesis.
[3] [Dimcovic, 2011] Zlatko Dimcovic. Framework for discrete-time quantum walks and a symmetric walk on a binary tree, *Arxiv.org*,2011
[4] [Grünbaum et al., 2003] F. A. Grünbaum, L. Velázquez. The Quantum Walk Of F. Riesz, *Arxiv.org*, 2011.
[5] [Linden et al.,2009] Noah Linden, James Sharam. Inhomogeneous Quantum Walks, *Physics Review A.*, 2009.
[6] [Nayak et al., 2000] Ashwin Nayak, Ashvin Vishwanath. Quantum Walk on the Line, *Arxiv.org*, 2000.

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