

# LARGE INDUCED OUTERPLANAR AND ACYCLIC SUBGRAPHS OF PLANAR GRAPHS

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ABSTRACT. Albertson and Berman [1] conjectured that every planar graph has an induced forest on half of its vertices; the current best result, due to Borodin [3], is an induced forest on two fifths of the vertices. We show that the Albertson-Berman conjecture holds, and is tight, for planar graphs of treewidth 3 (and, in fact, for any graph of treewidth at most 3). We also improve on Borodin’s bound for 2-outerplanar graphs by finding a large outerplanar induced subgraph and invoking Hosono’s result [9] that outerplanar graphs have large induced forests. Finally, we discuss potential extensions of this approach to  $k$ -outerplanar graphs and the related problem of the vertex arboricity of 2-outerplanar graphs.

## 1. INTRODUCTION

A graph  $G = (V(G), E(G))$  is a set  $V(G)$  of *vertices* and a set  $E(G)$  of pairs of vertices, called *edges*. If the graph being referred to is clear from context, we simply write  $V$  and  $E$ . In this paper, all graphs are assumed to be finite and simple (without loops or parallel edges). If  $V' \subseteq V(G)$ ,  $G[V'] = (V', E')$  is the subgraph *induced by*  $V'$ , where  $E' \subseteq E(G)$  consists of all edges in  $V'$ . A *proper  $m$ -coloring* of a graph  $G$  is an assignment of one of  $m$  colors to each vertex such that no adjacent vertices are the same color. An *acyclic  $m$ -coloring* of  $G$  is a proper  $m$ -coloring in which the union of any two color classes induces an acyclic subgraph (also called a *forest*).

A graph is *planar* if it can be drawn (embedded) in the plane without any edge crossings. A given planar graph could have many embeddings in the plane with no edge crossings, and some properties of the graph are dependent upon the embedding. Given a planar graph  $G$  and a planar embedding of  $G$ , we denote the set of vertices and edges in the infinite face of  $G$  by  $f_G^\infty$ . A planar graph  $G = (V, E)$  is *outerplanar* (or (1-outerplanar)) with respect to an embedding if all vertices are in  $f_G^\infty$ ; it is  *$k$ -outerplanar* if for  $k = 1$ ,  $G$  is outerplanar and for  $k > 1$ ,  $G[V \setminus V(f_G^\infty)]$  is  $(k - 1)$ -outerplanar.

It is well-known that planar graphs are *sparse*, meaning that the number of edges is linear in the number of vertices [6]. The sparsity of planar graphs suggests that they have “large” induced forests, as sparsity means that, on average, vertices are incident to relatively few edges, potentially allowing many vertices to be included in an induced subgraph without creating a cycle. As there are planar graphs on  $n$  vertices whose largest induced forests have

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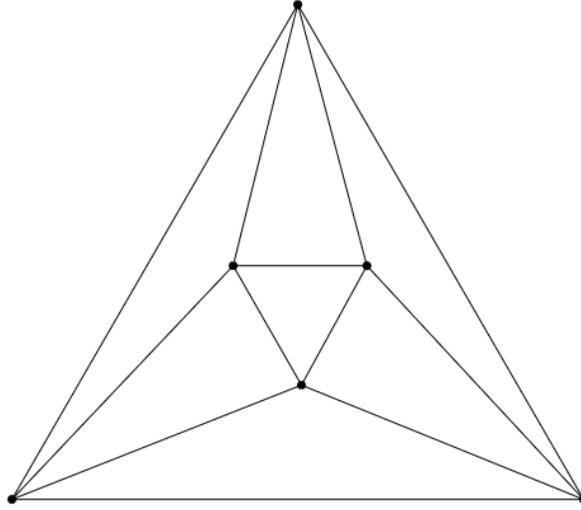


FIGURE 1.  $K_{2,2,2}$ , also called the octahedron, is not acyclically 4-colorable.

size exactly  $\frac{n}{2}$  (e.g.  $K_4$ ),  $\frac{n}{2}$  is a clear upper bound on the size of induced forests in planar graphs. Albertson and Berman [1] conjectured that this upper bound is tight:

**Conjecture 1.1.** *Every planar graph on  $n$  vertices has an induced forest on at least  $\frac{n}{2}$  vertices.*

Among the consequences of the Albertson-Berman conjecture would be that every planar graph has an independent set of size at least  $\frac{n}{4}$ , a fact implied by the Four Color Theorem.

The current best lower bound on the size of induced forests in planar graphs is  $\frac{2n}{5}$ , and is due to Borodin [3]. This lower bound is actually a corollary of the following stronger result by Borodin: planar graphs are acyclically 5-colorable. This implies the lower bound stated above, as the union of the largest two color classes of an acyclic 5-coloring contains at least two fifths of the vertices and induces a forest. This result is the best possible, as there are planar graphs which do not have an acyclic 4-coloring (see Figure 1). This implies that Conjecture 1.1 cannot be proven using acyclic colorings, as an acyclic 4-coloring is needed to guarantee that the union of two color classes has size at least  $\frac{n}{2}$ . However, acyclic colorings are a means to prove special cases of Conjecture 1.1. In section 2.1, we show that some classes of planar graphs are acyclically  $k$ -colorable for  $k \leq 4$ . The  $\frac{2n}{5}$  lower bound on induced forest size implied by Borodin's result is thought not to be tight, as the notion of an acyclic coloring is considerably more constrained than that of an induced forest.

Classes of planar graphs for which the conjecture has previously been proven include outerplanar graphs [9] and planar graphs without  $k$ -cycles, where  $k \in \{3, 4, 5, 6\}$  [10]. It is worth noting that the lower bound for induced forests in outerplanar and triangle-free planar graphs is in fact larger than  $\frac{n}{2}$ . Outerplanar graphs have induced forests on at least  $\frac{2}{3}$  of their vertices [9], and Salavatipour uses the discharging technique to show that triangle-free planar graphs on  $n$  vertices have induced forests on at least  $\frac{17n+24}{32}$  vertices [11]. The results

on planar graphs without cycles of length  $k$  are implied by the fact that the vertices of such graphs can be partitioned into two classes such that each class induces a forest, a notion which will be discussed further in section 3.

In this paper, we use acyclic colorings to prove that Conjecture 1.1 is true for planar graphs of treewidth at most 3 (defined below). We also improve the lower bound for 2-outerplanar graphs to  $\frac{4}{9}$  by giving an algorithm to find an outerplanar induced subgraph of a 2-outerplanar graph on at least  $\frac{2}{3}$  of the vertices.

**Definition 1.2.** A  $k$ -tree is a graph  $G = (V, E)$  such that:

- i. If  $|V| = k$ ,  $G = K_k$
- ii. If  $|V| > k$ , there exists a vertex  $v \in V$  such that  $d(v) = k$  and  $G - v$  is a  $k$ -tree.

A graph is a *partial  $k$ -tree* if it is the subgraph of a  $k$ -tree. We say that a graph  $G$  has *treewidth  $k$*  if  $k$  is the least integer such that  $G$  is a partial  $k$ -tree. Note that  $k$ -outerplanar graphs have treewidth at most  $3k - 1$  [7].

## 2. RESULTS

### 2.1. Low Treewidth Graphs.

**Proposition 2.1.** Let  $G=(V,E)$  be a  $k$ -tree. There exists an acyclic  $(k + 1)$ -coloring of  $G$ .

*Proof.* We proceed by induction on  $|V|$ .

The base case, when  $|V| \leq k$ , is trivial, as any coloring with exactly  $|V|$  colors is acyclic.

Suppose  $|V| = n$ . By the definition of a  $k$ -tree, there exists  $v \in V$  such that  $d(v) = k$  and  $G - v$  is a  $k$ -tree.  $G - v$  has an acyclic  $(k + 1)$ -coloring by the inductive hypothesis. The neighbors of  $v$  induce a  $k$ -clique in  $G$  and thus are colored  $k$  distinct colors in any proper coloring. Color  $v$  the remaining color, say  $i$ . This coloring is acyclic,  $v$  has only one edge with a vertex colored  $j$  for  $i \neq j$ .  $\square$

**Corollary 2.2.** Let  $G$  be a partial  $k$ -tree on  $n$  vertices.  $G$  has an induced forest on at least  $\frac{2n}{k+1}$  vertices.

*Proof.* By the above result,  $G$  has an acyclic  $(k + 1)$ -coloring. Let  $U$  be the union of the two largest color classes in this coloring. Clearly,  $|U| \geq \frac{2n}{k+1}$  and  $G[U]$  is a forest.  $\square$

The corollary implies Conjecture 1.1 for planar graphs of treewidth 3, and, as  $K_4$  has treewidth 3, the conjecture is tight for this class of planar graphs. For planar graphs of treewidth 2, which includes outerplanar graphs, the corollary implies the existence of an induced forest on  $\frac{2}{3}$  of the vertices. This bound is again tight, as the union of disjoint triangles is outerplanar.

### 2.2. 2-outerplanar Graphs.

Our next result requires some additional notation and terminology. In the following discussion, we will assume the embedding of a planar graph  $G$  is fixed and inherited by subgraphs. Properties of  $G$  and its subgraphs such as  $k$ -outerplanarity and the elements of the set  $f^\infty$  will be considered with respect to this fixed embedding.

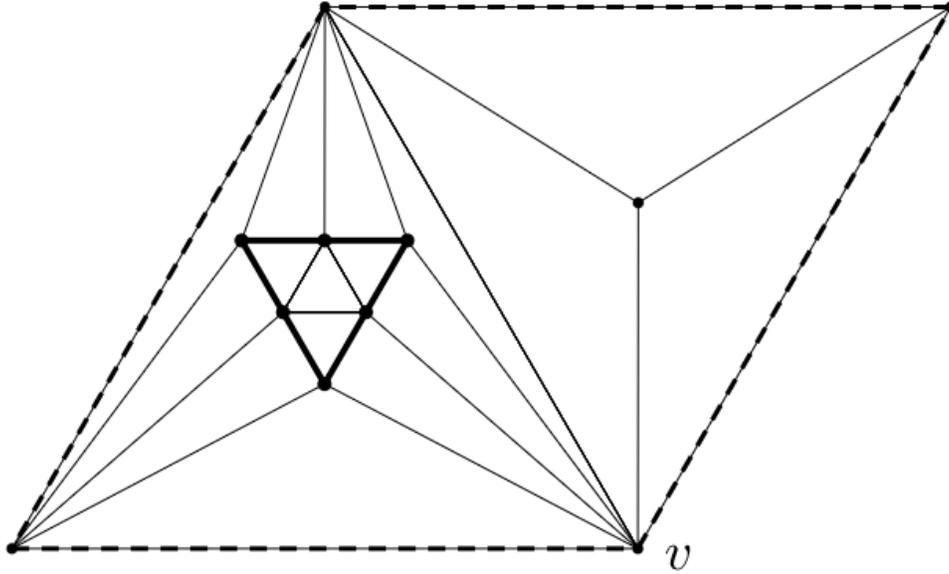


FIGURE 2. A MLT 2-outerplanar graph. External edges are dashed; 2-external edges are bolded. The vertex  $v$  has between degree 4.

For convenience, if  $G = (V, E)$  is 2-outerplanar, we write it as  $G = (L_1, L_2; E)$ , where  $L_1 := \{v \in V : v \in f_G^\infty\}$  and  $L_2 := \{v \in V : v \in f_{G[V \setminus L_1]}^\infty\}$ . Clearly,  $L_1 \cup L_2 = V$ . If a vertex is in  $L_1$ , we call it *external*; if a vertex is in  $L_2$ , we call it *internal*. The edges in  $f_G^\infty$  are called *external*; those in  $f_G^\infty[L_2]$  are *2-external* (see Figure 2). Note that  $f_{G[L_i]}^\infty$  is a cactus graph (every edge is in at most 1 cycle). Additionally, we say that a set of vertices  $A \subseteq V$  *encloses* another set of vertices  $B \subseteq V$  if  $B \cap f_{G[A \cup B]}^\infty = \emptyset$ . To simplify later arguments, we assume that the embedding of  $G$  is such that if  $H$  is a connected component of  $G$ ,  $H$  is not enclosed by another connected component of  $G$ .

We call  $B \subseteq L_1$  a *block* of  $G$  if  $G[B]$  is a cycle. We also introduce a weakened notion of triangulation, to simplify later arguments.  $G$  is *mid-layer triangulated (MLT)* if every face  $f$  of  $G$  that contains vertices of exactly 2 layers is a triangle. Finally, in order to speak about adjacencies of vertices in different layers, we define the *between degree* of  $v \in L_i$ , denoted  $d_b(v)$ , as  $|N(v) \cap L_j|$  where  $j \neq i$  (see Figure 2).

**Observation 1.** If a 2-outerplanar graph is MLT and  $uv$  is a 2-external edge, then there exists  $w \in L_2$  such that  $uvw$  is a face.

**Observation 2.** If a 2-outerplanar graph is MLT,  $d_b(v) \geq 1$  for all  $v \in L_2$ .

**Lemma 2.3.** Let  $G = (L_1, L_2; E)$  be a simple, MLT 2-outerplanar graph. If  $v \in L_2$  has between degree 1, then it is incident to exactly two 2-external edges.

*Proof.* Let  $u$  be the vertex in  $N(v) \cap L_1$ . By MLT, there exist 2 triangles,  $xuv$  and  $yuv$  containing the edge  $uv$ .  $x \neq y$  since  $G$  is simple. As  $d_b(v) = 1$ ,  $x$  and  $y$  are in  $L_2$ , and the edges  $xv$  and  $yv$  are 2-external. Therefore,  $v$  is incident to at least two 2-external edges.

Suppose for the sake of contradiction that  $v$  is incident to more than two 2-external edges. Let  $w, z$  be consecutive neighbors in the clockwise ordering of  $N(v) \cap f_{G[L_2]}^\infty$  in the embedding such that  $w$  and  $z$  are not in the same 2-connected component of  $f_{G[L_2]}^\infty$  and  $w \notin \{x, y\}$ . Then by Observation 1, there exists  $s \in L_1$  such that  $vws$  is a face. Since  $w \notin \{x, y\}$  and  $G$  is simple,  $s \neq u$ . This implies  $d_b(v) \geq 2$ , which is a contradiction.  $\square$

**Lemma 2.4.** *Let  $G = (L_1, L_2; E)$  be an MLT 2-outerplanar graph and let  $D$  be the set of 2-external edges. There exists a matching  $M \subseteq D$  with the following property:*

(1) 
$$\text{If } v \in L_2 \setminus V(M) \text{ then } d_b(v) \geq 2.$$

*Proof.* Without loss of generality, we assume the only edges between  $L_2$  vertices are in  $C$ .

Let  $L := \{v \in L_2 : d_b(v) = 1\}$ .

First, observe a block  $B$  of  $G$  enclosing a vertex in  $L_2$  in fact encloses a vertex with between degree at least 2: consider an edge  $uv$  in  $B$ : by MLT,  $u$  and  $v$  have a mutual neighbor  $w \in L_2$ , so  $d_b(w) \geq 2$ .

We proceed by strong induction on  $|L|$ . If  $|L| = 0$ , any matching  $M \subseteq D$  has property (1).

Suppose  $|L| = n$ . Let  $v \in L$  and  $u \in L_2 \setminus L$  be vertices such that  $uv \in D$ . Such vertices exist because every block enclosing a vertex in  $L_2$  encloses a vertex of between degree at least 2. By Lemma 2.3,  $v$  has exactly one other neighbor  $w$  such that  $vw \in D$ .

Contract  $u$  and  $w$  to  $v$  and delete parallel edges and loops; let the resulting graph be  $G'$ . Since  $v \notin L(G')$ , by the inductive hypothesis, there exists a matching  $M' \subseteq D(G')$  with property (1). Now, consider  $M'$  as a matching in  $D(G)$ .

If  $v$  is not covered by  $M'$  in  $G'$ , then  $u, v, w$  are not covered by  $M'$  in  $G$ , so let  $M := M' \cup \{vw\}$ .  $M \subseteq D(G)$  is a matching and has property (1), since  $d_b(u) > 1$  as argued above.

If  $vx \in M'$  for some  $x \in V(G')$ , then either  $ux \in D(G)$  or  $wx \in D(G)$ . In the first case, let  $M := (M' \setminus \{vx\}) \cup \{ux, vw\}$ ; in the second, let  $M := (M' \setminus \{vx\}) \cup \{wx, uv\}$ . In both cases,  $M$  is a subset of  $D(G)$ , is a matching and has property (1).  $\square$

**Theorem 2.5.** *Let  $G = (L_1, L_2; E)$  be a 2-outerplanar graph on  $n$  vertices.  $G$  has an induced outerplanar subgraph on at least  $\frac{2n}{3}$  vertices.*

*Proof.* Without loss of generality, we assume  $G$  is MLT by adding edges if necessary. To find the vertices inducing a large outerplanar graph in  $G$ , we delete vertices in  $L_1$  until all vertices in  $L_2$  are “exposed” to the external face. To ensure that the resulting outerplanar graph is sufficiently large, we delete vertices in  $L_1$  that expose 2 vertices in  $L_2$  or otherwise ensure 2 vertices will be included in the outerplanar graph.

Let  $M$  be a matching with property (1) of Lemma 2.4. We create a list  $K$  of triples such that each vertex in  $L_2$  occurs in exactly one triple. For each  $u \in L_2$  not covered by  $M$ ,  $d_b(u) \geq 2$ , and we add  $\{u, v, w\}$  to  $K$ , where  $v, w \in N(u) \cap L_1$ . For each edge  $xy \in M$ , by Observation 1, there exists  $z \in L_1$  such that  $xyz$  is a face, and we add  $\{x, y, z\}$  to  $K$ .

We then perform the following procedure:

while there exists  $\{u, v, w\} \in K$  such that  $\{u, v, w\} \cap L_1 = \{v\}$   
 delete  $v$  from  $G$  and delete all triples containing  $v$  from  $K$ ;  
 while there exists  $v \in L_1$  such that  $v$  is in 2 or more distinct triples of  $K$   
 delete  $v$  from  $G$  and delete all triples containing  $v$  from  $K$ ;  
 while  $\{u, v, w\} \in K$   
 delete  $v \in L_1$  from  $G$  and delete  $\{u, v, w\}$  from  $K$ .

Note that if  $v \in L_1$  is deleted from  $G$ , all  $L_2$  vertices in a triple with  $v$  are exposed. Therefore, the undeleted vertices induce an outerplanar subgraph in  $G$ .

In the first two steps, at least two  $L_2$  vertices were exposed for every deleted  $L_1$  vertex. In the final step, all triples are disjoint, so each deletion of an  $L_1$  vertex exposes one  $L_2$  vertex and ensures that one  $L_1$  vertex will not be deleted; again, 2 vertices are included in the induced outerplanar subgraph for every deleted vertex. This means that the subgraph contains at least two thirds of the vertices of  $G$ . □

This result is tight, as the disjoint union of multiple octahedrons (see Figure 1) is 2-outerplanar, and its largest induced outerplanar subgraph is on  $\frac{2}{3}$  of its vertices. The result is also tight for arbitrarily large connected 2-outerplanar graphs, as the same property holds for graphs constructed by connecting disjoint octahedrons as shown in Figure 3.

Theorem 2.5 has an immediate corollary for  $k$ -outerplanar graphs. We first must extend our notation for 2-outerplanar graphs. For a  $k$ -outerplanar graph  $G = (V, E)$ , let  $L_1$  and  $L_2$  be defined as for 2-outerplanar graphs. Let  $L_3 := V \cap f_{G[V \setminus (L_1 \cup L_2)]}^\infty$ ,  $L_4 := V \cap f_{G[V \setminus (L_1 \cup L_2 \cup L_3)]}^\infty$  and so on.

**Corollary 2.6.** *Let  $G = (L_1, L_2, \dots, L_k; E)$  be a  $k$ -outerplanar graph on  $n$  vertices. Then  $G$  has an induced  $\lceil \frac{k}{2} \rceil$ -outerplanar subgraph on at least  $\frac{2n}{3}$  vertices.*

*Proof.* We apply Theorem 2.5 to pairs of successive layers in  $G$ , finding large induced outerplanar subgraphs  $H_i \subseteq G[L_i \cup L_{i-1}]$  for  $i = 1, 3, 5, \dots, k-1$  (if  $k$  is even; if  $k$  is odd, we end at  $i = k-2$ ). Let  $V' := \cup_i V(H_i)$ .  $G[V']$  is  $\lceil \frac{k}{2} \rceil$ -outerplanar, as  $L_i(G[V']) = V(H_i)$ , and  $|V'| \geq \frac{2n}{3}$ , as  $|V(H_i)| \geq \frac{2}{3}|L_i \cup L_{i-1}|$ . □

In combination with Corollary 2.2 and the fact that outerplanar graphs have treewidth 2, Theorem 2.5 gives the following easy corollary.

**Corollary 2.7.** *Let  $G=(V, E)$  be a 2-outerplanar graph on  $n$  vertices.  $G$  has an induced forest on at least  $\frac{4n}{9}$  vertices.*

Though this bound is an improvement on Borodin's general bound, the fact that the induced forest found in the above process is in fact an induced forest in an induced subgraph of the original graph suggests that the bound need not be tight. It is possible that vertices deleted from the 2-outerplanar graph to find the outerplanar induced subgraph could be added to the induced forest found in that subgraph without creating cycles (see Figure 4).

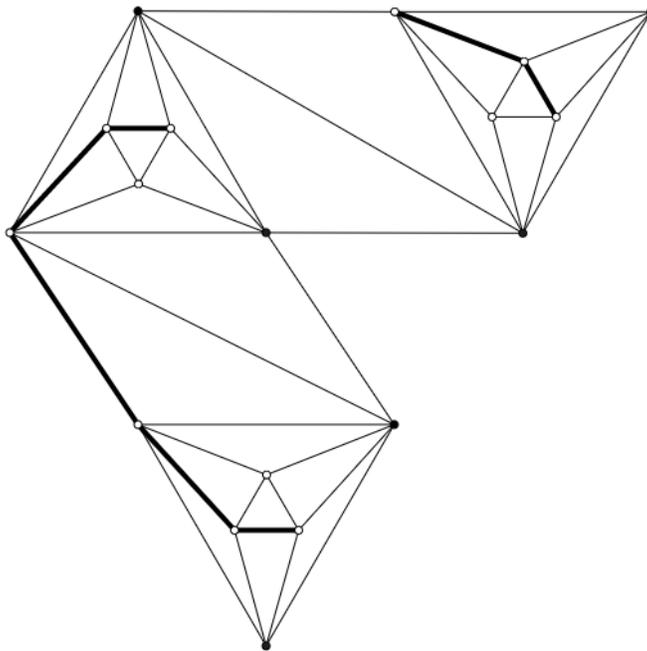


FIGURE 3. A 2-outerplanar graph whose largest induced outerplanar subgraph is on  $\frac{2}{3}$  of its vertices. The white vertices induce such a subgraph, found by the algorithm in Theorem 2.5. The bolded edges show an induced forest in this outerplanar subgraph whose size satisfies the Albertson-Berman conjecture for the 2-outerplanar graph.

### 3. CONJECTURES AND CURRENT DIRECTIONS

Our first conjecture is an extension of the above result on 2-outerplanar graphs.

**Conjecture 3.1.** *3-outerplanar graphs on  $n$  vertices contain induced outerplanar subgraphs on at least  $\frac{3n}{5}$  vertices.*

If this conjecture is true, it would provide an alternate, and likely much simpler, proof of Borodin's bound for 3-outerplanar graphs. It should be noted that no tight examples have been found for the conjecture; nor, indeed, have any examples been found by the authors of 3-outerplanar graphs with induced outerplanar subgraphs on less than  $\frac{2}{3}$  of the vertices.

A possible proof of this conjecture would take much the same form as the proof for 2-outerplanar graphs, in which appropriately selected vertices in the 2 outer layers are deleted to expose all vertices in the innermost layer and all remaining vertices in the middle layer. In the 2-outerplanar proof, one structure that heavily influenced our algorithm was internal vertices with between degree 1; this seems to remain true for this proof method for 3-outerplanar graphs. In particular, to expose vertices in the innermost layer with between degree 1, we must delete their neighbor in the next layer. This suggests that a step of our algorithm should be to delete middle-layer vertices adjacent to between-degree 1 inner vertices and then delete appropriate external vertices to expose the between-degree 1 inner vertices. For any path of middle-layer vertices adjacent to between-degree 1 inner vertices,

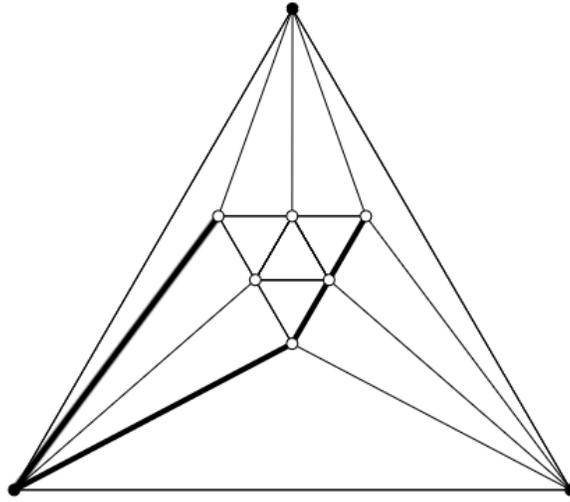


FIGURE 4. The outerplanar induced subgraph of this graph found by the algorithm in Theorem 2.5 is induced by the white vertices. Every induced forest on at least half of the vertices of this graph (an example is shown by the bolded edges) includes vertices not in this outerplanar subgraph.

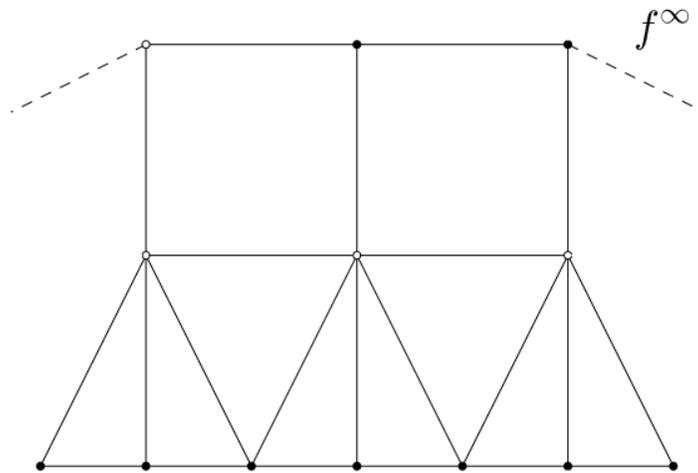


FIGURE 5. In this segment of a 3-outerplanar graph, the inner vertices (bottom row) are exposed by deleting the white external (top row) and middle-layer (middle row) vertices.

it seems we only need to delete 1 external vertex to expose them (see Figure 5); this saves us from deleting more than 2 of every 5 vertices.

Our second conjecture requires additional context. The *vertex-arboricity* of a graph  $G = (V, E)$ , denoted  $a(G)$ , is the least number of classes into which  $V$  can be partitioned such

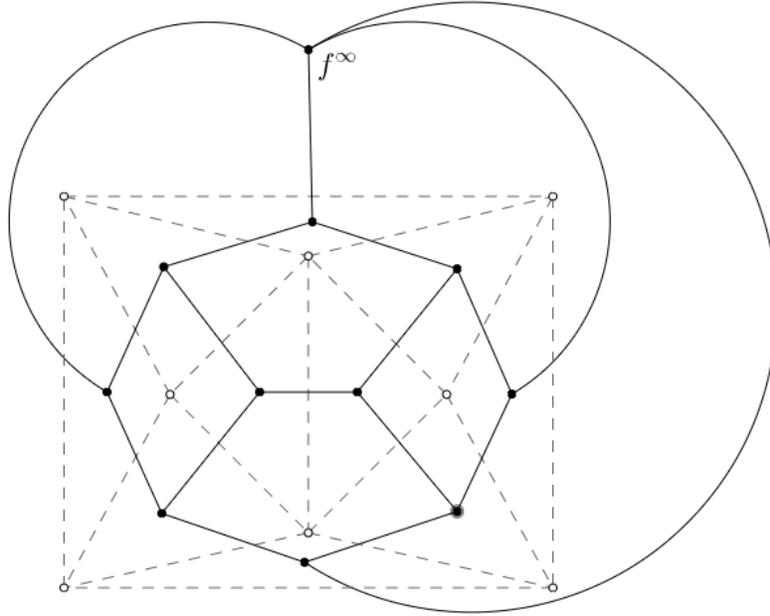


FIGURE 6. An example of an almost triangulated 2-outerplanar graph (dashed edges) and its planar dual, a modified Halin graph (solid edges).

that each partition class induces a forest. Clearly,  $a(G) = 2$  implies the existence of an induced forest on at least half of the vertices of  $G$ . While it is true that  $a(G) \leq 3$  for all planar graphs  $G$  [5], there are planar graphs with vertex arboricity 3 [10, 4]. However, all examples of planar graphs with vertex arboricity 3 known to the authors of this paper are at least 3-outerplanar. Additionally, all outerplanar graphs have vertex-arboricity at most 2, as they are acyclically 3-colorable. This leads us to the following conjecture.

**Conjecture 3.2.** *If  $G$  is 2-outerplanar, then  $a(G)=2$ .*

This conjecture, if true, would clearly imply the Albertson-Berman conjecture for 2-outerplanar graphs.

Hakimi and Schmeichel have shown that, for a planar graph  $G$ ,  $a(G) = 2$  if and only if  $G^*$  has a connected Eulerian spanning subgraph [8].  $G^* = (V^*, E^*)$  here is the *planar dual* of  $G$ , where  $V^*$  is the set of faces of  $G$ , including the infinite face, and, for  $f, g \in V^*$   $fg \in E^*$  if  $f$  and  $g$  share an edge in  $G$ .

A *Halin graph* is a planar graph constructed in the following manner: a tree in which no vertex has exactly 2 neighbors is drawn in the plane with no edge crossings; edges are then added to connect the leaves of the tree in a cycle in a manner that preserves planarity. If  $G = (L_1, L_2; E)$  is a 2-outerplanar graph with triangulated finite faces such that  $L_1$  is a block,  $G^*$  is similar to a 3-regular Halin graph; it is a Halin graph with some edges in the cycle subdivided by one or more vertices, each of which is adjacent to the dual vertex  $f_G^\infty$ . Additionally, no vertex in the tree is adjacent to  $f_G^\infty$  (see Figure 6), and all vertices except  $f_G^\infty$  are degree 3. For convenience, we will call such graphs modified Halin graphs.

Halin graphs are Hamiltonian, a result due to Bondy [2], and there is a fairly clear inductive proof of this fact. Modified Halin graphs are not necessarily Hamiltonian, but it is possible that the existence of a connected spanning Eulerian subgraph can be shown through an inductive argument similar to that used for Halin graphs. The presence of subdivided edges, however, complicates the inductive step.

As noted by Raspaud and Wang, if a counterexample to the conjecture exists, it must have at least 21 vertices [10].

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