RANK AND CRANK-LIKE FUNCTIONS GENERATED BY BAILEY PAIRS

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ABSTRACT. Jennings-Shaffer extends in [6] work done by Garvan in [5] to generalize certain *spt* functions by considering higher-order moments. These authors prove identities related to specific partition functions by applying Bailey's transform and related techniques to various Bailey pairs. By introducing a few restrictions, we find ordinary rank- and crank-like moment inequalities for multiple Bailey pairs as well as combinatorial interpretations.

1. INTRODUCTION

Following the work of Garvan in [5] and Jennings-Shaffer in [6], we examine rank- and crank-like moments generated by Bailey pairs and their relation to smallest parts-like functions. As a result we prove inequalities between the ordinary moments and find combinatorial interpretations for some higher order spt-like functions.

We use the standard product notation,

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j),$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

When q is understood we abbreviate $(a;q)_n = (a)_n$ and $(a;q)_\infty = (a)_\infty$.

We recall that two sequences of functions α_n, β_n are a Bailey pair with respect to (a,q) if they satisfy

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

Definition 1.1. *For a Bailey pair α_n, β_n relative to $(1,q)$, we define the following rank-like function*

$$R_\alpha(z,q) = \mathrm{prod}(\beta_n(1,q))\left(1+\sum_{n=1}^{\infty}\frac{\alpha_n(1-z)(1-z^{-1})q^n}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

The term $\mathrm{prod}(\beta_n(1,q))$ *is is dependent on each Bailey pair.*

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Definition 1.2. *For a Bailey pair α_n, β_n relative to $(1,q)$, we define an α-rank $N^{\alpha}(m,n)$ by*

$$R_{\alpha}(z,q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N^{\alpha}(m,n) z^m q^n,$$

and an α-crank $M^{\alpha}(m,n)$ by

$$C_{\alpha}(z,q) = \frac{\text{prod}(\beta(1,q))(q)_{\infty}^2}{(zq)_{\infty}(z^{-1}q)_{\infty}} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^{\alpha}(m,n) z^m q^n.$$

Note that $M^{\alpha}(m,n) = M^{\alpha}(-m,n)$ and $N^{\alpha}(m,n) = N^{\alpha}(-m,n)$ since $R_{\alpha}(z,q)$ and $C_{\alpha}(z,q)$ are symmetric in z and z^{-1}.

We define the ordinary moments, $N_k^{\alpha}(n)$, $M_k^{\alpha}(n)$, and the symmetrized moments, $\mu_k^{\alpha}(n)$, $\eta_k^{\alpha}(n)$, in the same way as in [5] and [6]:

$$N_k^{\alpha}(n) = \sum_{m=-\infty}^{\infty} m^k N^{\alpha}(m,n),$$

$$M_k^{\alpha}(n) = \sum_{m=-\infty}^{\infty} m^k M^{\alpha}(m,n),$$

$$\mu_k^{\alpha}(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M^{\alpha}(m,n),$$

$$\eta_k^{\alpha}(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N^{\alpha}(m,n).$$

Due to the symmetries $M^{\alpha}(m,n) = M^{\alpha}(-m,n)$ and $N^{\alpha}(m,n) = N^{\alpha}(-m,n)$, we have that the odd moments are zero and

$$\mu_{2k}^{\alpha}(n) = \sum_{m=-\infty}^{\infty} \binom{m+k-1}{2k} M^{\alpha}(m,n),$$

$$\eta_{2k}^{\alpha}(n) = \sum_{m=-\infty}^{\infty} \binom{m+k-1}{2k} N^{\alpha}(m,n).$$

Additionally, the sums are finite since $M^{\alpha}(m,n) = N^{\alpha}(m,n) = 0$ for $|m| > n$.

We can now introduce a higher order spt-like function for each Bailey pair given by

$$\alpha\text{spt}_k(n) = \mu_{2k}^{\alpha}(n) - \eta_{2k}^{\alpha}(n).$$

The following is a generalization of Theorem 4.3 from [5].

Theorem 1.3. *For $k \geq 1$,*

$$\mu_{2k}^{\alpha}(n) = \frac{1}{(2k)!} \sum_{m=-\infty}^{\infty} g_k(m) M^{\alpha}(m,n),$$

$$\eta_{2k}^{\alpha}(n) = \frac{1}{(2k)!}\sum_{m=-\infty}^{\infty} g_k(m)N^{\alpha}(m,n),$$

$$M_{2k}^{\alpha}(n) = \sum_{j=1}^{k}(2j)!S^*(k,j)\mu_{2j}(n),$$

$$N_{2k}^{\alpha}(n) = \sum_{j=1}^{k}(2j)!S^*(k,j)\eta_{2j}(n),$$

where

$$g_k(x) = \prod_{j=0}^{k-1}(x^2 - j^2),$$

and the sequence $S^*(n,k)$ is defined recursively by $S^*(n+1,k) = S^*(n,k-1) + k^2 S^*(n,k)$, $S^*(1,1) = 1$ and $S^*(n,k) = 0$ if $k \leq 0$ or $k > n$.

Proof. We note if $m+k-1$ is negative, then $\binom{m+k-1}{2k} = \binom{k-m}{2k}$. We then find that

$$\begin{aligned}
\mu_{2k}^{\alpha}(n) &= \sum_{m=-\infty}^{\infty}\binom{m+k-1}{2k}M^{\alpha}(m,n) \\
&= \frac{1}{(2k)!}\sum_{m=-\infty}^{\infty}(m+k-1)(m+k-2)\cdots(m-k)M^{\alpha}(m,n) \\
&= \frac{1}{(2k)!}\sum_{m=-\infty}^{\infty}(m^2-(k-1)^2)(m^2-(k-2)^2)\cdots(m^2-1)m(m-k)M^{\alpha}(m,n) \\
&= \frac{1}{(2k)!}\sum_{m=0}^{\infty}(m^2-(k-1)^2)(m^2-(k-2)^2)\cdots(m^2-1)2m^2M^{\alpha}(m,n) \\
&= \frac{1}{(2k)!}\sum_{m=-\infty}^{\infty}(m^2-(k-1)^2)(m^2-(k-2)^2)\cdots(m^2-1)m^2M^{\alpha}(m,n) \\
&= \frac{1}{(2k)!}\sum_{m=-\infty}^{\infty}g_k(m)M^{\alpha}(m,n),
\end{aligned}$$

since $M^{\alpha}(-m,n) = M^{\alpha}(m,n)$ for all m. This gives the first equality, and the second follows similarly. Using Lemma 4.2 of [5] we see that

$$\begin{aligned}
M_{2k}^{\alpha}(n) &= \sum_{m=-\infty}^{\infty} m^{2k}M^{\alpha}(m,n) \\
&= \sum_{m=-\infty}^{\infty}\left(\sum_{j=1}^{k}S^*(k,j)g_j(m)\right)M^{\alpha}(m,n) \\
&= \sum_{j=1}^{k}(2j)!S^*(k,j)\mu_{2j}^{\alpha}(n).
\end{aligned}$$

This gives the third equality, and $N_{2k}^{\alpha}(n)$ follows similarly. □

2. Theorems & Proofs

Lemma 2.1. *If α_n is a sequence such that $\alpha_n = \alpha_{-n}$, we have that*

$$1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = 1+\sum_{n\neq 0}\frac{\alpha_n q^n(1-z)}{(1+q^n)(1-zq^n)}.$$

Proof. We have that

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\\
&=1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n}{1+q^n}\left(\frac{1-z}{1-zq^n}+\frac{1-z^{-1}}{1-z^{-1}q^n}\right)\\
&=1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n(1-z)}{(1+q^n)(1-zq^n)}+\sum_{n=-\infty}^{-1}\frac{\alpha_{-n}q^{-n}(1-z^{-1})}{(1+q^{-n})(1-z^{-1}q^{-n})}\\
&=1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n(1-z)}{(1+q^n)(1-zq^n)}+\sum_{n=-\infty}^{-1}\frac{\alpha_n q^n(1-z)}{(1+q^n)(1-zq^n)}\\
&=1+\sum_{n\neq 0}\frac{\alpha_n q^n(1-z)}{(1+q^n)(1-zq^n)}.
\end{aligned}$$

□

Definition 2.2. *We define rank-like functions for the following Bailey Pairs given by*

$$R_{\alpha}(z,q) := \operatorname{prod}(\beta_n(1,q))\left(1+\sum_{n=1}^{\infty}\frac{\alpha_n q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

All Bailey pairs are relative to (1,q) unless otherwise noted.

(1) *A1 from* [7]

$$\beta_n = \frac{1}{(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0\\ -q^{6k^2-5k+1} & n=3k-1\\ q^{6k^2-k}+q^{6k^2+k} & n=3k\\ -q^{6k^2+5k+1} & n=3k+1\end{cases}$$

$$\begin{aligned}
R_{A1}(z,q) =& \frac{1}{(q)_\infty}\left[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{6k^2-2k}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})}\right.\\
&-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{6k^2+8k+2}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})}\\
&\left.+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(q^{6k^2+2k}+q^{6k^2+4k})}{(1-zq^{3k})(1-z^{-1}q^{3k})}\right]
\end{aligned}$$

(2) *A3 from* [7]

$$\beta_n = \frac{q^n}{(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ -q^{6k^2-2k} & n=3k-1 \\ q^{6k^2-2k}+q^{6k^2+2k} & n=3k \\ -q^{6k^2+2k} & n=3k+1 \end{cases}$$

$$\begin{aligned} R_{A3}(z,q) = &\frac{1}{(q)_\infty}\left[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{6k^2+k-1}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})}\right. \\ &-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{6k^2+5k+1}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})} \\ &\left.+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(q^{6k^2+k}+q^{6k^2+5k})}{(1-zq^{3k})(1-z^{-1}q^{3k})}\right] \end{aligned}$$

(3) *A5 from* [7]

$$\beta_n = \frac{q^{n^2}}{(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ -q^{3k^2-k} & n=3k-1 \\ q^{3k^2-k}+q^{3k^2+k} & n=3k \\ -q^{3k^2+k} & n=3k+1 \end{cases}$$

$$\begin{aligned} R_{A5}(z,q) = &\frac{1}{(q)_\infty}\left[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{3k^2+2k-1}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})}\right. \\ &-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{3k^2+4k+1}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})} \\ &\left.+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(q^{3k^2+2k}+q^{3k^2+4k})}{(1-zq^{3k})(1-z^{-1}q^{3k})}\right] \end{aligned}$$

(4) *A7 from* [7]

$$\beta_n = \frac{q^{n^2-n}}{(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ -q^{3k^2-4k+1} & n=3k-1 \\ q^{3k^2-2k}+q^{3k^2+2k} & n=3k \\ -q^{3k^2+4k+1} & n=3k+1 \end{cases}$$

$$\begin{aligned} R_{A7}(z,q) = &\frac{1}{(q)_\infty}\left[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{3k^2-k}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})}\right. \\ &-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{3k^2+7k+2}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})} \end{aligned}$$

$$+\sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(q^{3k^2+k}+q^{3k^2+5k})}{(1-zq^{3k})(1-z^{-1}q^{3k})}\Bigg]$$

(5) *B2 from* [7]

$$\beta_n = \frac{q^n}{(q)_n}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^n q^{3n(n-1)/2}(1+q^{3n}) & n \geq 1 \end{cases}$$

$$R_{B2}(z,q) = \frac{1}{(q)_\infty}\left[1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n-1)/2}(1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)}\right]$$

(6) *F1 from* [7]*, relative to* $(1,q^2)$

$$\beta_n = \frac{1}{(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ q^{2n^2-n}(1+q^{2n}) & n \geq 1 \end{cases}$$

$$R_{F1}(z,q) = \frac{1}{(q^2;q^2)_\infty}\left[1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{2n^2+n}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right]$$

(7) *F3 from* [7]*, relative to* $(1,q^2)$

$$\beta_n = \frac{1}{q^n(q)_{2n}}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ q^n+q^{-n} & n \geq 1 \end{cases}$$

$$R_{F3}(z,q) = \frac{1}{(q^2;q^2)_\infty}\left[1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(q^{3n}+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right]$$

(8) *L5 from* [8]*, however, the formula for* β_n *has been corrected by Jennings-Shaffer.*

$$\beta_n = \frac{(-1)_n}{(q)_n(q;q^2)_n}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ q^{n(n-1)/2}(1+q^n) & n \geq 1 \end{cases}$$

$$R_{L5}(z,q) = \frac{1}{(q)_\infty}\left[1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right]$$

(9) *J1 from* [7]

$$\beta_n = \begin{cases} 1 & n=0 \\ \frac{(q^3;q^3)_{n-1}}{(q)_{2n-1}(q)_n} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ 0 & n=3k-1 \\ (-1)^k q^{3k(3k-1)/2}(1+q^{3k}) & n=3k \\ 0 & n=3k+1 \end{cases}$$

$$R_{J1}(z,q) = \frac{1}{(q)_\infty(q^3;q^3)_\infty}\left[1+\sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k q^{3k(3k+1)/2}(1+q^{3k})}{(1-zq^{3k})(1-z^{-1}q^{3k})}\right]$$

(10) *J2 from* [8]

$$\beta_n = \begin{cases} 1 & n=0 \\ \frac{(q^3;q^3)_{n-1}}{(q)_{2n}(q)_{n-1}} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^{k+1}q^{9k(k-1)/2+1} & n=3k-1 \\ (-1)^k q^{3k(3k-1)/2}(1+q^{3k}) & n=3k \\ (-1)^{k+1}q^{9k(k+1)/2+1} & n=3k+1 \end{cases}$$

$$\begin{aligned} R_{J2}(z,q) = & \frac{1}{(q)_\infty (q^3;q^3)_\infty} \left[1 + \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{k+1}q^{3k(3k-1)/2}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})} \right. \\ & + \sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{k+1}q^{3k(3k+5)/2+2}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})} \\ & \left. + \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k q^{3k(3k+1)/2}(1+q^{3k})}{(1-zq^{3k})(1-z^{-1}q^{3k})} \right] \end{aligned}$$

(11) *J3 from* [8]

$$\beta_n = \begin{cases} 1 & n=0 \\ \frac{q^n(q^3;q^3)_{n-1}}{(q)_{2n}(q)_{n-1}} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^{k+1}q^{3k(3k-1)/2} & n=3k-1 \\ (-1)^k q^{3k(3k-1)/2}(1+q^{3k}) & n=3k \\ (-1)^{k+1}q^{3k(3k+1)/2} & n=3k+1 \end{cases}$$

$$\begin{aligned} R_{J3}(z,q) = & \frac{1}{(q)_\infty (q^3;q^3)_\infty} \left[1 + \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{k+1}q^{3k(3k+1)/2-1}}{(1-zq^{3k-1})(1-z^{-1}q^{3k-1})} \right. \\ & + \sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^{k+1}q^{9k(k+1)/2+1}}{(1-zq^{3k+1})(1-z^{-1}q^{3k+1})} \\ & \left. + \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k q^{3k(3k+1)/2}(1+q^{3k})}{(1-zq^{3k})(1-z^{-1}q^{3k})} \right] \end{aligned}$$

(12) *E4 from* [7]

$$\beta_n = \frac{q^n}{(q^2;q^2)_n}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^n q^{n^2-n}(1+q^{2n}) & n \geq 1 \end{cases}$$

$$R_{E4}(z,q) = \frac{(-q)_\infty}{(q)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2}(1+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)} \right]$$

(13) *G1 from* [7], *relative to* $(1,q^2)$

$$\beta_n = \frac{1}{(-q;q^2)_n (q^4;q^4)_n}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^n q^{n(3n-1)/2}(1+q^n) & n \geq 1 \end{cases}$$

$$R_{G1}(z,q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2}\left[1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{3n(n+1)/2}(1+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right]$$

(14) *G3 from* [7], *relative to* $(1,q^2)$

$$\beta_n = \frac{q^{2n}}{(-q;q^2)_n(q^4;q^4)_n}, \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^n q^{3n(n-1)/2}(1+q^{3n}) & n\geq 1 \end{cases}$$

$$R_{G3}(z,q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2}\left[1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^{3n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right]$$

(15) *C1 from* [7]

$$\beta_n = \frac{1}{(q;q^2)_n(q)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{3k^2-k}(1+q^{2k}) & n=2k \\ 0 & n=2k+1 \end{cases}$$

$$R_{C1}(z,q) = \frac{1}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{3k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right]$$

(16) *C2 from* [7]

$$\beta_n = \frac{q^n}{(q;q^2)_n(q)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{3k^2-k}(1+q^{2k}) & n=2k \\ (-1)^{k+1} q^{3k^2+k}(1-q^{4k+2}) & n=2k+1 \end{cases}$$

$$R_{C2}(z,q) = \frac{1}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{3k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^{k+1} q^{3k^2+3k+1}(1-q^{4k+2})}{(1-zq^{2k+1})(1-z^{-1}q^{2k+1})}\right]$$

(17) *C5 from* [7]

$$\beta_n = \frac{q^{n(n-1)/2}}{(q;q^2)_n(q)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{k^2-k}(1+q^{2k}) & n=2k \\ 0 & n=2k+1 \end{cases}$$

$$R_{C5}(z,q) = \frac{1}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right]$$

(18) *Y1, unlabeled in* [7], *relative to* $(1,q^2)$

$$\beta_n = \frac{q^{n^2-2n}}{(q^4;q^4)_n(q;q^2)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{2k^2-3k}(1+q^{6k}) & n=2k \\ (-1)^k q^{2k^2-k-1}(1-q^{6k+3}) & n=2k+1 \end{cases}$$

$$R_{Y1}(z,q) = \frac{1}{(q^2;q^2)_\infty^2}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+k}(1+q^{6k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+3k+1}(1-q^{6k+3})}{(1-zq^{4k+2})(1-z^{-1}q^{4k+2})}\right]$$

(19) *Y2 unlabeled in* [7]*, relative to* $(1,q^2)$

$$\beta_n = \frac{q^{n^2}}{(q^4;q^4)_n(q;q^2)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{2k^2-k}(1+q^{2k}) & n=2k \\ (-1)^{k+1} q^{2k^2+k}(1+q^{2k+1}) & n=2k+1 \end{cases}$$

$$R_{Y2}(z,q) = \frac{1}{(q^2;q^2)_\infty^2}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+3k}(1+q^{2k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^{k+1} q^{2k^2+5k+2}(1+q^{2k+1})}{(1-zq^{4k+2})(1-z^{-1}q^{4k+2})}\right]$$

(20) *Y3 unlabeled in* [7]*, relative to* $(1,q^2)$

$$\beta_n = \frac{1}{(q^4;q^4)_n(q;q^2)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{6k^2-k}(1+q^{2k}) & n=2k \\ (-1)^k q^{6k^2+5k+1}(1-q^{2k+1}) & n=2k+1 \end{cases}$$

$$R_{Y3}(z,q) = \frac{1}{(q^2;q^2)_\infty^2}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{6k^2+3k}(1+q^{2k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{6k^2+9k+3}(1-q^{2k+1})}{(1-zq^{4k+2})(1-z^{-1}q^{4k+2})}\right]$$

(21) *Y4 unlabeled in* [7]*, relative to* $(1,q^2)$

$$\beta_n = \frac{q^{2n}}{(q^4;q^4)_n(q;q^2)_n} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{6k^2-3k}(1+q^{6k}) & n=2k \\ (-1)^{k+1} q^{6k^2+3k}(1-q^{6k+3}) & n=2k+1 \end{cases}$$

$$R_{Y4}(z,q) = \frac{1}{(q^2;q^2)_\infty^2}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{6k^2+k}(1+q^{6k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^{k+1} q^{6k^2+7k+2}(1-q^{6k+3})}{(1-zq^{4k+2})(1-z^{-1}q^{4k+2})}\right]$$

(22) *X38 unlabeled in* [7]

$$\beta_n = \frac{(-1;q^2)_n}{(q)_{2n}} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{2k^2-k}(1+q^{2k}) & n=2k \\ (-1)^k q^{2k^2+k}(1-q^{2k+1}) & n=2k+1 \end{cases}$$

$$\begin{aligned} R_{X38}(z,q) = & \frac{1}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right. \\ & \left. +\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+3k+1}(1-q^{2k+1})}{(1-zq^{2k+1})(1-z^{-1}q^{2k+1})}\right] \end{aligned}$$

(23) *X39 unlabeled in* [7]*, the formula for* α_n *was corrected by Jennings-Shaffer*

$$\beta_n = \frac{q^n(-1;q^2)_n}{(q)_{2n}} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ (-1)^k q^{2k^2-k}(1+q^{2k}) & n=2k \\ (-1)^{k+1} q^{2k^2+k}(1-q^{2k+1}) & n=2k+1 \end{cases}$$

$$\begin{aligned} R_{X39}(z,q) = & \frac{1}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^k q^{2k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right. \\ & \left. +\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^{k+1} q^{2k^2+3k+1}(1-q^{2k+1})}{(1-zq^{2k+1})(1-z^{-1}q^{2k+1})}\right] \end{aligned}$$

(24) *X40 unlabeled in* [7]

$$\beta_n = \begin{cases} 1 & n=0 \\ \frac{(q^2;q^2)_{n-1}}{(q;q^2)_n(q)_n(q)_{n-1}} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ q^{2k^2-k}(1+q^{2k}) & n=2k \\ 0 & n=2k+1 \end{cases}$$

$$R_{X40}(z,q) = \frac{1}{(q)_\infty(q^2;q^2)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{2k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right]$$

(25) *X41 unlabeled in* [7]

$$\beta_n = \begin{cases} 1 & n=0 \\ \frac{(-q^2;q^2)_{n-1}}{(q)_{2n}} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n=0 \\ 0 & n=4k-2 \\ -q^{8k^2-6k+1} & n=4k-1 \\ q^{8k^2-2k}(1+q^{4k}) & n=4k \\ -q^{8k^2+6k+1} & n=4k+1 \end{cases}$$

$$R_{X41}(z,q) = \frac{1}{(q)_\infty}\left[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2-2k}}{(1-zq^{4k-1})(1-z^{-1}q^{4k-1})}\right.$$

$$+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2+2k}(1+q^{4k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}$$
$$-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2+10k+2}}{(1-zq^{4k+1})(1-z^{-1}q^{4k+1})}\Bigg]$$

(26) *X42 unlabeled in* [7]

$$\beta_n=\begin{cases}1 & n=0\\ \frac{q^n(-q^2;q^2)_{n-1}}{(q)_{2n}} & n\geq 1\end{cases}\qquad \alpha_n=\begin{cases}1 & n=0\\ 0 & n=4k-2\\ -q^{8k^2-2k} & n=4k-1\\ q^{8k^2-2k}(1+q^{4k}) & n=4k\\ -q^{8k^2+2k} & n=4k+1\end{cases}$$

$$R_{X42}(z,q)=\frac{1}{(q)_\infty}\Bigg[1-\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2+2k-1}}{(1-zq^{4k-1})(1-z^{-1}q^{4k-1})}$$
$$+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2+2}(1+q^{4k})}{(1-zq^{4k})(1-z^{-1}q^{4k})}$$
$$-\sum_{k=0}^{\infty}\frac{(1-z)(1-z^{-1})q^{8k^2+6k+1}}{(1-zq^{4k+1})(1-z^{-1}q^{4k+1})}\Bigg]$$

(27) *I14 from* [8]

$$\beta_n=\begin{cases}1 & n=0\\ \frac{(-q^2;q^2)_{n-1}}{(q;q^2)_n(q)_n(-q)_{n-1}} & n\geq 1\end{cases}\qquad \alpha_n=\begin{cases}1 & n=0\\ (-1)^kq^{2k^2-k}(1+q^{2k}) & n=2k\\ 0 & n=2k+1\end{cases}$$

$$R_{I14}(z,q)=\frac{(-q)_\infty}{(q)_\infty}\left[1+\sum_{k=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^kq^{2k^2+k}(1+q^{2k})}{(1-zq^{2k})(1-z^{-1}q^{2k})}\right]$$

(28) *L2/M1 from* [8], *relative to* $(1,q^4)$

$$\beta_n=\frac{(q,q^2)_{2n}}{(q^4;q^4)_{2n}},\qquad \alpha_n=\begin{cases}1 & n=0\\ (-1)^nq^{2n^2-n}(1+q^{2n}) & n\geq 1\end{cases}$$

$$R_{L2/M1}(z,q)=\frac{(-q)_\infty}{(q^4;q^4)_\infty}\left[1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{2n^2+3n}(1+q^{2n})}{(1-zq^{4n})(1-z^{-1}q^{4n})}\right]$$

(29) *X46 unlabeled in* [4]

$$\beta_n = \begin{cases} 1 & n = 0 \\ \frac{(-q^3;q^3)_{n-1}}{(-q)_n(q)_{2n-1}} & n \geq 1 \end{cases} \qquad \alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^k q^{\frac{3k(3k-1)}{2}}(1+q^{3k}) & n = 3k \\ -2q^{18k^2+9k+1} & n = 6k+1 \\ 2q^{18k^2+15k+3} & n = 6k+2 \\ 2q^{18k^2+21k+6} & n = 6k+4 \\ -2q^{18k^2+27k+10} & n = 6k+5 \end{cases}$$

$$\begin{aligned} R_{X46}(z,q) = \frac{(-q)_\infty}{(q)_\infty} \Bigg[1 &+ \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k q^{3k(3k+1)/2}(1+q^{3k})}{(1-zq^{3k})(1-z^{-1}q^{3k})} \\ &- 2\sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18k^2+15k+2}}{(1-zq^{6k+1})(1-z^{-1}q^{6k+1})} \\ &+ 2\sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18k^2+21k+5}}{(1-zq^{6k+2})(1-z^{-1}q^{6k+2})} \\ &+ 2\sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18k^2+27k+10}}{(1-zq^{6k+4})(1-z^{-1}q^{6k+4})} \\ &- 2\sum_{k=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{18k^2+33k+15}}{(1-zq^{6k+5})(1-z^{-1}q^{6k+5})} \Bigg] \end{aligned}$$

Lemma 2.3. *For a Bailey pair α_n and β_n relative to $(1,q)$ such that $\alpha_n = \alpha_{-n}$ and $\alpha_0 = \beta_0 = 1$, for $R_\alpha(z,q) = \mathrm{prod}(\beta_n(1,q))\left(1+\sum_{n=1}^{\infty} \frac{\alpha_n(1-z)(1-z^{-1})q^n}{(1-zq^n)(1-z^{-1}q^n)}\right)$ we have that*

$$R_\alpha(z,q)^{(j)} := \left(\frac{\partial}{\partial z}\right)^j R_\alpha(z,q) = -j!\mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \frac{\alpha_n q^{nj}(1-q^n)}{(1+q^n)(1-zq^n)^{j+1}}.$$

Proof. By Lemma 2.1 we have that

$$R_\alpha(z,q) = \mathrm{prod}(\beta_n(1,q)) \left[1 + \sum_{n\neq 0} \frac{(1-z)\alpha_n q^n}{(1+q^n)(1-zq^n)}\right].$$

So,

$$\begin{aligned} R_\alpha(z,q)^{(j)} &= \left(\frac{\partial}{\partial z}\right)^j \mathrm{prod}(\beta_n(1,q)) \left[1 + \sum_{n\neq 0} \frac{(1-z)\alpha_n q^n}{(1+q^n)(1-zq^n)}\right] \\ &= \mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \left[\frac{\alpha_n q^n}{1+q^n} \cdot \left(\frac{\partial}{\partial z}\right)^j \frac{1-z}{1-zq^n}\right] \end{aligned}$$

$$= \mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \left[\frac{\alpha_n q^n}{1+q^n} \cdot \frac{-j!(1-q^n)q^{n(j-1)}}{(1-zq^n)^{j+1}} \right]$$
$$= -j!\mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \frac{\alpha_n q^{nj}(1-q^n)}{(1+q^n)(1-zq^n)^{j+1}}.$$

□

Theorem 2.4. *For a Bailey pair α_n, β_n relative to $(1,q)$ with $\alpha_0 = \beta_0 = 1$ and $\alpha_n = \alpha_{-n}$,*

$$\sum_{n=1}^{\infty} \eta_{2k}^{\alpha}(n) q^n = -\mathrm{prod}(\beta_n(1,q)) \sum_{n=1}^{\infty} \frac{q^{kn}\alpha_n}{(1-q^n)^{2k}}.$$

Proof. By Lemma 2.3,

$$\sum_{n=1}^{\infty} \eta_{2k}^{\alpha}(n) q^n = \frac{1}{(2k)!} \left(\frac{\partial}{\partial z}\right)^{2k} z^{k-1} R_\alpha(z,q)\Big|_{z=1}$$
$$= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (k-1)\cdots(k-j) R_\alpha^{(2k-j)}(1,q)$$
$$= -\mathrm{prod}(\beta_n(1,q)) \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{n\neq 0} \frac{\alpha_n q^{n(2k-j)}(1-q^n)}{(1+q^n)(1-q^n)^{2k-j+1}}$$
$$= -\mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \frac{\alpha_n q^{2nk}}{(1-q^n)^{2k}(1+q^n)} \sum_{j=0}^{k-1} \binom{k-1}{j} (q^{-n}(1-q^n))^j$$
$$= -\mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \frac{\alpha_n q^{2nk}}{(1-q^n)^{2k}(1+q^n)} (1+q^{-n}(1-q^n))^{k-1}$$
$$= -\mathrm{prod}(\beta_n(1,q)) \sum_{n\neq 0} \frac{\alpha_n q^{n+nk}}{(1-q^n)^{2k}(1+q^n)}$$
$$= -\mathrm{prod}(\beta_n)(1,q)) \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}}.$$

□

Corollary 2.5. *For each Bailey pair in Definition 2.2, the symmetrized rank moments are as follows.*

(1) *A1*

$$\sum_{n=1}^{\infty} \eta_{2k}^{A1}(n) q^n = \frac{1}{(q)_\infty} \left[\sum_{m=1}^{\infty} \frac{q^{6m^2-5m+1+k(3m-1)}}{(1-q^{3m-1})^{2k}} + \sum_{m=0}^{\infty} \frac{q^{6m^2+5m+1+(3m+1)k}}{(1-q^{3m+1})^{2k}} - \sum_{m=1}^{\infty} \frac{(q^{6m^2-m}+q^{6m^2+m})q^{3mk}}{(1-q^{3m})^{2k}} \right]$$

(2) *A3*

$$\sum_{n=1}^{\infty}\eta_{2k}^{A3}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{q^{6m^2-2m+k(3m-1)}}{(1-q^{3m-1})^{2k}} + \sum_{m=0}^{\infty}\frac{q^{6m^2+2m+(3m+1)k}}{(1-q^{3m+1})^{2k}} - \sum_{m=1}^{\infty}\frac{(q^{6m^2-2m}+q^{6m^2+2m})q^{3mk}}{(1-q^{3m})^{2k}}\right]$$

(3) *A5*

$$\sum_{n=1}^{\infty}\eta_{2k}^{A5}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{q^{3m^2-m+k(3m-1)}}{(1-q^{3m-1})^{2k}} + \sum_{m=0}^{\infty}\frac{q^{3m^2+m+(3m+1)k}}{(1-q^{3m+1})^{2k}} - \sum_{m=1}^{\infty}\frac{(q^{3m^2-m}+q^{3m^2+m})q^{3mk}}{(1-q^{3m})^{2k}}\right]$$

(4) *A7*

$$\sum_{n=1}^{\infty}\eta_{2k}^{A7}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{q^{3m^2-4m+1+k(3m-1)}}{(1-q^{3m-1})^{2k}} + \sum_{m=0}^{\infty}\frac{q^{3m^2+4m+1+(3m+1)k}}{(1-q^{3m+1})^{2k}} - \sum_{m=1}^{\infty}\frac{(q^{3m^2-2m}+q^{3m^2+2m})q^{3mk}}{(1-q^{3m})^{2k}}\right]$$

(5) *B2*

$$\sum_{n=1}^{\infty}\eta_{2k}^{B2}(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{3n(n-1)/2+nk}(1+q^{3n})}{(1-q^n)^{2k}}$$

(6) *F1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{F1}(n)q^n = \frac{-1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n^2-n+2nk}(1+q^{2n})}{(1-q^{2n})^{2k}}$$

(7) *F3*

$$\sum_{n=1}^{\infty}\eta_{2k}^{F3}(n)q^n = \frac{-1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(q^n+q^{-n})q^{2nk}}{(1-q^{2n})^{2k}}$$

(8) *L5*

$$\sum_{n=1}^{\infty}\eta_{2k}^{L5}(n)q^n = \frac{-1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2+nk}(1+q^n)}{(1-q^{2n})^{2k}}$$

(9) *J1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{J1}(n)q^n = \frac{1}{(q)_\infty(q^3;q^3)_\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{3m(3m-1)/2+3mk}(1+q^{3m})}{(1-q^{3m})^{2k}}$$

(10) *J2*

$$\sum_{n=1}^{\infty}\eta_{2k}^{J2}(n)q^n = \frac{1}{(q)_\infty(q^3;q^3)_\infty}\left[\sum_{m=1}^{\infty}\frac{(-1)^m q^{9m(m-1)/2+1+(3m-1)k}}{(1-q^{3m-1})^{2k}}\right.$$

$$+\sum_{m=0}^{\infty}\frac{(-1)^m q^{9m(m+1)/2+1+(3m+1)k}}{(1-q^{3m+1})^{2k}}$$
$$-\sum_{m=1}^{\infty}\frac{(-1)^m q^{3m(3m-1)/2+3mk}(1+q^{3m})}{(1-q^{3m})^{2k}}\Bigg]$$

(11) *J3*

$$\sum_{n=1}^{\infty}\eta_{2k}^{J3}(n)q^n=\frac{1}{(q)_\infty(q^3;q^3)_\infty}\Bigg[\sum_{m=1}^{\infty}\frac{(-1)^m q^{3m(3m-1)/2+(3m-1)k}}{(1-q^{3m-1})^{2k}}$$
$$+\sum_{m=0}^{\infty}\frac{(-1)^m q^{3m(3m+1)/2+(3m+1)k}}{(1-q^{3m+1})^{2k}}$$
$$-\sum_{m=1}^{\infty}\frac{(-1)^m q^{3m(3m-1)/2+3mk}(1+q^{3m})}{(1-q^{3m})^{2k}}\Bigg]$$

(12) *E4*

$$\sum_{n=1}^{\infty}\eta_{2k}^{E4}(n)q^n=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2-n+kn}(1+q^{2n})}{(1-q^n)^{2k}}$$

(13) *G1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{G1}(n)q^n=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(3n-1)/2+2nk}(1+q^n)}{(1-q^{2n})^{2k}}$$

(14) *G3*

$$\sum_{n=1}^{\infty}\eta_{2k}^{G3}(n)q^n=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{3m(m-1)/2+2mk}(1+q^{3m})}{(1-q^{2m})^{2k}}$$

(15) *C1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{C1}(n)q^n=\frac{1}{(q)_\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{3m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}}$$

(16) *C2*

$$\sum_{n=1}^{\infty}\eta_{2k}^{C2}(n)q^n=\frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{3m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}}+\sum_{m=0}^{\infty}\frac{(-1)^m q^{3m^2+m+k(2m+1)}(1-q^{4m+2})}{(1-q^{2m+1})^{2k}}\right]$$

(17) *C5*

$$\sum_{n=1}^{\infty}\eta_{2k}^{C5}(n)q^n=\frac{1}{(q)_\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}}$$

(18) *Y1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{Y1}(n)q^n=\frac{1}{(q^2;q^2)_\infty^2}\Bigg[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-3m+4mk}(1+q^{6m})}{(1-q^{4m})^{2k}}$$
$$+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m-1+k(4m+2)}(1-q^{6m+3})}{(1-q^{4m+2})^{2k}}\Bigg]$$

(19) *Y2*

$$\sum_{n=1}^{\infty}\eta_{2k}^{Y2}(n)q^n = \frac{1}{(q^2;q^2)_\infty^2}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m+4mk}(1+q^{2m})}{(1-q^{4m})^{2k}} + \sum_{m=0}^{\infty}\frac{(-1)^{m}q^{2m^2+m+k(4m+2)}(1+q^{2m+1})}{(1-q^{4m+2})^{2k}}\right]$$

(20) *Y3*

$$\sum_{n=1}^{\infty}\eta_{2k}^{Y3}(n)q^n = \frac{1}{(q^2;q^2)_\infty^2}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{6m^2-m+4mk}(1+q^{2m})}{(1-q^{4m})^{2k}} + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{6m^2+5m+1+k(4m+2)}(1-q^{2m+1})}{(1-q^{4m+2})^{2k}}\right]$$

(21) *Y4*

$$\sum_{n=1}^{\infty}\eta_{2k}^{Y4}(n)q^n = \frac{1}{(q^2;q^2)_\infty^2}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{6m^2-3m+4mk}(1+q^{6m})}{(1-q^{4m})^{2k}} + \sum_{m=0}^{\infty}\frac{(-1)^{m}q^{6m^2+3m+k(4m+2)}(1-q^{6m+3})}{(1-q^{4m+2})^{2k}}\right]$$

(22) *X38*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X38}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}} + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}q^{2m^2+m+(2m+1)k}(1-q^{2m+1})}{(1-q^{2m+1})^{2k}}\right]$$

(23) *X39*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X39}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}} + \sum_{m=0}^{\infty}\frac{(-1)^{m}q^{2m^2+m+(2m+1)k}(1-q^{2m+1})}{(1-q^{2m+1})^{2k}}\right]$$

(24) *X40*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X40}(n)q^n = \frac{-1}{(q)_\infty(q^2;q^2)_\infty}\left[\sum_{m=1}^{\infty}\frac{q^{2m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}}\right]$$

(25) *X41*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X41}(n)q^n = \frac{1}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{q^{8m^2-6m+1+(4m-1)k}}{(1-q^{4m-1})^{2k}} - \sum_{m=1}^{\infty}\frac{q^{8m^2-2m+4mk}(1+q^{4m})}{(1-q^{4m})^{2k}}\right.$$

$$+\sum_{m=0}^{\infty}\frac{q^{8m^2+6m+1+(4m+1)k}}{(1-q^{4m+1})^{2k}}\Bigg]$$

(26) *X42*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X42}(n)q^n=\frac{1}{(q)_\infty}\Bigg[\sum_{m=1}^{\infty}\frac{q^{8m^2-2m+(4m-1)k}}{(1-q^{4m-1})^{2k}}-\sum_{m=1}^{\infty}\frac{q^{8m^2-2m+4mk}(1+q^{4m})}{(1-q^{4m})^{2k}}$$
$$+\sum_{m=0}^{\infty}\frac{q^{8m^2+2m+(4m+1)k}}{(1-q^{4m+1})^{2k}}\Bigg]$$

(27) *I14*

$$\sum_{n=1}^{\infty}\eta_{2k}^{I14}(n)q^n=\frac{(-q)_\infty}{(q)_\infty}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m+2mk}(1+q^{2m})}{(1-q^{2m})^{2k}}\right]$$

(28) *L2/M1*

$$\sum_{n=1}^{\infty}\eta_{2k}^{L2/M1}(n)q^n=\frac{(-q)_\infty}{(q^4;q^4)_\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{2m^2-m+4mk}(1+q^{2m})}{(1-q^{4m})^{2k}}$$

(29) *X46*

$$\sum_{n=1}^{\infty}\eta_{2k}^{X46}(n)q^n=\frac{(-q)_\infty}{(q)_\infty}\Bigg[\sum_{m=1}^{\infty}\frac{(-1)^{m+1}q^{\frac{3m(3m-1)}{2}+3mk}(1+q^{3m})}{(1-q^{3m})^{2k}}$$
$$+\sum_{m=0}^{\infty}\frac{2q^{18m^2+9m+1+(6m+1)k}}{(1-q^{6m+1})^{2k}}-\sum_{m=0}^{\infty}\frac{2q^{18m^2+15m+3+(6m+2)k}}{(1-q^{6m+2})^{2k}}$$
$$-\sum_{m=0}^{\infty}\frac{2q^{18m^2+21m+6+(6m+4)k}}{(1-q^{6m+4})^{2k}}+\sum_{m=0}^{\infty}\frac{2q^{18m^2+27m+10+(6m+5)k}}{(1-q^{6m+5})^{2k}}\Bigg]$$

Proof. The results follow immediately from Theorem 2.4. □

Definition 2.6. *We define the following crank and crank-like functions, so that for a Bailey pair α_n,β_n relative to $(1,q)$ with $\alpha_0=\beta_0=1$ and $\alpha_n=\alpha_{-n}$,*

$$\sum_{n=1}^{\infty}\mu_{2k}^{\alpha}(n)q^n=\text{prod}(\beta_n(1,q))\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}.$$

(1) *We use the standard*

$$\sum_{n=1}^{\infty}\mu_{2k}(n)q^n=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}$$

from [5].

(2)

$$\sum_{n=1}^{\infty}\mu_{2k}^{x10}(n)q^n=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}$$

We note this is $\sum_{n=1}^{\infty}\mu 2_{2k}(n)q^n$ *from* [6].

(3)
$$\sum_{n=1}^{\infty} \mu_{2k}^{J}(n)q^n = \frac{1}{(q^3;q^3)_\infty (q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}$$
(4)
$$\sum_{n=1}^{\infty} \mu_{2k}^{E}(n)q^n = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}.$$
We note this is $\sum_{n=1}^{\infty} \bar{\mu}_{2k}(n)q^n$ *from* [6].

(5)
$$\sum_{n=1}^{\infty} \mu_{2k}^{G}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}$$
(6)
$$\sum_{n=1}^{\infty} \mu_{2k}^{Y}(n)q^n = \frac{1}{(q^2;q^2)_\infty^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}$$
(7)
$$\sum_{n=1}^{\infty} \mu_{2k}^{X40}(n)q^n = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}$$
(8)
$$\sum_{n=1}^{\infty} \mu_{2k}^{L2/M1}(n)q^n = \frac{(-q)_\infty}{(q^4;q^4)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n(n-1)+4kn}(1+q^{4n})}{(1-q^{4n})^{2k}}$$

We make use of Theorem 3.3 from [5].

Theorem 2.7. *Suppose* α_n *and* β_n *are a Bailey pair relative to* $(1,q)$ *and* $\alpha_0 = \beta_0 = 1$. *Then*
$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 q^{n_1+n_2+\cdots+n_k} \beta_{n_1}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2}$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr}\alpha_r}{(1-q^r)^{2k}}.$$

Corollary 2.8. *For all of the Bailey pairs given, we have that*
$$\sum_{n=1}^{\infty} \alpha spt_k(n)q^n = \sum_{n=1}^{\infty} (\mu_{2k}^{\alpha}(n) - \eta_{2k}^{\alpha}(n))q^n$$
has non-negative coefficients.

(1)
$$\sum_{n=1}^{\infty} \mathrm{A1spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{A1}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+\cdots+n_k}}{(q^{n_1+1})_{n_1}(q^{n_1+1})_\infty (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(2)

$$\sum_{n=1}^{\infty} \mathrm{A3spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{A3}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\ldots+n_k}}{(q^{n_1+1})_{n_1}(q^{n_1+1})_\infty(1-q^{n_k})^2\ldots(1-q^{n_1})^2}$$

(3)

$$\sum_{n=1}^{\infty} \mathrm{A5spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{A5}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{n_1^2+n_1+n_2+\ldots+n_k}}{(q^{n_1+1})_{n_1}(q^{n_1+1})_\infty(1-q^{n_k})^2\ldots(1-q^{n_1})^2}$$

(4)

$$\sum_{n=1}^{\infty} \mathrm{A7spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{A7}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{n_1^2+n_2+\ldots+n_k}}{(q^{n_1+1})_{n_1}(q^{n_1+1})_\infty(1-q^{n_k})^2\ldots(1-q^{n_1})^2}$$

(5)

$$\sum_{n=1}^{\infty} \mathrm{B2spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{B2}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\ldots+n_k}}{(q^{n_1+1})_\infty(1-q^{n_k})^2\ldots(1-q^{n_1})^2}$$

(6)

$$\sum_{n=1}^{\infty} \mathrm{F1spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^{2n} - \sum_{n=1}^{\infty} \eta_{2k}^{F1}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{2n_1+\ldots+2n_k}}{(q^{2n_1+2};q^2)_\infty(q;q^2)_{n_1}(1-q^{2n_k})^2\ldots(1-q^{2n_1})^2}$$

here we note that $q \to q^2$ *in* $\sum_{n=1}^{\infty} \mu_{2k}(n)q^n$ *since F1 is relative to* $(1,q^2)$

(7)

$$\sum_{n=1}^{\infty} \mathrm{F3spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^{2n} - \sum_{n=1}^{\infty} \eta_{2k}^{F3}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{n_1+2n_2\ldots+2n_k}}{(q^{2n_1+2};q^2)_\infty(q;q^2)_{n_1}(1-q^{2n_k})^2\ldots(1-q^{2n_1})^2}$$

here we note that $q \to q^2$ *in* $\sum_{n=1}^{\infty} \mu_{2k}(n)q^n$ *since F3 is relative to* $(1,q^2)$

(8)

$$\sum_{n=1}^{\infty} \text{L5spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{L5}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{(-1)_{n_1} q^{n_1+n_2\ldots+n_k}}{(q^{n_1+1})_\infty (q;q^2)_{n_1} (1-q^{n_k})^2 \ldots (1-q^{n_1})^2}$$

(9)

$$\sum_{n=1}^{\infty} \text{J1spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{J}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{J1}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(q)_{2n_1-1}(q^{n_1+1})_\infty (q^{3n_1};q^3)_\infty (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(10)

$$\sum_{n=1}^{\infty} \text{J2spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{J}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{J2}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1} (q)_{n_1-1} (q^{3n_1};q^3)_\infty (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(11)

$$\sum_{n=1}^{\infty} \text{J3spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{J}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{J3}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1} (q)_{n_1-1} (q^{3n_1};q^3)_\infty (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(12)

$$\sum_{n=1}^{\infty} \text{E4spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{E}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{E4}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_\infty q^{2n_1+n_2+\ldots+n_k}}{(q^{n_1+1})_\infty (1-q^{n_k})^2 \ldots (1-q^{n_1})^2}$$

(13)

$$\sum_{n=1}^{\infty} \text{G1spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{G}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{G1}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{(-q^{2n_1+1};q^2)_\infty q^{2n_1+2n_2\ldots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1} (1-q^{2n_k})^2 \ldots (1-q^{2n_1})^2}$$

(14)

$$\sum_{n=1}^{\infty} \text{G3spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{G}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{G3}(n)q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{(-q^{2n_1+1};q^2)_\infty q^{4n_1+2n_2\ldots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1} (1-q^{2n_k})^2 \ldots (1-q^{2n_1})^2}$$

(15)

$$\sum_{n=1}^{\infty} \mathrm{C1spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{C1}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(q^{n_1+1})_\infty (q;q^2)_{n_1} (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(16)

$$\sum_{n=1}^{\infty} \mathrm{C2spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{C2}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{2n_1+n_2+\cdots+n_k}}{(q^{n_1+1})_\infty (q;q^2)_{n_1} (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(17)

$$\sum_{n=1}^{\infty} \mathrm{C5spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{C5}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1(n_1+1)/2+n_2+\cdots+n_k}}{(q^{n_1+1})_\infty (q;q^2)_{n_1} (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(18)

$$\sum_{n=1}^{\infty} \mathrm{Y1spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}^{Y}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{Y1}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1^2+2n_2+2n_3+\cdots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1} (q;q^2)_{n_1} (1-q^{2n_k})^2 \cdots (1-q^{2n_1})^2}$$

(19)

$$\sum_{n=1}^{\infty} \mathrm{Y2spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}^{Y}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{Y2}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1^2+2n_1+2n_2+2n_3+\cdots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1} (q;q^2)_{n_1} (1-q^{2n_k})^2 \cdots (1-q^{2n_1})^2}$$

(20)

$$\sum_{n=1}^{\infty} \mathrm{Y3spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}^{Y}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{Y3}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+2n_3+\cdots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1} (q;q^2)_{n_1} (1-q^{2n_k})^2 \cdots (1-q^{2n_1})^2}$$

(21)

$$\sum_{n=1}^{\infty} \mathrm{Y4spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{Y}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{Y4}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{4n_1+2n_2+2n_3+\cdots+2n_k}}{(q^{2n_1+2};q^2)_\infty^2 (q^4;q^4)_{n_1}(q;q^2)_{n_1}(1-q^{2n_k})^2 \cdots (1-q^{2n_1})^2}$$

(22)

$$\sum_{n=1}^{\infty} \mathrm{X38spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X38}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1;q^2)_{n_1} q^{n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1}(1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(23)

$$\sum_{n=1}^{\infty} \mathrm{X39spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X39}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1;q^2)_{n_1} q^{2n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1}(1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(24)

$$\sum_{n=1}^{\infty} \mathrm{X40spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{X40}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X40}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(q)_{n_1-1}(q^{n_1+1})_\infty (q^{2n_1};q^2)_\infty (q;q^2)_{n_1}(1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(25)

$$\sum_{n=1}^{\infty} \mathrm{X41spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X41}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-q^2;q^2)_{n_1-1} q^{n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1}(1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(26)

$$\sum_{n=1}^{\infty} \mathrm{X42spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X42}(n)q^n$$
$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-q^2;q^2)_{n_1-1} q^{2n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1}(1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(27)

$$\sum_{n=1}^{\infty} \mathrm{I14spt}_k(n)q^n = \sum_{n=1}^{\infty} \mu_{2k}^{E}(n)q^n - \sum_{n=1}^{\infty} \eta_{2k}^{I14}(n)q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-q^{n_1})_\infty (-q^2;q^2)_{n_1-1} q^{n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q;q^2)_{n_1} (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

(28)

$$\sum_{n=1}^{\infty} \mathrm{L2/M1spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}^{L2/M1}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{L2/M1}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{q^{4n_1+4n_2\ldots+4n_k}}{(q^{4n_1+4};q^4)_\infty (q^{4n_1+1};q^2)_\infty (q^{4n_1+4};q^4)_{n_1} (1-q^{4n_k})^2 \ldots (1-q^{4n_1})^2}$$

(29)

$$\sum_{n=1}^{\infty} \mathrm{X46spt}_k(n) q^n = \sum_{n=1}^{\infty} \mu_{2k}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{X46}(n) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-q^{n_1+1})_\infty (-q^3,q^3)_{n_1-1} q^{n_1+n_2+n_3+\cdots+n_k}}{(q^{n_1+1})_\infty (q^{n_1+1})_{n_1-1} (1-q^{n_k})^2 \cdots (1-q^{n_1})^2}$$

Proof. By Corollary 2.5, Theorem 2.7, and Corollary 3.4 from [5],

$$\sum_{n=1}^{\infty} \alpha\mathrm{spt}_k(n) q^n = \sum_{n=1}^{\infty} (\mu_{2k}^{\alpha}(n) - \eta_{2k}^{\alpha}(n)) q^n$$

$$= \mathrm{prod}(\beta_n(1,q)) \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}} + \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}} \right]$$

$$= \mathrm{prod}(\beta_n(1,q)) \left[\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} + \sum_{n=1}^{\infty} \frac{\alpha_n q^{nk}}{(1-q^n)^{2k}} \right]$$

$$= \mathrm{prod}(\beta_n(1,q)) \left[\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 q^{n_1+n_2+\cdots+n_k} \beta_{n_1}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} \right]$$

At this stage we note that for each Bailey Pair, $\mathrm{prod}(\beta_n(1,q))$ has been chosen such that the resulting sum has non-negative coefficients. It follows that

$$\sum_{n=1}^{\infty} \alpha spt_k(n) q^n = \sum_{n=1}^{\infty} (\mu_{2k}^{\alpha}(n) - \eta_{2k}^{\alpha}(n)) q^n$$

has non-negative coefficients. □

As a result, for all $k \geq 1$ we have the following table to summarize for which values of n it holds that $M_{2k}^{\alpha}(n) > N_{2k}^{\alpha}(n)$.

TABLE 1. Ordinary Rank Moment Inequalities

BP	n	BP	n	BP	n
$A1$	$n \geq 1$	$J3$	$n \geq 2$	$Y4$	$n \geq 4$
$A3$	$n \geq 2$	$E4$	$n \geq 2$	$X38$	$n \geq 1$
$A5$	$n \geq 2$	$G1$	$n = 2, n \geq 4$	$X39$	$n \geq 2$
$A7$	$n \geq 1$	$G3$	$n = 4, n \geq 6$	$X40$	$n \geq 1$
$B2$	$n \geq 2$	$C1$	$n \geq 1$	$X41$	$n \geq 1$
$F1$	$n \geq 2$	$C2$	$n \geq 2$	$X42$	$n \geq 1$
$F3$	$n \geq 1$	$C5$	$n \geq 1$	$I14$	$n \geq 2$
$L5$	$n \geq 1$	$Y1$	$n \geq 1$	$L2/M1$	$n = 4, n = 8,$
$J1$	$n \geq 1$	$Y2$	$n \geq 3$		$n = 9, n \geq 11$
$J2$	$n \geq 1$	$Y3$	$n \geq 2$	$X46$	$n \geq 1$

Proof. We now know that

$$\sum_{n=1}^{\infty} \mu_{2k}^{\alpha}(n) q^n - \sum_{n=1}^{\infty} \eta_{2k}^{\alpha}(n) q^n$$

has nonnegative coefficients. We examine each sum to see for which values it holds that $\mu_{2j}^{\alpha}(n) - \eta_{2j}^{\alpha}(n) > 0$. Since the $S^*(k,j)$ are integers and positive for $1 \leq j \leq k$, it follows that

$$\begin{aligned} M_{2k}^{\alpha}(n) - N_{2k}^{\alpha}(n) &= \sum_{j=1}^{k} (2j)! S^*(k,j) (\mu_{2j}^{\alpha}(n) - \eta_{2j}^{\alpha}(n)) \\ &\geq \mu_2^{\alpha}(n) - \eta_2^{\alpha}(n) \\ &> 0 \end{aligned}$$

when the inequality between the symmetrized moments holds. □

3. COMBINATORIAL INTERPRETATIONS

In agreement with [5] and [6], for a partition π with parts $n_1 < n_2 < \cdots < n_m$, we take $f_j = f_j(\pi)$ to be the frequency of the part n_j.

Definition 3.1. *We define:*

- S^{A1} *- The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2)$ *where* π_1 *has smallest part* n_1*, and the parts* n_j *of* π_2 *satisfy* $n_1 + 1 \leq n_j \leq 2n_1$.
- S^{A3} *- The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where* π_1 *has smallest part* n_1*, and the parts* n_j *of* π_2 *satisfy* $n_1 + 1 \leq n_j \leq 2n_1$*, and* π_3 *contains exactly one part which is* n_1.
- S^{A5} *- The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where* π_1 *has smallest part* n_1*, and the parts* n_j *of* π_2 *satisfy* $n_1 + 1 \leq n_j \leq 2n_1$*, and* π_3 *contains exactly one part which is* n_1^2.
- S^{A7} *- The set of partitions described by* (π_1, π_2, π_3) *where* π_1 *has smallest part* n_1*, each part* n_j *of* π_2 *satisfies* $n_1 + 1 \leq n_j \leq 2n_1$*, and* π_3 *contains exactly one part, which is* $n_1^2 - n_1$. *We note that if* $n_1 = 1$ *this implies that* π_3 *is empty.*
- S^{B2} *- The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2)$ *where* π_1 *has smallest part* n_1*, and* π_2 *contains exactly one part which is* n_1.

- S^{F1} - *The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2)$ *where* π_1 *has even parts and the smallest part is* $2n_1$*, and the parts of* π_2 *are odd and less than* $2n_1 + 1$.
- S^{F3} - *The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where* π_1 *has even parts and smallest part* n_1*,* π_2 *has odd parts* $\leq 2n_1 - 1$*, and* π_3 *consists of a single part which is* $-n_1$.
- S^{L5} - *The set of partitions described by* (π_1, π_2, π_3) *where* π_1 *has smallest part* n_1*, each part of* π_2 *is odd and less than or equal to* $2n_1 - 1$ *and where the parts* π_3 *are distinct and* $\leq n_1 - 1$.
- S^{J1} - *The set of partitions described by* (π_1, π_2, π_3)*, where* π_1 *has smallest part* n_1*, each part of* π_2 *is* $< 2n_1$ *and the parts* n_j *in* π_3 *are* $\geq 3n_1$ *and for each j,* $3 \big| n_j$.
- S^{J2} - *The set of partitions described by* (π_1, π_2, π_3)*, where* π_1 *has smallest part* n_1*, each part of* π_2 *is* $\leq 2n_1$ *and* $\neq n_1$*, and the parts* n_j *in* π_3 *are* $\geq 3n_1$ *and for each j,* $3 \big| n_j$.
- S^{J3} - *The set of partitions described by* $(\pi_1, \pi_2, \pi_3, \pi_4)$*, where* π_1 *has smallest part* n_1*, each part of* π_2 *is* $\leq 2n_1$ *and* $\neq n_1$*, the parts* n_j *in* π_3 *are* $\geq 3n_1$ *and for each j,* $3 \big| n_j$*, and* π_4 *has exactly one part which is equal to* n_1.
- S^{E4} - *The set of partitions described by* (π_1, π_2, π_3)*, where* π_1 *has smallest part* n_1*, the parts of* π_2 *are distinct and* $> n_1$*, and* π_3 *has exactly one part which is equal to* n_1.
- S^{G1} - *The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where* π_1 *has even parts and the smallest part is* $2n_1$*,* π_2 *is comprised of distinct odd parts greater than* $2n_1$*,* π_3 *is comprised of even parts greater than* $2n_1$*, and the parts* n_j *in* π_4 *are* $\leq 4n_1$ *and for each j,* $4 \big| n_j$.
- S^{G3} - *The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ *where* π_1 *has even parts and the smallest part is* $2n_1$*,* π_2 *is comprised of distinct odd parts greater than* $2n_1$*,* π_3 *is comprised of even parts greater than* $2n_1$*, the parts* n_j *in* π_4 *are* $\leq 4n_1$ *and for each j,* $4 \big| n_j$*, and* π_5 *has exactly one part, which is* $2n_1$.
- S^{C1}*: The set of partitions described by* (π_1, π_2)*, where the smallest part of* π_1 *is* n_1*, and* π_2 *has odd parts* $< 2n_1 + 1$.
- S^{C2}*: The set of partitions described by* (π_1, π_2, π_3)*, where the smallest part of* π_1 *is* n_1*,* π_2 *has odd parts* $< 2n_1 + 1$*, and* π_3 *contains exactly one part, which is* n_1.
- S^{C5} - *The set of partitions described by* (π_1, π_2, π_3)*, where the smallest part of* π_1 *is* n_1*,* π_2 *has odd parts* $< 2n_1 + 1$*, and* π_3 *contains exactly one part, which is* $\frac{n_1(n_1-1)}{2}$.
- S^{Y1}*: For* $n_1 > 1$*, the set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where all parts in* π_1 *are even and the smallest is* $2n_1$*,* π_2 *is comprised of odd parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $n_j \leq 4n_1$ *and* $4 \big| n_j$ *for all j, and* π_4 *has exactly one part, which is equal to* $n_1^2 - 2n_1$*. Note that* π_4 *is empty when* $n_1 = 2$.

  For $n_1 = 1$*, the set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where all parts in* π_1 *are even and the smallest part is* 2*,* π_2 *is comprised of ones only, and* π_3 *is comprised of fours only. The weight is then shifted so that the coefficient on* q^n *is* $\omega_k(n+1)$
- S^{Y2}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ *where all parts in* π_1 *are even and the smallest is* $2n_1$*,* π_2 *is comprised of odd parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $n_j \leq 4n_1$ *and* $4 \big| n_j$ *for all j,* π_4 *has exactly one part, which is equal to* n_1^2*, and* π_5 *consists of even parts* $\geq 2n_1 + 2$.
- S^{Y3}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where all parts in* π_1 *are even and the smallest is* $2n_1$*,* π_2 *is comprised of odd parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $n_j \leq 4n_1$ *and* $4 \big| n_j$ *for all j, and* π_4 *consists of even parts* $\geq 2n_1 + 2$.

- S^{Y4}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ *where all parts in* π_1 *are even and the smallest is* $2n_1$*,* π_2 *is comprised of odd parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $4n_1 < n_j \leq 8n_1$ *and* $4 \big| n_j$ *for all j,* π_4 *has exactly one part, which is equal to* $2n_1$*, and* π_5 *consists of even parts* $\geq 2n_1 + 2$*..*
- S^{X38}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3,)$ *where the smallest part in* π_1 *is* n_1*,* π_2 *is comprised of distinct even parts less than* $2n_1$*, and the parts* n_j *in* π_3 *are such that* $n_1 < n_j \leq 2n_1$ *for all j.*
- S^{X39}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where the smallest part in* π_1 *is* n_1*,* π_2 *is comprised of distinct even parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $n_1 < n_j \leq 2n_1$ *for all j, and* π_4 *has exactly one part, which is equal to* n_1*.*
- S^{X40}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where the smallest part in* π_1 *is* n_1*,* π_2 *is comprised of parts less than* n_1*, and* π_3 *is comprised of even parts* $\geq 2n_1$*.*
- S^{X41}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where the smallest part in* π_1 *is* n_1*,* π_2 *is comprised of distinct even parts less than* $2n_1$*, and the parts* n_j *in* π_3 *are such that* $n_1 < n_j \leq 2n_1$ *for all j.*
- S^{X42}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where the smallest part in* π_1 *is* n_1*,* π_2 *is comprised of distinct even parts less than* $2n_1$*, the parts* n_j *in* π_3 *are such that* $n_1 < n_j \leq 2n_1$ *for all j, and* π_4 *has exactly one part, which is* n_1*.*
- S^{I14}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where the smallest part in* π_1 *is* n_1*, the parts in* π_2 *are distinct and* $\geq n_1$*, the parts* n_j *in* π_3 *are distinct, even and such that* $n_j \leq 2n_1$ *for all j, and each part* n_k *in* π_4 *is odd and such that* $1 < n_k < 2n_1 + 1$ *for all k.*
- $S^{L2/M1}$*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ *where the smallest part in* π_1 *is* $4n_1$ *and all parts are divisible by 4,* π_2 *is comprised of odd parts* $\geq 4n_1 + 1$*, and* π_3 *has parts* n_j *such that for each j,* $4 \big| n_j$ *and* $4n_1 < n_j \leq 8n_1$*.*
- S^{X46}*: The set of partitions described by* $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ *where the smallest in* π_1 *is* n_1*, the parts in* π_2 *are distinct and* $> n_1$*, the parts in* π_3 *are distinct, divisible by 3 and also less than* $3n_1$*, and each part* n_j *in* π_4 *is such that* $n_1 < n_j < 2n_1$*.*

Note that for each Bailey-Pair (α_n, β_n) considered,

$$\alpha\mathrm{spt}_1(n) = \sum_{\vec{\pi} \in S^{\alpha}, |\vec{\pi}| = n} f_1^1(\vec{\pi}).$$

For $k \geq 1$ we use the weight ω_k of [6] for vector partitions $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ where $\omega_k(\vec{\pi}) = \omega_k(\pi_1)$, and $\omega_k(\pi)$ is the weight from [5].

Definition 3.2.

$$\omega_k(\vec{\pi}) := \sum_{\substack{m_1 + m_2 + \cdots + m_r = k \\ 1 \leq r \leq k}} \binom{f_1^1 + m_1 - 1}{2m_1 - 1} \times \sum_{2 \leq j_2 < j_3 < \cdots < j_r} \binom{f_{j_2}^1 + m_2}{2m_2} \binom{f_{j_3}^1 + m_3}{2m_3} \cdots \binom{f_{j_r}^1 + m_r}{2m_r}$$

Definition 3.3. *Relative to* $(1, q)$*, we define*

$$\beta_n'(q) = (q^{n+1})_\infty (q)_n^2 \mathrm{prod}(\beta(1,q)) \beta_n(q)$$

We note that $\sum_{n=1}^{\infty} \alpha\text{spt}_k(n)q^n = \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 1} \frac{\beta'_{n_1}(q)q^{n_1+n_2+\ldots+n_k}}{(q^{n_1+1})_\infty(1-q^{n_k})^2\ldots(1-q^{n_1})^2}$

Theorem 3.4. *For all Bailey pairs (α_n, β_n) considered except for L5, X38, and X39, for all $k \geq 1$ and $n \geq 1$ we have*

$$\alpha\text{spt}_k(n) = \sum_{\vec{\pi} \in S^\alpha} \omega_k(\vec{\pi}).$$

For L5, X38, and X39, we have that

$$\alpha\text{spt}_k(n) = 2\sum_{\vec{\pi} \in S^\alpha} \omega_k(\vec{\pi}).$$

Proof. What follows is a generalized version of the proof of 5.6 in [5]. We write the general case for $k = 3$ and $k = 4$ and then explain the general procedure for all k.

We use that

$$\sum_{n=j}^{\infty} \binom{n+j-1}{2j-1} x^n = \frac{x^j}{(1-x)^{2j}}$$
$$\sum_{n=j}^{\infty} \binom{n+j}{2j} x^n = \frac{x^j}{(1-x)^{2j+1}}.$$

For $k = 3$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha\text{spt}_3(n)q^n &= \sum_{n_{j_3} \geq n_{j_2} \geq n_1 \geq 1} \frac{\beta'_{n_1}(q)q^{n_1+n_{j_2}+n_{j_3}}}{(q^{n_1+1})_\infty(1-q^{n_{j_3}})^2(1-q^{n_{j_2}})^2(1-q^{n_1})^2} \\
&= \sum_{1 \leq n_1 = n_{j_2} = n_{j_3}} + \sum_{1 \leq n_1 = n_{j_2} < n_{j_3}} + \sum_{1 \leq n_1 < n_{j_2} = n_{j_3}} \\
&+ \sum_{1 \leq n_1 < n_{j_2} < n_{j_3}} \left[\frac{\beta'_{n_1}(q)q^{n_1+n_{j_2}+n_{j_3}}}{(q^{n_1+1})_\infty(1-q^{n_{j_3}})^2(1-q^{n_{j_2}})^2(1-q^{n_1})^2} \right] \\
&= \sum_{1 \leq n_1} \frac{q^{3n_1}}{(1-q^{n_1})^6} \beta'_{n_1}(q) \prod_{i > n_1} \frac{1}{1-q^i} \\
&+ \sum_{1 \leq n_1 < n_{j_3}} \frac{q^{2n_1}}{(1-q^{n_1})^4} \frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_3}}} \frac{1}{1-q^i} \\
&+ \sum_{1 \leq n_1 < n_{j_2}} \frac{q^{n_1}}{(1-q^{n_1})^2} \frac{q^{2n_{j_2}}}{(1-q^{n_{j_2}})^5} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}}} \frac{1}{1-q^i} \\
&\sum_{1 \leq n_1 < n_{j_2} < n_{j_3}} \frac{q^{n_1}}{(1-q^{n_1})^2} \frac{q^{n_{j_2}}}{(1-q^{n_{j_2}})^3} \frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3} \beta'_{n_1}(q) \prod_{\substack{i > n_1 \\ i \neq n_{j_2}, n_{j_3}}} \frac{1}{1-q^i}
\end{aligned}$$

$$= \sum_{1\le n_1} \sum_{f_1=3}^{\infty} \binom{f_1+3-1}{6-1} q^{n_1 f_1} \beta'_{n_1}(q) \prod_{i>n_1} \frac{1}{1-q^i}$$
$$+ \sum_{1\le n_1<n_{j_3}} \sum_{f_1=2}^{\infty} \binom{f_1+2-1}{4-1} q^{n_1 f_1} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{n_{j_3} f_{j_3}} \beta'_{n_1}(q) \prod_{\substack{i>n_1\\ i\neq n_{j_3}}} \frac{1}{1-q^i}$$
$$+ \sum_{1\le n_1<n_{j_2}} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{n_1 f_1} \sum_{f_{j_2}=2}^{\infty} \binom{f_{j_2}+2}{4} q^{n_{j_2} f_{j_2}} \beta'_{n_1}(q) \prod_{\substack{i>n_1\\ i\neq n_{j_2}}} \frac{1}{1-q^i}$$
$$+ \sum_{1\le n_1<n_{j_2}<n_{j_3}} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{n_1 f_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{n_{j_2} f_{j_2}} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{n_{j_3} f_{j_3}}$$
$$\times \beta'_{n_1}(q) \prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_3}}} \frac{1}{1-q^i}$$

The set of compositions of 3 is $A = \{(3),(2,1),(1,2),(1,1,1)\}$, so we have

$$\sum_{n=1}^{\infty} \alpha\mathrm{spt}_3(n)q^n = \sum_{(m_1,\dots m_r)=\vec{m}\in A} \sum_{1\le n_1<n_{j_2}<\dots<n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1}$$
$$\times \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} \beta'(q) \prod_{\substack{i>n_1\\ i\notin\{n_{j_2},\dots,n_{j_r}\}}} \frac{1}{1-q^i}.$$

For $k = 4$:

$$\sum_{n=1}^{\infty} \alpha\mathrm{spt}_4(n)q^n$$
$$= \sum_{1\le n_1\le n_{j_2}\le n_{j_3}\le n_{j_4}} \frac{\beta'_{n_1}(q) q^{n_1+n_{j_2}+n_{j_3}+n_{j_4}}}{(q^{n_1+1})_\infty (1-q^{n_1})^2 (1-q^{n_{j_2}})^2 (1-q^{n_{j_3}})^2 (1-q^{n_{j_4}})^2}$$
$$= \Bigg(\sum_{1\le n_1=n_{j_2}=n_{j_3}=n_{j_4}} + \sum_{1\le n_1=n_{j_2}=n_{j_3}<n_{j_4}} + \sum_{1\le n_1=n_{j_2}<n_{j_3}=n_{j_4}} + \sum_{1\le n_1=n_{j_2}<n_{j_3}<n_{j_4}}$$
$$+ \sum_{1\le n_1<n_{j_2}=n_{j_3}=n_{j_4}} + \sum_{1\le n_1<n_{j_2}=n_{j_3}<n_{j_4}} + \sum_{1\le n_1<n_{j_2}<n_{j_3}=n_{j_4}}$$
$$+ \sum_{1\le n_1<n_{j_2}<n_{j_3}<n_{j_4}} \Bigg) \frac{\beta'_{n_1}(q) q^{n_1+n_{j_2}+n_{j_3}+n_{j_4}}}{(q^{n_1+1})_\infty (1-q^{n_1})^2 (1-q^{n_{j_2}})^2 (1-q^{n_{j_3}})^2 (1-q^{n_{j_4}})^2}$$
$$= \sum_{1\le n_1} \frac{q^{4n_1}}{(1-q^{n_1})^8} \beta'_{n_1}(q) \prod_{i>n_1} \frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_4}} \frac{q^{3n_1}}{(1-q^{n_1})^6}\frac{q^{n_{j_4}}}{(1-q^{n_{j_4}})^3}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_4}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_3}} \frac{q^{2n_1}}{(1-q^{n_1})^4}\frac{q^{2n_{j_3}}}{(1-q^{n_{j_3}})^5}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_3}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_3}<n_{j_4}} \frac{q^{2n_1}}{(1-q^{n_1})^4}\frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3}\frac{q^{n_{j_4}}}{(1-q^{n_{j_4}})^3}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_3},n_{j_4}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_2}} \frac{q^{n_1}}{(1-q^{n_1})^2}\frac{q^{3n_{j_2}}}{(1-q^{n_{j_2}})^7}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_2}<n_{j_4}} \frac{q^{n_1}}{(1-q^{n_1})^2}\frac{q^{2n_{j_2}}}{(1-q^{n_{j_2}})^5}\frac{q^{n_{j_4}}}{(1-q^{n_{j_4}})^2}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_4}}}$$

$$+\sum_{1\le n_1<n_{j_2}<n_{j_3}} \frac{q^{n_1}}{(1-q^{n_1})^2}\frac{q^{n_{j_2}}}{(1-q^{n_{j_2}})^3}\frac{q^{2n_{j_3}}}{(1-q^{n_{j_3}})^5}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_3}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_2}<n_{j_3}<n_{j_4}} \frac{q^{n_1}}{(1-q^{n_1})^2}\frac{q^{n_{j_2}}}{(1-q^{n_{j_2}})^3}\frac{q^{n_{j_3}}}{(1-q^{n_{j_3}})^3}\frac{q^{n_{j_4}}}{(1-q^{n_{j_4}})^3}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_3},n_{j_4}}}\frac{1}{1-q^i}$$

$$=\sum_{1\le n_1}\sum_{f_1=4}^{\infty}\binom{f_1+4-1}{7}q^{n_1f_1}\beta'_{n_1}(q)\prod_{i>n_1}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_4}}\sum_{f_1=3}^{\infty}\binom{f_1+3-1}{5}q^{n_1f_1}\sum_{f_{j_4}=1}^{\infty}\binom{f_{j_4}+1}{2}q^{n_{j_4}f_{j_4}}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_4}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_3}}\sum_{f_1=2}^{\infty}\binom{f_1+2-1}{3}q^{n_1f_1}\sum_{f_{j_3}=2}^{\infty}\binom{f_{j_3}+2}{4}q^{n_{j_3}f_3}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_3}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_3}<n_{j_4}}\sum_{f_1=2}^{\infty}\binom{f_1+2-1}{3}q^{n_1f_1}\sum_{f_{j_3}=1}^{\infty}\binom{f_{j_3}+1}{2}q^{n_{j_3}f_{j_3}}\sum_{f_{j_4}=1}^{\infty}\binom{f_{j_4}+1}{2}q^{n_{j_4}f_{j_4}}\beta'_{n_1}(q)$$

$$\times\prod_{\substack{i>n_1\\ i\neq n_{j_3},n_{j_4}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_2}}\sum_{f_1=1}^{\infty}\binom{f_1+1-1}{1}q^{n_1f_1}\sum_{f_{j_2}=2}^{\infty}\binom{f_{j_2}+3}{6}q^{f_{j_2}n_{j_2}}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2}}}\frac{1}{1-q^i}$$

$$+\sum_{1\le n_1<n_{j_2}<n_{j_4}}\sum_{f_1=1}^{\infty}\binom{f_1+1-1}{1}q^{n_1f_1}\sum_{f_{j_2}=2}^{\infty}\binom{f_{j_2}+2}{4}q^{n_2f_2}\sum_{f_{j_4}=1}^{\infty}\binom{f_{j_4}+1}{2}q^{n_4f_4}\beta'_{n_1}(q)$$
$$\times\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_4}}}\frac{1}{1-q^i}$$
$$+\sum_{1\le n_1<n_{j_2}<n_{j_3}}\sum_{f_1=1}^{\infty}\binom{f_1+1-1}{1}q^{n_1f_1}\sum_{f_{j_2}=1}^{\infty}\binom{f_{j_2}+1}{2}q^{n_{j_2}f_{j_2}}\sum_{f_{j_3}=2}^{\infty}\binom{f_{j_3}+2}{4}q^{n_{j_3}f_{j_3}}\beta'_{n_1}(q)$$
$$\times\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_3}}}\frac{1}{1-q^i}$$
$$+\sum_{1\le n_1<n_{j_2}<n_{j_3}<n_{j_4}}\sum_{f_1=1}^{\infty}\binom{f_1+1-1}{1}q^{n_1f_1}\sum_{f_{j_2}=1}^{\infty}\binom{f_2+1}{2}q^{n_{j_2}f_{j_2}}$$
$$\times\sum_{f_{j_3}=1}^{\infty}\binom{f_{j_3}+1}{2}q^{n_{j_3}f_{j_3}}\sum_{f_{j_4}=1}^{\infty}\binom{f_{j_4}+1}{2}q^{n_{j_4}f_{j_4}}\beta'_{n_1}(q)\prod_{\substack{i>n_1\\ i\neq n_{j_2},n_{j_3},n_{j_4}}}\frac{1}{1-q^i}.$$

In order, the above eight terms correspond to the compositions of 4:

$$(4),(3,1),(2,2),(2,1,1),(1,3),(1,2,1),(1,1,2),(1,1,1,1).$$

Thus for each composition $m_1+\cdots+m_r=4$ we have a sum of the form:

$$\sum_{1\le n_1<n_2<\cdots<n_{j_r}}\frac{q^{n_1m_1}}{(1-q^{n_1})^{2m_1}}\frac{q^{n_{j_2}m_2}}{(1-q^{n_{j_2}})^{2m_2+1}}\cdots\frac{q^{n_{j_r}m_r}}{(1-q^{n_{j_r}})^{2m_{j_r}+1}}\beta'_{n_1}(q)$$
$$\times\prod_{\substack{i>n_1\\ i\notin\{n_{j_2},\dots,n_{j_r}\}}}\frac{1}{1-q^i}$$

$$=\sum_{n_1<n_{j_2}<\cdots<n_{j_r}}\sum_{f_1=m_1}^{\infty}\sum_{f_2=m_2}^{\infty}\cdots\sum_{f_{j_r}=m_r}^{\infty}\binom{f_1+m_1-1}{2m_1-1}$$
$$\times\binom{f_{j_2}+m_2}{2m_2}\cdots\binom{f_{j_r}+m_r}{2m_r}q^{n_1f_1+n_{j_2}f_{j_2}+\cdots n_{j_r}f_{j_r}}\beta'_{n_1}(q)$$
$$\times\prod_{\substack{i>n_1\\ i\notin\{n_{j_2},\dots,n_{j_r}\}}}\frac{1}{1-q^i}$$

For general k we take the expression for $\sum_{n=1}^{\infty}\alpha\mathrm{spt}_k(n)q^n$ in Corollary 2.8 and split it into 2^{k-1} sums by turning the index bounds into $<$ or $=$, each of which corresponds to a composition of k. If we let A be the set of all compositions of k, with the manipulations illustrated above, we have that

$$\sum_{n=1}^{\infty} \alpha\mathrm{spt}_k(n)q^n = \sum_{(m_1,\dots m_r)=\vec{m}\in A} \sum_{1\le n_1<n_{j_2}<\dots<n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1}$$
$$\times \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} q^{n_1f_1+n_{j_2}f_{j_2}+\dots+n_{j_r}f_{j_r}} \beta'_{n_1}(q) \prod_{\substack{i>n_1 \\ i\notin\{n_{j_2},\dots,n_{j_r}\}}} \frac{1}{1-q^i}.$$

This we recognize as the generating function for vector partitions $\vec{\pi}=(\pi_1,\dots,\pi_r)$ counted according to the weight $\omega_k(\vec{\pi})$ where $\beta'_{n_1}(q)$ determines the types of partitions in $(\pi_2,\dots,\pi_r)$. □

3.1. **Examples.** To help describe what is being counted, we give the details regarding $\alpha\mathrm{spt}_k(n)$ for $k=1,2,3$ and the Bailey pairs A1 and B2.

We note that the first three weights are given by

$$\omega_1(\pi) = f_1^1(\pi)$$
$$\omega_2(\pi) = \binom{f_1^1(\pi)+1}{3} + f_1^1(\pi)\sum_{2\le j}\binom{f_j^1(\pi)+1}{2}$$
$$\omega_3(\pi) = \binom{f_1^1(\pi)+2}{5} + \binom{f_1^1(\pi)+1}{3}\sum_{2\le j}\binom{f_j^1(\pi)+1}{2} + f_1^1(\pi)\sum_{2\le j}\binom{f_j+2}{4}$$
$$+ f_1^1(\pi)\sum_{2\le j<k}\binom{f_j^1(\pi)+1}{2}\binom{f_k^1(\pi)+1}{2}.$$

TABLE 2. A1 Partitions of 4

	f_1^1	f_2^1	f_3^1	ω_1	ω_2	ω_3
$(4,0)$	1	0	0	1	0	0
$(1+3,0)$	1	1	0	1	1	0
$(2+2,0)$	2	0	0	2	1	0
$(1+1+2,0)$	2	1	0	2	3	1
$(1+1+1+1,0)$	4	0	0	4	10	6
$(1+1,2)$	2	0	0	2	1	0
Total				12	16	7

TABLE 3. A1 Partitions of 5

	f_1^1	f_2^1	f_3^1	ω_1	ω_2	ω_3
$(5,0)$	1	0	0	1	0	0
$(1+4,0)$	1	1	0	1	1	0
$(2+3,0)$	1	1	0	1	1	0
$(1+1+3,0)$	2	1	0	2	3	1
$(1+2+2,0)$	1	2	0	1	3	0
$(1+1+1+2,0)$	3	1	0	3	7	5
$(1+1+1+1+1,0)$	5	0	0	5	20	21
$(1+1+1,2)$	3	0	0	3	4	1
$(1+2,2)$	1	1	0	1	1	0
$(1,2+2)$	1	0	0	1	0	0
$(2,3)$	1	0	0	1	0	0
Total				20	40	28

$$\sum_{n=1}^{\infty} \mathrm{A1spt}_1(n)q^n = q+3q^2+6q^3+12q^4+20q^5+36q^6+\cdots$$

$$\sum_{n=1}^{\infty} \mathrm{A1spt}_2(n)q^n = q^2+5q^3+16q^4+40q^5+90q^6+\cdots$$

$$\sum_{n=1}^{\infty} \mathrm{A1spt}_3(n)q^n = q^3+7q^4+28q^5+92q^6+\cdots$$

TABLE 4. B2 Partitions of 4

	f_1^1	f_2^1	f_3^1	ω_1	ω_2	ω_3
$(2,2)$	1	0	0	1	0	0
$(1+2,1)$	1	1	0	1	1	0
$(1+1+1,1)$	3	0	0	3	4	1
Total				5	5	1

TABLE 4. B2 Partitions of 5

	f_1^1	f_2^1	f_3^1	ω_1	ω_2	ω_3
$(1+3,1)$	1	1	0	1	1	0
$(1+1+2,1)$	2	1	0	2	3	1
$(1+1+1+1,1)$	4	0	0	4	10	6
Total				7	14	7

$$\sum_{n=1}^{\infty} \mathrm{B2spt}_1(n)q^n = q^2+2q^3+5q^4+7q^5+15q^6+20q^7+\cdots$$

$$\sum_{n=1}^{\infty} \mathrm{B2spt}_2(n)q^n = q^3+5q^4+14q^5+35q^6+70q^7+\cdots$$

$$\sum_{n=1}^{\infty} \mathrm{B2spt}_3(n)q^n = q^4+7q^5+28q^6+84q^7+\cdots$$

4. CONGRUENCES

We conjecture the following congruences for all nonnegative n.

A1:

$$\mathrm{A1spt}_2(5n) \equiv 0 \bmod 5$$
$$\mathrm{A1spt}_2(5n+1) \equiv 0 \bmod 5$$

A3:

$$\mathrm{A3spt}_2(5n+1) \equiv 0 \bmod 5$$
$$\mathrm{A3spt}_2(5n+2) \equiv 0 \bmod 5$$
$$\mathrm{A3spt}_2(5n+4) \equiv 0 \bmod 5$$
$$\mathrm{A3spt}_2(9n) \equiv 0 \bmod 3$$

A5:

$$\mathrm{A5spt}_2(5n) \equiv 0 \bmod 5$$
$$\mathrm{A5spt}_2(5n+4) \equiv 0 \bmod 5$$
$$\mathrm{A5spt}_3(7n) \equiv 0 \bmod 7$$
$$\mathrm{A5spt}_3(7n+1) \equiv 0 \bmod 7$$
$$\mathrm{A5spt}_3(7n+3) \equiv 0 \bmod 7$$
$$\mathrm{A5spt}_3(7n+5) \equiv 0 \bmod 7$$
$$\mathrm{A5spt}_6(7n+5) \equiv 0 \bmod 7$$

A7:

$$\mathrm{A7spt}_2(5n+1) \equiv 0 \bmod 5$$
$$\mathrm{A7spt}_2(5n+4) \equiv 0 \bmod 5$$
$$\mathrm{A7spt}_2(7n) \equiv 0 \bmod 7$$
$$\mathrm{A7spt}_2(7n+1) \equiv 0 \bmod 7$$
$$\mathrm{A7spt}_3(7n) \equiv 0 \bmod 7$$
$$\mathrm{A7spt}_3(7n+1) \equiv 0 \bmod 7$$
$$\mathrm{A7spt}_3(7n+2) \equiv 0 \bmod 7$$
$$\mathrm{A7spt}_3(7n+4) \equiv 0 \bmod 7$$

B2:

$$\mathrm{B2spt}_2(5n+1) \equiv 0 \bmod 5$$
$$\mathrm{B2spt}_2(5n+2) \equiv 0 \bmod 5$$
$$\mathrm{B2spt}_2(5n+4) \equiv 0 \bmod 5$$
$$\mathrm{B2spt}_2(7n+1) \equiv 0 \bmod 7$$
$$\mathrm{B2spt}_2(7n+5) \equiv 0 \bmod 7$$

$$\mathrm{B2spt}_2(11n+1) \equiv 0 \bmod 11$$
$$\mathrm{B2spt}_3(4n+3) \equiv 0 \bmod 2$$
$$\mathrm{B2spt}_3(7n) \equiv 0 \bmod 7$$
$$\mathrm{B2spt}_3(7n+1) \equiv 0 \bmod 7$$
$$\mathrm{B2spt}_3(7n+3) \equiv 0 \bmod 7$$
$$\mathrm{B2spt}_3(7n+5) \equiv 0 \bmod 7$$
$$\mathrm{B2spt}_4(3n) \equiv 0 \bmod 3$$
$$\mathrm{B2spt}_6(7n+5) \equiv 0 \bmod 7$$

E4:

$$\mathrm{E4spt}_2(8n+7) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_2(16n+1) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_2(17n) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_2(18n+5) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_2(16n+14) \equiv 0 \bmod 4$$
$$\mathrm{E4spt}_2(5n) \equiv 0 \bmod 5$$
$$\mathrm{E4spt}_2(5n+2) \equiv 0 \bmod 5$$
$$\mathrm{E4spt}_3(16n+3) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_3(16n+13) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_3(17n) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_4(16n+7) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_4(17n) \equiv 0 \bmod 2$$
$$\mathrm{E4spt}_{16}(16n) \equiv 0 \bmod 2$$

C5:

$$\mathrm{C5spt}_2(5n) \equiv 0 \bmod 5$$
$$\mathrm{C5spt}_2(5n+1) \equiv 0 \bmod 5$$
$$\mathrm{C5spt}_2(5n+4) \equiv 0 \bmod 5$$

Y1:

$$\mathrm{Y1spt}_{5k}(10n+3) \equiv 0 \bmod 5,\ k \geq 1$$

Y2:

$$\mathrm{Y2spt}_{5k}(10n+3) \equiv 0 \bmod 5,\ k \geq 1$$

X40:

$$\mathrm{X40spt}_2(17n+15) \equiv 0 \bmod 2$$
$$\mathrm{X40spt}_3(4n) \equiv 0 \bmod 2$$

$$\begin{aligned}
\mathrm{X40spt}_3(17n+15) &\equiv 0 \bmod 2\\
\mathrm{X40spt}_3(18n+2) &\equiv 0 \bmod 2\\
\mathrm{X40spt}_3(18n+14) &\equiv 0 \bmod 2\\
\mathrm{X40spt}_6(8n) &\equiv 0 \bmod 2\\
\mathrm{X40spt}_6(8n+7) &\equiv 0 \bmod 2
\end{aligned}$$

X41:

$$\mathrm{X41spt}_3(4n+2) \equiv 0 \bmod 2$$

X46:

$$\begin{aligned}
\mathrm{X46spt}_2(9n) &\equiv 0 \bmod 3\\
\mathrm{X46spt}_3(16n+11) &\equiv 0 \bmod 2\\
\mathrm{X46spt}_4(16n+7) &\equiv 0 \bmod 2\\
\mathrm{X46spt}_8(16n+5) &\equiv 0 \bmod 2\\
\mathrm{X46spt}_{20}(16n+7) &\equiv 0 \bmod 2\\
\mathrm{X46spt}_{23}(16n) &\equiv 0 \bmod 2\\
\mathrm{X46spt}_{24}(13n+4) &\equiv 0 \bmod 2
\end{aligned}$$

5. Concluding Remarks

Working with a list of 29 suitable Bailey Pairs, we defined rank and crank like functions, from which we derived symmetrized rank and crank-like moments. Using the relation between the symmetrized moments and the ordinary moments, we prove inequalities for the ordinary moments. By defining an spt-like function in the same way as Garvan in [5] and using the extended weight statistic as Jennings-Shaffer in [6], we find combinatorial interpretations for the spt-like functions.

As in the work of Garvan in [5] and Jennings-Shaffer in [6], it is likely that there will be congruences for some of the higher order spt-like functions. We have numerical evidence to support the previous conjectures.

There were some Bailey pairs that did not meet the $\alpha_n = \alpha_{-n}$ criteria in our formulas, but it is possible that with further manipulation they could also yield spt-like functions of some interest.

We also expect that the ordinary rank moments should be quasi-mock modular forms, and that the ordinary crank moments should be quasi-modular forms.

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