STRENGTHENED EULER'S INEQUALITY IN SPHERICAL AND HYPERBOLIC GEOMETRIES

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ABSTRACT. Many formulas and inequalities which hold for polygons in Euclidean geometry have analogous non-Euclidean versions that hold in spherical and hyperbolic geometry. In this paper we prove a unified strengthening of the well-known Euler's inequality which holds for triangles in Euclidean, spherical, and hyperbolic geometry. We also prove a generalization of Svrtan and Veljan's strengthening of Euler's inequality [4] into spherical geometry, examine a symmetrized version of that inequality in Euclidean and spherical geometry, and show that neither strengthening can be extended into hyperbolic geometry.

1. INTRODUCTION

Euler's inequality states that, for a triangle in Euclidean geometry with circumradius R and inradius r,

$$R \geq 2r,$$

with equality only in the case of an equilateral triangle. Svrtan and Veljan showed analogous versions of this well-known inequality in non-Euclidean geometries of constant curvature, namely,

$$\tan R \geq 2 \tan r$$

in spherical geometry, and

$$\tanh R \geq 2 \tanh r$$

in hyperbolic geometry [4]. They also provided a strengthened version of Euler's inequality for Euclidean geometry in the form of the following theorem:

Theorem 1.1 ([3], [5]).

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \tag{1a}$$

$$\geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 1 \tag{1b}$$

$$\geq \frac{2}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \tag{1c}$$

$$\geq 2 \tag{1d}$$

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In this paper, we examine how the spherical and hyperbolic versions of Euler's inequality can be strengthened in the same sense as this Euclidean strengthening. We begin in Section 2 with unified results that hold in all three of these geometries. In Section 3 we extend Theorem 1.1 to a directly analogous theorem in spherical geometry, and show that the same cannot be done in hyperbolic geometry. Finally, in Section 4 we discuss an alternative strengthening of Euler's inequality in Euclidean geometry, and its analogues in spherical and hyperbolic geometry.

2. Unified Results

In this section, we prove results that unify the three geometries, culminating in Theorem 2.7, which provides an inequality that holds in each geometry. Throughout, we define a function s as used by Guo and Sönmez [1]

$$s(x) = \begin{cases} \frac{x}{2} & \text{in Euclidean geometry} \\ \sinh \frac{x}{2} & \text{in hyperbolic geometry} \\ \sin \frac{x}{2} & \text{in spherical geometry} \end{cases}$$

Our first result will be a lemma, modeled after the central result of [1], which will allow us to easily show spherical and hyperbolic analogues of inequalities in Euclidean geometry which take a specific form.

Lemma 2.1. *Let $f(x,y,z) \geq 0$ be an inequality which holds for all Euclidean triangles with side lengths x,y,z. Then $f(s(a),s(b),s(c)) \geq 0$, with s as defined above, for all spherical and hyperbolic triangles with side lengths a,b,c, .*

Proof. We begin with the hyperbolic case. Consider a hyperbolic triangle with sidelengths a,b,c and vertices A,B,C in the Poincaré disk model of the hyperbolic plane. Without loss of generality, we may assume that the triangle is positioned so that its circumcenter is located at the origin. When we consider the unit disk as a subset of the Euclidean plane, we may consider the Euclidean triangle T_1 with vertices A,B,C. This triangle will have side lengths $(1-R^2)\sinh\frac{a}{2}, (1-R^2)\sinh\frac{b}{2}$, and $(1-R^2)\sinh\frac{c}{2}$, where R is the Euclidean radius of the circumcircle of T_1. By similarity, this implies the existence of a Euclidean triangle T_2 with side lengths $\sinh\frac{a}{2}, \sinh\frac{b}{2}, \sinh\frac{c}{2}$. Since the inequality $f(x,y,z) \geq 0$ holds for all Euclidean triangles, we may apply it to T_2 to get $f(\sinh\frac{a}{2}, \sinh\frac{b}{2}, \sinh\frac{c}{2}) \geq 0$.

We now prove the spherical case. Consider a spherical triangle with side lengths a,b,c and vertices A,B,C. As in the hyperbolic case, we consider the Euclidean triangle T_1 with vertices A,B,C. The Euclidean distance between B and C will be $2\sin\frac{a}{2}$, since the leg of the Euclidean triangle will be a chord of the great circular arc of length a. The other two side lengths can be found in the same fashion. So T_1 is a Euclidean triangle with side lengths $2\sin\frac{a}{2}, 2\sin\frac{b}{2}$ and $2\sin\frac{c}{2}$. This implies that there is another Euclidean triangle T_2 similar to T_1 with side lengths $\sin\frac{a}{2}, \sin\frac{b}{2}, \sin\frac{c}{2}$. Since the inequality $f(x,y,z) \geq 0$ holds for any Euclidean triangle, we may apply it to T_2 to get $f(\sin\frac{a}{2}, \sin\frac{b}{2}, \sin\frac{c}{2}) \geq 0$, as required.

□

Next we prove a lemma which relates in a unified manner two quantities which appear frequently in inequalities relating circumradius and inradius. This lemma will be instrumental in the proof of Theorem 2.7.

Lemma 2.2. *For a triangle in Euclidean, spherical, or hyperbolic geometry, with side-lengths a,b,c, let*

$$\begin{aligned}\bar{B} &:= (s(a)+s(b)-s(c))\,(s(a)+s(c)-s(b))\,(s(b)+s(c)-s(a))\\ B &:= s(a+b-c)s(a+c-b)s(b+c-a)\end{aligned}$$

Then

$$B-\bar{B}\begin{cases}=0 & \text{in Euclidean geometry}\\ \geq 0 & \text{in hyperbolic geometry}\\ \leq 0 & \text{in spherical geometry}\end{cases}$$

Proof. In Euclidean geometry,

$$\begin{aligned}B-\bar{B} &= \left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{b+c-a}{2}\right)-\left(\frac{a}{2}+\frac{b}{2}-\frac{c}{2}\right)\left(\frac{a}{2}+\frac{c}{2}-\frac{b}{2}\right)\left(\frac{b}{2}+\frac{c}{2}-\frac{a}{2}\right)\\ &= \frac{1}{8}(a+b-c)(a+c-b)(b+c-a)-\frac{1}{8}(a+b-c)(a+c-b)(b+c-a)\\ &= 0.\end{aligned}$$

To show that $B \geq \bar{B}$ in hyperbolic geometry, we assume, without loss of generality, that $a \geq b \geq c$. Then it is sufficient to verify the following two propositions:

Proposition 2.3. $\sinh\frac{b+c-a}{2} \geq \sinh\frac{b}{2}+\sinh\frac{c}{2}-\sinh\frac{a}{2}$

Proof. Since $2a-b-c \geq b-c$, and cosh is even function which increases on positive real numbers,

$$\cosh\frac{b+c-2a}{4} = \cosh\frac{2a-b-c}{4} \geq \cosh\frac{b-c}{4},$$

and therefore

$$\sinh\frac{b+c-a}{2}+\sinh\frac{a}{2} = 2\sinh\frac{b+c}{4}\cosh\frac{b+c-2a}{4} \geq 2\sinh\frac{b+c}{4}\cosh\frac{b-c}{4} = \sinh\frac{b}{2}+\sinh\frac{c}{2}$$

□

Proposition 2.4. $\sinh\frac{a+b-c}{2}\sinh\frac{a+c-b}{2} \geq \left(\sinh\frac{a}{2}+\sinh\frac{b}{2}-\sinh\frac{c}{2}\right)\left(\sinh\frac{a}{2}+\sinh\frac{c}{2}-\sinh\frac{b}{2}\right)$

Proof.

$$\sinh\frac{a+b-c}{2}\sinh\frac{a+c-b}{2} \geq \left(\sinh\frac{a}{2}+\sinh\frac{b}{2}-\sinh\frac{c}{2}\right)\left(\sinh\frac{a}{2}+\sinh\frac{c}{2}-\sinh\frac{b}{2}\right)$$

if and only if

$$
\begin{aligned}
0 \le & \sinh\frac{a+b-c}{2}\sinh\frac{a+c-b}{2} - (\sinh\frac{a}{2}+\sinh\frac{b}{2}-\sinh\frac{c}{2})(\sinh\frac{a}{2}+\sinh\frac{c}{2}-\sinh\frac{b}{2}) \\
=&\frac{1}{4}(e^{\frac{a+b-c}{2}}-e^{\frac{c-a-b}{2}})(e^{\frac{a+c-b}{2}}-e^{\frac{b-a-c}{2}}) - \frac{1}{4}(e^{\frac{a}{2}}-e^{\frac{-a}{2}}+e^{\frac{b}{2}}-e^{\frac{-b}{2}}-e^{\frac{c}{2}}+e^{\frac{-c}{2}}) \\
&(e^{\frac{a}{2}}-e^{\frac{-a}{2}}+e^{\frac{c}{2}}-e^{\frac{-c}{2}}-e^{\frac{b}{2}}+e^{\frac{-b}{2}}) \\
=&\frac{1}{4}(e^{a}+e^{-a}-e^{b-c}-e^{c-b}) - \frac{1}{4}(2+e^{a}+e^{-a}-e^{b}-e^{-b}-e^{c}-e^{-c}+2(e^{\frac{b+c}{2}}+e^{\frac{-b-c}{2}})-2(e^{\frac{b-c}{2}}+e^{\frac{c-b}{2}})) \\
=&\frac{1}{4}(e^{b}+e^{-b}+e^{c}+e^{-c}+2e^{\frac{b-c}{2}}+2e^{\frac{c-b}{2}}-2e^{\frac{b+c}{2}}-2e^{\frac{-b-c}{2}}-e^{b-c}-e^{c-b}-2) \\
=&\frac{1}{4}(e^{\frac{b}{2}}-e^{\frac{-b}{2}})(e^{\frac{b}{2}}+e^{\frac{2c-b}{2}}-2e^{\frac{c}{2}}-e^{\frac{b-2c}{2}}-e^{\frac{-b}{2}}+2e^{\frac{-c}{2}}) \\
=&\frac{1}{2}\sinh\frac{b}{2}(e^{\frac{c}{2}}-e^{\frac{-c}{2}})(e^{\frac{b-c}{2}}+e^{\frac{c-b}{2}}-2) \\
=&2\sinh\frac{b}{2}\sinh\frac{c}{2}\left(\cosh\left(\frac{b-c}{2}\right)-1\right) \\
=&4\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh^{2}\left(\frac{b-c}{4}\right)
\end{aligned}
$$

□

Thus, in hyperbolic geometry,

$$
\begin{aligned}
B &= \sinh\frac{a+b-c}{2}\sinh\frac{a+c-b}{2}\sinh\frac{b+c-a}{2} \\
&\ge (\sinh\frac{a}{2}+\sinh\frac{b}{2}-\sinh\frac{c}{2})(\sinh\frac{a}{2}+\sinh\frac{c}{2}-\sinh\frac{b}{2})(\sinh\frac{b}{2}+\sinh\frac{c}{2}-\sinh\frac{a}{2}) \\
&= \bar{B}
\end{aligned}
$$

To show that $B \le \bar{B}$ in spherical geometry, we assume, without loss of generality, that $a \ge b \ge c$. Then it is sufficient to verify the following two propositions:

Proposition 2.5. $\sin\frac{b+c-a}{2} \le \sin\frac{b}{2}+\sin\frac{c}{2}-\sin\frac{a}{2}$

Proof. Since $2a-b-c \ge b-c$, and cos is a decreasing function on the interval $[0,\pi]$,

$$\cos\frac{b+c-2a}{4} = \cos\frac{2a-b-c}{4} \le \cos\frac{b-c}{4},$$

and therefore

$$\sin\frac{b+c-a}{2}+\sin\frac{a}{2} = 2\sin\frac{b+c}{4}\cos\frac{b+c-2a}{4} \le 2\sin\frac{b+c}{4}\cos\frac{b-c}{4} = \sin\frac{b}{2}+\sin\frac{c}{2}.$$

□

Proposition 2.6. $\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2} \le \left(\sin\frac{a}{2}+\sin\frac{b}{2}-\sin\frac{c}{2}\right)\left(\sin\frac{a}{2}+\sin\frac{c}{2}-\sin\frac{b}{2}\right)$

Proof. This follows directly from the proof of (2.4), since

$$\sinh ix = i\sin x,$$

so

$$\left(\sin\frac{a}{2}+\sin\frac{b}{2}-\sin\frac{c}{2}\right)\left(\sin\frac{a}{2}+\sin\frac{c}{2}-\sin\frac{b}{2}\right)-\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}$$
$$=i\sin\frac{a+b-c}{2}i\sin\frac{a+c-b}{2}-\left(i\sin\frac{a}{2}+i\sin\frac{b}{2}-i\sin\frac{c}{2}\right)\left(i\sin\frac{a}{2}+i\sin\frac{c}{2}-i\sin\frac{b}{2}\right)$$
$$=\sinh\frac{i(a+b-c)}{2}\sinh\frac{i(a+c-b)}{2}-\left(\sinh\frac{ia}{2}+\sinh\frac{ib}{2}-\sinh\frac{ic}{2}\right)\left(\sinh\frac{ia}{2}+\sinh\frac{ic}{2}-\sinh\frac{ib}{2}\right)$$
$$=4\sinh\frac{ib}{2}\sinh\frac{ic}{2}\sinh^2\frac{i(b-c)}{4}$$
$$=4\sin\frac{b}{2}\sin\frac{c}{2}\sin^2\frac{b-c}{4}\geq 0$$

□

Thus, in spherical geometry,

$$\begin{aligned}B&=\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}\\&\leq(\sin\frac{a}{2}+\sin\frac{b}{2}-\sin\frac{c}{2})(\sin\frac{a}{2}+\sin\frac{c}{2}-\sin\frac{b}{2})(\sin\frac{b}{2}+\sin\frac{c}{2}-\sin\frac{a}{2})\\&=\bar{B}\end{aligned}$$

□

The equations for inradius and circumradius in Euclidean, hyperbolic, and spherical geometry can be unified as follows: for a triangle with side-lengths a,b,c, circumradius R, and inradius r,

$$\frac{2s(a)s(b)s(c)}{\sqrt{s(a+b-c)s(a+c-b)s(b+c-a)s(a+b+c)}}=\begin{cases}R & \text{in Euclidean geometry}\\ \tan R & \text{in spherical geometry}\\ \tanh R & \text{in hyperbolic geometry}\end{cases}$$

and

$$\sqrt{\frac{s(a+b-c)s(a+c-b)s(b+c-a)}{s(a+b+c)}}=\begin{cases}r & \text{in Euclidean geometry}\\ \tan r & \text{in spherical geometry}\\ \tanh r & \text{in hyperbolic geometry}\end{cases}$$

The following is a theorem which provides a unified inequality dealing with the inradius and circumradius of triangles in all three of these geometries.

Theorem 2.7. *A triangle with side-lengths a,b,c has the following property in Euclidean geometry:*

$$\frac{R}{r}\geq\frac{2s(\frac{a+b}{2})s(\frac{a+c}{2})s(\frac{b+c}{2})}{s(a)s(b)s(c)}\geq 2, \tag{2}$$

while in hyperbolic geometry,

$$\frac{\tanh R}{\tanh r} \geq \frac{2s(\frac{a+b}{2})s(\frac{a+c}{2})s(\frac{b+c}{2})}{s(a)s(b)s(c)} \geq 2, \tag{3}$$

and in spherical geometry,

$$\frac{\tan R}{\tan r} \geq \frac{2s(\frac{a+b}{2})s(\frac{a+c}{2})s(\frac{b+c}{2})}{s(a)s(b)s(c)} \geq 2, \tag{4}$$

with equality if and only if $a = b = c$.

Proof. We begin with the right-most inequality of the Euclidean case.

$$\frac{2s(\frac{a+b}{2})s(\frac{a+c}{2})s(\frac{b+c}{2})}{s(a)s(b)s(c)} = \frac{2(\frac{a+b}{4})(\frac{a+c}{4})(\frac{b+c}{4})}{\frac{abc}{8}} \geq 2$$

is equivalent to

$$\begin{aligned} & \frac{1}{8}(a+b)(b+c)(a+c) \geq abc \\ \iff \quad & ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c + 2abc \geq 8abc \\ \iff \quad & ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c \geq 6abc. \end{aligned}$$

This in turn follows from the inequality of arithmetic and geometric means, since

$$\frac{ab^2 + bc^2 + ca^2}{3} \geq \sqrt[3]{a^3b^3c^3} \iff ab^2 + bc^2 + ca^2 \geq 3abc$$

and, similarly,

$$a^2b + b^2c + c^2a \geq 3abc.$$

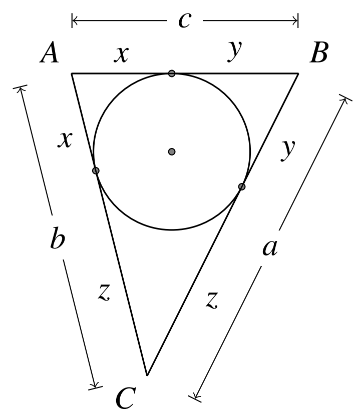
The left inequality in expression (2) is equivalent to

$$\frac{2abc}{(a+b-c)(a+c-b)(b+c-a)} \geq \frac{(a+b)(a+c)(b+c)}{4abc},$$

or

$$8a^2b^2c^2 \geq (a+b-c)(a+c-b)(b+c-a)(a+b)(a+c)(b+c).$$

Now, note that for any circle and any point external to that circle there are two lines tangent to the circle which pass through the point, with the point equidistant from the two points of tangency corresponding to those lines, so we have, for some $x, y, z > 0$, that $a = y + z$, $b = x + z$, and $c = x + y$, as depicted in the following diagram:



Then the left inequality of (2) is equivalent to

$$8(x+y)^2(x+z)^2(y+z)^2 - 8xyz(2x+y+z)(x+2y+z)(x+y+2z) \geq 0,$$

or

$$(x+y+z)(x^3y^2+x^2y^3-2x^2y^2z+x^3z^2-2x^2yz^2-2xy^2z^2+y^3z^2+x^2z^3+y^2z^3) \geq 0$$

Which is true, since

$$\begin{aligned} & x^3y^2+x^2y^3-2x^2y^2z+x^3z^2-2x^2yz^2-2xy^2z^2+y^3z^2+x^2z^3+y^2z^3 \\ &= x^2(y+z)(y-z)^2+y^2(x+z)(x-z)^2+z^2(x+y)(x-y)^2 \\ &\geq 0 \end{aligned}$$

for all non-negative x, y, z.

In spherical geometry, the left inequality of (4) is equivalent to

$$\frac{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{B} \geq \frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}},$$

or

$$(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2 \geq B\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}.$$

As discussed in [1], we have

$$\sin^2 R = \frac{4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2}{\bar{B}(\sin\frac{a}{2}+\sin\frac{b}{2}+\sin\frac{c}{2})}$$

$$\tan^2 R = \frac{4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2}{B\sin\frac{a+b+c}{2}}$$

so

$$\begin{aligned} B\sin\frac{a+b+c}{2}+4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2 &= \frac{4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2}{\tan^2 R}+4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2 \\ &= \frac{4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2}{\sin^2 R} \\ &= \bar{B}(\sin\frac{a}{2}+\sin\frac{b}{2}+\sin\frac{c}{2}), \end{aligned}$$

since

$$\frac{1}{\sin^2 R}=\frac{1}{\tan^2 R}+1.$$

Now, since $B \leq \bar{B}$,

$$B\left(\sin\frac{a}{2}+\sin\frac{b}{2}+\sin\frac{c}{2}-\sin\frac{a+b+c}{2}\right) \leq 4(\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2})^2,$$

and since

$$\sin\frac{a}{2}+\sin\frac{b}{2}+\sin\frac{c}{2}-\sin\frac{a+b+c}{2}=4\left(\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}\right)$$

this is precisely equivalent to the left inequality of (4).

The right inequality of (4) is equivalent to

$$\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4} \geq \sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2},$$

which is shown on page 636 of [2].

We now consider the hyperbolic case. As in the spherical case, the left-most inequality in (3) is equivalent to

$$\frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{B} \geq \frac{2\sinh\frac{a+b}{4}\sinh\frac{a+c}{4}\sinh\frac{b+c}{4}}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}$$

or

$$\iff (\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2})^2 \geq B\sinh\frac{a+b}{4}\sinh\frac{a+c}{4}\sinh\frac{b+c}{4}$$

We also have the following formulas, similar to those used in the spherical case:

$$\sinh^2 R = \frac{4(\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2})^2}{\bar{B}(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2})}$$

$$\tanh^2 R = \frac{4(\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2})^2}{B\sinh\frac{a+b+c}{2}}$$

as discussed in [1]. Using the identity

$$\frac{1}{\sinh^2 x}=\frac{1}{\tanh^2 x}-1,$$

we get

$$\bar{B}\left(\sinh\frac{a}{2}+\sinh\frac{b}{2}+\sinh\frac{c}{2}\right)=B\sinh\frac{a+b+c}{2}-4\left(\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\right)^2.$$

Using the fact that $B \geq \bar{B}$ and

$$\sinh\frac{a+b+c}{2}-\sinh\frac{a}{2}-\sinh\frac{b}{2}-\sinh\frac{c}{2}=4\sinh\frac{a+b}{4}\sinh\frac{a+c}{4}\sinh\frac{b+c}{4},$$

we are left with

$$B\sinh\frac{a+b}{4}\sinh\frac{a+c}{4}\sinh\frac{b+c}{4}\leq\left(\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\right)^2$$

which, as was shown above, is equivalent to the leftmost inequality of expression (3). □

3. Original Strengthening

3.1. **Euclidean.** We will here prove Theorem 1.1 for the reader's convenience.

Proof. The proof of (1.1) will be divided into proofs of its four parts:

(1d) $\frac{2}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})\geq 2$:

(1d) $\iff \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$. Now, by the inequality of arithmetic and geometric means,

$$\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\sqrt[3]{\frac{a}{b}\frac{b}{c}\frac{c}{a}}=1,$$

so

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

(1c) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\geq\frac{2}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})$:

This follows directly from the inequality proven above: $\frac{2}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})\geq 2 \iff \frac{1}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})\geq 1 \iff \frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{1}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})=\frac{2}{3}(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})$.

(1b) $\frac{abc+a^3+b^3+c^3}{2abc}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1$:

$$\frac{abc+a^3+b^3+c^3}{2abc}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1=\frac{a^2c+b^2a+c^2b-abc}{abc},$$

or

$$abc+a^3+b^3+c^3\geq 2(a^2c+b^2a+c^2b-abc)$$

if and only if

$$\begin{aligned}0&\leq a^3+b^3+c^3+3abc-2a^2c-2b^2a-2c^2b\\&=(a^3+ac^2-2a^2c)+(b^3+a^2b-2ab^2)+(c^3+b^2c-2bc^2)-ac^2-a^2b-b^2c+3abc\\&=a(a-c)^2+b(b-a)^2+c(c-b)^2+ac(b-c)+ab(c-a)+bc(a-b).\end{aligned}$$

Without loss of generality, suppose $a \geq b \geq c$. Now

$$\begin{aligned}
&a(a-c)^2 + b(b-a)^2 + c(c-b)^2 + ac(b-c) + ab(c-a) + bc(a-b) \\
&\geq a(a-c)^2 + ac(b-c) + ab(c-a) + bc(a-b) \\
&\geq a(a-c)^2 + bc(b-c) + ab(c-a) + bc(a-b) \\
&= (a-c)(a^2 - ac + bc - ab) \\
&= (a-c)^2(a-b) \\
&\geq 0
\end{aligned}$$

(1a) $\frac{R}{r} \geq \frac{abc+a^3+b^3+c^3}{2abc}$:
Recall that we have $a = y+z$, $b = x+z$, and $c = x+y$ for some $x,y,z > 0$. Now,

$$\frac{R}{r} = \frac{abc}{4(\frac{a+b-c}{2})(\frac{a+c-b}{2})(\frac{b+c-a}{2})} = \frac{2abc}{(a+b-c)(a+c-b)(b+c-a)},$$

so

$$\frac{R}{r} \geq \frac{abc+a^3+b^3+c^3}{2abc}$$

if and only if

$$(y+z)^2(x+z)^2(x+y)^2 \geq 2xyz((y+z)(x+z)(x+y) + (y+z)^3 + (x+z)^3 + (x+y)^3),$$

which is equivalent to

$$2(x^2y^2(x-z)(y-z) + y^2z^2(y-x)(z-x) + x^2z^2(x-y)(z-y)) + x^4(y-z)^2 + y^4(x-z)^2 + z^4(y-z)^2 \geq 0.$$

So it will suffice to show

$$x^2y^2(x-z)(y-z) + y^2z^2(y-x)(z-x) + x^2z^2(x-y)(z-y) \geq 0$$

Suppose, without loss of generality, that $x \geq y \geq z$. Then $y^2 \geq z^2$ and $(x-z) \geq (x-y)$, so

$$\begin{aligned}
0 &\leq y^2z^2(y-x)(z-x) \\
&= y^2z^2(y-x)(z-x) + z^2x^2(y-z)(x-y) + z^2x^2(z-y)(x-y) \\
&\leq x^2y^2(x-z)(y-z) + y^2z^2(y-x)(z-x) + x^2z^2(x-y)(z-y)
\end{aligned}$$

□

3.2. **Spherical.** Theorem 1.1 has the following direct analogue in spherical geometry:

Theorem 3.1. *Let a,b, and c be the side-lengths of a spherical triangle with circumradius R and inradius r. Then:*

$$\frac{\tan R}{\tan r} \geq \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}+\sin^3\frac{a}{2}+\sin^3\frac{b}{2}+\sin^3\frac{a}{2}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \tag{5a}$$

$$\geq \frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}-1 \tag{5b}$$

$$\geq \frac{2}{3}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}\right) \tag{5c}$$

$$\geq 2 \tag{5d}$$

Proof. (5b), (5c), and (5d) follow from (1b), (1c), and (1d), respectively, as an application of Lemma 2.1, so it remains only to show (5a).

Now, note that

$$\tan R = \frac{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\sqrt{\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}}}$$

$$\tan r = \sqrt{\frac{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}}{\sin\frac{a+b+c}{2}}}$$

so

$$\frac{\tan R}{\tan r} = \frac{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}} = \frac{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{B},$$

and since $B \leq \bar{B}$, we have

$$\frac{\tan R}{\tan r} \geq \frac{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\bar{B}} \geq \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}+\sin^3\frac{a}{2}+\sin^3\frac{b}{2}+\sin^3\frac{a}{2}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}},$$

by Lemma 2.1. □

3.3. **Hyperbolic.** Although we have

$$\begin{aligned}\frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh^3\frac{a}{2}+\sinh^3\frac{b}{2}+\sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}} &\geq \frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}-1\\ &\geq \frac{2}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}\right)\\ &\geq 2\end{aligned}$$

for triangles in hyperbolic geometry with side-lengths a,b,c, circumradius R, and inradius r as an application of Lemma 2.1, there is no direct generalization of Theorem 1.1 into hyperbolic geometry. Take, for example, a triangle in hyperbolic geometry with side-lengths $a = b = 2$ and

$c = 0.4$. Then

$$\begin{aligned}
0 &\geq -0.00923 \\
&\approx \frac{\tanh R}{\tanh r} - \frac{2}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}\right) \\
&\geq \frac{\tanh R}{\tanh r} - \left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}} - 1\right) \\
&\geq \frac{\tanh R}{\tanh r} - \frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} + \sinh^3\frac{a}{2} + \sinh^3\frac{b}{2} + \sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}.
\end{aligned}$$

In fact, for triangles in hyperbolic geometry, $\frac{\tanh R}{\tanh r}$ is not comparable with the hyperbolic version of any of the terms in Theorem 1.1, apart from that given by Euler's inequality itself. To see this, consider a triangle with edge-lengths $a = b = 2$ and $c = 0.5$. Then we have

$$\begin{aligned}
0 &\leq 0.037107 \\
&\approx \frac{\tanh R}{\tanh r} - \frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} + \sinh^3\frac{a}{2} + \sinh^3\frac{b}{2} + \sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}} \\
&\leq \frac{\tanh R}{\tanh r} - \left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}} - 1\right) \\
&\leq \frac{\tanh R}{\tanh r} - \frac{2}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}\right)
\end{aligned}$$

4. Symmetric Strengthening

4.1. **Euclidean.** We here suggest a new strengthening of Euler's inequality in Euclidean geometry, which includes the term from our unified strengthening.

Theorem 4.1. *A Euclidean triangle with edge-lengths a, b, c, circumradius R, and inradius r has the following property:*

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc} \tag{6a}$$

$$\geq \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}\right) - 1 \tag{6b}$$

$$\geq \frac{1}{3}\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}\right) \tag{6c}$$

$$\geq \frac{(a+b)(a+c)(b+c)}{4abc} \tag{6d}$$

$$\geq 2 \tag{6e}$$

Proof. (6a) is equivalent to (1a). (6b) follows from (1b), since

$$\frac{abc+a^3+b^3+c^3}{2abc} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1$$
$$\frac{abc+a^3+b^3+c^3}{2abc} \geq \frac{b}{a}+\frac{a}{c}+\frac{c}{b}-1$$

so

$$\frac{abc+a^3+b^3+c^3}{2abc} \geq \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}-2\right).$$

Similarly, (6c) follows from (1c).

(6d) and (6e) are both equivalent to

$$a^2b+b^2a+a^2c+c^2a+b^2c+c^2b \geq 6abc$$

which is also equivalent to the right-most inequality of (2) as proven in Section 2. □

4.2. **Spherical.** We have the following theorem for triangles in spherical geometry:

Theorem 4.2. *A triangle in spherical geometry with edge-lengths a,b,c, circumradius R, and inradius r has the following property:*

$$\text{(7a)} \qquad \frac{\tan R}{\tan r} \geq \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}+\sin^3\frac{a}{2}+\sin^3\frac{b}{2}+\sin^3\frac{c}{2}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}$$

$$\text{(7b)} \qquad \geq \frac{1}{2}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right)-1$$

$$\text{(7c)} \qquad \geq \frac{1}{3}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right)$$

$$\text{(7d)} \qquad \geq 2$$

Proof. (7a) is equivalent to (5a). (7b) and (7c) follow from (6b) and (6c), respectively, as an application of Lemma 2.1, while (7d) likewise follows from the combination of (6d) and (6e). □

We cannot, however, include an inequality analogous to (6d) in spherical geometry. Take, for example, a triangle in spherical geometry with edge-lengths $a=b=3$ and $c=1.5$. Then,

$$\begin{aligned}
0 &\leq 0.19775 \\
&\approx \frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} - \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}+\sin^3\frac{a}{2}+\sin^3\frac{b}{2}+\sin^3\frac{c}{2}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} \\
&\leq \frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} - \left(\frac{1}{2}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right)-1\right) \\
&\leq \frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} - \frac{1}{3}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right).
\end{aligned}$$

In fact, $\frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}$ is not comparable with any of the quantities expressed in Theorem 4.2, except for those given in Theorem 2.7. To see this, consider a triangle with side-lengths $a=b=0.75$, $c=1$, which has

$$\begin{aligned}
0 &\leq 0.00418\\
&\approx \frac{1}{3}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right)-\frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.\\
&\leq \frac{1}{2}\left(\frac{\sin\frac{a}{2}}{\sin\frac{b}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{a}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{a}{2}}+\frac{\sin\frac{b}{2}}{\sin\frac{c}{2}}+\frac{\sin\frac{c}{2}}{\sin\frac{b}{2}}\right)-1-\frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}\\
&\leq \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}+\sin^3\frac{a}{2}+\sin^3\frac{b}{2}+\sin^3\frac{c}{2}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}-\frac{2\sin\frac{a+b}{4}\sin\frac{a+c}{4}\sin\frac{b+c}{4}}{\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.
\end{aligned}$$

4.3. **Hyperbolic.** Although we have

$$\begin{aligned}
&\frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh^3\frac{a}{2}+\sinh^3\frac{b}{2}+\sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}\\
&\geq \frac{1}{2}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}}\right)-1\\
&\geq \frac{1}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}}\right)\\
&\geq 2
\end{aligned}$$

for triangles of edge-lengths a,b,c in hyperbolic geometry as an application of Lemma 2.1 to Theorem 4.1, there is no theorem in hyperbolic geometry which is directly analogous to Theorem 4.1. To show this, consider a triangle with side-lengths $a=b=2.5$ and $c=2$, then we have

$$\begin{aligned}
0 &\geq -0.0457201\\
&\approx \frac{\tanh R}{\tanh r}-\frac{1}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}}\right)\\
&\geq \frac{\tanh R}{\tanh r}-\left(\frac{1}{2}(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}}+\frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}})-1\right)\\
&\geq \frac{\tanh R}{\tanh r}-\frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh^3\frac{a}{2}+\sinh^3\frac{b}{2}+\sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}.
\end{aligned}$$

As we saw with the failure of Theorem 1.1 to generalize into hyperbolic geometry, $\frac{\tanh R}{\tanh r}$ is not in fact comparable with any of these quantities, since, for example, a triangle with edge-lengths

$a = b = 1$, $c = 1.5$ has

$$\begin{aligned}
0 &\le 0.23557 \\
&\approx \frac{\tanh R}{\tanh r} - \frac{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} + \sinh^3\frac{a}{2} + \sinh^3\frac{b}{2} + \sinh^3\frac{c}{2}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}} \\
&\le \frac{\tanh R}{\tanh r} - \left(\frac{1}{2}(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}}) - 1\right) \\
&\le \frac{\tanh R}{\tanh r} - \frac{1}{3}\left(\frac{\sinh\frac{a}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{a}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{b}{2}}{\sinh\frac{c}{2}} + \frac{\sinh\frac{c}{2}}{\sinh\frac{b}{2}}\right).
\end{aligned}$$

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