

# A QUANTUM COMPUTING ALGORITHM FOR CARD SHUFFLING

BEN CORNELIS AND SANGPO SHERPA

ADVISOR: YEVGENIY KOVCHEGOV  
OREGON STATE UNIVERSITY

ABSTRACT. In this paper we investigate a quantum computing algorithm for top-to-random card shuffling. First we show that using a unitary shuffling matrix  $U$ , the probabilities for each deck state in the two-card case are periodic and then we find the mixing time. This is a walk on the Cayley graph of  $S_2$ , an abelian group. Then we investigate the matrix for shuffling three cards which represents a walk on  $S_3$ , a nonabelian group. We show where the method for the two-card case breaks down when walking on a nonabelian group, and attempt to uncover some of the structure of the matrix. Then we derive a limit law for the converging averages of the probabilities in the three-card case. Lastly, we make a conjecture about the general spectral structure for a walk on  $S_n$ .

## 1. INTRODUCTION

The classical analog to our quantum shuffling algorithm works as follows:

- (1) For two cards: take the top card, flip a normal coin, and based on the outcome of the coin toss, place the top card back in the top spot or transpose it with the second card.
- (2) For  $n$  cards: take the top card, flip an  $n$ -sided coin, move the card to that spot.

Thus for classical card shuffling with  $n$  cards, the top card will have probability  $\frac{1}{n}$  of being placed in each spot in the deck. And therefore the deck, upon each shuffling, will have probability  $\frac{1}{n!}$  of being in any unique configuration.

For the quantum algorithm, however, the probability of each deck state is not constant at  $\frac{1}{n!}$ , but varies with each shuffle (application of  $U$ ). In fact, since all matrices in quantum computing are unitary, and unitary matrices are length preserving, the probabilities will not even converge with time to their classical values, but instead converge in the average. We will find that in the three-card case, the averages will converge to a different distribution depending on the initial state of the system.

Thus far, quantum computing researchers have investigated quantum walks on graphs, and found the differences between those algorithms and the classical versions. Much work has been done in particular on walks on the Cayley graphs of abelian groups. To our knowledge however, no work has been done so far on walks on the Cayley graphs of non-abelian groups.

In the two-card case, the card shuffling matrix  $U$  can be seen as a walk on the Cayley graph of  $S_2$ , an abelian group. For that case the literature is helpful in predicting the behaviour of the walk. But

---

*Date:* October 6, 2011.

This work was done during the Summer 2011 REU program in Mathematics at Oregon State University.

since for three cards  $U$  represents a walk on  $S_3$ , the smallest non-abelian group, the probabilities turn out to be aperiodic and to converge to a nonuniform distribution. In order to prove a law for the limiting distribution of the probabilities, we attempt to discover the eigenstructure of  $U$ .

**1.1. Definitions and Notation.** A qubit  $|q\rangle$  is a quantum system whose state lies in a finite Hilbert space  $\mathcal{H}$  over  $\mathbb{C}$ . A vector written in this notation is called a **ket** and its conjugate transpose  $|q\rangle^* = \langle q|$  is called a **bra**. We use the conventions:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |01\rangle = |0\rangle \otimes |1\rangle$$

and we write the labels in binary, so that, for instance,  $|1\rangle \otimes |0\rangle = |10\rangle = |2\rangle$ . We represent a deck of  $n$  cards by the system  $|i\rangle \otimes |q\rangle$ ,  $q = \sigma(0\dots n-1)$ ,  $\sigma \in S_n$  where  $|i\rangle$  is a qubit to be randomized which lies in the coin space  $\mathcal{H}^C$  of dimension  $n$  and  $|q\rangle$  is the state of the deck, which lies in the deck space  $\mathcal{H}^D$ , of dimension  $n!$ . The state of the quantum system at any time  $t$  is then:

$$|\psi_t\rangle = U^t |\psi_0\rangle$$

where  $|\psi_0\rangle$  is the initial state of the system (some pure state). It lies in  $\mathcal{H}^C \otimes \mathcal{H}^D \cong \mathbb{C}^n \otimes \mathbb{C}^{n!}$ . We can write  $|\psi_t\rangle$  can be write as  $|\psi_t\rangle = \sum_{i, \sigma \in S_n} \alpha_{i, \sigma} |i\rangle \otimes |\sigma(0\dots n-1)\rangle$  where  $\sum_i |\alpha_{i, \sigma}|^2 = 1$ .

To shuffle the deck, we first apply a quantum coin  $C$  to the coin space  $|i\rangle$  to put the qubit into a superposition of all possible states with equal probability amplitudes. We want to do this while leaving the deck state unchanged, so we apply the tensor product  $C \otimes I$ . Then we apply a matrix  $V$  that reads the labelling of the first qubit then changes the deck state to  $|\sigma_c \sigma(0\dots n-1)\rangle$  where  $\sigma_c$  corresponds to shuffling the top card to the position indicated by the first qubit. Thus we have  $U = V(C \otimes I)$ . From here on, when we write  $|\sigma\rangle$  we use  $\sigma = \sigma(0\dots n-1)$ , which we also use as the nodes of the Cayley graphs.

**Definition 1.1. Mixing Time:**

$$M_\varepsilon = \min \{T \mid \forall t \geq T : |A_t - \pi| \leq \varepsilon\}$$

where  $A_T = \frac{1}{T} \sum_{t=1}^T P_t(\sigma)$  and  $\pi$  is the limiting distribution of the averages. [Ah]

**Definition 1.2. Measurement Probability** The probability of measuring the state  $|\sigma\rangle$  at a time  $t$  given that the system is in the initial state  $|\psi_0\rangle$  is

$$P_t(\sigma|\psi_0) = \sum_{i \in \{1 \dots n\}} |\langle i, \sigma | \psi_t \rangle|^2$$

where  $\langle \cdot | \cdot \rangle$  denotes the usual inner product.

## 2. 2-CARD CASE

**2.1. Computing the Mixing-Time.** We simulate a 2-card deck with a 3-qubit system, so that at any time  $t$ , the state of the system is

$$\begin{aligned} |\psi_t\rangle &= U^t |\psi_0\rangle \\ &= a_t |0\rangle \otimes |01\rangle + b_t |1\rangle \otimes |01\rangle + c_t |0\rangle \otimes |10\rangle + d_t |1\rangle \otimes |10\rangle \end{aligned}$$

For our first computation, we choose the 2x2 Hadamard matrix as our coin:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It maps a qubit in states  $|0\rangle, |1\rangle$  to a superposition of both basis states, so either is measured with probability  $\frac{1}{2}$ . In order to build our shuffling matrix  $U$ , we also need the measurement operators:

$$\Pi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we define  $V = \Pi_0 \otimes I + \Pi_1 \otimes X$ , where  $X$  swaps the second and third qubits. Thus  $V$  applies the identity to the last two qubits if the first qubit reads zero (which is equivalent to shuffling the top card back into the same spot), and shuffles the top card to the last spot if the first qubit reads one. Hence we write  $U = V(H \otimes I)$ .

Since  $U$  only acts on the subspace of  $\mathbb{C}^8$  with basis  $f = \{f_1, f_2, f_3, f_4\} = \{|001\rangle, |010\rangle, |101\rangle, |110\rangle\}$ , we can rewrite it in this basis. From here on, any shuffling matrix  $U$  will be written in the basis of the possible pure states of the system. Now we have:

$$\begin{aligned} U_f &= (Uf_1 \quad Uf_2 \quad Uf_3 \quad Uf_4) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

which has eigenvalues

$$e^{\pi i}, e^{2\pi i}, e^{\frac{\pi}{4}i}, e^{\frac{-\pi}{4}i}$$

and eigenvectors

$$\delta \begin{pmatrix} 1 \\ -1 - \sqrt{2} \\ 1 \\ -1 - \sqrt{2} \end{pmatrix}, -\delta \begin{pmatrix} 1 + \sqrt{2} \\ 1 \\ 1 + \sqrt{2} \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

with  $\delta = \frac{1}{2\sqrt{2+\sqrt{2}}}$ . Then, for  $|\psi_0\rangle = |001\rangle$ , we have

$$\begin{pmatrix} a_t \\ b_t \\ c_t \\ d_t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

To make calculations simpler, we diagonalize U and obtain the coefficient vector:

$$\begin{pmatrix} a_t \\ b_t \\ c_t \\ d_t \end{pmatrix} = \begin{pmatrix} \delta^2 e^{\pi it} + \delta^2 (1 + \sqrt{2})^2 e^{2\pi it} + \frac{1}{4} e^{\frac{\pi}{4} it} + \frac{1}{4} e^{-\frac{\pi}{4} it} \\ -\delta^2 (1 + \sqrt{2}) e^{\pi it} + \delta^2 (1 + \sqrt{2}) e^{2\pi it} + \frac{1}{4} i e^{\frac{\pi}{4} it} - \frac{1}{4} i e^{-\frac{\pi}{4} it} \\ \delta^2 e^{\pi it} + \delta^2 (1 + \sqrt{2})^2 e^{2\pi it} - \frac{1}{4} e^{\frac{\pi}{4} it} - \frac{1}{4} e^{-\frac{\pi}{4} it} \\ -\delta (1 + \sqrt{2}) e^{\pi it} + \delta^2 (1 + \sqrt{2}) e^{2\pi it} - \frac{1}{4} i e^{\frac{\pi}{4} it} + \frac{1}{4} i e^{-\frac{\pi}{4} it} \end{pmatrix}$$

By Definition 1.2,  $P_t(01|001) = |a_t|^2 + |b_t|^2 = (\delta_t)^2 + (\beta_t)^2 + 1/4 + (\delta_t)(\cos \pi t/4) - (\beta_t)(\sin \pi t/4)$  where  $\delta_t = \delta^2((-1)^t + 3 + 2\sqrt{2})$ ,  $\beta_t = \delta^2(1 + \sqrt{2})((-1)^{t+1} + 1)$ . Thus, because of the  $\sin \pi t/4$ ,  $\cos \pi t/4$  terms, we see that the sequence of probabilities has period 8, repeating the values:  $\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1$ . Hence we obtain:

$$A_T = \begin{cases} \frac{1}{2}, & T \equiv 0, 1, 2 \pmod{8} \\ \frac{1}{2} - \frac{1}{2T}, & T \equiv 3, 7 \pmod{8} \\ \frac{1}{2} - \frac{1}{T}, & T \equiv 4, 5, 6 \pmod{8} \end{cases}$$

Thus  $|A_T - 1/2| \leq \frac{1}{T}$ . Hence by definition 1.1, we have the mixing time  $M_\epsilon = \lceil \frac{1}{\epsilon} \rceil$ .

**2.2. Application of Representation Theory.** For the two-card case, it is relatively easy to calculate the coefficient formulas by hand and determine the mixing-time. But in order to generalize the shuffling algorithm to three, four, and eventually arbitrary cards we need to know the form of all the eigenvectors and eigenvalues. One important observation about any eigenvector  $|v\rangle$  of U is that permuting cards yields a multiple  $e^{i\theta} |v\rangle$  of  $|v\rangle$ . The eigenvalue of  $|v\rangle$  will be a root of unity since all eigenvalues of a unitary matrix lie on the unit circle. For example, one of the eigenvectors of U is:

$$|v\rangle = \frac{1}{2}(|001\rangle + i|101\rangle - |010\rangle - i|110\rangle)$$

If we switch the last two qubits (i.e. the cards), we get

$$\frac{1}{2}(|010\rangle + i|110\rangle - |001\rangle - i|101\rangle) = e^{i\pi} |v\rangle$$

In other words, an eigenvector should be invariant with respect to card permutations. Because of this observation, it is convenient to look at the evolution of a state under application of U as a

walk on the Cayley graph  $\Gamma = \Gamma(S_n, G)$  of  $S_n$ , where  $G$  is the generating set of the Cayley graph. A generator is an element of  $S_n$  that determines which nodes are connected by an edge and what direction that edge has. For instance, for  $\Gamma = \Gamma(S_3, \{id, (01), (012)\})$ , the Cayley graph that we will be considering for the three-card case, the generators correspond to the three different ways the top card can be placed into the deck. Hence for instance, there is a directed edge from  $id$  to  $(012)$  but not from  $(012)$  to  $id$ .

For two cards, shuffling is equivalent to walking on  $S_2$ . Since  $S_2$  is an abelian group, we can use the same method for finding the spectral structure as presented in the paper *Quantum Walks on Graphs*, where the authors dealt with general quantum walks on the graphs of abelian groups. [Ah]

The authors of that paper noticed that the matrix  $V$  could be broken up into  $n$  blocks of dimension  $n!$ , each corresponding to applying a generator on the group ( $S_2$  in our case). For 2-card shuffling, the generators are just the elements of  $S_2$  and correspond to the different types of shuffling. In other words, for any node in the graph only certain nodes are one edge distance away. Hence we rewrite the shuffling matrix  $V$  as

$$V = \left[ \begin{array}{c|c} I & \\ \hline & X \end{array} \right]$$

Where  $I$  corresponds to applying the generator  $g_0 = id$  to the group, and  $X$  corresponds to applying  $g_1 = (01)$ .

We can use representation theory to guess the eigenvectors. There are two irreducible representations of  $S_2$ :

- (1) Trivial Representation:  $\{1\}$  - all elements are mapped to 1
- (2) Alternating Representation:  $\{1, -1\}$  - all elements that are the product of an even number of transpositions are mapped to 1; all that are the product of an odd number of transpositions are mapped to -1. Then the characters (the trace of the representations) of each group element under each representation are

$$\chi_1(\sigma_0) = \chi_1(\sigma_1) = 1$$

$$\chi_2(\sigma_0) = 1, \chi_2(\sigma_1) = -1$$

The characters show us that we can group the deck states with respect to coefficients in two different ways such that they are invariant under both of the generators. These are

$$|\chi_1\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^1 \chi_1(\sigma_i) |\sigma_i\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$$

$$|\chi_2\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^1 \chi_2(\sigma_i) |\sigma_i\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

Then  $|\chi_1\rangle$  has corresponding eigenvalues 1, 1 and  $|\chi_2\rangle$  has eigenvalues 1, -1 under  $I$  and  $X$ . For the coin space's contribution to the vector, we have the form  $\sum_{i=0}^1 c_i |i\rangle$ . Hence an educated guess for the general form of the eigenvectors is

$$|v_k\rangle = \left( \sum_{i=0}^1 c_i |i\rangle \right) \otimes |\chi_k\rangle$$

where  $k = 1, 2$ . Applying  $U$ , we obtain

$$U|v_1\rangle = \left(\frac{1}{\sqrt{2}}c_0 + \frac{1}{\sqrt{2}}c_1\right)|0\rangle \otimes |\chi_1\rangle + \left(\frac{1}{\sqrt{2}}c_0 - \frac{1}{\sqrt{2}}c_1\right)|1\rangle \otimes |\chi_1\rangle$$

and

$$U|v_2\rangle = \left(\frac{1}{\sqrt{2}}c_0 + \frac{1}{\sqrt{2}}c_1\right)|0\rangle \otimes |\chi_2\rangle + \left(-\frac{1}{\sqrt{2}}c_0 + \frac{1}{\sqrt{2}}c_1\right)|1\rangle \otimes |\chi_2\rangle$$

so that  $|v_1\rangle$  is an eigenvector of  $U$  if and only if

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

is an eigenvector of  $H$ , and  $|v_2\rangle$  is an eigenvector if and only if

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

is an eigenvector of

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Thus for each  $k$ , we obtain two eigenvectors of  $U$ .

**2.3. Block Structure for 2-cards.** This method gives us all of the eigenvalues and eigenvectors of  $U$ . But since for three cards the underlying group is  $S_3$ , which is a non-abelian group, the same method does not yield the entire eigenstructure. Therefore we need to find a more general form for the eigenvectors. We can decompose  $U$  into blocks as follows:

$$U = V(H \otimes I) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c|c} I & \\ \hline & X \end{array} \right] \left[ \begin{array}{c|c} I & I \\ \hline I & -I \end{array} \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{c|c} I & I \\ \hline I & -I \end{array} \right] = M_I \otimes I + M_X \otimes X$$

where

$$M_I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad M_X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = N \text{ (NOT gate)}$$

We observe that we can find two linearly independent vectors  $|v_1\rangle, |v_2\rangle$  that are invariant under  $X$ . (NOTE: ANY KET IS AN EIGENVECTOR OF  $I$ )

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then if we set  $|v_\alpha\rangle = |\alpha\rangle \otimes |v_1\rangle$  and  $|v_\beta\rangle = |\beta\rangle \otimes |v_2\rangle$  as our eigenvector guess and apply  $U$ , we obtain

$$U|v_\alpha\rangle = (M_I + M_X)|\alpha\rangle \otimes |v_1\rangle, \quad U|v_\beta\rangle = (M_I - M_X)|\beta\rangle \otimes |v_2\rangle$$

so that  $|v_\alpha\rangle$  and  $|v_\beta\rangle$  are eigenvectors of  $U$  if and only if the relations  $(M_I + M_X)|\alpha\rangle = \lambda_1|\alpha\rangle$ ,  $(M_I - M_X)|\beta\rangle = \lambda_2|\beta\rangle$  hold. This method gives us a full set of eigenvectors for  $U$ , without using the representation theory method of the paper.

Next we test the shuffling algorithm with a different coin than the Hadamard matrix. We use

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Using this coin we obtain the same period and sequence of repeating probabilities and hence the same mixing time. We observe that  $U$  has a similar block structure:

$$U = V(C \otimes I) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c|c} I & iI \\ \hline iX & X \end{array} \right]$$

but in this case

$$M_I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \quad M_X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ i & 1 \end{pmatrix}$$

Using similar guesses for the forms of the eigenvectors, we get that the eigenvalues of  $U$  come from  $M_I + M_X$ ,  $M_I - M_X$ . This leads us to believe that the block decomposition method works for any coin and that this method for guessing eigenvectors is very general.

### 3. 3-CARD CASE

Moving to applying the shuffling algorithm to a deck of three cards, we immediately note that the underlying group,  $S_3$  is nonabelian, and therefore the methods of the paper cannot be applied to find the entire eigenstructure of  $U$ . For our coin we use

$$C = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$$

with  $a = e^{i\frac{2\pi}{3}}$ . We have 18 possible pure states, so that the orthonormal basis for  $\mathcal{H}^C$  is  $\{|0\rangle, |1\rangle, |2\rangle\}$  and for  $\mathcal{H}^D$  is  $\{|012\rangle, |120\rangle, |201\rangle, |102\rangle, |021\rangle, |210\rangle\}$ .

**3.1. Attempting to Apply Representation Theory.** We label the elements of  $S_3$  as  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\} = \{id, (012), (021), (02), (12), (01)\}$  respectively. If we try to use representation theory here, we have three irreducible representations of  $S_3$ :

(1) Trivial:  $\{1\}$

(2) Alternating:  $\{1, -1\}$

(3) 2-Dimensional:  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$   
- each element  $\sigma_i$  is mapped to the  $i$ th matrix in the set

We obtain the following characters:

$$\chi_1(\sigma_0) = \chi_1(\sigma_1) = \dots = \chi_1(\sigma_5) = 1$$

$$\chi_2(\sigma_0) = \chi_2(\sigma_1) = \chi_2(\sigma_2) = 1, \chi_2(\sigma_3) = \chi_2(\sigma_4) = \chi_2(\sigma_5) = -1$$

$$\chi_3(\sigma_0) = 2, \chi_3(\sigma_1) = \chi_3(\sigma_2) = -1, \chi_3(\sigma_3) = \chi_3(\sigma_4) = \chi_3(\sigma_5) = 0$$

Here the generators, as mentioned earlier, are  $\{g_0, g_1, g_2\} = \{id, (01), (012)\}$ . In a similar manner to the two-card case, we obtain the invariant vectors

$$|\chi_1\rangle = \frac{1}{\sqrt{6}} \sum_{i=0}^5 \chi_1(\sigma_i) |\sigma_i\rangle = \frac{1}{\sqrt{6}} (|012\rangle + |120\rangle + |201\rangle + |210\rangle + |021\rangle + |102\rangle)$$

which has eigenvalue under each of  $g_0, g_1, g_2$ , and

$$|\chi_2\rangle = \frac{1}{\sqrt{6}} \sum_{i=0}^5 \chi_2(\sigma_i) |\sigma_i\rangle = \frac{1}{\sqrt{6}} (|012\rangle + |120\rangle + |201\rangle - |210\rangle - |021\rangle - |102\rangle)$$

which has eigenvalues 1, -1, 1 under  $g_0, g_1, g_2$ . That these eigenvectors also work for the three-card case reflects the fact that  $S_3$  retains the structure of  $S_2$  while adding on another layer. But this method breaks down for the third representation, where we get the vector:

$$|\chi_3\rangle = \frac{1}{\sqrt{6}} \sum_{i=0}^5 \chi_3(\sigma_i) |\sigma_i\rangle = \frac{1}{\sqrt{6}} (2|012\rangle - |120\rangle - |201\rangle)$$

which is not invariant under  $g_1$  or  $g_2$ .

If we now assume the same form for the eigenvectors as in the two-card case:

$$|v_k\rangle = \left( \sum_{i=0}^2 c_i |i\rangle \right) \otimes |\chi_k\rangle, k = 1, 2$$

and apply U, we only find a set of six orthonormal eigenvectors for U, far short of the 18 we need. Thus there does not seem to be a way to use representation theory to find the spectral decomposition of the matrix.



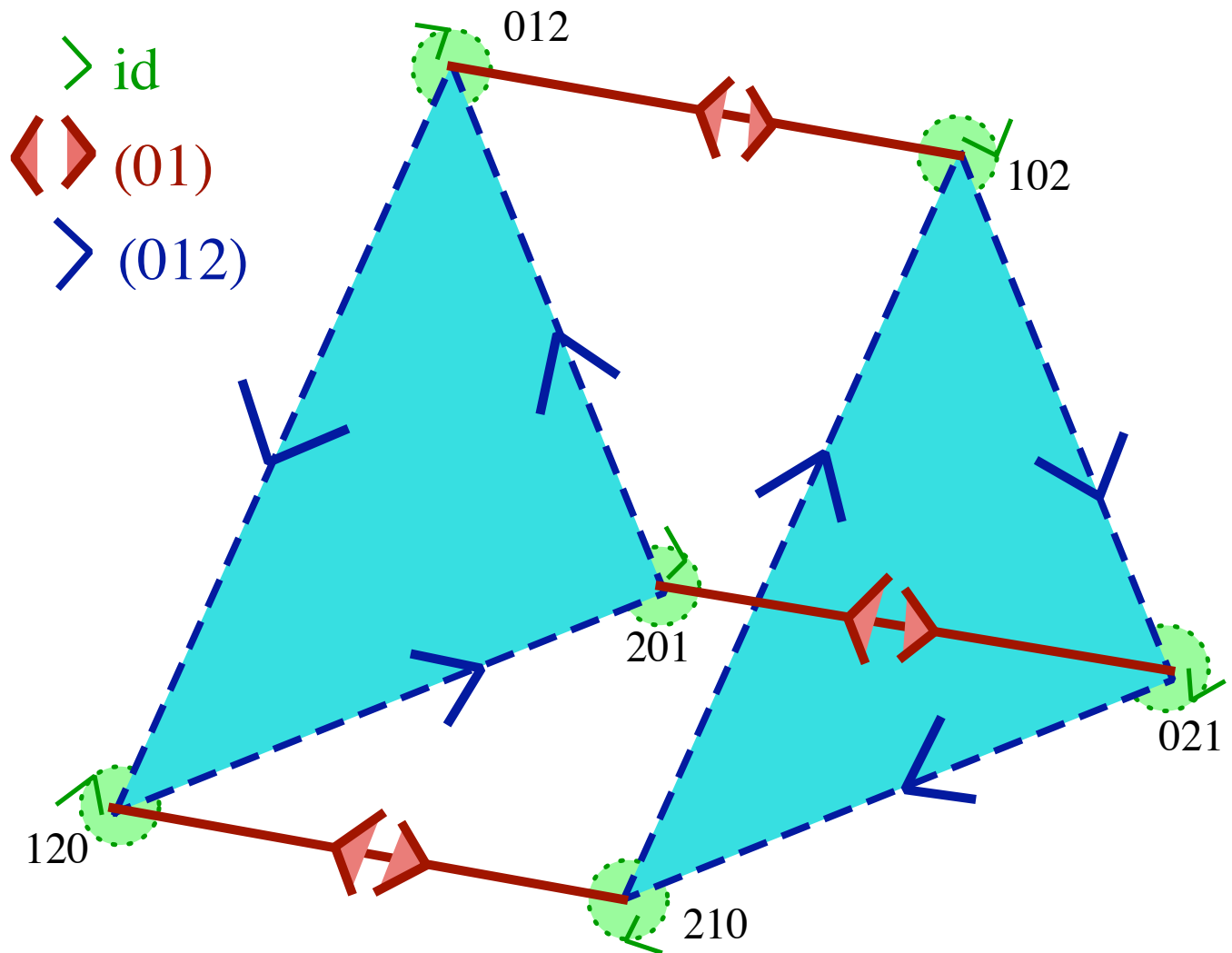


FIGURE 1. A three-dimensional version of the walk on the Cayley graph  $\Gamma = \Gamma(S_3, \{id, (01), (012)\})$ .

3.2. **Walk on  $S_3$ .** The cayley graph represents the walk on  $S_3$  that the shuffling algorithm for 3-cards represents. The vertices are labelled as  $\sigma(012)$  so that each vertex is associated with both a group element  $\sigma$  and a deck state  $|\sigma(012)\rangle$ . The three two-cycles (corresponding to applying the generator  $g_0$  twice) are colored red and the two three-cycles (corresponding to applying the generator  $g_1$  three times) are colored blue.

**3.3. Block Structure for 3-cards.** Returning to the method of block decomposition described earlier, we can write

$$U = V(C \otimes I) = \frac{1}{\sqrt{3}} \begin{bmatrix} I & & \\ & R & \\ & & B \end{bmatrix} \begin{bmatrix} I & aI & aI \\ aI & I & aI \\ aI & aI & I \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} I & aI & aI \\ aR & R & aR \\ aB & aB & B \end{bmatrix}$$

where R corresponds to applying the generator  $g_1$  to the group and B corresponds to applying  $g_2$  to the group. We can decompose U even further into blocks of dimension three if we set

$$R = \begin{bmatrix} & X \\ X & \end{bmatrix} \quad B = \begin{bmatrix} S & \\ & S \end{bmatrix} \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and also

$$M_I = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & a & 0 & a & 0 \\ 0 & 1 & 0 & a & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad M_X = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & a \\ a & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad M_S = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & 0 & 1 & 0 \\ 0 & a & 0 & a & 0 & 1 \end{pmatrix}$$

We can then decompose U into

$$U = M_I \otimes I + M_X \otimes X + M_S \otimes S$$

This form of U is useful because we can now make eigenvectors out of the form  $|v\rangle = |\alpha\rangle \otimes |v_1\rangle + |\beta\rangle \otimes |v_2\rangle + |\gamma\rangle \otimes |v_3\rangle$ , where we choose  $|v_1\rangle, |v_2\rangle, |v_3\rangle$  and  $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$  are arbitrary. We

immediately observe that the only simultaneous eigenvector of I, X, and S is  $|v_1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

Then we set

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and notice the following relations:

$$\begin{aligned} X|e_1\rangle &= |e_3\rangle & X|e_2\rangle &= |e_2\rangle & X|e_3\rangle &= |e_1\rangle \\ S|e_1\rangle &= |e_3\rangle & S|e_2\rangle &= |e_1\rangle & S|e_3\rangle &= |e_2\rangle \end{aligned}$$

Letting  $|v\rangle = |\alpha\rangle \otimes |v_1\rangle + |\beta\rangle \otimes |e_1\rangle + |\gamma\rangle \otimes |e_2\rangle + |\zeta\rangle \otimes |e_3\rangle$  we apply U and find that  $|v\rangle$  is an eigenvector of U if and only if the following equations hold:

- (1)  $(M_I + M_X + M_S)|\alpha\rangle = \lambda|\alpha\rangle$
- (2)  $M_I|\beta\rangle + M_X|\zeta\rangle + M_S|\gamma\rangle = \lambda|\beta\rangle$
- (3)  $M_I|\gamma\rangle + M_X|\gamma\rangle + M_S|\zeta\rangle = \lambda|\gamma\rangle$
- (4)  $M_I|\zeta\rangle + M_X|\beta\rangle + M_S|\beta\rangle = \lambda|\zeta\rangle$

Equation (1) dictates that  $|\alpha\rangle$  must be an eigenvector of  $M_I + M_X + M_S$  with eigenvalue  $\lambda$ , which allows us to solve for  $|\beta\rangle$ ,  $|\gamma\rangle$  and  $|\zeta\rangle$ . Since there will be multiple choices for each of these vectors, we are able to find the correct number of eigenvectors corresponding to the multiplicity of each eigenvalue.

Another choice of an eigenvector is to make the entries of  $|v_2\rangle, |v_3\rangle$  third roots of unity corresponding to walking in different directions on a graph of  $\mathbb{Z}_3$ , i.e.

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ a^2 \\ a \end{pmatrix}$$

With these vectors, we utilize a symmetry of U where we have the relations:  $X|v_2\rangle = a^2|v_3\rangle$ ,  $X|v_3\rangle = a|v_2\rangle$  and  $S|v_2\rangle = a|v_2\rangle$ ,  $S|v_3\rangle = a^2|v_3\rangle$ . In other words,  $|v_2\rangle, |v_3\rangle$  have the property that they are eigenvectors of S and are mapped to multiples of each other by X. In a certain sense, they are the next best thing to a simultaneous eigenvector under all the blocks.

If we then use the eigenvector form  $|v\rangle = |\beta\rangle \otimes |v_2\rangle + |\gamma\rangle \otimes |v_3\rangle + |\omega\rangle \otimes |e_1\rangle + |\kappa\rangle \otimes |e_2\rangle + |\zeta\rangle \otimes |e_3\rangle$  and apply U, we see that  $|v\rangle$  will be an eigenvector if

$$(5) \quad (M_I + a^2 M_X + a M_S) |\beta\rangle = \lambda |\beta\rangle$$

$$(6) \quad (M_I + a M_X + a^2 M_S) |\gamma\rangle = \lambda |\gamma\rangle$$

which again allows us to solve for  $|\omega\rangle$ ,  $|\kappa\rangle$  and  $|\zeta\rangle$ . Thus the eigenvalues of U are those of the matrix  $M_I + M_X + M_S$  and the shared eigenvalues of  $M_I + a^2 M_X + a M_S$  and  $M_I + a M_X + a^2 M_S$ .

We can also set:

$$C_I = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C_R = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & a \\ 0 & 0 & 0 \end{pmatrix} \quad C_B = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & a & 1 \end{pmatrix}$$

in order to decompose U as

$$U = C_I \otimes I + C_R \otimes R + C_B \otimes B$$

This decomposition has the advantage that the eigenvalues we obtain will come from some polynomial in  $C_I$ ,  $C_R$ ,  $C_B$ . That is, the eigenvalues will come from the coin with different multiples of its original rows. For example, we can return to our use of characters in the first and second representations of  $S_3$  to formulate the following eigenvector guesses:  $|v_\alpha\rangle = |\alpha\rangle \otimes |v_1\rangle$ ,  $|v_\beta\rangle = |\beta\rangle \otimes |v_2\rangle$  where

$$|v_1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad |v_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Applying U, we see that  $|v_\alpha\rangle$  and  $|v_\beta\rangle$  will be eigenvectors of U if  $|\alpha\rangle$  is an eigenvector of  $C_I + C_R + C_B$  and  $|\beta\rangle$  is an eigenvector of  $C_I - C_R + C_B$ . It follows from this that the eigenvalues of

$$C = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a & a \\ -a & -1 & -a \\ a & a & 1 \end{pmatrix}$$

are also eigenvalues of  $U$ . It turns out that combining the eigenvalues from these two matrices, we obtain the full set of eigenvalues of the matrix  $M_I + M_X + M_S$ , so that in some sense,  $M_I + M_X + M_S$  combines the information contained in the two versions of the coin. But these matrices still do not yield the missing eigenvalues. We notice that multiplying  $C_R$  by  $-1$ , a second root of unity, gave us a matrix with eigenvalues of  $U$ . This is a carry-over from the two-card case where one of the matrices that provided eigenvalues was obtained in the same way. Thus a natural guess for the three card case is to weight  $C_I, C_R, C_B$  with third roots of unity. It turns out that  $C_I + a^2 C_R + a C_B$  and  $C_I + a C_R + a^2 C_B$  have the missing eigenvalues. We also observe that multiplying the columns of  $C$  with  $1, a$ , and  $a^2$  (ordered left to right) yields the missing eigenvalues. These can be written:

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a & a \\ 1 & a^2 & 1 \\ a^2 & a^2 & a \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & a^2 & 1 \\ a & a & 1 \\ a & a^2 & a^2 \end{pmatrix}$$

As of now we cannot show how the eigenvalues come from these specific matrices. But for the two-card case and now the three-card case as well, all of the eigenvalues seem to come from multiplying the rows/columns of the coin by second roots of unity (for two-cards) and third roots (for three-cards). Because of this observation, we make the following conjecture:

**Conjecture 3.1.** *Let  $U$  be the quantum shuffling matrix for an  $n$ -card deck. Then the eigenvalues of  $U$  will be the same as the coin itself and those of the forms of the coin where the rows or columns are multiplied by 2nd to  $n$ th roots of unity.*

Once we have the spectral decomposition, we can use the following limit law to determine the limiting distribution of the averages of the probabilities of each deck state:

**Theorem 3.2.** *Let  $\mathcal{B} = \{|e_k\rangle \mid k = 0, \dots, \dim(U) - 1\}$  be an orthonormal basis for  $\mathcal{H}^C \otimes \mathcal{H}^D$  and let  $|e_0\rangle \in \mathcal{B}$  be the initial state of the system. Additionally let  $|v_j\rangle$ ,  $\lambda_j$ ,  $j = 0, \dots, \dim(U) - 1$  be corresponding eigenvectors and eigenvalues of  $U$ . Then set  $\alpha_j(k) = \langle e_k \mid v_j \rangle \langle v_j \mid e_0 \rangle$  for any  $0 \leq k \leq \dim(U) - 1$ . Now let  $E = \{\lambda_i \mid 0 \leq i \leq \dim(U) - 1, \lambda_i \text{ is an eigenvalue of } U\}$ ,  $E_d = \{\lambda \mid \lambda \text{ is a distinct eigenvalue of } U\}$ . Partition  $E$  into  $|E_d|$  sets  $E(\lambda)$  of size  $m(\lambda)$  where  $\lambda \in E_d$ ,  $m(\lambda)$  denotes the multiplicity of  $\lambda$ , and  $\lambda_j \in E(\lambda)$  implies  $\lambda_j = \lambda$ .*

Then,

$$\lim_{T \rightarrow \infty} A_T(P(\sigma \mid e_0)) = \sum_{\lambda \in E_d} \sum_{\substack{j, i, j \neq i \\ \lambda_i, \lambda_j \in E(\lambda)}} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) + \\ \sum_j (|\alpha_j(k_1)|^2 + |\alpha_j(k_2)|^2 + |\alpha_j(k_3)|^2)$$

where  $|e_{k_1}\rangle, |e_{k_2}\rangle, |e_{k_3}\rangle \in \mathcal{B}$  share the same deck state  $|\sigma\rangle$ .

*Proof.* By the spectral decomposition theorem, we have

$$U^t = \sum_j \lambda_j^t |v_j\rangle \langle v_j|$$

Then

$$|\zeta_t\rangle = U^t |e_0\rangle = \sum_j \lambda_j^t |v_j\rangle \langle v_j \mid e_0\rangle$$

so that

$$\begin{aligned} \zeta_t(k) &= \langle e_k \mid \zeta_t \rangle \\ &= \sum_j \lambda_j^t \langle e_k \mid v_j \rangle \langle v_j \mid e_0 \rangle \\ &= \sum_j \lambda_j^t \alpha_j(k) \end{aligned}$$

Hence

$$|\zeta_t(k)|^2 = \sum_{j, i} (\lambda_j^* \lambda_i)^t (\alpha_j^*(k) \alpha_i(k))$$

and

$$\begin{aligned} &\frac{1}{T} \sum_{t=1}^T \sum_{j, i} (\lambda_j^* \lambda_i)^t (\alpha_j^*(k) \alpha_i(k)) = \sum_{j, i} (\alpha_j^*(k) \alpha_i(k)) \left( \frac{1}{T} \sum_{t=1}^T (\lambda_j^* \lambda_i)^t \right) \\ &= \sum_{j, i, j \neq i, \lambda_j \neq \lambda_i} \frac{(\lambda_j^* \lambda_i)(1 - (\lambda_j^* \lambda_i)^T)}{T(1 - \lambda_j^* \lambda_i)} (\alpha_j^*(k) \alpha_i(k)) + \sum_{j, i, j \neq i, \lambda_j = \lambda_i} (\alpha_j^*(k) \alpha_i(k)) + \sum_j |\alpha_j(k)|^2 \end{aligned}$$

Then

$$\begin{aligned} A_T(P(\boldsymbol{\sigma} | e_0)) &= \frac{1}{T} \sum_{t=1}^T P_t(\boldsymbol{\sigma} | e_0) \\ &= \frac{1}{T} \sum_{t=1}^T (|\zeta_t(k_1)|^2 + |\zeta_t(k_2)|^2 + |\zeta_t(k_3)|^2) \end{aligned}$$

so that

$$\begin{aligned} A_T(P(\boldsymbol{\sigma} | e_0)) &= \sum_{j,i} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) \left( \frac{1}{T} \sum_{t=1}^T (\lambda_j^* \lambda_i)^t \right) \\ &= \sum_{j,i, j \neq i, \lambda_j \neq \lambda_i} \frac{(\lambda_j^* \lambda_i)(1 - (\lambda_j^* \lambda_i)^T)}{T(1 - \lambda_j^* \lambda_i)} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) \\ &+ \sum_{j,i, j \neq i, \lambda_j = \lambda_i} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) + \sum_j (|\alpha_j(k_1)|^2 + |\alpha_j(k_2)|^2 + |\alpha_j(k_3)|^2) \end{aligned}$$

This implies

$$\begin{aligned} A_T(P(\boldsymbol{\sigma} | e_0)) &= \sum_{\lambda \in E_d} \sum_{\substack{j,i \\ \lambda_i \in E(\lambda), \lambda_j \notin E(\lambda)}} \frac{(\lambda_j^* \lambda_i)(1 - (\lambda_j^* \lambda_i)^T)}{T(1 - \lambda_j^* \lambda_i)} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) \\ &+ \sum_{\lambda \in E_d} \sum_{\substack{j,i, j \neq i \\ \lambda_i, \lambda_j \in E(\lambda)}} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) + \sum_j (|\alpha_j(k_1)|^2 + |\alpha_j(k_2)|^2 + |\alpha_j(k_3)|^2) \end{aligned}$$

Hence

$$\begin{aligned} \lim_{T \rightarrow \infty} A_T(P(\boldsymbol{\sigma} | e_0)) &= \sum_{\lambda \in E_d} \sum_{\substack{j,i, j \neq i \\ \lambda_i, \lambda_j \in E(\lambda)}} (\alpha_j^*(k_1) \alpha_i(k_1) + \alpha_j^*(k_2) \alpha_i(k_2) + \alpha_j^*(k_3) \alpha_i(k_3)) + \\ &\sum_j (|\alpha_j(k_1)|^2 + |\alpha_j(k_2)|^2 + |\alpha_j(k_3)|^2). \end{aligned}$$

□

## 4. NUMERICAL RESULTS AND OBSERVATIONS

From matlab calculations, we see that the limiting distribution of the averages is not uniform in the 3-card case. The distribution is instead weighted toward the initial deck state of the system and the other states that it shares a cycle with. More specifically,  $id > (01) > (012)$  in terms of how the probabilities are weighted. For example, with the initial state  $|1120\rangle$ , deck state  $s = 120$  is weighted the most, followed by  $210 = (01)s$  (the other state in the two cycle with  $s$ ) and  $201, 012 = (012)^n s, n \in \mathbb{N}$  (the states in the 3-cycle with  $s$ ) in descending order. We list the averages of the probabilities of different deck-states along the rows at  $t = 10000$  beginning with initial state  $|j\rangle \otimes |q\rangle$ . **Pure states** to the left of each row indicate the initial state and the corresponding row indicates the probability distribution in the ordering  $q = 012, 120, 201, 102, 021, 210$ .

*Initial state (pure)*

$ 0012\rangle$	0.3946	0.1245	0.1245	0.0703	0.0703	0.2157
$ 0120\rangle$	0.1245	0.3946	0.1245	0.0703	0.2157	0.0703
$ 0201\rangle$	0.1245	0.1245	0.3946	0.2157	0.0703	0.0703
$ 0102\rangle$	0.0703	0.0703	0.2157	0.3946	0.1245	0.1245
$ 0021\rangle$	0.0703	0.2157	0.0703	0.1245	0.3946	0.1245
$ 0210\rangle$	0.2157	0.0703	0.0703	0.1245	0.1245	0.3946
$ 1012\rangle$	0.3375	0.0456	0.1169	0.1169	0.0456	0.3375
$ 1120\rangle$	0.1169	0.3375	0.0456	0.0456	0.3375	0.1169
$ 1201\rangle$	0.0456	0.1169	0.3375	0.3375	0.1169	0.0456
$ 1102\rangle$	0.1169	0.0456	0.3375	0.3375	0.0456	0.1169
$ 1021\rangle$	0.0456	0.3375	0.1169	0.1169	0.3375	0.0456
$ 1210\rangle$	0.3375	0.1169	0.0456	0.0456	0.1169	0.3375
$ 2012\rangle$	0.2640	0.2639	0.1927	0.0456	0.1169	0.1169
$ 2120\rangle$	0.1927	0.2640	0.2639	0.1169	0.1169	0.0456
$ 2201\rangle$	0.2639	0.1927	0.2640	0.1169	0.0456	0.1169
$ 2102\rangle$	0.0456	0.1169	0.1169	0.2640	0.2639	0.1927
$ 2021\rangle$	0.1169	0.1169	0.0456	0.1927	0.2640	0.2639
$ 0210\rangle$	0.1169	0.0456	0.1169	0.2639	0.1927	0.2640

Based on these numbers, we make an observation: keeping the coin state fixed, sum over all of the different deck states, then

$$\sum_{\sigma_o \in S_3} P_t^{|i\rangle \otimes |\sigma_o\rangle}(\sigma) = 1$$

Now we sum over all possible initial, **pure** states. In other words, we try the new average defined as

$$B_T = \frac{1}{|S_3|} \frac{1}{n} \frac{1}{T} \sum_{t=1}^T \sum_{i=0}^2 \sum_{\sigma_o \in S_3} P_t^{|i\rangle \otimes |\sigma_o\rangle}(\sigma)$$

Then instead of converging to  $\frac{1}{6}$ , we find that  $B_T = \frac{1}{6}$  for any T. This implies

$$\frac{1}{|S_3|} \frac{1}{3} \sum_{i=0}^2 \sum_{\sigma_o \in S_3} P_t^{|i\rangle \otimes |\sigma_o\rangle}(\sigma) = \frac{1}{6}$$

**Conjecture 4.1.** *From this observation for the three card case, we conjecture that for  $n$  cards, we will obtain the invariance*

$$\frac{1}{|S_n|} \sum_{\sigma_o \in S_n} P_t^{|i\rangle \otimes |\sigma_o\rangle}(\sigma) = \frac{1}{n!}$$

## 5. CONCLUSION

We first investigated the shuffling algorithm for a 2-card deck, which represents a quantum walk on the abelian group  $S_2$ . We found that the limiting distribution of probabilities averaged over time is uniform, which is a direct consequence of the elements of this group being self-inverses. In applying the shuffling algorithm to a 3-card deck, which is a walk on the nonabelian group  $S_3$ , we confirmed that the limiting distribution is nonuniform and also depends on the initial state. We proved a limit theorem which would allow us to explicitly calculate the limiting distribution, given the full spectral structure of the  $U$ . We conjectured that the eigenvalues of  $U$  come from different forms of the coin used (rows or columns of the coin multiplied by 2nd to  $n$ th roots of unity).

Next we plan to formalize the spectral structure extraction used in the 3-card case so that we can generalize it to any arbitrary  $n$ -card deck. This would allow us to obtain the complete picture for a coined, quantum walk on  $S_n$ . This can provide further insight into quantum walks on non-abelian groups in general, a field which is mostly unexplored.

## REFERENCES

[Ah] Aharonov, Dorit, Andris Ambainis, Julia Kempe and Umesh Vazirani. *Quantum Walks on Graphs*

UNIVERSITY OF BRITISH COLUMBIA  
*E-mail address:* cneb@interchange.ubc.ca

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
*E-mail address:* sangpos@mit.edu