EXTENDING A CATALOG OF QUANTUM MODULAR FORMS TO AN INFINITE CLASS

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Abstract. Quantum modular forms, which are particular complex-valued functions with interesting "modular"-like properties, have fascinated mathematicians since their discovery by Don Zagier in 2010. In 2016, Folsom, Garthwaite, Kang, Swisher, and Treneer gave a catalog of explicit examples of quantum modular forms, in this case arising from mock modular forms which have eta-theta function shadows. In this paper, we will introduce a generalization of their functions, and show explicitly that this generalization is a quantum modular form as well.

1. Background, Motivation, and Statement of Results

In 2016, Folsom, Garthwaite, Kang, Swisher, and Treneer [2] were able to construct mock modular forms that unified all eta-theta functions, as classified by Lemke Oliver [5], whose shadows are given by eta-theta functions with odd characters. By Zagier [7], a mock modular form is simply the holomorphic part of a harmonic Maass form, a complex-valued function on the upper half-plane that can be uniquely decomposed into a holomorphic part as well as a non-holomorphic part[1]. It is interesting to note that in [2] they were able to show these mock modular forms are also *quantum* modular forms, which is a property not necessarily all mock modular forms inherit. Simply put, after Zagier [7], a quantum modular form is an "extension" of a modular form in the sense they are defined on a particular subset of the rationals, and in which they have transformation properties similar to modular forms, with the exception of an an error term that is required to be real analytic. Now, quantum modular forms are a recent phenomenon, so examples of such constructions are a particular interest of study.

One of our primary motivations in studying the work of Folsom, et al., is to see if it is possible to construct a generalization that at least constitutes a small class of quantum modular forms covered in [2], and use this as a catalyst for further extensions. As [2] cataloged a list 59 quantum modular forms, we wanted to find patterns that would enable us to find a class of quantum modular forms that would encompass these. An important achievement is that our functions of interest contains a subclass of quantum modular forms explored by the authors in [2], and is itself a quantum modular form. Throughout the paper, we use the notation $e(a) := e^{2\pi i a}$ and $\zeta_a := e^{\frac{2\pi i}{a}}$.

Definition 1. *For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$,*

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i u} q^n}, \tag{1.1}$$

where, for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$,

$$\vartheta(z; \tau) := \sum_{v \in \mathbb{Z} + \frac{1}{2}} e^{\pi i v^2 \tau + 2\pi i v \left(z + \frac{1}{2}\right)}.$$

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[1]For more information on mock modular forms, see [3], and for more on harmonic Maass forms, see [6].

Definition 2. *Let* $\alpha = \frac{A}{2C}\tau + \frac{a}{b}$*,* $A, a \in \mathbb{Z}$ *and* $C, b \in \mathbb{N}$ *such that* $0 \le a \le b-1$ *and* $\gcd(A,C) = 1$*; then let*

$$V_\alpha(\tau) := i\zeta_b^a q^{-\frac{(2A-C)^2}{8C^2}} \mu\left(2\alpha, \frac{\tau}{2}; \tau\right).$$

One point we ought to emphasize is that although this generalizes a subclass of functions of interest, we are straying away from the study of eta-theta functions, a goal explored in [2].

Now, the subclass of functions of interest are when $b = 4$. In this paper, we were able to prove quantum modularity for the above function in that case for appropriate subsets of the rationals, denoted as S_α, and appropriate groups G_α, which will be defined in Section 3. In particular, we prove the following theorem.

Theorem 1. *For* $\alpha = \frac{A}{2C}\tau + \frac{a}{4}$ *where* $a \in \{0,1,2,3\}$*,* $\gcd(A,C) = 1$*,* $0 < \frac{A}{C} < 1$*, the functions* V_α *are quantum modular forms on the sets* S_α *for the groups* G_α*. In particular, the following are true.*

(1) *For all* $\tau \in \mathbb{H} \cup S_\alpha$*, we have that*

$$V_\alpha(\tau) - (2\tau+1)^{-1/2} V_\alpha(M_2\tau) = \frac{-i}{2}\int_{1/2}^{i\infty} \frac{g_{A/C,0}(z)}{\sqrt{-i(z+\tau)}} dz,$$

when a *= 1,3 and*

$$V_\alpha(\tau) - \frac{1}{i}(2\tau+1)^{-1/2} V(M_2\tau) = \frac{1}{2i} e\left(\frac{-A}{2C}\right) \int_{1/2}^{i\infty} \frac{g_{A/C,1/2}(z)}{\sqrt{-i(\tau+z)}} dz,$$

when a *= 0,2.*

(2) *For all* $\tau \in \mathbb{H} \cup S_\alpha$*, we have that*

$$V_\alpha(\tau) - \zeta_8^{-1}(\tau+1)^{-1/2} V_\alpha(M_1\tau) = \frac{-i}{2} e\left(\frac{-A}{2C}\right) \int_{1}^{i\infty} \frac{g_{A/C,1/2}(z)}{\sqrt{-i(z+\tau)}} dz,$$

when a *= 0,2.*

(3) *For all* $\tau \in \mathbb{H}$*, we have that*

$$V(\tau) - (-1)^{A+\frac{C}{2}} e\left(\frac{C}{8}\right) e\left(\frac{C(2A-C)^2}{8C^2}\right) V(\tau + C) = 0,$$

for all even C*, and*

$$V(\tau) - (-1)^{2A+C} e\left(\frac{C}{4}\right) e\left(\frac{C(2A-C)^2}{4C^2}\right) V(\tau + 2C) = 0,$$

for all odd C*.*

Now, proving quantum modularity for arbitrary $b \in \mathbb{Z}$ has been shown to be an excessively difficult task, and so investigation on this has been postponed for another time. In the next section, we will review previous work of Zwegers [8], Kang [4], and Folsom et. al. [2] to provide necessary background and lemmas. In Section 3, we define our quantum sets and groups. In Section 4, we show our transformations on $V_\alpha(M\tau)$ where $M \in G_\alpha$. In Section 5, we prove quantum modularity for our function V_α, and finally, in Section 6, we discuss lingering questions and possible ways to extend our results.

2. Preliminaries

Here, we begin to define functions and present lemmas, propositions, and theorems that we will need to prove Theorem 1. We begin by outlining the tools we need in order to compute our transformations and converting our functions to integral form, and follow that by explaining the tools we used to determine our quantum sets and quantum groups. We also need to define the h function, which occurs when we compute the transformations on $V_\alpha(M_r\tau)$.

Zwegers defines for $u \in \mathbb{C}$ and $\tau \in \mathbb{H}$ the Mordell integral h by

$$h(u) = h(u;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 - 2\pi u x}}{\cosh \pi x} dx. \tag{2.1}$$

The following lemma will be used in transforming μ in our $V_\alpha(M_r\tau)$ back to the form of $V_\alpha(\tau)$. This is where we see the h function appear. We later transform these terms that appear into integrals.

Lemma 1 (Zwegers, Prop. 1.4 and 1.5 of [8]). *Let $\mu(u,v) := \mu(u,v;\tau)$ and $h(u;\tau)$ be defined as in (1.1) and (2.1). Then we have*

(1) $\mu(u+1,v) = -\mu(u,v)$,
(2) $\mu(u,v+1) = -\mu(u,v)$,
(3) $\mu(-u,-v) = \mu(u,v)$,
(4) $\mu(u+z,v+z) - \mu(u,v) = \frac{1}{2\pi i}\frac{\vartheta'(0)\vartheta(u+v+z)\vartheta(z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}$, *for* $u, v, u+z, v+z \notin \mathbb{Z}\tau + \mathbb{Z}$,

and the modular transformation properties,

(5) $\mu(u,v;\tau+1) = e^{-\frac{\pi i}{4}}\mu(u,v;\tau)$,
(6) $\frac{1}{\sqrt{-i\tau}}e^{\pi i(u-v)^2/\tau}\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) + \mu(u,v;\tau) = \frac{1}{2i}h(u-v;\tau)$.

In some cases the h functions are not in the correct form to us one of our Theorems. Therefore, we utilize the following proposition in order to shift them into the correct form before we can change them into integrals.

Proposition 1 (Zwegers, Prop. 1.2 of [8]). *The function h has the following properties:*

(1) $h(z) + h(z+1) = \frac{2}{\sqrt{-i\tau}}e^{\pi i\left(z+\frac{1}{2}\right)^2/\tau}$
(2) $h(z) + e^{2\pi iz-\pi i\tau}h(z+\tau) = 2e^{-\pi iz-\pi i\tau/4}$
(3) *$z \to h(z;\tau)$ is the unique holomorphic function satisfying (1) and (2)*
(4) *h is an even function of z,*
(5) $h(\frac{z}{\tau};-\frac{1}{\tau}) = \sqrt{-i\tau}e^{-\pi iz^2/\tau}h(z;\tau)$
(6) $h(z;\tau) = e^{\frac{\pi i}{4}}h(z;\tau+1) + e^{\frac{-\pi i}{4}}\frac{e^{\pi iz^2/(\tau+1)}}{\sqrt{\tau+1}}h(\frac{z}{\tau+1};\frac{\tau}{\tau+1})$

Later in our paper, this g function occurs and occasionally is not in the correct form. We needed to shift it to match our desired result. We were able to use the following lemma to accomplish this.

Definition 3. *Let $a, b \in \mathbb{R}$ and $\tau \in \mathbb{H}$; then*

$$g_{a,b}(\tau) := \sum_{v \in a+\mathbb{Z}} ve^{\pi iv^2\tau + 2\pi ivb}.$$

Lemma 2 (Zwegers, Prop. 1.15 of [8]). *The function $g_{a,b}$ satisfies the following:*

(1) $g_{a+1,b}(\tau) = g_{a,b}(\tau)$,
(2) $g_{a,b+1}(\tau) = e^{2\pi ia}g_{a,b}(\tau)$,
(3) $g_{-a,-b}(\tau) = -g_{a,b}(\tau)$,
(4) $g_{a,b}(\tau+1) = e^{-\pi ia(a+1)}g_{a,a+b+\frac{1}{2}}(\tau)$,
(5) $g_{a,b}(-\frac{1}{\tau}) = ie^{2\pi iab}(-i\tau)^{3/2}g_{b,-a}(\tau)$.

Finally, we use the following theorem and lemma when we need to cahnge a certain function to a specific integral form.

Theorem 2 (Zwegers, Thm. 1.16 of [8]). *For $\tau \in \mathbb{H}$, we have the following two results.*

When $a \in (-\frac{1}{2},\frac{1}{2})$ and $b \in \mathbb{R}$,

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz = -e^{2\pi ia(b+\frac{1}{2})}q^{-\frac{a^2}{2}}R(a\tau - b;\tau). \tag{2.2}$$

Also, when $a, b \in (-\frac{1}{2},\frac{1}{2})$,

$$\int_{0}^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz = -e^{2\pi ia(b+\frac{1}{2})}q^{-\frac{a^2}{2}}h(a\tau - b;\tau). \tag{2.3}$$

Lemma 3 (Lemma 2.8 of [2]). *Let $\tau \in \mathbb{H}$.*

i) *For $b \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$,*

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{1,b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz = -ie\left(-\frac{\tau}{8}+\frac{b}{2}\right)R\left(\frac{\tau}{2}-b;\tau\right)+i.$$

ii) *For* $b \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$,

$$\int_0^{i\infty} \frac{g_{1,b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz = -ie\left(-\frac{\tau}{8} + \frac{b}{2}\right) h\left(\frac{\tau}{2} - b; \tau\right) + i.$$

iii) *For* $a \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$,

$$\int_0^{i\infty} \frac{g_{a+1/2,1}(z)}{\sqrt{-i(z+\tau)}} dz = -e\left(-\frac{a^2}{2}\tau + a\right) h\left(a\tau - \frac{1}{2}; \tau\right) + \frac{e(a)}{\sqrt{-i\tau}}.$$

We also need to determine for which sets and groups our function is well defined. We used the following theorem to determine the form of our function and decipher when it would vanish.

Theorem 3 (Kang [4] [2])**.** *If* $\alpha \in \mathbb{C}$ *such that* $\alpha \notin \frac{1}{2}\mathbb{Z}\tau + \frac{1}{2}\mathbb{Z}$*, then*

$$\mu\left(2\alpha, \frac{\tau}{2}; \tau\right) = iq^{\frac{1}{8}} g_2(e(\alpha); q^{\frac{1}{2}}) - e(-\alpha) q^{\frac{1}{8}} \frac{\eta(\tau)^4}{\eta(\frac{\tau}{2})^2 \vartheta(2\alpha; \tau)},$$

where g_2 *is the universal mock theta function defined by*

$$g_2(z; q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{n(n+1)/2}}{(z; q)_{n+1} (z^{-1}q; q)_{n+1}}.$$

The following lemma aids us in determining when our function is well defined.

Lemma 4. *Fix* $\frac{A}{2C}, \frac{a}{b} \in \mathbb{Q}$ *such that* $\frac{A}{2C}$ *and* $\frac{a}{b}$ *are not both in* $\frac{1}{2}\mathbb{Z}$*. Suppose* $\widetilde{S} \subseteq S \subseteq \mathbb{Q}$ *is a set of rationals such that for all* $n \geq 1$ *and all* $\frac{h}{k} \in \widetilde{S}$*,*

$$\frac{nh}{k} \pm 2\left(\frac{Ah}{2Ck} + \frac{a}{b}\right) \notin \mathbb{Z}.$$

Then

$$\mu\left(2\left(\frac{Ah}{2Ck} + \frac{a}{b}\right), \frac{h}{2k}; \frac{h}{k}\right) = ie^{\frac{\pi i h}{4k}} g_2(\zeta_b^a e^{2\pi i \frac{Ah}{2Ck}}; e^{\frac{\pi i h}{k}}),$$

and has a well-defined value in $\mathbb{C}$*.*

Recall Theorem 3. Lemma 4 states that when the conditions of Theorem 3 are satisfied, both the second eta-quotient term arising from the Theorem vanishes, and the g_2 sum terminates. The conditions in Lemma 4 arise from considering when the denominator in the second term from Theorem 3 is nonvanishing.

Proof. First, suppose $\alpha \notin \frac{1}{2}\mathbb{Z}\tau + \frac{1}{2}\mathbb{Z}$. We observe that using the definitions of $\eta(\tau)$ and $\vartheta(2\alpha; \tau)$ (see equation (17) in [2]) we can rewrite the eta-quotient as

$$\begin{aligned}
\frac{\eta(\tau)^4}{\eta(\frac{\tau}{2})^2 \vartheta(2\alpha; \tau)} &= ie^{2\pi i \alpha} q^{-\frac{1}{12}} \frac{(q; q)_\infty^4}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^2 (q; q)_\infty (e^{4\pi i \alpha}; q)_\infty (e^{-4\pi i \alpha} q; q)_\infty} \\
&= \frac{ie^{2\pi i \alpha} q^{-\frac{1}{12}}}{(1 - e^{4\pi i \alpha})} \cdot \frac{(q; q)_\infty (-q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^2}{(e^{4\pi i \alpha} q; q)_\infty (e^{-4\pi i \alpha} q; q)_\infty}.
\end{aligned}$$

Now when we look at μ, this gives us,

$$\mu\left(2\alpha, \frac{\tau}{2}; \tau\right) = iq^{\frac{1}{8}} g_2(e(\alpha); q^{\frac{1}{2}}) - iq^{\frac{1}{24}} \frac{1}{(1 - e^{4\pi i \alpha})} \cdot \frac{(q; q)_\infty (-q^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty^2}{(e^{4\pi i \alpha} q; q)_\infty (e^{-4\pi i \alpha} q; q)_\infty}.$$

Now, we consider when $\tau = \frac{h}{k} \in \mathbb{Q}$, and $\alpha = \frac{Ah}{2Ck} + \frac{a}{b}$, where $\frac{A}{2C}$, and $\frac{a}{b}$ are both $\notin \frac{1}{2}\mathbb{Z}$. We first consider the second term and see that for any $\tau = \frac{h}{k} \in \mathbb{Q}$, the numerator will be zero. Therefore, the hypotheses of the lemma tell us that $(e^{4\pi i \alpha} q; q)_\infty (e^{-4\pi i \alpha} q; q)_\infty \neq 0$. The hypotheses of the lemma also tell us that $(1 - e^{4\pi i \alpha}) \neq 0$. Therefore, the denominator of our η-quotient is not zero. Now, we need to show that the term $g_2(e(\alpha); q^{\frac{1}{2}})$ terminates under the hypotheses of our lemma.

[2] The notation used here is slightly different than Kang's, and is taken from Zwegers [8].

Suppose there exists an integer $j \geq 1$ such that

$$r = \frac{jh}{2k} \pm \left(\frac{Ah}{2Ck} + \frac{a}{b}\right) \in \mathbb{Z}.$$

Then, by simply multiplying through by 2, we see that

$$2r = \frac{jh}{k} \pm \left(\frac{Ah}{Ck} + \frac{2a}{b}\right) \in \mathbb{Z},$$

which contradicts the hypotheses of the lemma. Therefore, we can conclude that no such integer exists, and we must have that

$$(e^{4\pi i\alpha}q;q)_\infty(e^{-4\pi i\alpha}q;q)_\infty \neq 0.$$

We can also conclude that $(1-e^{4\pi i\alpha}) \neq 0$, because $(1-e^{4\pi i\alpha}) = 0$ can only occur when $2\alpha \in \mathbb{Z}$. However, since we are assuming that for all $n \geq 1$, $\frac{nh}{k} \pm \left(\frac{Ah}{Ck} + \frac{2a}{b}\right) \notin \mathbb{Z}$, then when $n = k$, this is the also true, and so,

$$h \pm 2\left(\frac{Ah}{2Ck} + \frac{a}{b}\right) \notin \mathbb{Z} \iff 2\left(\frac{Ah}{2Ck} + \frac{a}{b}\right) \notin \mathbb{Z}$$

and so we can conclude that $\alpha = \frac{Ah}{2Ck} + \frac{a}{b} \notin \mathbb{Z}$. Thus, we must have that $g_2(e(\alpha);q^{\frac{1}{2}})$ terminates, which completes the proof of our lemma. □

3. Quantum Sets and Groups

Recall our function from Definition 2. In this section, we must specify $b = 4$. With that specified, we have that

$$V_\alpha(x) = i^{a+1}q^{\frac{-(C-2A)^2}{8C^2}}\mu\left(2\alpha, \frac{\tau}{2}; \tau\right)$$

where $\alpha = \frac{A}{2C}\tau + \frac{a}{4}$, $\gcd(A,C) = 1$, and $a \in \{0,1,2,3\}$. When considering quantum modularity, we want to find quantum sets and quantum groups for this function. We use the definition for quantum sets from [2]. We call a subset $S \subseteq \mathbb{Q}$ a *quantum set* for a function F with respect to the group $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ if both $F(x)$ and $F(Mx)$ exist (are non-singular) for all $x \in S$ and $M \in G$.

So, the following theorem is regarding the quantum sets and quantum groups of V_α. First, we need to define our sets and groups. Let

$$\begin{aligned} S &= \left\{\frac{h}{k} \in \mathbb{Q} \,\middle|\, h \in \mathbb{Z}, k \in \mathbb{N}, \gcd(h,k) = 1, h \equiv 1 \pmod 2\right\}, \\ S_{C1} &= \left\{\frac{h}{k} \in S \,\middle|\, C \nmid h\right\}, \\ S_{C2} &= \left\{\frac{h}{k} \in S \,\middle|\, C \nmid 2h\right\}, \\ S_{ev} &= \left\{\frac{h}{k} \in S_{C1} \,\middle|\, k \equiv 0 \pmod 2\right\}. \end{aligned}$$

We will now define our sets and groups.

Definition 4. *Fix* $\alpha = \frac{A}{2C} + \frac{a}{4}$. *We define*

$$S_\alpha = \begin{cases} S_{C1}, & \text{if } a = 0,2 \\ S_{C2} \cup S_{ev}, & \text{if } a = 1,3. \end{cases} \tag{3.1}$$

Definition 5. *Fix $\alpha = \frac{A}{2C} + \frac{a}{4}$. We define*

$$G_\alpha = \begin{cases} \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a = 0, 2, \text{ and even } C \\ \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a = 1, 3, \text{ and even } C \\ \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a = 0, 2, \text{ and odd } C \\ \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a = 1, 3, \text{ and odd } C. \end{cases}$$

Theorem 4. *Let $\alpha = \frac{A}{2C}x + \frac{a}{4}$; the set S_α is a quantum set for V_α, and the group G_α is a quantum group for V_α. Moreover, for all $x \in S_\alpha$ and $M \in G_\alpha$ we have*

$$V_\alpha(x) = i^{a+1} e\left(\frac{(C-2A)^2}{8C^2} x\right) \mu\left(2\alpha, \frac{x}{2}; x\right)$$

and

$$V_\alpha(Mx) = i^{a+1} e\left(\frac{(C-2A)^2}{8C^2} Mx\right) \mu\left(2\alpha, \frac{Mx}{2}; Mx\right)$$

are well-defined.

Proof. First consider the general case. By Lemma 4, we get that $\mu(2\alpha, \frac{x}{2}; x)$ is well defined for $x \in S_\alpha$ if $\frac{nh}{k} \pm \frac{Ah}{Ck} + \frac{2a}{4}$ is never an integer for all $n \in \mathbb{N}$. Therefore, for any $n \geq 1$, we want to avoid the existence of an $r \in \mathbb{Z}$ such that

$$r = \frac{nh}{k} + \frac{Ah}{Ck} + \frac{a}{2} \iff Ck(2r - a) = 2h(Cn + A)$$

and

$$r = \frac{nh}{k} - \frac{Ah}{Ck} - \frac{a}{2} \iff Ck(2r + a) = 2h(Cn - A).$$

Consider $a = 0$. Suppose $\frac{h}{k} \in S_{C1}$, and suppose for sake of contradiction that there exists an $r \in \mathbb{Z}$ such that

$$2Ckr = 2h(Cn \pm A),$$

and so

$$Ckr = h(Cn \pm A).$$

There is a factor of C on the left-hand side. So, there must be a factor of C on the right-hand side. Because $\gcd(A, C) = 1$, we know that C does not divide $(Cn \pm A)$. By definition of S_{C1}, $C \nmid h$, so there is no factor of C on the right-hand side. This is a contradiction. Therefore, no such integer can exist.

Consider $a = 2$. Suppose $\frac{h}{k} \in S_{C1}$, and suppose for sake of contradiction that there exists an $r \in \mathbb{Z}$ such that

$$Ck(2r \pm 2) = 2h(Cn \pm A),$$

and so

$$Ck(r \pm 1) = h(Cn \pm A).$$

The left-hand side has a factor of C. Therefore, the right-hand side must have a factor of C as well. Because $\gcd(A, C) = 1$, we know that $(Cn \pm A)$ is not divisible by C. We also know that because $\frac{h}{k} \in S_{C1}$, $C \nmid h$. Therefore, there are no factors of C on the right-hand side and this is a contradiction. Thus, no such integer exist.

Consider $a = 1, 3$. Suppose $\frac{h}{k} \in S_{C2} \cup S_{ev}$. Suppose for sake of contradiction there exists an $r \in \mathbb{Z}$ such that

$$Ck(2r \pm a) = 2h(Cn \pm A).$$

The left-hand side has a factor of C. Therefore, the right-hand side must also have a factor of C. Because $gcd(A, C) = 1$, we know that $(Cn \pm A)$ is not divisible by C. We also know that because $\frac{h}{k} \in S_{C2} \cup S_{ev}$, $\frac{h}{k} \in S_{C2}$ or $\frac{h}{k} \in S_{ev}$. If $\frac{h}{k} \in S_{C2}$, then $C \nmid 2h$, and there is no factor of C in the right-hand side. If $\frac{h}{k} \in S_{ev}$, then k is even, and cancels the factor of 2 on the right-hand side. In this case, we also know that $C \nmid h$. So, there is no factor of C in the right-hand side. Therefore, this is a contradiction and no such integer exists.

Next, consider $M_r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ for $r \in \{1, 2\}$. Then,

$$M_r \frac{h}{k} = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \frac{h}{k} = \frac{\frac{h}{k}}{r\frac{h}{k} + 1} = \frac{h}{hr + k}$$

First, we make a note that h and $hr + k$ are relatively prime because it is given that h and k are relatively prime. Next, suppose $\frac{h}{k} \in S_{C1}$. Let $r \in \mathbb{N}$. So, h is both odd and $C \nmid h$. Therefore, if $x \in S_{C1}$, $M_r x \in S_{C1}$ as well.

Suppose $\frac{h}{k} \in S_{C2} \cup S_{ev}$. Let $r \in 2\mathbb{N}$. We know that becuase $\frac{h}{k} \in S_{C2} \cup S_{ev}$, $\frac{h}{k}$ must either be in S_{C2} or S_{ev}. If $\frac{h}{k} \in S_{C2}$, h is odd and $C \nmid 2h$. So, $M_r x$ remains in the set. If $\frac{h}{k} \in S_{ev}$, we must look at $hr + k$. Because r is even and k is also even, $hr + k$ is even. Therefore, if $x \in S_{C2} \cup S_{ev}$, then $M_r x$ must remain in $S_{C2} \cup S_{ev}$.

Now, consider $T_r := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$.

$$T_r \frac{h}{k} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \frac{h}{k} = \frac{h}{k} + r = \frac{h + rk}{k}$$

We first note that $h + rk$ and k are relatively prime, because h and k are relatively prime. Next, suppose $x \in S_{C1}$. Let r be an even multiple of C. Therefore, $h + rk$ is odd, because h is odd and r is even. Also, C does not divide $h + rk$ because $C \nmid h$ and r is a multiple of C. So, if $x \in S_{C1}$, $T_r x \in S_{C1}$.

Suppose $x \in S_{C2} \cup S_{ev}$. Let r be an even multiple of C. We know $x \in S_{C2}$, or $x \in S_{ev}$. Suppose $x \in S_{ev}$. So, $h + rk$ is odd because h is odd and r is even. Also, C does not divide $h + rk$ because $C \nmid h$ and r is a multiple of C. We also know k is even. Therefore, if $x \in S_{ev}$, $T_r x \in S_{ev}$. Suppose $x \in S_{C2}$. We see $h + rk$ remains odd because h is odd and r is even. Consider $2(h + rk) = 2h + 2rk$. Because $C \nmid 2h$ and r is a multiple of C, $C \nmid 2(h + rk)$. Therefore, $T_r x$ remains in the set. Therefore, if $x \in S_{C2} \cup S_{ev}$, $T_r x \in S_{C2} \cup S_{ev}$ for all $r \in \mathbb{N}$ that are an even multiple of C.

We must consider the inverses as well. So,

$$M_r^{-1} x = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \frac{h}{k} = \frac{h}{-rh + k}.$$

We note once again that h and $k - rh$ are relatively prime because h and k are relatively prime. Next, suppose $x \in S_{C1}$ and suppose $r \in 2\mathbb{N}$. So, h is odd and $C \nmid h$. Therefore, if $x \in S_{C1}$, $M_r^{-1} x$ is in the set S_{C1}.

Suppose $x \in S_{C2} \cup S_{ev}$. Suppose $r \in 2\mathbb{N}$. We know $x \in S_{C2}$ or $x \in S_{ev}$. If $x \in S_{C2}$, h is odd and $C \nmid 2h$. So, $M_r^{-1} x \in S_{C2}$. Suppose $x \in S_{ev}$. Then, h is odd and $C \nmid h$. Also, $-hr + k$ is even because r is even and k is even. Therefore, $M_r^{-1} x \in S_{ev}$. So, if $x \in S_{C2} \cup S_{ev}$, then $M_r^{-1} x \in S_{C2} \cup S_{ev}$.

Next, we look at

$$T_r^{-1} x = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \frac{h}{k} = \frac{h - rk}{k}.$$

We again note that $h - rk$ and k are relatively prime because h and k are relatively prime. Suppose $x \in S_{C1}$, and $r \in \mathbb{N}$ and is an even multiple of C. Therefore, $h - rk$ is odd because h is odd and r is even. Also, $C \nmid h - rk$ because $C \nmid h$ and r is a multiple of C. Therefore, if $x \in S_{C1}$, then $T_r^{-1} x \in S_{C1}$.

Suppose $x \in S_{C2} \cup S_{ev}$ and $r \in \mathbb{N}$ and is an even multiple of C. Consider $x \in S_{C2}$. Then, $h - rk$ is odd because h is odd and r is even. Also, $2(h - rk) = 2h - rk$ is not divisible by C because $C \nmid 2h$ and r is a multiple of C. Consider the alternative, $x \in S_{ev}$. So, k is even. Then, $h - rk$ remains odd as explained. $C \nmid h - rk$ because $C \nmid h$ and r is a multiple of C. So, if $x \in S_{C2} \cup S_{ev}$, then $T_r^{-1}x \in S_{C2} \cup S_{ev}$.

So, we have covered all of the cases and have proved Theorem 12. □

4. Transformations

Now that we have defined our quantum sets S_α and quantum groups G_α, we will need to explore the transformation properties of V_α with respect to matrices $M_r, T_r \in \mathrm{SL}_2(\mathbb{Z})$, where

$$T_r := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \qquad M_r := \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

We have derived in full generality the transformation properties of V_α for matrices T_r and M_r for arbitrary b. Because the quantum sets and quantum groups are defined only in the case $b = 4$, we will close this section by specializing the transformations to this case. Recall

$$V_\alpha(\tau) := i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}\tau + \frac{a}{b}\right), \frac{\tau}{2}; \tau\right).$$

We will now present the transformation properties of V_α under T_r and M_r.

Lemma 5. *Let $A, a \in \mathbb{Z}$, and $C, b, r \in \mathbb{N}$ with the restriction $(A, C) = 1$. The function V_α satisfies the following transformation properties:*

$$V_\alpha(T_r\tau) = \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}} e\left(-\frac{r}{2}\left(\frac{A}{C} - \frac{1}{2}\right)^2\right) V_\alpha(\tau) \tag{4.1}$$

$$V_\alpha(M_r\tau) = (-1)^{-\frac{2ar}{b}} \zeta_8^r e\left(\frac{2a^2}{b^2}r\right) \sqrt{r\tau+1}V_\alpha(\tau) + I_\alpha(\tau) + J_\alpha(\tau), \tag{4.2}$$

where, in the T_r-transformation, r is chosen such that $C|r$ and $2|r$, and in M_r-transformation, r is chosen such that $b|2ar$, in addition to $r = 1$ whenever $a = 0$. Furthermore,

$$I_\alpha(\tau) := i\zeta_b^a \sqrt{-i\tau_r} e\left(\frac{2a^2}{b^2\tau}\right) \frac{\zeta_{a,b}^{A,C,r}}{2i} h\left(\frac{2a}{b}\tau_r - \left(\frac{A}{C} - \frac{1}{2}\right); \tau_r\right), \tag{4.3}$$

$$J_\alpha(\tau) := -i\zeta_b^a \sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}} \zeta_8^r e\left(\frac{2a^2}{b^2}r\right) q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2} \frac{1}{2i} h\left(\left(\frac{1}{2} - \frac{A}{C}\right)\tau - \frac{2a}{b}; \tau\right). \tag{4.4}$$

Proof. Consider the first transformation; we have

$$V_\alpha(T_r\tau) = i\zeta_b^a e\left(-T_r\tau\frac{(A/C-1/2)^2}{2}\right) \mu\left(2\left(\frac{A}{2C}T_r\tau + \frac{a}{b}\right), \frac{T_r\tau}{2}; T_r\tau\right).$$

Suppose $C|r$ and $2|r$; then a simple computation quickly yields

$$\begin{aligned} V_\alpha(T_r\tau) &= e\left(-\frac{r}{2}\left(\frac{A}{C} - \frac{1}{2}\right)^2\right) i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}\tau + \frac{a}{b}\right) + \frac{Ar}{C}, \frac{\tau}{2} + \frac{r}{2}; \tau + r\right) \\ &= (-1)^{\frac{Ar}{C}+\frac{r}{2}} e\left(-\frac{r}{2}\left(\frac{A}{C} - \frac{1}{2}\right)^2\right) i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}\tau + \frac{a}{b}\right), \frac{\tau}{2}; \tau + r\right) \qquad (4.5) \\ &= \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}} e\left(-\frac{r}{2}\left(\frac{A}{C} - \frac{1}{2}\right)^2\right) i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}\tau + \frac{a}{b}\right), \frac{\tau}{2}; \tau\right) \qquad (4.6) \\ &= \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}} e\left(-\frac{r}{2}\left(\frac{A}{C} - \frac{1}{2}\right)^2\right) V_\alpha(\tau), \end{aligned}$$

where we have applied Lemma 1.1 $\frac{Ar}{C}$ times and Lemma 1.2 $\frac{r}{2}$ times in (4.5), and Lemma 1.5 r times in (4.6).

Now consider the second transformation; using a similar approach to the authors in [2], we let $M_r\tau = S\tau_r = -\frac{1}{\tau_r}$, where $\tau_r := -\frac{1}{\tau} - r$, and apply Lemma 1.6 so that we have

$$\begin{aligned}
V_\alpha(M_r\tau) &= i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}M_r\tau + \frac{a}{b}\right), \frac{M_r\tau}{2}; M_r\tau\right) \\
&= i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\left(\frac{A}{2C}\left(\frac{-1}{\tau_r}\right) + \frac{a}{b}\right), \frac{-1}{2\tau_r}; \frac{-1}{\tau_r}\right) \\
&= i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \mu\left(2\frac{\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right)}{\tau_r}, \frac{-1}{2\tau_r}; \frac{-1}{\tau_r}\right) \\
&= i\zeta_b^a q^{-\frac{(A/C-1/2)^2}{2}} \sqrt{-i\tau_r}\, e\left(-\frac{\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right) + \frac{1}{2}\right)^2}{2\tau_r}\right) \times \\
&\left[\frac{1}{2i}h\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right) + \frac{1}{2}; \tau_r\right) - \mu\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right), -\frac{1}{2}; \tau_r\right)\right].
\end{aligned}$$

Now, recalling that $\tau_r := -\frac{1}{\tau} - r$, the roots of unity can be simplified such that

$$\begin{aligned}
&q^{-\frac{(A/C-1/2)^2}{2}} e\left(-\frac{\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right) + \frac{1}{2}\right)^2}{2\tau_r}\right) = e\left(\frac{\left(\frac{A}{C} - \frac{1}{2}\right)^2}{2\tau_r}\right) e\left(-\frac{\left(\frac{2a}{b}\tau_r - \left(\frac{A}{C} - \frac{1}{2}\right)\right)^2}{2\tau_r}\right) \\
&= e\left(\frac{\left(\frac{A}{C} - \frac{1}{2}\right)^2}{2\tau_r}\right) e\left(-\frac{2a^2}{b^2}\tau_r + \frac{2a}{b}\left(\frac{A}{C} - \frac{1}{2}\right) - \frac{\left(\frac{A}{C} - \frac{1}{2}\right)^2}{2\tau_r}\right) \\
&= e\left(\frac{2a^2}{b^2}\left(\frac{1}{\tau} + r\right) + \frac{2a}{b}\left(\frac{A}{C} - \frac{1}{2}\right)\right) = \zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right)
\end{aligned}$$

where $\zeta_{a,b}^{A,C,r} := e\left(\frac{2a^2}{b^2}r + \frac{2a}{b}\left(\frac{A}{C} - \frac{1}{2}\right)\right)$. If we let

$$I_\alpha(\tau) := i\zeta_b^a \sqrt{-i\tau_r}\, e\left(\frac{2a^2}{b^2\tau}\right) \frac{\zeta_{a,b}^{A,C,r}}{2i} h\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right) + \frac{1}{2}; \tau_r\right),$$

we have then

$$V_\alpha(M_r\tau) = -i\zeta_b^a \sqrt{-i\tau_r}\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) \mu\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right), -\frac{1}{2}; \tau_r\right) + I_\alpha(\tau).$$

Consider focusing on the former term; recalling $\tau_r := -\frac{1}{\tau} - r$, we can rewrite this such that

$$\begin{aligned}
&- i\zeta_b^a \sqrt{-i\tau_r}\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) \mu\left(2\left(\frac{a}{b}\tau_r - \frac{A}{2C}\right), -\frac{1}{2}; \tau_r\right) \\
&= -i\zeta_b^a \sqrt{-i\tau_r}\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) \mu\left(-2\frac{\left(\frac{A}{2C}\tau + \frac{a}{b}\right)}{\tau} - \frac{2ar}{b}, -\frac{1}{2}; -\frac{1}{\tau} - r\right)
\end{aligned}$$

$$= -i\zeta_b^a \sqrt{-i\tau_r}\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) (-1)^{-\frac{2ar}{b}} \zeta_8^r \mu\left(-2\frac{\left(\frac{A}{2C}\tau + \frac{a}{b}\right)}{\tau}, -\frac{\tau/2}{\tau}; -\frac{1}{\tau}\right) \tag{4.7}$$

$$\begin{aligned}
&= -i\zeta_b^a \sqrt{-i\tau_r}\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) (-1)^{-\frac{2ar}{b}} \zeta_8^r \sqrt{-i\tau}\, e\left(-\frac{\left(-2\left(\frac{A}{2C}\tau + \frac{a}{b}\right) + \frac{\tau}{2}\right)^2}{2\tau}\right) \times
\end{aligned}$$

$$\left[\frac{1}{2i}h\left(-2\left(\frac{A}{2C}\tau + \frac{a}{b}\right) + \frac{\tau}{2}; \tau\right) - \mu\left(-2\left(\frac{A}{2C}\tau + \frac{a}{b}\right), -\frac{\tau}{2}; \tau\right)\right], \tag{4.8}$$

where we have applied Lemma 1.1 (supposing $b|2ar$), Lemma 1.5 in (4.7), and Lemma 1.6 in (4.8). Now, we would like to further reduce the roots of unity shown in (4.8). First, notice that $\sqrt{-i\tau_r}\sqrt{-i\tau} = \sqrt{r\tau + 1}$, which trivially follows. In addition, the roots of unity can be reduced such that

$$\zeta_{a,b}^{A,C,r} e\left(\frac{2a^2}{b^2\tau}\right) e\left(-\frac{\left(-2\left(\frac{A}{2C}\tau+\frac{a}{b}\right)+\frac{\tau}{2}\right)^2}{2\tau}\right)$$
$$= e\left(\frac{2a^2}{b^2}r+\frac{2a}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right) e\left(\frac{2a^2}{b^2\tau}\right) e\left(-\frac{\tau}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2-\frac{2a}{b}\left(\frac{A}{C}-\frac{1}{2}\right)-\frac{2a^2}{b^2\tau}\right) = e\left(\frac{2a^2}{b^2}r\right) q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}.$$

If we let

$$J_\alpha(\tau) := -i\zeta_b^a\sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right) q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\frac{1}{2i}h\left(-2\left(\frac{A}{2C}\tau+\frac{a}{b}\right)+\frac{\tau}{2};\tau\right),$$

we have then

$$V_\alpha(M_r\tau) = i\zeta_b^a\sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right) q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\mu\left(-2\left(\frac{A}{2C}\tau+\frac{a}{b}\right), -\frac{\tau}{2};\tau\right) + I_\alpha(\tau) + J_\alpha(\tau).$$

Applying Lemma 1.3 on the μ-function, this reduces to

$$V_\alpha(M_r\tau) = (-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)\sqrt{r\tau+1}V_\alpha(\tau) + I_\alpha(\tau) + J_\alpha(\tau).$$

□

Now that we have completed the proof for Lemma 5, we are interested in converting our functions I_α and J_α into Mordell integrals, which will help in establishing real analyticity once we establish a correspondence between the quantum sets and groups for the case $b = 4$. Before we proceed, we will make use of the following definition.

Definition 6. *Let a, b be such that both a and b are not in $\mathbb{Z}+\frac{1}{2}$, and $\tau \in \mathbb{H}$; then*

$$\delta_{a,b}(\tau) := \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}\right)\right) q^{-\frac{a^2}{2}} h\left(a\tau-b;\tau\right).$$

It can be easily shown applying Lemma 6.1 and Lemma 6.3, the δ-function is mostly constant in the interval $[-\frac{1}{2}, \frac{1}{2}]$, with slight changes at the end points. In addition, δ-function satisfies simple shift properties, akin to the g and h functions from Section 2.

Lemma 6. *Let a, b such that both a and b are not in $\mathbb{Z}+\frac{1}{2}$, and $\tau \in \mathbb{H}$; then*

$$\delta_{a,b}(\tau) = \begin{cases} i, & \text{if } a = \frac{1}{2}, & b \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\} \\ 0, & \text{if } a \in (-\frac{1}{2}, \frac{1}{2}), & b \in (-\frac{1}{2}, \frac{1}{2}) \\ e(a)/\sqrt{-i\tau}, & \text{if } a \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}, & b = \frac{1}{2}. \end{cases}$$

In addition, $\delta_{a,b}(\tau)$ satisfies

$$\delta_{a,b+1}(\tau) = e^{2\pi i a}\left(\frac{2}{\sqrt{-i\tau}}e^{\pi i(b+\frac{1}{2})^2/\tau} - \delta_{a,b}(\tau)\right), \tag{4.9}$$

$$\delta_{a+1,b}(\tau) = \delta_{a,b}(\tau) - 2e^{\pi i(a+b)}e^{2\pi iab}q^{-(a+\frac{1}{2})^2/2}, \tag{4.10}$$

Proof. It can be easily shown that, by Theorem 2.3, Lemma 3.1, and Lemma 3.2, $\delta_{a,b}(\tau)$ must be one of the above three cases. Now consider the first shift property, or (4.9); we have

$$\delta_{a,b+1}(\tau) = \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}+1}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}+1\right)\right) q^{-\frac{a^2}{2}} h\left(a\tau-b-1;\tau\right).$$

Applying Lemma 2.2 and Proposition 1.1, we have

$$\begin{aligned}\delta_{a,b+1}(\tau) &= \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}+1}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}+1\right)\right) q^{-\frac{a^2}{2}} h\left(a\tau-b-1;\tau\right)\\
&= e^{2\pi i(a+1/2)}\int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}+1\right)\right) q^{-\frac{a^2}{2}}\\
&\times\left(\frac{2}{\sqrt{-i\tau}}e^{\pi i(a\tau-b-1/2)^2/\tau} - h\left(a\tau-b;\tau\right)\right)\\
&= -e^{2\pi ia}\int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz - e^{2\pi ia}e\left(a\left(b+\frac{1}{2}\right)\right) q^{-\frac{a^2}{2}} h\left(a\tau-b;\tau\right)\\
&+\frac{2}{\sqrt{-i\tau}}e^{\pi i(a\tau-b-1/2)^2/\tau}e^{2\pi ia\left(b+\frac{1}{2}+1\right)}q^{-\frac{a^2}{2}}\\
&= -e^{2\pi ia}\delta_{a,b}(\tau)+\frac{2e^{2\pi ia}}{\sqrt{-i\tau}}e^{\pi i(b+\frac{1}{2})^2/\tau}.\end{aligned}$$

Now consider the second shift property, or (4.10); we have

$$\delta_{a+1,b}(\tau) = \int_0^{i\infty} \frac{g_{a+\frac{1}{2}+1,b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left((a+1)\left(b+\frac{1}{2}\right)\right) q^{-\frac{(a+1)^2}{2}} h\left(a\tau-b+\tau;\tau\right).$$

Applying Lemma 2.1 and Proposition 1.2, we have

$$\begin{aligned}\delta_{a+1,b}(\tau) &= \int_0^{i\infty} \frac{g_{a+\frac{1}{2}+1,b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left((a+1)\left(b+\frac{1}{2}\right)\right) q^{-\frac{(a+1)^2}{2}} h\left(a\tau-b+\tau;\tau\right)\\
&= \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}\right)\right) q^{-\frac{a^2}{2}} e^{2\pi i(b+1/2)}q^{-a-\frac{1}{2}}e^{2\pi i((a+1/2)\tau-b)}\\
&\times\left(2e^{-\pi i((a+1/4)\tau-b)} - h\left(a\tau-b;\tau\right)\right)\\
&= \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz + e\left(a\left(b+\frac{1}{2}\right)\right) q^{-\frac{a^2}{2}} h\left(a\tau-b;\tau\right)\\
&- 2e^{\pi ib}e^{2\pi ia\left(b+\frac{1}{2}\right)}q^{-\frac{1}{2}\left(a^2+a+\frac{1}{4}\right)}\\
&= \delta_{a,b}(\tau) - 2e^{\pi i(a+b)}e^{2\pi iab}q^{-(a+\frac{1}{2})^2/2}.\end{aligned}$$

□

The benefit of applying this δ-function is it contains the necessary information such that it eases computations with respect to manipulating the transformations of V_α, since δ contains information about how the transformation properties of V_α behave over $(-\frac{1}{2},\frac{1}{2})$ and at the endpoints. This is essential to the proof of transforming the functions I_α and J_α to Mordell integrals.

Lemma 7. *Let $A,a\in\mathbb{Z}$, and $C,b,r\in\mathbb{N}$ with the restriction $(A,C)=1$. We have then*

$$I_\alpha(\tau) := \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_{\frac{1}{r}}^{0}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i\left(u+\tau\right)}}\,du + \frac{\sqrt{-i\tau_r}}{2}\delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r).$$

$$J_\alpha(\tau) := \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_0^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}\,du + \sqrt{r\tau+1}\zeta_C^A\frac{i\epsilon_{a,b}^{A,C,r}}{2}\delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau),$$

where

$$\epsilon_{a,b}^{A,C,r} := e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2+\frac{A}{C}\left(\frac{2a}{b}-\frac{1}{2}\right)\right),$$

and where r is chosen such that $b|2ar$, in addition to $r=1$ whenever $a=0$.

Proof. Recall from (4.2)

$$V_\alpha(M_r\tau) = (-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)\sqrt{r\tau+1}V_\alpha(\tau) + I_\alpha(\tau) + J_\alpha(\tau),$$

where

$$I_\alpha(\tau) := i\zeta_b^a\sqrt{-i\tau_r}e\left(\frac{2a^2}{b^2\tau}\right)\frac{\zeta_{a,b}^{A,C,r}}{2i}h\left(\frac{2a}{b}\tau_r - \left(\frac{A}{C}-\frac{1}{2}\right);\tau_r\right), \tag{4.11}$$

$$J_\alpha(\tau) := -i\zeta_b^a\sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\frac{1}{2i}h\left(\left(\frac{1}{2}-\frac{A}{C}\right)\tau - \frac{2a}{b};\tau\right). \tag{4.12}$$

First, consider (4.11); we have

$$\begin{aligned}I_\alpha(\tau) &= i\zeta_b^a\sqrt{-i\tau_r}e\left(\frac{2a^2}{b^2\tau}\right)\frac{\zeta_{a,b}^{A,C,r}}{2i}h\left(\frac{2a}{b}\tau_r - \left(\frac{A}{C}-\frac{1}{2}\right);\tau_r\right)\\ &= -i\zeta_b^a\sqrt{-i\tau_r}e\left(\frac{2a^2}{b^2\tau}\right)\frac{\zeta_{a,b}^{A,C,r}}{2i}e\left(\frac{\tau_r}{2}\left(\frac{2a}{b}\right)^2 - \frac{2aA}{bC}\right)\times\left(\int_0^{i\infty}\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}(z)}{\sqrt{-i(z+\tau_r)}}\,dz - \delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r)\right).\end{aligned}$$

Recalling $\tau_r = -\frac{1}{\tau} - r$, we can reduce our roots of unity such that

$$\begin{aligned}&\zeta_{a,b}^{A,C,r}e\left(\frac{2a^2}{b^2\tau}\right)e\left(\frac{\tau_r}{2}\left(\frac{2a}{b}\right)^2 - \frac{2aA}{bC}\right)\\ &= e\left(\frac{2a^2}{b^2}r + \frac{2a}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)e\left(\frac{2a^2}{b^2\tau}\right)e\left(-\frac{2a^2}{b^2}\left(\frac{1}{\tau}+r\right) - \frac{2aA}{bC}\right)\\ &= e\left(\frac{2a^2}{b^2}r + \frac{2aA}{bC} - \frac{a}{b}\right)e\left(\frac{2a^2}{b^2\tau}\right)e\left(-\frac{2a^2}{b^2\tau} - \frac{2a^2}{b^2}r - \frac{2aA}{bC}\right) = \zeta_b^{-a},\end{aligned}$$

so that

$$I_\alpha(\tau) = \frac{-\sqrt{-i\tau_r}}{2}\int_0^{i\infty}\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}(z)}{\sqrt{-i(z+\tau_r)}}\,dz + \frac{\sqrt{-i\tau_r}}{2}\delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r).$$

Consider the integral part of $I_\alpha(\tau)$, i.e.

$$\int_0^{i\infty}\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}(z)}{\sqrt{-i(z+\tau_r)}}\,dz.$$

Substituting $z = r - u^{-1} \to dz = u^{-2}du$, we have

$$\int_0^{i\infty}\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}(z)}{\sqrt{-i(z+\tau_r)}}\,dz = \int_{\frac{1}{r}}^0\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}\left(r-\frac{1}{u}\right)}{\sqrt{-i\left(r-\frac{1}{u}-\frac{1}{\tau}-r\right)}}\frac{du}{u^2} = \sqrt{\tau}\int_{\frac{1}{r}}^0\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}\left(r-\frac{1}{u}\right)}{\sqrt{i\left(u+\tau\right)}}\frac{du}{u^{3/2}}$$

$$= \sqrt{\tau}e\left(-\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{3}{2}\right)\right)\int_{\frac{1}{r}}^0\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}+r\left(\frac{2a}{b}+1\right)}\left(-\frac{1}{u}\right)}{\sqrt{i\left(u+\tau\right)}}\frac{du}{u^{3/2}} \tag{4.13}$$

$$= \sqrt{\tau}e\left(-\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{3}{2}\right)\right)e\left(r\left(\frac{2a}{b}+\frac{1}{2}\right)\left(\frac{2a}{b}+1\right)\right)\int_{\frac{1}{r}}^0\frac{g_{\frac{2a}{b}+\frac{1}{2},\frac{A}{C}}\left(-\frac{1}{u}\right)}{\sqrt{i\left(u+\tau\right)}}\frac{du}{u^{3/2}} \tag{4.14}$$

$$= \sqrt{\tau}i(-i)^{3/2}e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2\right)e\left(\frac{A}{C}\left(\frac{2a}{b}+\frac{1}{2}\right)\right)\int_{\frac{1}{r}}^0\frac{g_{\frac{A}{C},-\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{i\left(u+\tau\right)}}\,du \tag{4.15}$$

$$= \sqrt{\tau}i(-i)^{3/2}e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2\right)e\left(\frac{A}{C}\left(\frac{2a}{b}+\frac{1}{2}\right)\right)e\left(-\frac{A}{C}\right)\int_{\frac{1}{r}}^0\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{i\left(u+\tau\right)}}\,du \tag{4.16}$$

$$= -i\sqrt{-i\tau}e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{2a}{b}-\frac{1}{2}\right)\right)\int_{\frac{1}{r}}^0\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i\left(u+\tau\right)}}\,du, \tag{4.17}$$

where we have applied Lemma 2.4 in (4.13), Lemma 2.2 in (4.14) (supposing $b|2ar$), and Lemma 2.5 in (4.15), each r times. Furthermore, trivial simplification of the roots of unity immediately follow (4.13)-(4.17). If we denote

$$\epsilon_{a,b}^{A,C,r} := e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{2a}{b}-\frac{1}{2}\right)\right),$$

and recall $\sqrt{-i\tau_r}\sqrt{-i\tau} = \sqrt{r\tau+1}$, then $I_\alpha(\tau)$ reduces to

$$I_\alpha(\tau) = \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_{\frac{1}{r}}^{0} \frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}\,du + \frac{\sqrt{-i\tau_r}}{2}\delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r). \tag{4.18}$$

Now consider (4.12), or

$$J_\alpha(\tau) := -i\zeta_b^a\sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\frac{1}{2i}h\left(\left(\frac{1}{2}-\frac{A}{C}\right)\tau - \frac{2a}{b};\tau\right).$$

Applying the definition for δ, we have

$$J_\alpha(\tau) := -\frac{\zeta_b^a}{2}\sqrt{r\tau+1}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\left[-q^{\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}e\left(\left(\frac{A}{C}-\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{1}{2}\right)\right)\right]$$
$$\times\left[\int_0^{i\infty}\frac{g_{1-\frac{A}{C},\frac{2a}{b}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du - \delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau)\right].$$

Simplifying our roots of unity such that

$$\begin{aligned}
&-\frac{\zeta_b^a}{2}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)q^{-\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}\left[-q^{\frac{1}{2}\left(\frac{1}{2}-\frac{A}{C}\right)^2}e\left(\left(\frac{A}{C}-\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{1}{2}\right)\right)\right]\\
&=\frac{\zeta_b^a}{2}(-1)^{-\frac{2ar}{b}}\zeta_8^r e\left(\frac{2a^2}{b^2}r\right)e\left(\left(\frac{A}{C}-\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{1}{2}\right)\right)\\
&=\frac{\zeta_b^a}{2}e\left(\frac{r}{2}\left(\frac{2a}{b}+\frac{1}{2}\right)^2\right)e\left(\left(\frac{A}{C}-\frac{1}{2}\right)\left(\frac{2a}{b}+\frac{1}{2}\right)\right) = \frac{-i}{2}\zeta_C^A\epsilon_{a,b}^{A,C,r}
\end{aligned}$$

then we have

$$J_\alpha(\tau) := -\sqrt{r\tau+1}\zeta_C^A\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_0^{i\infty}\frac{g_{1-\frac{A}{C},\frac{2a}{b}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du + \sqrt{r\tau+1}\zeta_C^A\frac{i\epsilon_{a,b}^{A,C,r}}{2}\delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau).$$

Consider the integral part of $J_\alpha(\tau)$, i.e.

$$\int_0^{i\infty}\frac{g_{1-\frac{A}{C},\frac{2a}{b}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du.$$

We have

$$\int_0^{i\infty}\frac{g_{1-\frac{A}{C},\frac{2a}{b}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du = \int_0^{i\infty}\frac{g_{-\frac{A}{C},\frac{2a}{b}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du \tag{4.19}$$

$$= -\int_0^{i\infty}\frac{g_{\frac{A}{C},-\frac{2a}{b}-\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du \tag{4.20}$$

$$= -\zeta_C^{-A}\int_0^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}\,du \tag{4.21}$$

where we have applied Lemma 2.1 in (4.19), Lemma 2.3 in (4.20), and Lemma 2.2 in (4.21). Thus, our $J_\alpha(\tau)$ reduces to

$$J_\alpha(\tau) := \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_0^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}\,du + \sqrt{r\tau+1}\zeta_C^A\frac{i\epsilon_{a,b}^{A,C,r}}{2}\delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau). \tag{4.22}$$

□

We can now state the transformations of V_α as follows.

Theorem 5. *Let $A \in \mathbb{Z}$, and $C, b, r \in \mathbb{N}$, $a \in \{0, 1, \ldots, b-1\}$, and $\frac{A}{C} \in (0,1)$; then*

$$V_\alpha(T_r\tau) = \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}} e\left(-\frac{r}{2}\left(\frac{A}{C}-\frac{1}{2}\right)^2\right) V_\alpha(\tau) \tag{4.23}$$

$$V_\alpha(M_r\tau) = (-1)^{\frac{2ar}{b}} \zeta_8^r e\left(\frac{2a^2}{b^2}r\right)\sqrt{r\tau+1}V_\alpha(\tau) + \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_{\frac{1}{r}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}du. \tag{4.24}$$

where in (4.23), $C|r$ *and* $2|r$, *and in* (4.24), $b|2ar$.

Proof. The first transformation (4.23) follows from Lemma 5. For the second transformation, or (4.24), we combine (4.18) and (4.22) so that we have

$$V_\alpha(M_r\tau) = (-1)^{\frac{2ar}{b}} \zeta_8^r e\left(\frac{2a^2}{b^2}r\right)\sqrt{r\tau+1}V_\alpha(\tau) + \sqrt{r\tau+1}\frac{i\epsilon_{a,b}^{A,C,r}}{2}\int_{\frac{1}{r}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{2a}{b}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$+\frac{\sqrt{-i\tau_r}}{2}\left[\delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A\epsilon_{a,b}^{A,C,r}\sqrt{-i\tau}\delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau)\right].$$

By Definition 2, we have that $0 \le \frac{2a}{b} < 2$, or otherwise $\frac{2a}{b}$ is no larger than 2, and thus must account for the single integer shift. It is enough to account for this integer shift in order to show

$$\delta_{\frac{2a}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A\epsilon_{a,b}^{A,C,r}\sqrt{-i\tau}\delta_{\frac{1}{2}-\frac{A}{C},\frac{2a}{b}}(\tau) = 0$$

for $\frac{A}{C} \in (0,1)$ and $a \in \{0,1,\ldots,b-1\}$. We may write $\frac{2a}{b} = p + \frac{s}{b}$, where $p \in \{0,1\}$ and $s \in \{0,1,\ldots,b-1\}$. Suppose $p=1$; applying the δ-shifts, and recalling $\tau_r := -\frac{1}{\tau} - r$, we have

$$\delta_{1+\frac{s}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A\epsilon_{a,b}^{A,C,r}\sqrt{-i\tau}\delta_{\frac{1}{2}-\frac{A}{C},1+\frac{s}{b}}(\tau)$$
$$= \left(\delta_{\frac{s}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) - 2\exp\left(\pi i\left(\frac{s}{b}+\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(2\pi i\frac{s}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right)\right)$$
$$+ i\zeta_C^A\epsilon_{a,b}^{A,C,r}\sqrt{-i\tau}\exp\left(2\pi i\left(\frac{1}{2}-\frac{A}{C}\right)\right)\left(\frac{2}{\sqrt{-i\tau}}\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right) - \delta_{\frac{1}{2}-\frac{A}{C},\frac{s}{b}}(\tau)\right)$$
$$= \delta_{\frac{s}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A\epsilon_{s,2b}^{A,C,r}\sqrt{-i\tau}\delta_{\frac{1}{2}-\frac{A}{C},\frac{s}{b}}(\tau)$$
$$+ 2i\zeta_C^A\epsilon_{a,b}^{A,C,r}\exp\left(2\pi i\left(\frac{1}{2}-\frac{A}{C}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right)$$
$$- 2\exp\left(\pi i\left(\frac{s}{b}+\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(2\pi i\frac{s}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right).$$

We want to show

$$2i\zeta_C^A\epsilon_{a,b}^{A,C,r}\exp\left(2\pi i\left(\frac{1}{2}-\frac{A}{C}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right)$$
$$- 2\exp\left(\pi i\left(\frac{s}{b}+\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(2\pi i\frac{s}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right) = 0. \tag{4.25}$$

Notice we can rewrite the former term such that

$$
\begin{aligned}
&2i\zeta_C^A \epsilon_{a,b}^{A,C,r} \exp\left(2\pi i\left(\frac{1}{2}-\frac{A}{C}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right)\\
&= -2i\zeta_C^A \epsilon_{s,2b}^{A,C,r} \exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right)\\
&= -2i\zeta_C^A \exp\left(2\pi i\left(\frac{r}{2}\left(\frac{s}{b}+\frac{1}{2}\right)^2+\frac{A}{C}\left(\frac{s}{b}-\frac{1}{2}\right)\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2/\tau\right)\\
&= -2i\zeta_C^A \exp\left(2\pi i\frac{A}{C}\left(\frac{s}{b}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right)\\
&= 2\exp\left(-\frac{\pi i}{2}\right)\exp\left(2\pi i\frac{A}{C}\right)\exp\left(\pi i\left(\frac{s}{b}-\frac{A}{C}\right)\right)\exp\left(2\pi i\frac{s}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right)\\
&= 2\exp\left(\pi i\left(\frac{s}{b}+\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(2\pi i\frac{s}{b}\left(\frac{A}{C}-\frac{1}{2}\right)\right)\exp\left(\pi i\left(\frac{s}{b}+\frac{1}{2}\right)^2\left(\frac{1}{\tau}+r\right)\right),
\end{aligned}
$$

which implies (4.25) is in fact 0. Thus, this reduces to showing[3]

$$
\delta_{\frac{s}{b},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A \epsilon_{s,2b}^{A,C,r}\sqrt{-i\tau}\,\delta_{\frac{1}{2}-\frac{A}{C},\frac{s}{b}}(\tau) = 0. \tag{4.26}
$$

Since $\left|\frac{A}{C}-\frac{1}{2}\right| \in \left(-\frac{1}{2},\frac{1}{2}\right)$, we only need to consider the case when $\frac{s}{b}\in[0,\frac{1}{2}]$. If $\frac{s}{b}\in[0,\frac{1}{2})$, then it is trivial (4.26) is 0. Now suppose $\frac{s}{b}=\frac{1}{2}$; because $b|2ar$ must be satisfied, this implies $b|sr$, and therefore $2|r$. With a bit of simplification, we have

$$
\begin{aligned}
&\delta_{\frac{1}{2},\frac{A}{C}-\frac{1}{2}}(\tau_r) + i\zeta_C^A \epsilon_{s,2b}^{A,C,r}\sqrt{-i\tau}\,\delta_{\frac{1}{2}-\frac{A}{C},\frac{1}{2}}(\tau)\\
&= i + i\zeta_C^A \epsilon_{s,2b}^{A,C,r} e^{2\pi i\left(\frac{1}{2}-\frac{A}{C}\right)} = i - i = 0,
\end{aligned}
$$

which implies our result. □

We can see that the functions described in [2] are special cases of this transformation whenever $b=4$, for select r, and the restriction $\frac{A}{C}\in(0,1)$, of which we will now present as corollaries.

Corollary 1. *For $b=4$, $r=2$, $\frac{A}{C}\in(0,1)$ and $a\in\{0,1,2,3\}$,*

$$
V_\alpha(M_2\tau) = (-1)^a i\zeta_4^{a^2}\sqrt{2\tau+1}V_\alpha(\tau) + ie\left(\left(\frac{a+1}{2}\right)^2+\frac{A}{2C}(a-1)\right)\frac{\sqrt{2\tau+1}}{2}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{a}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du.
$$

Now, we would like to analyze the nature of the second transformation of V_α whenever $b=4$ and $r=1$. However, because the condition $b|2ar$ must be satisfied, a is forced to be even.

Corollary 2. *For $b=4$, $r=1$, $\frac{A}{C}\in(0,1)$, and $a\in\{0,2\}$,*

$$
V_\alpha(M_1\tau) = \zeta_8\sqrt{\tau+1}V_\alpha(\tau) + \zeta_8\zeta_{2C}^{-A}\frac{\sqrt{\tau+1}}{2}\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\,du.
$$

5. Proof of Theorem 1

Proof. Now, in order to prove quantum modularity, we must make the restrictions outlined in Theorem 1. Therefore, we need to set $b=4$ from our equations in Theorem 5. For $V_\alpha(M_r\tau)$, we get

[3] Notice this would have been the case if we had let $p=0$, and thus $s=2a$.

$$V_\alpha(M_r\tau) = (-1)^{\frac{ar}{2}}\zeta_8^r e\left(\frac{a^2}{8}r\right)\sqrt{r\tau+1}V_\alpha(\tau)$$

$$+\sqrt{r\tau+1}\frac{i}{2}e\left(\frac{r}{2}\left(\frac{a}{2}+\frac{1}{2}\right)^2+\frac{A}{C}\left(\frac{a}{2}-\frac{1}{2}\right)\right)\int_{\frac{1}{r}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{a}{2}}(u)}{\sqrt{-i(u+\tau)}}du. \tag{5.1}$$

Following this, we are going to compute what each $V_\alpha(M_2\tau)$ looks like for $a = 0, 1, 2, 3$ and what each $V_\alpha(M_1\tau)$ looks like for $a = 0, 2$. and show that the equations we state in Theorem 1 hold. Then, we will do the same for $V_\alpha(T_r\tau)$ for a general a and and r that is an even multiple of C. Following that, we will argue why this makes our functions quantum modular.

We first begin by focusing on the M_2 transformations. So, we can simplify (5.1) further by setting $r = 2$ which looks like

$$V_\alpha(M_2\tau) = (-1)^a ie\left(\frac{a^2}{4}\right)\sqrt{2\tau+1}V_\alpha(\tau) \tag{5.2}$$

$$+\sqrt{2\tau+1}\frac{i}{2}e\left(\left(\frac{a}{2}+\frac{1}{2}\right)^2+\frac{A}{C}\left(\frac{a}{2}-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{a}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

From here, we can go through the cases of $a = 0, 1, 2, 3$. We start with $a = 0$. We get

$$\begin{aligned}
V_\alpha(M_r\tau) &= i\sqrt{2\tau+1}V_\alpha(\tau)+\sqrt{2\tau+1}\frac{i}{2}e\left(\left(\frac{1}{2}\right)^2+\frac{A}{C}\left(-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.\\
&= i\sqrt{2\tau+1}V_\alpha(\tau)+\sqrt{2\tau+1}\frac{i}{2}e\left(\frac{1}{4}\right)e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}\\
&= i\sqrt{2\tau+1}V_\alpha(\tau)-\frac{1}{2}\sqrt{2\tau+1}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}.
\end{aligned}$$

Now we compute $\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$ for $a = 0$. We get

$$\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_r\tau) = V_\alpha(\tau)-\frac{1}{2i}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Finally, we compute $V_\alpha(\tau)-\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$. This looks like

$$V_\alpha(\tau)-\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau) = \frac{1}{2i}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

This clearly matches our second equation from Theorem 1 for $a = 0$ and M_2. Now, we follow the same procedure for $a = 2$ and M_2. So, we plug in $a = 2$ to our equation (5.2), which already accounts for $r = 2$. We get

$$V_\alpha(M_r\tau) = (-1)^a ie\left(\frac{a^2}{4}\right)\sqrt{2\tau+1}V_\alpha(\tau) + \sqrt{2\tau+1}\frac{i}{2}e\left(\left(\frac{a}{2}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{a}{2}-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{a}{2}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= i\sqrt{2\tau+1}V_\alpha(\tau) + \sqrt{2\tau+1}\frac{i}{2}e\left(\left(1+\frac{1}{2}\right)^2 + \frac{A}{C}\left(1-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-1}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= i\sqrt{2\tau+1}V_\alpha(\tau) + \sqrt{2\tau+1}\frac{i}{2}e\left(\frac{9}{4}+\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-1}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= i\sqrt{2\tau+1}V_\alpha(\tau) + -\frac{1}{2}\sqrt{2\tau+1}e\left(\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{-1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Now we use Lemma 2 part (2) in order to shift our g function to the desired form. Therefore, we get

$$V_\alpha(M_2\tau) = i\sqrt{2\tau+1}V_\alpha(\tau) + -\frac{1}{2}\sqrt{2\tau+1}e\left(\frac{A}{2C}-\frac{A}{C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= i\sqrt{2\tau+1}V_\alpha(\tau) + -\frac{1}{2}\sqrt{2\tau+1}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Now, we compute $\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$ which is

$$\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau) = V_\alpha(\tau) + -\frac{1}{2i}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

From here, we compute $V_\alpha - \frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$. We get

$$\frac{1}{i}(2\tau+1)^{-1/2}V_\alpha(M_2\tau) = \frac{1}{2i}e\left(-\frac{A}{2C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

This clearly matches our second equation from Theorem 1. Now we continue to look at our M_2 transformations, but we aim to match $a = 1$ and $a = 3$ to our first equation in Theorem 1. We begin with plugging in $a = 1$ in equation (5.2), which already accounts for $r = 2$. We obtain that

$$V_\alpha(M_2\tau) = -ie\left(\frac{1}{4}\right)\sqrt{2\tau+1}V_\alpha(\tau) + \sqrt{2\tau+1}\frac{i}{2}e\left(\left(\frac{1}{2}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{1}{2}-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= \sqrt{2\tau+1}V_\alpha(\tau) + \frac{i}{2}\sqrt{2\tau+1}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

Now we compute $(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$. It looks like

$$(2\tau+1)^{-1/2}V_\alpha(M_2\tau) = V_\alpha(\tau) + \frac{i}{2}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

Finally, we are able to compute $V_\alpha(\tau) - (2\tau+1)^{-1/2}V_\alpha(M_2\tau)$. We find that

$$V_\alpha(\tau) - (2\tau+1)^{-1/2}V_\alpha(M_2\tau) = -\frac{i}{2}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

We see this matches our first equation from Theorem 1. Now we need to do the same for $a = 3$. So, we plug in $a = 3$ in our equation (5.2), which accounts for $r = 2$, to obtain

$$V_\alpha(M_2\tau) = -ie\left(\frac{9}{4}\right)\sqrt{2\tau+1}V_\alpha(\tau) + \sqrt{2\tau+1}\frac{i}{2}e\left(\left(\frac{3}{2}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{3}{2}-\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{3}{2}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= \sqrt{2\tau+1}V_\alpha(\tau) + \frac{i}{2}\sqrt{2\tau+1}e\left(\frac{A}{C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},-1}(u)}{\sqrt{-i(u+\tau)}}du.$$

Now, we need to use Lemma 2 in order to get our g function in the form that we desire. We see that

$$V_\alpha(M_2\tau) = \sqrt{2\tau+1}V_\alpha(\tau) + \frac{i}{2}\sqrt{2\tau+1}e\left(\frac{A}{C}-\frac{A}{C}\right)\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= \sqrt{2\tau+1}V_\alpha(\tau) + \frac{i}{2}\sqrt{2\tau+1}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

We now can compute $(2\tau+1)^{-1/2}V_\alpha(M_2\tau)$ and we find

$$(2\tau+1)^{-1/2}V_\alpha(M_2\tau) = V_\alpha(\tau) + \frac{i}{2}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

Finally, we compute $V_\alpha(\tau) - (2\tau+1)^{-1/2}V_\alpha(M_2\tau)$. We see that

$$V_\alpha(\tau) - (2\tau+1)^{-1/2}V_\alpha(M_2\tau) = -\frac{i}{2}\int_{\frac{1}{2}}^{i\infty}\frac{g_{\frac{A}{C},0}(u)}{\sqrt{-i(u+\tau)}}du.$$

This clearly matches our first equation from Theorem 1, and we have finished the M_2 transformations.

Now we move on to the M_1 transformations for $a = 0$ and $a = 2$. We need to plug in $r = 1$ and $b = 4$ in our equation from Theorem 5, (5.1). When we do this, we get

$$V_\alpha(M_1\tau) = (-1)^{\frac{a}{2}}e\left(\frac{a^2+1}{8}\right)\sqrt{\tau+1}V_\alpha(\tau)$$
$$+ \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{2}\left(\frac{a}{2}+\frac{1}{2}\right)^2 + \frac{A}{C}\left(\frac{a}{2}-\frac{1}{2}\right)\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}-\frac{a}{2}}(u)}{\sqrt{-i(u+\tau)}}du. \tag{5.3}$$

And now we can begin with $a = 0$. Because (5.3) already accounts for $r = 1$, we first need to plug in $a = 0$ to obtain

$$V_\alpha(M_1\tau) = \zeta_8\sqrt{\tau+1}V_\alpha(\tau) + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{A}{C}\left(-\frac{1}{2}\right)\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du$$
$$= \zeta_8\sqrt{\tau+1}V_\alpha(\tau) + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{8}\right)e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Now we compute $\zeta_8^{-1}(1+\tau)^{-1/2}V_\alpha(M_1\tau)$. We find

$$\zeta_8^{-1}(1+\tau)^{-1/2}V_\alpha(M_1\tau) = V_\alpha(\tau) + \frac{i}{2}e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Finally, we can compute $V_\alpha(\tau) - \zeta_8^{-1}(1+\tau)^{-1/2}V_\alpha(M_1\tau)$. We get

$$V_\alpha(\tau) - \zeta_8^{-1}(1+\tau)^{-1/2}V_\alpha(M_1\tau) = -\frac{i}{2}e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

We see this matches our third equation from Theorem 1. Now, we need to show that our M_1 transformation for $a = 2$ does as well. Therefore, we need to plug in $a = 2$ in (5.3), which already accounts for $r = 1$. We get that

$$\begin{aligned} V_\alpha(M_1\tau) &= -e\left(\frac{5}{8}\right)\sqrt{\tau+1}V_\alpha(\tau) \\ &\quad + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{2}\left(1+\frac{1}{2}\right)^2 + \frac{A}{C}\left(1-\frac{1}{2}\right)\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},-\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du \\ &= e\left(\frac{1}{8}\right)\sqrt{\tau+1}V_\alpha(\tau) + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{8}\right)e\left(\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},-\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du. \end{aligned}$$

From here, we must use Lemma 2 part (2) in order to shift our g to the form we want. Doing this, we see

$$\begin{aligned} V_\alpha(M_1\tau) &= e\left(\frac{1}{8}\right)\sqrt{\tau+1}V_\alpha(\tau) + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{8}\right)e\left(\frac{A}{2C}-\frac{A}{C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du \\ &= e\left(\frac{1}{8}\right)\sqrt{\tau+1}V_\alpha(\tau) + \sqrt{\tau+1}\frac{i}{2}e\left(\frac{1}{8}\right)e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du. \end{aligned}$$

Now we must calculate $\zeta_8^{-1}(\tau+1)^{1/2}V_\alpha(M_1\tau)$. We find

$$\zeta_8^{-1}(\tau+1)^{-1/2}V_\alpha(M_1\tau) = V_\alpha(\tau) + \frac{i}{2}e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

Finally, we must find $V_\alpha(\tau) - \zeta_8^{-1}(\tau+1)^{-1/2}V_\alpha(M_1\tau)$, which is

$$V_\alpha(\tau) - \zeta_8^{-1}(\tau+1)^{-1/2}V_\alpha(M_1\tau) = -\frac{i}{2}e\left(-\frac{A}{2C}\right)\int_1^{i\infty}\frac{g_{\frac{A}{C},\frac{1}{2}}(u)}{\sqrt{-i(u+\tau)}}du.$$

This clearly matches the third equation from Theorem 1. Finally, we must match up our T transformations with the forth equation in our Theorem 1. We begin by looking at the T transformation in integral form from our Theorem 5. Here, r is an even multiple of C, as we recall from Theorem 4. We see

$$\begin{aligned} V_\alpha(T_r\tau) &= \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(-\frac{r}{2}\left(\frac{A}{C}-\frac{1}{2}\right)^2\right)V_\alpha(\tau) \\ &= \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(-\frac{r}{2}\left(\frac{2A-C}{2C}\right)^2\right)V_\alpha(\tau) \\ &= \zeta_8^{-r}(-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(\frac{r\left(2A-C\right)^2}{8C^2}\right)V_\alpha(\tau). \end{aligned}$$

Now, we calculate $(-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(\frac{r}{8}\right)e\left(\frac{r(2A-C)^2}{8C^2}\right)V(\tau+r)$. This looks like

$$(-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(\frac{r}{8}\right)e\left(\frac{r(2A-C)^2}{8C^2}\right)V_\alpha(T_r\tau) = V_\alpha(\tau).$$

And from here, we can clearly see

$$V_\alpha(\tau) - (-1)^{\frac{Ar}{C}+\frac{r}{2}}e\left(\frac{r}{8}\right)e\left(\frac{r(2A-C)^2}{8C^2}\right)V(\tau+r) = 0.$$

So, if C is even, we have $r = C$, and

$$V_\alpha(\tau) - (-1)^{A+\frac{C}{2}} e\left(\frac{C}{8}\right) e\left(\frac{C(2A-C)^2}{8C^2}\right) V(\tau + C) = 0.$$

If C is odd, we have $r = 2C$, and

$$V_\alpha(\tau) - (-1)^{2A+C} e\left(\frac{C}{4}\right) e\left(\frac{C(2A-C)^2}{4C^2}\right) V(\tau + 2C) = 0.$$

These match our last two equations in Theorem 1.

Now we have shown each of the equations in Theorem 1 hold. These equations tell us that the difference between $V_\alpha(\tau)$ and a root of unity multiplied by $(r\tau+1)^{-1/2} V_\alpha(M_r\tau)$ is a Mordell integral. Where M_r is from G_α. Because of the shape of the integral, we know the difference is real analytic as discussed in Folsom, et al [2]. The equations from Theorem 1 also tell us that the difference between $V_\alpha(\tau)$ and a root of unity multiplied by $V_\alpha(T_r\tau)$ is 0; this is clearly real analytic. We also know that by Theorem 4, we have quantum sets and quantum groups for these functions.

Therefore, the function V_α is quantum modular on the set S_α with respect to the group G_α, and this concludes our proof of Theorem 1. □

6. Conclusion

Although we were able to determine the transformation properties of V_α for arbitrary $b \in \mathbb{Z}$ satisfying particular conditions, it is still under investigation determining which quantum sets and groups will follow with respect to b. One approach that may yield a result is if b is restricted to be powers of two. In addition, it is curious what the structure of a strong generalization of our current work would appear such that it constitutes all classes of quantum modular forms seen in the work of [2]. A possible area to expand our work in order to encompass the work of [2] would be to investigate shifting our function, so we not only generalize the $n = 1$ case, but also the $n = 2, 3, 4, 5, 6$ cases. We are also interested in proving mock modularity for V_α as that is something that is accomplished in [2].

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