Investigations into Stable Marriages in Cases of Indifference

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Introduction

The basic stable marriage problem of matching men and women can be extended to include cases of indifference--where one or more individuals' preference lists contain elements of equal rank. This requires adaptations to the standard Gale-Shapley algorithm for finding stable matchings. These changes and their results will be investigated in cases of symmetric and identical ties, and observations will be made about the general case.

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In the standard stable marriage problem, equal numbers of men and women are matched together in a one-to-one correspondence, forming man-woman, (m,w), pairs, who are thus considered *engaged*. Each man submits a *preference list* containing the names of all the women, ranked in order of preference, with first place being the most preferred. The women submit preference lists of men, as well. From these lists, engagements can be made, with the goal of a *stable matching* in mind.

A *stable matching* is a matching for which there can be found no *blocking pair* among the men and women. A *blocking pair*, (m_o,w_o), consists of a man m_o who prefers woman w_o to his current partner, while woman w_o prefers m_o to her current partner. Since m_o and w_o would both prefer to be matched to each other, they break their previous engagements. Therefore, (m_o,w_o) creates instability in the matching. As long as there exists no such blocking pair, a matching is stable.

In 1962 Gale and Shapley introduced an algorithm which would produce a stable matching in any *instance*. (An *instance* is a set of n men and n women together with their complete preference lists.) Additional steps were added to eliminate blocking pairs by appropriately reducing the preference lists. The resulting algorithm proceeds in the following way:

* The first man proposes to the first woman on his list. She accepts and deletes from her list all men who are ranked lower than her new partner.

* The men deleted must then delete the now engaged woman from their lists, as she is no longer a possible choice for them.

* The next man proposes to the first woman on his list. If she is not currently engaged then she immediately accepts. If she is currently engaged, she chooses between her partner and the proposer, selecting the one she prefers and declining the other. If the proposer is declined, he moves on to the next woman on his list and continues in the same manner.

* Appropriate deletions are again made from the newly engaged woman's list, and she from the lists of those she deleted.

* This process continues until all men and women are engaged, and the pairs necessarily form a stable matching.

* In order to further reduce the preference lists and discover any additional stable matchings that can be made from the instance, the process can be repeated by beginning with the already shortened preference lists and allowing the women to be the proposers.

An example of the Gale-Shapley algorithm is as follow

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	1 2 3	1	2 3 1
2	2 1 3	2	1 2 3
3	2 3 1	3	3 2 1

* Man 1 proposes to woman 1 and the pair (1,1) is formed.
* Woman 1 has no deletions to make from her list.
* Man 2 proposes to woman 2 and the pair (2,2) is formed.
* Woman 2 deletes man 3 from her list, and so man 3 deletes woman 2 from his list.

The resulting lists are:

1	1 2 3	1	2 3 1
2	2 1 3	2	1 2
3	3 1	3	3 2 1

* Man 3 proposes to woman 3 and the pair (3,3) is formed.
* Woman 3 deletes men 2 and 1 from her list, and she is deleted from theirs.

The resulting lists are:

1	1 2	1	2 3 1
2	2 1	2	1 2
3	3 1	3	3

The matching (1,1), (2,2), (3,3) is stable.

To continue reducing the lists, the roles reverse and the women propose, and the result is the stable matching (1,2), (2,1), (3,3) , with the following reduced lists:

1	1	2	1	2	1
2	2	1	2	1	2
3	3		3	3	

It is evident by looking at the reduced preference lists that the only possible stable matchings in this instance are the two already mentioned, because the reduced lists contain all possible stable matchings.

To simplify notation, the matchings will henceforth be represented by the women's numbers only, in order of their engagements to man 1, 2, ...,n. For example, 2341 represents the matching of (1,2), (2,3), (3,4), (4,1).

All of the previous information on stable matchings and the Gale-Shapley algorithm has served as a foundation for the primary focus of this paper--cases of *indifference.*

Simply stated, a man is *indifferent* if in his preference list he has ranked two or more women equally. This, of course, generalizes to women ranking men as well. The equally ranked individuals are then considered to be *tied*. Thus, the term *tie instance* refers to a set of n men and n women together with their preference lists, at least one of which contains a tie. Henceforth a *standard instance* will refer to an instance in which there are no ties.

It is now necessary to cover a number of definitions concerning different types of ties and different types of stability that should be considered, since the stable matching (now to be referred to as *standard-stable*) does not readily apply to cases of indifference.

The term *random tie* refers to an instance in which there can be any number of ties of any number of individuals (up to n in an instance of size n). For example, the instance

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	(1 2) 3 4	1	4 3 1 2
2	2 1 4 3	2	(1 2 4) 3
3	3 (4 2 1)	3	1 2 3 4
4	1 4 3 2	4	(3 2 1 4)

is considered a random tie instance, as there is no pattern to the placement of the ties in the lists. Note that the parentheses represent ties--they surround elements which are equally ranked.

The number of possible random ties for an arbitrary instance of size ($n > 2$) is quite large. For $n=3$, for example, there are over 3500 random tie combinations possible. It is not practical to work with such a group of variations, and thus the observations to follow will focus on *symmetric tie* and *identical tie* instances. In both symmetric and identical tie instances, each individual's preference list contains the same number of equally ranked elements. For a symmetric instance of size n, if each man has equally ranked elements $i, i+1, \ldots, k$ on his list, then each woman has equally ranked elements $n-i, n-(i+1), \ldots, n-k$ on her list. An example of a symmetric tie instance is as follows:

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	(1 2 3) 4	1	4 (3 1 2)
2	(2 1 4) 3	2	1 (2 4 3)
3	(3 4 2) 1	3	1 (2 3 4)
4	(1 4 3) 2	4	3 (2 1 4)

In an identical instance, if each man has equally ranked elements i, i+1, ..., k on his list, then each woman has also equally ranked elements i, i+1, ..., k on her list. An example of an identical tie instance is as follows:

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	(1 2) 3 4	1	(4 3) 1 2
2	(2 1) 4 3	2	(1 2) 4 3
3	(3 4) 2 1	3	(1 2) 3 4
4	(1 4) 3 2	4	(3 2) 1 4

Before describing what is meant by stability in a tie instance, it is necessary to introduce some new notation.

Let $p_w(m)$ represent the position of man m on woman w's list, and let $p_m(w)$ represent the position of woman w on man m's list. In an instance of size n, each individual's list contains n positions, regardless of ties. If $p_w(m)=1$ then m is in the first position on w's list. Likewise, $p_w(m)=n$ implies that m is in the last position on w's list. Position number increases from left to right in a list.

Let $r_w(m)$ represent the rank of man m on woman w's list, and let $r_m(w)$ represent the rank of woman w on man m's list. Each individual of an instance of size n has <= n ranks on his/her list; n ranks if there are no ties in the list and less than n ranks if there are ties. If $r_w(m)=1$ then m is ranked the highest (is the most preferable) on w's list. Rank number increases (preference decreases) from left to right in a list. In an instance containing no ties, $p_w(m) = r_w(m)$ and $p_m(w) = r_m(w)$.

There are three kinds of stability associated with matchings of a tie instance. The first, and strongest, is *super-stability.* A matching M is not super-stable if there exist pairs (m',w') and (m*,w*) in M such that

$r_{w^*}(m') <= r_{w^*}(m^*)$ and $r_{m'}(w^*) <= r_{m'}(w')$. The individuals m' and w* are considered to form a blocking pair against super-stability. If no such pair exists, the matching is super-stable.

The second type of stability in a tie instance is *strong-stability.* A matching M is not strongly-stable if there exist pairs (m',w') and (m*,w*) in M such that either $r_{w^*}(m') < r_{w^*}(m^*)$ and $r_{m'}(w^*) <= r_{m'}(w')$ holds or

$r_{w^*}(m') <= r_{w^*}(m^*)$ and $r_{m'}(w^*) < r_{m'}(w')$ holds. In this case m' and w* block strong-stability. Without such a blocking pair, M is strongly stable.

A third type of stability in a tie instance is *strict-stability.* A matching M is not strictly-stable if there exist pairs (m',w') and (m*,w*) in M such that $r_{w^*}(m') < r_{w^*}(m^*)$ and $r_{m'}(w^*) < r_{m'}(w')$. Here m' and w*, if they exist, block strict-stability, and if they do not exist, M is strictly-stable.

Super-stability and strong-stability will be focused on and compared to each other and to standard-stability from this point. Gusfield and Irving assert that strictly-stable matchings are basically trivial, because, by breaking ties arbitrarily, any matching which is standard-stable in the standard instance can be found to be strictly-stable in the corresponding tie instance. Super-stable and strongly-stable matchings do not always exist for arbitrary tie instances.

It should be clear that the standard Gale-Shapley algorithm for finding standard-stable matchings is not adequate for finding super-stable and strongly-stable matchings in tie instances. The algorithm must therefore be amended to provide for these differences in stability.

First, however, it is helpful to find relationships between the types of stability, in order to later simplify the search for all of the stable matchings. Four relationships are presented as formal statements with their corresponding proofs.

I. In a tie instance, all matchings which are super-stable are necessarily strongly-stable.

<u>proof</u>:

* Let I be a tie instance with a super-stable matching M.

* Then for all pairs (m,w) in M there do not exist pairs (m*,w*) and (m',w') in M such that $r_{w^*}(m') <= r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) <= r_{m'}(w')$.

* So for all (m*,w*) and (m',w') in M,

(i) $r_{w^*}(m') > r_{w^*}(m^*)$ <u>or</u> $r_{m'}(w^*) > r_{m'}(w')$.

* In order for M to be strongly-stable, there must <u>not</u> exist (m*,w*) and (m',w') in M such that:

$r_{w^*}(m') < r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) <= r_{m'}(w')$

<u>or</u>

$r_{w^*}(m') <= r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) < r_{m'}(w')$.

* So for all (m*,w*) and (m',w') in M,

(ii) $r_{w^*}(m') >= r_{w^*}(m^*)$ <u>or</u> $r_{m'}(w^*) > r_{m'}(w')$

<u>and</u>

(iii) $r_{w^*}(m') > r_{w^*}(m^*)$ <u>or</u> $r_{m'}(w^*) >= r_{m'}(w')$

must hold for strong-stability.

* It is clear that whenever (i) holds, (ii) and (iii) necessarily hold.

* Therefore if M is super-stable it is necessarily strongly-stable.

II. (follows from I.)

In a tie instance, any pair which blocks the strong-stability of a matching necessarily blocks the super-stability of that matching.

<u>proof</u>:

* Let I be a tie instance with a matching M which is not strongly-stable.

* Then there exist (m*,w*) and (m',w') in M such that:

(i) $r_{w^*}(m') < r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) <= r_{m'}(w')$

<u>or</u>

(ii) $r_{w^*}(m') <= r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) < r_{m'}(w')$.

* Thus m' and w* form a blocking pair against strong-stability.

* In order for (m',w*) to be a blocking pair against super-stability, it must hold that:

(iii) $r_{w^*}(m') <= r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) <= r_{m'}(w')$.

*It is clear that given (i) or (ii), (iii) necessarily follows.

* Therefore, if m' and w* block the strong-stability of a matching, then they also block the super-stability of that matching.

For the proofs of statements III and IV it must be defined that a tie instance which *corresponds* to a particular standard instance simply consists of the exact elements in the exact positions as the standard instance, but with ties placed in the lists. This may change some of the ranks, yet the positions remain fixed.

III. For a given standard instance of n men and n women, any matching which is not standard-stable is therefore not super-stable in any corresponding tie instance.

<u>proof</u>:

* Let I_s be a standard instance of size n.

* Let M be a matching of the n men and n women which is not standard-stable.

* Then there exists a blocking pair.

* So for some pairs (m^*,w^*) and (m',w') in M,

(i) $p_{w^*}(m') < p_{w^*}(m^*)$ <u>and</u> $p_{m'}(w^*) < p_{m'}(w')$.

* Let I_t be an arbitrary tie instance which corresponds to I_s.

* Assume M is super-stable in I_t.

* Then there does not exist a blocking pair against super-stability.

* So for any pairs (m^*,w^*) and (m',w') in M,

(ii) $r_{w^*}(m') <= r_{w^*}(m^*)$ <u>and</u> $r_{m'}(w^*) <= r_{m'}(w')$

does <u>not</u> occur.

* All of the positions $p_w(m)$ and $p_m(w)$ in I_t are the same as they are in I_s.

* However, for any I_t, $r_w(m) <= p_w(m)$ and $r_m(w) <= p_m(w)$ for pair (m,w) in any matching in I_t.

* Therefore, (from (i)), since $p_{w^*}(m') < p_{w^*}(m^*)$ <u>and</u> $p_{m'}(w^*) < p_{m'}(w')$

it is clear that $r_{w^*}(m') <= r_{w^*}(m^*)$ and $r_{m'}(w^*) <= r_{m'}(w')$.

* This contradicts (ii).

* Therefore M is not super-stable in any tie instance corresponding to I_s.

IV. For a given standard instance of n men and n women, any matching which is not standard-stable may or may not be strongly-stable in any corresponding tie instance.

proof:

* Referring to the proof of III where M is a matching which is not standard-stable, the result stands that there exist (m^*,w^*) and (m',w') in M such that

(i) $r_{w^*}(m') <= r_{w^*}(m^*)$ and $r_{m'}(w^*) <= r_{m'}(w')$.

* In order to prove that M is not strongly-stable in an arbitrary corresponding tie instance I_t, there must be found a blocking pair such that

(ii) $r_{w^*}(m') < r_{w^*}(m^*)$ and $r_{m'}(w^*) <= r_{m'}(w')$

or

$r_{w^*}(m') <= r_{w^*}(m^*)$ and $r_{m'}(w^*) < r_{m'}(w')$.

* Given a matching M for which (i) holds, (ii) may or may not hold.

* Therefore, M may or may not be strongly-stable in I_t when it is not standard-stable in I_s.

Now that some fundamental relationships have been presented about stability in tie instances, it is appropriate to introduce the adapted algorithm for finding super-stable and strongly-stable

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matchings, provided they exist. The first five steps in the algorithm are the same for both types of stability, and thereafter it branches to satisfy the differences between super- and strong-stability.

Gale-Shapley Algorithm for Indifference Cases:

* Begin with the men proposing, following the standard Gale-Shapley algorithm.

* An unengaged man who has two or more women tied at the head of his list must propose to all simultaneously.

* When a woman receives a proposal, all men on her list who she prefers strictly less than the proposer must be deleted from her list, and she from their lists.

* A woman may hold more than one engagement if the proposers are tied on her list.

* If the proposing process concludes with one or more men's lists empty, no super-stable or strongly-stable matching exists.

Continued algorithm for super-stability:

* No multiply engaged woman can have stability with any of her partners or those tied with them on her list. Delete such pairs, and then continue with the proposals, until there is a one-to-one correspondence between men and women in the matching. This will be a super-stable matching.

* If some man's list becomes empty, there exists no super-stable matching.

Continued algorithm for strong-stability:

* If the bipartite graph of pairs contains a *perfect* (one-to-

one correspondence) matching, then it is a strongly-stable matching.

* Otherwise there is a set of m men collectively engaged to fewer than m women (a *deficient set*).

* Let X be the *minimal deficient set* (a deficient set that contains no smaller deficient set).

* Any woman engaged to more than one man in X cannot have any of these partners or those tied with them. Delete the pairs and continue proposals.

* Continue until there is a perfect matching in the bipartite graph. The matching will be strongly-stable.

* If some man's list becomes empty, there is no strongly-stable matching.

Since much theory has thus far been presented, examples will now be given to demonstrate how to apply this adapted algorithm.

Random tie instance of size 3:

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	1 2 3	1	2 (3 1)
2	(2 1) 3	2	1 2 3
3	1 3 2	3	2 1 3

* Form pair (1,1). Delete nothing.

* Form pairs (2,2) and (2,1). Delete pairs (3,1) and (1,1) (eliminating the already engaged pair (1,1)).

The resulting lists are:

1	2 3	1	2
2	(2 1) 3	2	1 2
3	3	3	2 1 3

* Since man 1 no longer has a partner, go back to him and form

(1,2). Delete (2,2).

The resulting lists are:

1	2 3	1	2
2	1 3	2	1
3	3	3	2 1 3

The current pairs are (1,2) and (2,1).

* Form pair (3,3). No deletions result.

Following the first few steps of the adapted algorithm has led to the matching 213. It can easily be verified that this matching is indeed super- and strongly-stable. The continued algorithms for the two types of stability were not needed in this case.

A second example follows in which the continued adapted algorithms are needed.

Identical instance of size 4:

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	(1 2) 3 4	1	(3 1) 2 4
2	(2 3) 1 4	2	(2 4) 3 1
3	(3 4) 2 1	3	(1 3) 2 4
4	(1 4) 2 3	4	(1 2) 3 4

* Form pairs (1,1) and (1,2). Delete (2,1) and (4,1).

Reduced lists:

1	(1 2) 3 4	1	(3 1)
2	(2 3) 4	2	(2 4) 3 1
3	(3 4) 2 1	3	(1 3) 2 4
4	4 2 3	4	(1 2) 3 4

* Form pairs (2,2) and (2,3). Delete (3,2), (1,2), (4,3).

Reduced lists:

1	2 3 4	1	(3 1)
2	(2 3) 4	2	(2 4)
3	(3 4) 1	3	(1 3) 2
4	4 2	4	(1 2) 3 4

Current pairs are (1,1), (2,2), (2,3).

* Form pairs (3,3) and (3,4). Delete (2,3) and (4,4).

Reduced lists:

1	2 3 4	1	(3 1)
2	2 4	2	(2 4)
3	(3 4) 1	3	(1 3)
4	2	4	(1 2) 3

Current pairs are (1,1), (2,2), (3,3), (3,4).

* Form pair (4,2). No deletions result.

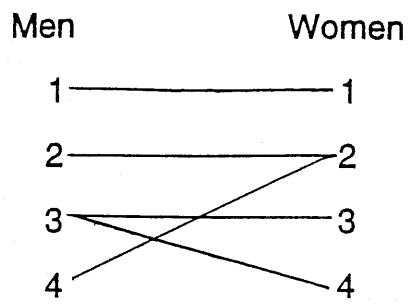
Reduced lists:

1	2 3 4	1	(3 1)
2	2 4	2	(2 4)
3	(3 4) 1	3	(1 3)
4	2	4	(1 2) 3

Current pairs are (1,1), (2,2), (3,3), (3,4), (4,2).

At this point in the algorithm woman 2 is engaged to two men. According to the continued algorithm for super-stability, woman 2 cannot have stability with 2 or 4. Thus the pairs (4,2) and (2,2) are deleted, and man 4's list becomes empty. Therefore there is no super-stable matching in this tie instance.

As far as strong-stability is concerned, the bipartite graph



is <u>not</u> a perfect matching.

X = men 4,2 (collectively engaged to woman 2).

Since woman 2 cannot, therefore, be engaged to either 2 or 4, man 4's list becomes empty, just as it did in the super-stability case, and there is no strongly-stable matching.

The previous examples have served to demonstrate the algorithmic process of finding at least one matching. The remainder of the paper will focus on attemping to make comparisons between super- and strong-stability. All of the identical and symmetric tie instances corresponding to a particular standard instance are tested for stability, and observations are made thereafter.

Although it is by no means adequate to draw conclusions based on one particular example, the goal is to reach more of an understanding of the occurrences of stability by looking at concrete cases.

Standard instance of size 4:

Men	Men's Pref. Lists	Women	Women's Pref. Lists
1	1 2 3 4	1	4 3 2 1
2	2 1 4 3	2	3 4 1 2
3	3 4 1 2	3	2 1 4 3
4	4 3 2 1	4	1 2 3 4

This instance was chosen as the primary example due to the fact that it contains ten standard-stable matchings, the maximum number for an instance of size 4. The high degree of symmetry makes it easier to recognize patterns in stability and predict results. Since the primary goal is to make comparisons between super- and strong-stability, and not to locate all matchings of those stabilities, statement IV will be ignored (the fact that there may be additional strongly-stable matchings to consider), and stability will be tested on only the ten standard-stable matchings.

The ten standard-stable matchings are:

1234	3142
1243	3412
2134	3421
2143	4312
2413	4321

All of the results are located in table 1 (identical ties) and table 2 (symmetric ties)--located at the end of the paper. It is important to note the following regarding the structure of the tables:

* Ties are labeled according to which columns of the instance are included in the tie. For example, "1,2 & 3,4" represents the tie of the first two women in the men's lists and the last two men in the women's lists.

* "sup" is an abbreviation for super-stable; "str" is an abbreviation for strongly-stable.

* "yes" indicates the matching is stable; if "no", then one blocking pair is given.

* The tie "none" corresponds to the standard instance.

Observations from the identical ties:

* Tie 1,2: yes to sup and str for pair (m,w) if m is in a tie on w's list and w is not in a tie on m's list, and conversely.

* Tie 2,3: yes to sup when no member of a pair is involved in a tie, yes to str when no member of a pair is involved in a tie <u>or</u> all are involved in ties.

* Tie 3,4: yes to sup and str when all w in a tie <u>and</u> no m in a tie, and conversely.

* Tie 1,2,3: no sup's; yes to str when all members are in the tie.

* Tie 2,3,4: yes to sup and str when all m are in a tie <u>and</u> all w are not, and conversely.

* Tie 1,2,3,4: no sup's; yes to all str.

* Tie "none": yes to all sup's and str's.

* Standard stable matchings that are never sup in a tie (other than "none") and are never str in a tie (other than "none" and 1,2,3,4): 1243, 2134, 3421, 4312.

Observations from the symmetric ties:

* Super-stable matchings occur where none of the men or women are involved in ties.

* There is symmetry of stabilities in matchings between 1,2 & 3,4 and 3,4 & 1,2, and between 1,2,3 & 2,3,4 and 2,3,4 & 1,2,3. (To be expected due to the symmetry of the standard instance itself.)

* Strongly-stable matchings which are not super-stable occur where: (1) it is not the case that both partners of some pairs are in a tie while neither partners of the remaining pairs are in a tie; (2) all

members of the pairs are in ties.

It would seem logical that different instances would produce widely varying results in regard to stability, and each case would be analyzed differently. These examples have merely tipped the iceberg of possibilities for cases of indifference. Further explorations could be made into numerous areas, such as finding all super-stable and strongly-stable matchings, investigating strict-stability and confirming its relation to standard-stability, searching for properties of random tie instance stability, and much more. The adapted algorithm is more or less a launching pad into this area of stable marriages. The first page on indifference cases has been turned, but a whole library remains.

Matching\Tie	1 & 2 (sup)	1 & 2 (str)	2 & 3 (sup)	2 & 3 (str)	3 & 4 (sup)
1 2 3 4	no (1,2)	no (1,2)	yes	yes	yes
1 2 4 3	no (1,2)	no (1,2)	no (3,1)	no (3,1)	no (3,3)
2 1 3 4	no (3,4)	no (3,4)	no (1,3)	no (1,3)	no (1,1)
2 1 4 3	yes	yes	no (1,3)	yes	no (1,1)
2 4 1 3	yes	yes	no (1,3)	yes	no (2,2)
3 1 4 2	yes	yes	no (1,2)	yes	no (1,1)
3 4 1 2	yes	yes	no (1,2)	yes	no (4,1)
3 4 2 1	no (3,1)	no (3,1)	no (1,2)	no (1,2)	no (2,3)
4 3 1 2	no (1,3)	no (1,3)	no (1,2)	no (3,4)	no (4,1)
4 3 2 1	no (1,3)	no (1,3)	yes	yes	yes

Matching\Tie	3 & 4 (str)	1,2,3 (sup)	1,2,3 (str)	2,3,4 (sup)	2,3,4 (str)
1 2 3 4	yes	no (1,2)	no (1,2)	yes	yes
1 2 4 3	no (3,3)	no (1,2)	no (1,2)	no (3,3)	no (3,3)
2 1 3 4	no (1,1)	no (3,4)	no (3,4)	no (1,1)	no (1,1)
2 1 4 3	no (1,1)	no (1,3)	yes	no (1,1)	no (1,1)
2 4 1 3	no (2,2)	no (1,3)	yes	no (1,1)	no (1,1)
3 1 4 2	no (1,1)	no (1,2)	yes	no (1,1)	no (1,1)
3 4 1 2	no (4,1)	no (1,2)	yes	no (1,1)	no (1,1)
3 4 2 1	no (2,3)	no (3,4)	no (3,4)	no (1,1)	no (1,1)
4 3 1 2	no (4,1)	no (1,2)	no (1,2)	no (4,1)	no (4,1)
4 3 2 1	yes	no (1,2)	no (1,2)	yes	yes

Matching\Tie	1,2,3,4 (sup)	1,2,3,4 (str)	none (sup)	none (str)	
1 2 3 4	no (1,2)	yes	yes	yes	
1 2 4 3	no (1,2)	yes	yes	yes	
2 1 3 4	no (1,1)	yes	yes	yes	
2 1 4 3	no (1,1)	yes	yes	yes	
2 4 1 3	no (1,1)	yes	yes	yes	
3 1 4 2	no (1,1)	yes	yes	yes	
3 4 1 2	no (1,1)	yes	yes	yes	
3 4 2 1	no (1,1)	yes	yes	yes	
4 3 1 2	no (1,1)	yes	yes	yes	
4 3 2 1	no (1,1)	yes	yes	yes	

TABLE 1
Identical Ties

Matching\Tie	1,2 & 3,4	1,2 & 3,4	3,4 & 1,2	3,4 & 1,2	2,3 & 2,3
	(sup)	(str)	(sup)	(str)	(sup)
1 2 3 4	no (1,2)	yes	yes	yes	yes
1 2 4 3	no (1,2)	yes	yes	yes	no (3,1)
2 1 3 4	no (1,1)	yes	yes	yes	no (1,3)
2 1 4 3	no (1,1)	yes	yes	yes	no (1,3)
2 4 1 3	no (2,2)	no (2,2)	no (2,3)	no (2,3)	no (1,3)
3 1 4 2	no (1,1)	no (1,1)	no (1,4)	no (1,4)	no (1,2)
3 4 1 2	yes	yes	no (1,4)	yes	no (1,2)
3 4 2 1	yes	yes	no (1,4)	yes	no (4,3)
4 3 1 2	yes	yes	no (1,3)	yes	no (1,2)
4 3 2 1	yes	yes	no (1,3)	yes	yes

Matching\Tie	2,3 & 2,3	1,2,3 & 2,3,4	1,2,3 & 2,3,4	2,3,4 & 1,2,3	2,3,4 & 1,2,3
	(str)	(sup)	(str)	(sup)	(str)
1 2 3 4	yes	no (1,2)	yes	yes	yes
1 2 4 3	no (3,1)	no (1,2)	yes	no (3,1)	no (3,1)
2 1 3 4	no (1,3)	no (1,1)	yes	no (1,3)	no (1,3)
2 1 4 3	yes	no (1,1)	yes	no (1,3)	yes
2 4 1 3	yes	no (1,1)	yes	no (1,3)	yes
3 1 4 2	yes	no (1,1)	yes	no (1,2)	yes
3 4 1 2	yes	no (1,1)	yes	no (1,2)	yes
3 4 2 1	no (4,3)	no (3,3)	no (3,3)	no (1,2)	yes
4 3 1 2	no (1,2)	no (1,1)	no (1,1)	no (1,2)	yes
4 3 2 1	yes	yes	yes	no (1,2)	yes

Matching\Tie	1234 & 1234	1234 & 1234	none	none	
	(sup)	(str)	(sup)	(str)	
1 2 3 4	no (1,2)	yes	yes	yes	
1 2 4 3	no (1,2)	yes	yes	yes	
2 1 3 4	no (1,1)	yes	yes	yes	
2 1 4 3	no (1,1)	yes	yes	yes	
2 4 1 3	no (1,1)	yes	yes	yes	
3 1 4 2	no (1,1)	yes	yes	yes	
3 4 1 2	no (1,1)	yes	yes	yes	
3 4 2 1	no (1,1)	yes	yes	yes	
4 3 1 2	no (1,1)	yes	yes	yes	
4 3 2 1	no (1,1)	yes	yes	yes	

TABLE 2
Symmetric Ties

Acknowledgements

The foundation for the ideas in this paper, including the standard Gale-Shapley algorithm, the stabilities possible in indifference cases, and an outline of the adapted algorithm, was laid by:

Dan Gusfield and Robert W. Irving, The stable Marriage Problem- Structure and Algorithms, MIT Press, Cambridge, Massachusetts, 1989.