Extensions of Euclidean Relations and Inequalities to Spherical and Hyperbolic Geometry

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August 2017

Abstract

We aim to find analogues in hyperbolic and spherical space for geometric relations known in Euclidean space. First we prove a theorem which provides a technique for determining whether an inequality involving a triangle's circumradius, inradius, and side lengths easily generalizes to hyperbolic or spherical geometry, and apply this technique to known strengthenings of Euler's inequality ($R \geq 2r$). We also extend, to spherical geometry, the generalization of Euler's inequality to simplices in n-dimensional space. We prove a simple relation between the pairwise distances generated by $n+2$ points in n-dimensional space of any curvature, using Cayley-Menger determinants. Finally, the analogue for Euler's theorem relating the circumradius and inradius with the distance between the circumcenter and incenter is provided for hyperbolic and spherical space.

1 Introduction

Many geometric relations and inequalities are known primarily in the Euclidean plane, characterized by dimension $n = 2$ and curvature $K = 0$. As a simple example, take Euler's inequality

$$R \geq 2r$$

which is true for all Euclidean triangles with circumradius R and inradius r. Given such a fact, we may ask: is there an analogue in spherical geometry ($K > 0$) or in hyperbolic geometry ($K < 0$)? Furthermore, does it apply to higher dimensions? To continue with our example, Euler's inequality generalizes to two-dimensional curved space as follows [6]:

$$2 \leq \begin{cases} \frac{R}{r} & \text{in Euclidean geometry } (K = 0) \\ \frac{\tan R}{\tan r} & \text{in spherical geometry } (K = 1) \\ \frac{\tanh R}{\tanh r} & \text{in hyperbolic geometry } (K = -1) \end{cases}$$

It is also known that Euler's inequality generalizes to an n-dimensional Euclidean simplex with $n+1$ vertices as

$$R \geq nr.$$

In this paper we extend this fact to n-dimensional spherical geometry ($K = 1$) as

$$\tan R \geq n \tan r.$$

Although the curvature K may take on any real-number value, for most of this report we assume $K = 1$ and $K = -1$ for spherical and hyperbolic geometry, respectively. With this assumption, n-dimensional spherical space $\mathbb{S}^n$ is modeled by the sphere $\mathbf{x} \cdot \mathbf{x} = 1$ in $\mathbb{R}^{n+1}$. Likewise, $\mathbb{H}^n$ corresponds to the hyperboloid $\mathbf{x}*\mathbf{x} = -1$ in $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$, where the inner product $*$ is defined as

$$\mathbf{x} * \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

In general, an appropriate geometric model for a manifold of constant curvature is a sphere or hyperboloid of square radius $R^2 = \frac{1}{K}$.

As noted in [3], some relations involving triangles can be generalized to spherical and hyperbolic geometry with the unifying function

$$s(x) := \begin{cases} \frac{x}{2} & \text{in Euclidean geometry} \\ \sin\frac{x}{2} & \text{in spherical geometry} \\ \sinh\frac{x}{2} & \text{in hyperbolic geometry} \end{cases} \tag{1}$$

(we will include a subscript e, s, or h to avoid ambiguity as needed). For example, we have the following lemma from [1]:

Lemma 1.1. *If $f(s(a), s(b), s(c)) \geq 0$ is an inequality which holds for all triangles with side lengths a, b, c in Euclidean geometry, then $f(s(a'), s(b'), s(c')) \geq 0$ holds for all triangles with side lengths a', b', c' in spherical and hyperbolic geometry.*

2 Inequalities Involving Circumradius and Inradius

In this section we provide an algebraic method to determine whether an inequality involving a triangle's circumradius R and inradius r along with its side lengths can be easily generalized to spherical or hyperbolic geometry. For a triangle T in Euclidean, spherical or hyperbolic geometry with side lengths a, b, c define the quantities

$$\begin{aligned} H(T) &:= s(a+b+c) \\ \bar{H}(T) &:= s(a) + s(b) + s(c) \\ J(T) &:= s(a+b-c)s(a+c-b)s(b+c-a) \\ \bar{J}(T) &:= (s(a)+s(b)-s(c))(s(a)+s(c)-s(b))(s(b)+s(c)-s(a)) \end{aligned}$$

(we will omit the parameter T when it is unambiguous to do so). The circumradius and inradius can be expressed using these quantities as follows:

$$\frac{2s(a)s(b)s(c)}{\sqrt{HJ}} = \frac{2s(a)s(b)s(c)}{\sqrt{s(a+b-c)s(a+c-b)s(b+c-a)s(a+b+c)}} = \begin{cases} R & \text{e.} \\ \tan R & \text{s.} \\ \tanh R & \text{h.} \end{cases} \tag{2}$$

$$\sqrt{\frac{J}{H}} = \sqrt{\frac{s(a+b-c)s(a+c-b)s(b+c-a)}{s(a+b+c)}} = \begin{cases} r & \text{e.} \\ \tan r & \text{s.} \\ \tanh r & \text{h.} \end{cases} \tag{3}$$

In algebraic proofs involving a triangle with side lengths a, b, c we often use the quantities

$$\begin{aligned} x &:= \frac{a+b-c}{2} \\ y &:= \frac{a+c-b}{2} \\ z &:= \frac{b+c-a}{2} \end{aligned} \tag{4}$$

in order to make the substitution $a = x + y$, $b = x + z$, $c = y + z$. Note that $x, y, z \geq 0$ due to the triangle inequality; geometrically, these values represent distances between vertices and tangent points of the incircle (see Fig. 1). Note that in spherical geometry we have $x, y, z \leq \pi$ as π is the maximum distance between two points.

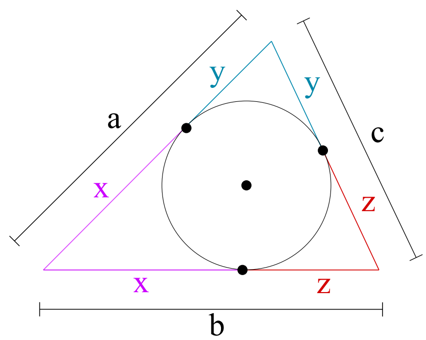

Figure 1: The geometric meaning of x, y, z.

2.1 Lemmas

The following was proved in [1]:

$$J(T) - \bar{J}(T) \begin{cases} = 0 & \text{if } T \text{ is Euclidean} \\ \leq 0 & \text{if } T \text{ is spherical} \\ \geq 0 & \text{if } T \text{ is hyperbolic.} \end{cases} \tag{5}$$

We show that the same relation holds between H and $\bar{H}$.

Lemma 2.1.

$$H(T) - \bar{H}(T) \begin{cases} = 0 & \textit{if } T \textit{ is Euclidean} \\ \leq 0 & \textit{if } T \textit{ is spherical} \\ \geq 0 & \textit{if } T \textit{ is hyperbolic.} \end{cases} \tag{6}$$

Proof. In Euclidean geometry,

$$H - \bar{H} = \frac{a+b+c}{2} - \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right) = 0.$$

In spherical geometry we have $0 \leq a, b, c \leq \pi$. Then

$$\begin{aligned} H - \bar{H} = \sin\frac{a+b+c}{2} &- \left(\sin\frac{a}{2} + \sin\frac{b}{2} + \sin\frac{c}{2}\right) \\ &= \sin\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\cos\frac{a}{2}\cos\frac{c}{2} + \sin\frac{c}{2}\cos\frac{a}{2}\cos\frac{b}{2} - \sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \\ &\quad - \left(\sin\frac{a}{2} + \sin\frac{b}{2} + \sin\frac{c}{2}\right) \\ &= \sin\frac{a}{2}\left(\cos\frac{b}{2}\cos\frac{c}{2} - 1\right) + \sin\frac{b}{2}\left(\cos\frac{a}{2}\cos\frac{c}{2} - 1\right) + \sin\frac{c}{2}\left(\cos\frac{a}{2}\cos\frac{b}{2} - 1\right) - \sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \\ &\leq 0. \end{aligned}$$

In hyperbolic geometry, noting that $a, b, c \geq 0$,

$$\begin{aligned}
H - \bar{H} &= \sinh \frac{a+b+c}{2} - \left(\sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} \right) \\
&= \sinh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2} + \sinh \frac{b}{2} \cosh \frac{a}{2} \cosh \frac{c}{2} + \sinh \frac{c}{2} \cosh \frac{a}{2} \cosh \frac{b}{2} + \sinh \frac{a}{2} \sinh \frac{b}{2} \sinh \frac{c}{2} \\
&\quad - \left(\sinh \frac{a}{2} + \sinh \frac{b}{2} + \sinh \frac{c}{2} \right) \\
&= \sinh \frac{a}{2} \left(\cosh \frac{b}{2} \cosh \frac{c}{2} - 1 \right) + \sinh \frac{b}{2} \left(\cosh \frac{a}{2} \cosh \frac{c}{2} - 1 \right) + \sinh \frac{c}{2} \left(\cosh \frac{a}{2} \cosh \frac{b}{2} - 1 \right) \\
&\quad + \sinh \frac{a}{2} \sinh \frac{b}{2} \sinh \frac{c}{2} \\
&\geq 0.
\end{aligned}$$

□

2.2 Main Theorem

Here we prove the main theorem of this section.

Theorem 2.2. *Let $f(s(a), s(b), s(c), H, J) \geq 0$ be an inequality which holds for all triangles with side lengths a, b, c in Euclidean geometry, where f is a decreasing (increasing) function of H and J. Then the same inequality holds for all triangles in spherical (hyperbolic) geometry.*

Proof. Spherical case: let T be a triangle with side lengths a, b, c in spherical geometry. This implies the existence of a Euclidean triangle T' with side lengths $a' = 2s_s(a)$, $b' = 2s_s(b)$ and $c' = 2s_s(c)$ (see the proof of Lemma 2.1 in [1]). Observe that $s_e(a') = \frac{a'}{2} = \frac{2s_s(a)}{2} = s_s(a)$; similarly, $s_e(b') = s_s(b)$ and $s_e(c') = s_s(c)$. Since the quantities $\bar{H}$ and $\bar{J}$ are functions of $s(a)$, $s(b)$, and $s(c)$ it follows that $\bar{H}(T) = \bar{H}(T')$ and $\bar{J}(T) = \bar{J}(T')$. Also, as T' is Euclidean, $\bar{H}(T') = H(T')$ and $\bar{J}(T') = J(T')$. Thus

$$\bar{H}(T) = H(T')$$

and

$$\bar{J}(T) = J(T').$$

From (5) and (6) we get $H \leq \bar{H}$ and $J \leq \bar{J}$ in spherical geometry, so if f is a decreasing function of H and J then

$$f(s(a), s(b), s(c), H(T), J(T)) \geq f(s(a), s(b), s(c), \bar{H}(T), \bar{J}(T)), \tag{7}$$

but notice that, as all arguments are equal,

$$f(s(a), s(b), s(c), \bar{H}(T), \bar{J}(T)) = f(s(a'), s(b'), s(c'), H(T'), J(T')). \tag{8}$$

Since the inequality holds for all Euclidean triangles we have

$$f(s(a'), s(b'), s(c'), H(T'), J(T')) \geq 0 \tag{9}$$

and finally, the inequality

$$f(s(a), s(b), s(c), H(T), J(T)) \geq 0$$

follows from (7), (8), and (9). The proof for hyperbolic triangles is directly analogous, as a hyperbolic triangle with side lengths a, b, c implies the existence of a Euclidean triangle with side lengths $2s_h(a), 2s_h(b), 2s_h(c)$. □

Theorem 2.2 immediately gives rise to a pair of corollaries. In fact, the statement of Corollary 2.3 can be proven not only for triangles in $\mathbb{S}^2$ and $\mathbb{H}^2$, but for n-dimensional simplices in $\mathbb{S}^n$ and $\mathbb{H}^n$. We provide a geometric proof in Section 3.

Corollary 2.3.

(a) *Let* $\frac{R}{r} \geq f(s(a), s(b), s(c))$ *be an inequality which holds for all Euclidean triangles with side lengths* a, b, c*, circumradius* R*, and inradius* r*. Then* $\frac{\tan R'}{\tan r'} \geq f(s(a'), s(b'), s(c'))$ *holds for all spherical triangles with side lengths* a', b', c'*, circumradius* R'*, and inradius* r'*.*

(b) *Let* $\frac{R}{r} \leq f(s(a), s(b), s(c))$ *be an inequality which holds for all Euclidean triangles with side lengths* a, b, c*. then* $\frac{\tanh R'}{\tanh r'} \leq f(s(a'), s(b'), s(c'))$ *for all hyperbolic triangles with side lengths* a', b', c'*, circumradius* R'*, and inradius* r'*.*

Proof. Let f be a function such that $\frac{R}{r} \geq f(s(a), s(b), s(c))$ holds for all Euclidean triangles. By (2) and (3) we can express $\frac{R}{r}$ as $\frac{2s(a)s(b)s(c)}{J}$. Now define $g(s(a), s(b), s(c), H, J) := \frac{2s(a)s(b)s(c)}{J} - f(s(a), s(b), s(c))$ and we have

$$g(s(a), s(b), s(c), H, J) \geq 0$$

for all Euclidean triangles with side lengths a, b, c. As g is a decreasing function of J we can apply Theorem 2.2, so the inequality is true for all spherical triangles as well, where $\frac{R}{r}$ is replaced by $\frac{\tan R}{\tan r}$. If we instead have $\frac{R}{r} \leq f(s(a, s(b), s(c))$, then $-g \geq 0$ (with g as defined above) and $-g$ is an increasing function of J. By Theorem 2.2 the inequality $\frac{\tanh R}{\tanh r} \leq f(s(a), s(b), s(c))$ is true for all hyperbolic triangles. ☐

The same proof but replacing J with H yields Corollary 2.4.

Corollary 2.4.

(a) *Let* $rR \geq f(s(a), s(b), s(c))$ *be an inequality which holds for all Euclidean triangles with side lengths* a, b, c*. Then* $\tan r' \tan R' \geq f(s(a'), s(b'), s(c'))$ *holds for all spherical triangles with side lengths* a', b', c'*.*

(b) *Let* $rR \leq f(s(a), s(b), s(c))$ *be an inequality which holds for all Euclidean triangles with side lengths* a, b, c*. Then* $\tanh r' \tanh R' \leq f(s(a'), s(b'), s(c'))$ *for all hyperbolic triangles with side lengths* a', b', c'*.*

2.3 Examples

The original goal was to determine which of the following strengthenings of Euler's inequality [5] generalize to hyperbolic and spherical geometry. In the theorems to follow, we modify the expressions based on the geometry according to the philosophy that equality should hold if and only if $a = b = c$.

$$\frac{R}{2r} \geq \frac{(a+b+c)(a^3+b^3+c^3)}{(ab+bc+ca)^2} \geq 1 \tag{10}$$

$$2R^2 + r^2 \geq \frac{1}{4}(a^2+b^2+c^2) \geq 3r(2R-r) \tag{11}$$

$$\frac{1}{4r^2} \geq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \geq \frac{1}{2rR} \tag{12}$$

Theorem 2.5 (Generalization of (10) to spherical triangles)**.** *The inequalities*

$$\frac{\tan R}{2\tan r} \geq \frac{(s(a)+s(b)+s(c))(s(a)^3+s(b)^3+s(c)^3)}{(s(a)s(b)+s(b)s(c)+s(c)s(a))^2} \geq 1 \tag{13}$$

hold for all spherical triangles with side lengths a, b, c*, circumradius* R*, and inradius* r*.*

Proof. Given that (10) is true for Euclidean triangles, the left inequality follows directly from Corollary 2.3, while the right inequality is a consequence of Lemma 1.1. □

Theorem 2.6 (Generalization of (11))**.**

(a) The inequality

$$2\tan^2 R + \tan^2 r \geq \left(s(a)^2 + s(b)^2 + s(c)^2\right)\left(\frac{s(a)+s(b)+s(c)}{s(a+b+c)}\right) \tag{14}$$

holds for all triangles in spherical geometry with side lengths a, b, c, circumradius R, and inradius r.

(b) The inequality

$$\left(s(a)^2 + s(b)^2 + s(c)^2\right)\left(\frac{s(a)+s(b)+s(c)}{s(a+b+c)}\right) \geq 3\tanh r(2\tanh R - \tanh r) \tag{15}$$

holds for all triangles in hyperbolic geometry with side lengths a, b, c, circumradius R, and inradius r.

Proof. Both parts follow from Theorem 2.2. The original Euclidean inequality (11) can be rewritten as

$$\frac{8s(a)^2s(b)^2s(c)^2}{HJ} + \frac{J}{H} \geq (s(a)^2 + s(b)^2 + s(c)^2) \geq \frac{12s(a)s(b)s(c)}{H} - \frac{3J}{H}. \tag{16}$$

Part (a). In spherical geometry, (14) reduces as follows:

$$\begin{aligned} & 2\tan^2 R + \tan^2 r \geq (s(a)^2 + s(b)^2 + s(c)^2)\left(\frac{s(a)+s(b)+s(c)}{s(a+b+c)}\right) \\ \iff \quad & \frac{8s(a)^2s(b)^2s(c)^2}{HJ} + \frac{J}{H} \geq (s(a)^2 + s(b)^2 + s(c)^2) \cdot \frac{s(a)+s(b)+s(c)}{H}. \end{aligned}$$

This inequality is true for Euclidean triangles as it is equivalent to the left inequality in (16), with the observation that in Euclidean geometry, $s(a) + s(b) + s(c) = H$. We cannot directly apply Theorem 2.2 since the left hand side is not a decreasing function of J for all values of J; however, noting the proof of Theorem 2.2, we only need to show

$$\frac{8s(a)^2s(b)^2s(c)^2}{J} + J \geq \frac{8s(a)^2s(b)^2s(c)^2}{\bar{J}} + \bar{J}.$$

Observe that

$$\frac{\partial}{\partial J}\left[\frac{8s(a)^2s(b)^2s(c)^2}{J} + J\right] = 1 - \frac{8s(a)^2s(b)^2s(c)^2}{J^2};$$

we can show this derivative is not positive for values of J less than $\bar{J}$. It is enough to show

$$\bar{J} \leq s(a)s(b)s(c)$$

or

$$(s(a) + s(b) - s(c))(s(a) + s(c) - s(b))(s(b) + s(c) - s(a)) \leq s(a)s(b)s(c).$$

Since $s(a), s(b), s(c)$ satisfy the triangle inequality, we can make the substitution $s(a) = x + y$, $s(b) = x + z$, $s(c) = y + z$ which leaves us to prove

$$(2x)(2y)(2z) \leq (x + y)(x + z)(y + z)$$

or

$$xyz \le \frac{2xyz + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2}{8}$$

which is a direct application of the arithmetic mean-geometric mean inequality.

Part (b). In hyperbolic geometry, (15) reduces as

$$\begin{aligned} & \left(s(a)^2 + s(b)^2 + s(c)^2\right)\left(\frac{s(a)+s(b)+s(c)}{s(a+b+c)}\right) \ge 3\tanh r(2\tanh R - \tanh r) \\ \iff \quad & (s(a)^2 + s(b)^2 + s(c)^2) \cdot \frac{s(a)+s(b)+s(c)}{H} \ge \frac{12 s(a)s(b)s(c)}{H} - \frac{3J}{H} \end{aligned}$$

This is true for Euclidean triangles as it is the right inequality in (16), again noting that in Euclidean geometry $s(a) + s(b) + s(c) = H$. Moving all terms to the left hand side and multiplying by H we are left with an increasing function of J, so Theorem 2.2 proves (15) for hyperbolic triangles. □

Theorem 2.2 does not directly apply to (12), so we utilize a different method for the following inequalities.

Theorem 2.7 (Generalization of (12) to hyperbolic geometry)**.**

(a) The inequality

$$\frac{1}{\tanh^2 r} - 4 \ge \frac{1}{s(a)^2} + \frac{1}{s(b)^2} + \frac{1}{s(c)^2} \tag{17}$$

holds for all triangles in hyperbolic geometry with side lengths a, b, c, circumradius R, and inradius r.

(b) The inequality

$$\frac{1}{3}\left(\frac{1}{s(a)} + \frac{1}{s(b)} + \frac{1}{s(c)}\right)^2 \ge \frac{2}{\tan r \tan R} + 4 \tag{18}$$

holds for all triangles in spherical geometry with side lengths a, b, c, circumradius R, and inradius r.

Proof. The original Euclidean inequality (12) can be rewritten as

$$\frac{H}{J} \ge \frac{1}{s(a)^2} + \frac{1}{s(b)^2} + \frac{1}{s(c)^2} \ge \frac{1}{3}\left(\frac{1}{s(a)} + \frac{1}{s(b)} + \frac{1}{s(c)}\right)^2 \ge \frac{H}{s(a)s(b)s(c)}. \tag{19}$$

Part (a). Let $a = x + y$, $b = x + z$, $c = y + z$ (see (2)); the left hand side of (17) is then equivalent to

$$\begin{aligned} \frac{1}{\tanh^2 r} - 4 &= \frac{s(a+b+c)}{s(a+b-c)s(a+c-b)s(b+c-a)} - 4 \\ &= \frac{\sinh \frac{a+b+c}{2}}{\sinh \frac{a+b-c}{2} \sinh \frac{a+c-b}{2} \sinh \frac{b+c-a}{2}} - 4 \\ &= \frac{\sinh(x+y+z) - 4\sinh x \sinh y \sinh z}{\sinh x \sinh y \sinh z} \\ &= \frac{\sinh x \cosh y \cosh z + \sinh y \cosh x \cosh z + \sinh z \cosh x \cosh y - 3\sinh x \sinh y \sinh z}{\sinh x \sinh y \sinh z} \\ &= \frac{\sinh x \cosh(y-z) + \sinh y \cosh(x-z) + \sinh z \cosh(x-y)}{\sinh x \sinh y \sinh z} \\ &= \frac{\cosh(y-z)}{\sinh y \sinh z} + \frac{\cosh(x-z)}{\sinh x \sinh z} + \frac{\cosh(x-y)}{\sinh x \sinh y} \end{aligned}$$

while the right hand side is

$$\frac{1}{s(a)^2}+\frac{1}{s(b)^2}+\frac{1}{s(c)^2} = \frac{1}{\sinh^2(\frac{x+y}{2})}+\frac{1}{\sinh^2(\frac{x+z}{2})}+\frac{1}{\sinh^2(\frac{y+z}{2})}$$
$$= 2\left(\frac{1}{\cosh(x+y)-1}+\frac{1}{\cosh(x+z)-1}+\frac{1}{\cosh(y+z)-1}\right)$$

due to the identity $2\sinh^2\alpha = \cosh(2\alpha)-1$. Now we compare term-by-term; it is sufficient to show

$$\frac{\cosh(x-y)}{\sinh x \sinh y} \geq \frac{2}{\cosh(x+y)-1}.$$

This is equivalent to

$$\begin{aligned} & \cosh(x-y)((\cosh(x+y)-1)-2\sinh x\sinh y \geq 0 \\ \iff & \cosh(x-y)((\cosh(x+y)-1)-(\cosh(x+y)-\cosh(x-y)) \geq 0 \\ \iff & (\cosh(x-y)-1)\cosh(x+y) \geq 0 \end{aligned}$$

which is true since $\cosh\alpha \geq 1$ for all α.

Part (b). The right hand side of (18) is

$$\begin{aligned} \frac{2}{\tan r \tan R}+4 &= \frac{s(a+b+c)}{s(a)s(b)s(c)}+4 \\ &= \frac{\sin(\frac{a+b+c}{2})+4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \\ &= \frac{\sin\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}+\sin\frac{b}{2}\cos\frac{a}{2}\cos\frac{c}{2}+\sin\frac{c}{2}\cos\frac{a}{2}\cos\frac{b}{2}+3\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \\ &= \frac{\sin\frac{a}{2}\cos\frac{b-c}{2}+\sin\frac{b}{2}\cos\frac{a-c}{2}+\sin\frac{c}{2}\cos\frac{a-b}{2}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \\ &\leq \frac{\sin\frac{a}{2}+\sin\frac{b}{2}+\sin\frac{c}{2}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \\ &= \frac{s(a)+s(b)+s(c)}{s(a)s(b)s(c)} \end{aligned}$$

so in order to prove (18) for spherical triangles, it is sufficient to show

$$\frac{1}{3}\left(\frac{1}{s(a)}+\frac{1}{s(b)}+\frac{1}{s(c)}\right)^2 \geq \frac{s(a)+s(b)+s(c)}{s(a)s(b)s(c)}.$$

This is the right inequality in (19), so it is true for spherical geometry by Lemma 1.1.

□

Remark: each inequality in (10), (11), and (12) extends to either spherical or hyperbolic geometry, but not both. Whenever we were able to extend an inequality to a particular curved geometry, we found a counterexample for the other.

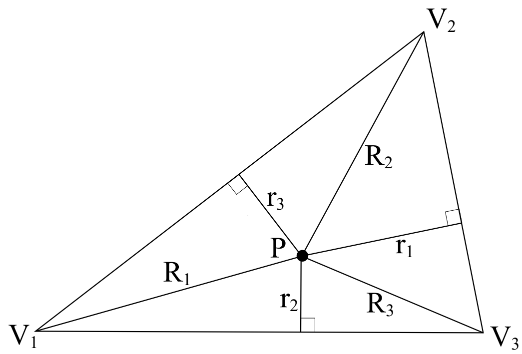

Figure 2: Drawing associated with the Erdős-Mordell Inequality.

2.4 The Erdős-Mordell Inequality

Given a point P in the interior of a triangle, let R_i denote the distance from P to a vertex and r_i denote the distance to the opposite edge (see Fig. 2). The Erdős-Mordell Inequality tells us that in Euclidean geometry,

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

We generalize this inequality to spherical and hyperbolic triangles as follows.

Theorem 2.8. *Given any point P in the interior of a triangle,*

$$\begin{aligned} R_1 + R_2 + R_3 &\geq 2(r_1 + r_2 + r_3) && \textit{in Euclidean geometry} \\ \tan R_1 + \tan R_2 + \tan R_3 &\geq 2(\tan r_1 + \tan r_2 + \tan r_3) && \textit{in spherical geometry} \\ \tanh R_1 + \tanh R_2 + \tanh R_3 &\geq 2(\tanh r_1 + \tanh r_2 + \tanh r_3) && \textit{in hyperbolic geometry.} \end{aligned}$$

Proof. Consider the projection of a spherical triangle T along with its R_i and r_i lengths (as in Fig. 2) onto the plane tangent to the sphere at point P. The image T' will be a Euclidean triangle. Note that R_i will have Euclidean length $\tan R_i$ in the projection (see Fig. 3). Likewise, r_i will have projected length $\tan r_i$. Since r_i is a ray from P that is perpendicular to side i on the sphere, the image r_i' will be perpendicular to the projected side i'. Then we apply the Erdős-Mordell Inequality to the Euclidean triangle T' to obtain the desired result.

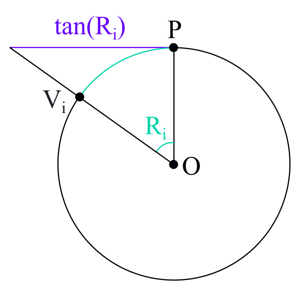

Figure 3: Intersection of the sphere and tangent plane with the plane containing O, P, and V_i.

For the hyperbolic case, consider a triangle T in the Klein model. We may assume without loss of generality that P is located at the origin. In the Klein model, a line segment of hyperbolic length x with one endpoint at the origin has Euclidean length $\tanh x$. Thus we have a Euclidean triangle T' with the same vertices as T and distances $\tanh R_i$ and $\tanh r_i$ from the vertices and edges, respectively, to point P. Applying the Erdős-Mordell Inequality to T' yields the desired result. $\square$

3 Euler's Inequality in $\mathbb{S}^n$

As stated previously, Corollary 2.3 applies to arbitrary dimension n. First we provide a more general statement.

Theorem 3.1.

(a) Let $\frac{R}{r} \geq f(\{s_e(d_{ij})\})$ be an inequality which holds for all Euclidean simplices with vertices $P_0, \ldots, P_n$ in $\mathbb{R}^n$ (where R is the circumradius, r is the inradius, and d_{ij} is the distance between P_i and P_j). Then $\frac{\tan R'}{\tan r'} \geq f(\{s_s(d'_{ij})\})$ holds for all spherical simplices in $\mathbb{S}^n$ with vertices $P'_0, \ldots, P'_n$.

(b) Let $\frac{R}{r} \leq f(\{s_e(d_{ij})\})$ be an inequality which holds for all Euclidean simplices with vertices $P_0, \ldots, P_n$ in $\mathbb{R}^n$ (where R is the circumradius, r is the inradius, and d_{ij} is the distance between P_i and P_j). Then $\frac{\tanh R'}{\tanh r'} \leq f(\{s_h(d'_{ij})\})$ holds for all hyperbolic simplices in $\mathbb{H}^n$ with vertices $P'_0, \ldots, P'_n$.

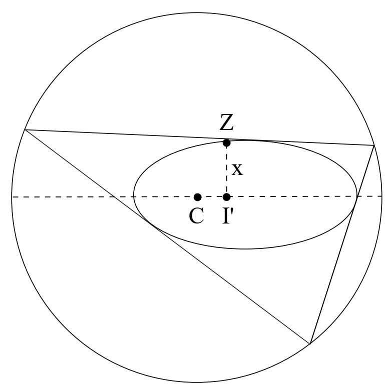

Figure 4: The projection of the simplex, circumsphere, and insphere onto the tangent plane $x_{n+1} = 1$.

Proof. Part (a). Assume without loss of generality that a spherical n-simplex (with circumradius R and inradius r) has its circumcenter at the north pole $C = (0, \ldots, 0, 1)$ of the unit sphere in $\mathbb{R}^{n+1}$. We may also assume the incenter lies on the (two-dimensional) $x_1 x_{n+1}$ plane. Project the simplex onto the n-dimensional hyperplane tangent to the unit sphere at C (described by $x_{n+1} = 1$); its image is an n-dimensional Euclidean simplex with circumradius $\tan R$ and some inradius r'. On the other hand, the image of its insphere under the same projection is an inscribed ellipsoid of the Euclidean simplex. By symmetry, this ellipsoid has the same radius b in all directions except along the x_1-axis in which it is longer.

The vertices on the sphere, in addition to determining a spherical simplex, also determine a Euclidean simplex where the distance between P_i and P_j is $2\sin\frac{d_{ij}}{2} = 2s_s(d_{ij})$. By simplex similarity, its circumradius-to-inradius ratio is also $\frac{\tan R}{r'}$. Thus (since the inequality is true for a Euclidean simplex) we have

$$\frac{\tan R}{r'} \geq f(\{s_e(2s_s(d_{ij}))\}) = f(\{s_s(d_{ij})\})$$

and so it is sufficient to show

$$\frac{\tan R}{\tan r} \geq \frac{\tan R}{r'}.$$

or

$$r' \geq \tan r.$$

Let O be the origin, I the image of the spherical incenter and Z the point where the ray starting at I and heading in the x_2-direction intersects the inscribed ellipse (see Fig. 4). Observe that OZI is a right triangle yielding $\tan r = \frac{IZ}{IO}$. It is clear that $IO \geq 1$ (as the x_{n+1}-coordinate of I is equal to 1) so we have $\tan r \leq IZ$. But $IZ \leq b$, and $b \leq r'$ as a sphere of radius b with the same center as the inscribed ellipse would be contained in the ellipse and also the simplex. Thus $\tan r \leq r'$.

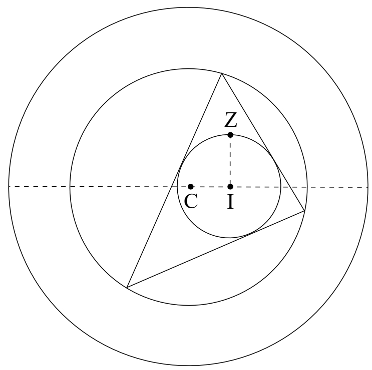

Figure 5: A triangle, its circumcircle, and its Euclidean incircle which corresponds to an inellipse of the hyperbolic triangle in the Klein model. Although this figure shows a triangle, the proof works for any n-dimensional simplex.

To prove part (b) we utilize the Klein model of the hyperbolic plane. Let a hyperbolic simplex have circumradius R and inradius r, and without loss of generality assume its circumcenter lies at the origin O. Its vertices determine a Euclidean simplex with circumradius $\tan R$ and some inradius r'. The Euclidean distance between P_i and P_j can be shown to equal $\frac{2}{\cosh R} s_h(d_{ij})$. By similarity this implies the existence of a Euclidean simplex with side lengths $\{2s_h(d_{ij})\}$ and the same circumradius-to-inradius ratio, so in analogue to part (a) it is sufficient to show $r' \leq \tanh r$ or $\tanh^{-1}(r') \leq r$.

Assume without loss of generality the incenter I of the Euclidean simplex lies on the x_1-axis. The Euclidean insphere is a hyperbolic inellipsoid with the same radius b in all directions except along the x_1-axis in which it is longer. Let Z be the point where the ray starting at I and heading in the x_2-direction intersects the hyperbolic inellipse/Euclidean insphere (see Fig. 5). The hyperbolic distance IZ cannot be larger than b, and $b \leq r$ as a circle of hyperbolic radius b with the same center as the ellipsoid would be contained in the simplex. So it is sufficient to show $\tanh^{-1} r' \leq IZ$. Let d be the Euclidean distance from O to I; the Klein model metric yields

$$IZ = \tanh^{-1}\left(\frac{r'}{\sqrt{1-d^2}}\right) \geq \tanh^{-1}(r').$$

□

The generalized Euler's inequality for spherical geometry follows from its known analogue in Euclidean geometry along with Theorem 3.1(a):

Theorem 3.2. *Let an n-dimensional simplex in spherical geometry (curvature $K = 1$) have circumradius R and inradius r. Then*

$$\tan R \geq n \tan r.$$

4 Pairwise Distances of $n+2$ points in $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$

Given points P_1, P_2, and P_3 on a Euclidean line, it is easy to see that the three pairwise distances d_{12}, d_{13}, and d_{23} satisfy

$$(d_{12} + d_{13} - d_{23})(d_{12} + d_{23} - d_{13})(d_{13} + d_{23} - d_{12}) = 0.$$

In general, pairwise distances between $n+2$ points in $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$ must satisfy some relation. Here we prove a general relation that works for any dimension n in all three geometries. In fact the relation is true for any curvature K, if we extend the unifying function s as follows:

$$s_K(x) := \begin{cases} \frac{1}{\sqrt{K}} \sin\left(\frac{x\sqrt{K}}{2}\right) & \text{if } K \neq 0, \\ \frac{x}{2} & \text{if } K = 0. \end{cases}$$

One can check that s_K is continuous and equivalent to (1) for $K \in \{0, 1, -1\}$, noting that $\sin(ix) = i \sinh x$.

Given $n+1$ points $P_0, \ldots, P_n$ in n-dimensional space of curvature K, let d_{ij} represent the distance between point i and point j. We use $\hat{M}_n(P_0, \ldots, P_n)$ to denote the $(n+1) \times (n+1)$ matrix whose entry in row i, column j is $s_K(d_{ij})^2$. Let $M_n(P_0, \ldots, P_n)$ be the $(n+2) \times (n+2)$ matrix obtained from $\hat{M}_n$ by adjoining a top row $(0, 1, \ldots, 1)$ and left column $(0, 1, \ldots, 1)^T$, and take D_n and $\hat{D}_n$ to be the determinants of M_n and $\hat{M}_n$, respectively. Observe that D_n is related to the Cayley-Menger determinant; the volume of the convex hull determined by $n+1$ points in n-dimensional Euclidean space is given by

$$V^2 = \frac{(-1)^{n+1} 2^n}{(n!)^2} \cdot D_n \quad \text{(in Euclidean space)} \tag{20}$$

[2]. On the other hand, $\hat{D}_n$ is the $(1,1)$-minor of M_n, and it is known in Euclidean geometry that $\hat{D}_n = 0$ if and only if the $n+1$ points lie on the surface of an $(n-2)$-dimensional sphere or hyperplane [2]. It is also given in [2] that $n+1$ points in $\mathbb{R}^n$ which do not lie in a proper affine subspace satisfy

$$R^2 = -2 \frac{\hat{D}_n}{D_n} \tag{21}$$

where R is the radius of the $(n-1)$-dimensional sphere containing all $n+1$ points (and thus circumscribing their determined simplex).

Now suppose instead that we have $n+2$ points in n-dimensional Euclidean space. These determine an $(n+1)$-dimensional simplex with volume zero, so by (20) the pairwise distances $\{d_{ij}\}$ satisfy the relation

$$D_{n+1} = 0. \tag{22}$$

The following is a generalization of (22) to spherical and hyperbolic geometry:

Theorem 4.1. *The pairwise distances between $n+2$ points $P_0, \ldots, P_{n+1}$ in n-dimensional space of curvature K satisfy the relation*

$$D_{n+1} + 2K\hat{D}_{n+1} = 0. \tag{23}$$

Proof. The Euclidean case $K = 0$ reduces to Equation (22). If $K > 0$, n-dimensional space is modeled by an n-sphere of radius $R = 1/\sqrt{K}$ embedded in $\mathbb{R}^{n+1}$. Then $n+2$ points on this sphere define a Euclidean $(n+1)$-simplex circumscribed by a sphere of radius R. Since the Euclidean distance between point i and point j is $2s_K(d_{ij})$, the Cayley-Menger matrices M_{n+1} and $\hat{M}_{n+1}$ associated with the Euclidean simplex have entries of the form $s_e(2s_K(d_{ij})) = \frac{2s_K(d_{ij})}{2} = s_K(d_{ij})$. That is, the matrices associated with the Euclidean simplex are equivalent to those associated with the spherical simplex (whose determinants are D_{n+1} and $\hat{D}_{n+1}$). By (21) we have

$$\frac{1}{K} = -2\frac{\hat{D}_{n+1}}{D_{n+1}}$$

which is equivalent to (23). The hyperbolic case is analogous (the proof of (21) provided in [4] also applies to a set of points in Minkowski space at uniform distance R from the origin, $R^2 < 0$). □

5 Generalization of Poncelet's Porism and Euler's Theorem

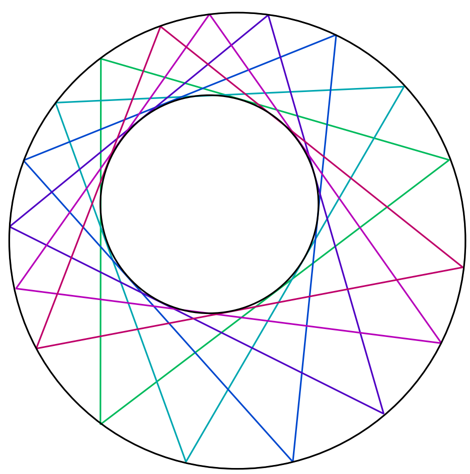

Figure 6: A colorful family of triangles sharing a circumcircle and incircle.

Poncelet's Porism states that if there exists a Euclidean n-sided polygon circumscribed by a conic and inscribed by another conic, then there is an infinite family of n-sided polygons circumscribed and inscribed by the same conics. Every triangle has a circumcircle and incircle, and Poncelet's Porism gives us infinitely many triangles with the same circumcircle and incircle (preserving the distance between their centers).

On the other hand, if we start with two circles, one contained by the other, how do we know whether there exist triangles that have the outer circle as a circumcircle and the inner circle as an incircle? The answer is given by Euler's Theorem. If d represents the distance between the circumcenter and incenter of a Euclidean triangle, then

$$d^2 = R(R - 2r). \tag{24}$$

If the outer circle has radius R and the inner circle has radius r, we can draw an appropriate family of triangles if and only if (24) is satisfied.

As both sides of (24) must be nonnegative, it follows that $R - 2r \geq 0$ or $R \geq 2r$, so Euler's Theorem implies Euler's inequality. We can generalize (24) to both spherical and hyperbolic geometry. First we provide a small generalization of Poncelet's Porism.

Lemma 5.1. *(Generalization of Poncelet's Porism) Let C and D be non-intersecting circles in 2-dimensional spherical or hyperbolic space. If there exists one n-sided polygon ($n > 2$) circumscribed by C and inscribed by D, then there is an infinite family of such polygons circumscribed by C and inscribed by D. Moreover, every point of C or D is a vertex or point of tangency respectively of a polygon in the family.*

Proof. First consider the spherical case. Say we have two non-intersecting conics C and D on a sphere with a polygon P that lays between them. Consider the projection of C, D, and P onto the plane tangent to the sphere at the center of C. Call the images C', D', and P' respectively.

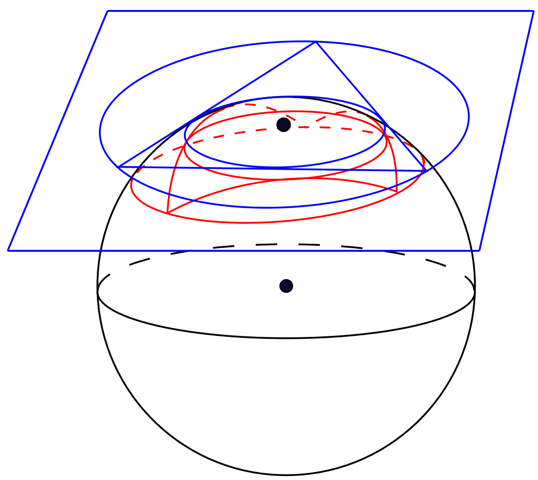

Figure 7: Projection of spherical triangle, circumcircle, and incircle (red) onto tangent plane (blue).

Since geodesics on the sphere map to lines on the projection, P' is a Euclidean polygon. Likewise, C' and D' are conics on the tangent plane which circumscribe and inscribe P'. Then we can apply Poncelet's Porism to C', D', and P', so we know that P' belongs to an infinite family of polygons $\bar{\mathcal{P}}'$. Since projection is a homeomorphism, we can apply the inverse operation to any polygon in $\bar{\mathcal{P}}'$ to obtain a corresponding spherical polygon that lies between C and D. In fact, for any point x of C or D, we have a polygon in $\bar{\mathcal{P}}'$ that touches the image of x.

For the hyperbolic case, we may consider C, D, and P drawn on the Klein model. Since conics and lines in the Klein model are represented as Euclidean conics and lines, the result follows directly. □

Next we present an general result which encapsulates (24).

Theorem 5.2. *Given a triangle in Euclidean, spherical, or hyperbolic geometry with circumradius R and inradius r, the distance d between its circumcenter and incenter can be expressed as*

$$d^2 = (R - r)^2 - r^2$$
$$\sin^2 d = \sin^2(R - r) - \sin^2 r \cos^2 R$$
$$\sinh^2 d = \sinh^2(R - r) - \sinh^2 r \cosh^2 R$$

in Euclidean, spherical, and hyperbolic geometry, respectively. The above equations can also be unified with the s-function (see (1)) as follows:

$$s(2d)^2 = s(2R-2r)^2 - s(2r)^2(1 - Ks(2R)^2). \tag{25}$$

Proof. The Euclidean case is equivalent to Euler's Theorem. For spherical and hyperbolic geometry, we prove the relation for an isosceles triangle. This also proves it for general triangles, as Lemma 5.1 implies that given any triangle, there exists an isosceles triangle with the same values of R, r and d.

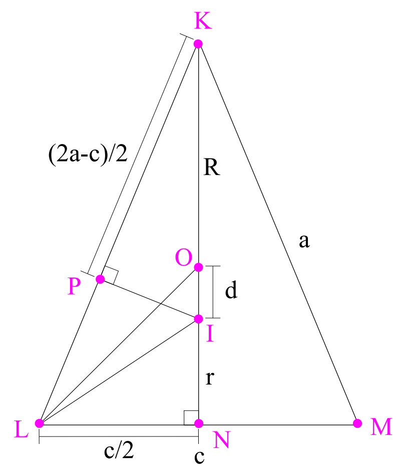

Figure 8: An isosceles triangle with circumcenter O, incenter I, and legs of length a, a, and c.

Consider a spherical isosceles triangle with lengths as indicated in (8). Note the following are right triangles: $\triangle LNK$, $\triangle LNO$, and $\triangle IPK$. For each one, we have a Pythagorean relation:

$$\cos\frac{c}{2}\cos(R+r+d) = \cos a \tag{26}$$

$$\cos\frac{c}{2}\cos(r+d) = \cos R \tag{27}$$

$$\cos r \cos\left(a - \frac{c}{2}\right) = \cos(R+d). \tag{28}$$

From (28) we get

$$\cos r\left(\cos a \cos\frac{c}{2} + \sin a \sin\frac{c}{2}\right) = \cos(R+d)$$

$$\sin a \sin\frac{c}{2} = \frac{\cos(R+d)}{\cos r} - \cos a \cos\frac{c}{2}$$

and squaring both sides yields

$$(1-\cos^2 a)\left(1 - \cos^2\frac{c}{2}\right) = \left(\frac{\cos(R+d)}{\cos r} - \cos a\cos\frac{c}{2}\right)^2 \tag{29}$$

$$1 - \cos^2 a - \cos^2\frac{c}{2} = \frac{\cos^2(R+d)}{\cos^2 r} - \frac{2\cos(R+d)\cos a\cos\frac{c}{2}}{\cos r}. \tag{30}$$

From (26) and (27) we can write $\cos a$ and $\cos \frac{c}{2}$ in terms of R, r, and d:

$$\cos \frac{c}{2} = \frac{\cos R}{\cos(r+d)}$$
$$\cos a = \frac{\cos R \cos(R+r+d)}{\cos(r+d)}.$$

Finally, substitution into (30) gives

$$1 - \frac{\cos^2 R \cos^2(R+r+d)}{\cos^2(r+d)} - \frac{\cos^2 R}{\cos^2(r+d)} = \frac{\cos^2(R+d)}{\cos^2 r} - \frac{2\cos^2 R \cos(R+d)\cos(R+r+d)}{\cos r \cos^2(r+d)},$$

an equation relating R, r, and d. One can show that it is equivalent to (25). The same can be done for a hyperbolic isosceles triangle. $\square$

Acknowledgement

This work was done during the 2017 REU program in mathematics at Oregon State University, with support by National Science Foundation Grant DMS-1359173.

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