HORSESHOES, ATTRACTORS, AND GOOD REDUCTION: DYNAMICS OF THE p-ADIC HÉNON MAP

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Abstract. While quite a bit of study has been put towards classifying the dynamics of the Hénon map over the real and complex numbers, little is known about its action on the p-adic plane $\mathbb{Q}_p^2$. In this paper, we present a partition of the map's parameter space and compile several results describing the Hénon map's dynamics when parametrized by values in each region of the partition. Similarly to the Euclidean case, there exists a region of the parameter space under which a conjugacy to the Smale horseshoe arises, and another region under which the Hénon map induces a trapped attracting set. We conclude by defining and conjecturing the existence of a p-adic strange attractor.

1. Introduction

Since Hénon [7] introduced his namesake mapping as an object of study in 1976, it has been the object of no small amount of study in the Euclidean contexts of $\mathbb{R}^2$ and $\mathbb{C}^2$. While a surprising number of features have been found in the deceptively simple plane automorphism (see for instance Robinson [12] for a survey of results), the two it is likely most known for are its horseshoe dynamics in one region of its parameter space and the strange attractor it admits in another. While the horseshoe dynamics have been known for quite a while, the existence of a strange attractor admitted by the Hénon map was not proven for any values in the parameter space until Benedicks and Carleson's work in 1991 [4], which was followed up by Mora and Viana's proof of the existence of an infinite class of parameter values inducing a strange attractor [10]. While some work has been put into the arithmetic aspects of the Hénon map by Silverman [14], Ingram [8] and others, there is little to be found on the dynamics of the Hénon map over non-Archimedean fields. In this paper, we initiate the project of classifying the dynamics over such a field, in particular the field of p-adic numbers $\mathbb{Q}_p$. In the spirit of Devaney-Nitecki [5] and Bedford-Smillie [2], we begin by performing a filtration in order to establish which regions of the parameter space may be of dynamical interest. We then prove that the p-adic Hénon map does indeed take on Horseshoe dynamics for certain values in the parameter space using elementary means, inspired loosely by the methods of Benedetto, Briend and Perdry [3]. We also compile many miscellaneous results regarding the dynamics over several regions in the parameter space, with some of these results number-theoretic in nature. Finally, like Hénon before us, we define and conjecture

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the existence of a strange attractor for some values of the parameter space, and compile some preliminary results inspired in part by Anashin and Khrennikov's work on Algebraic Dynamics [1] characterizing what such an attractor may look like and under which conditions one may be found. We hope our work leads to further research on the p-adic Hénon map, in particular towards confirming the existence of such an attractor.

We begin with a discussion of the field $\mathbb{Q}_p$ and its aspects most relevant to our work. For a more in-depth survey of p-adic analysis, we refer to Gouvêa [6] or Robert [11].

1.1. **Absolute values over fields.** An absolute value over a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ that satisfies the conditions that $|x| = 0$ if and only if $x = 0$, $|xy| = |x||y|$, and $|x + y| \leq |x| + |y|$.

1.2. **The field of p-adic numbers.** For a fixed prime number p, and a non-zero rational $x \in \mathbb{Q}$, we can write $x = p^v \frac{m}{n}$, where m and n are integers not divisible by p. The p-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ is defined by $|x|_p = p^{-v}$ if $x \neq 0$ and $|x|_p = 0$ if $x = 0$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is called the field of p-adic numbers, denoted by $\mathbb{Q}_p$, and $|\cdot|_p$ can be extended uniquely to $\mathbb{Q}_p$. From now on we will only refer to the p-adic absolute value, so we will omit the p and write $|\cdot|_p$ as $|\cdot|$.

Similarly, the p-adic valuation $v_p : \mathbb{Q}_p \to \mathbb{R}$ is defined: $v_p(x) = v$ when $x \neq 0$ and $v_p(x) = \infty$ when $x = 0$.

Proposition 1.1. *The absolute value* $|\cdot|$ *obeys the strong triangle inequality:* $|x + y| \leq \max\{|x|, |y|\}$.

Proposition 1.2. *The absolute value* $|\cdot|$ *obeys the strongest wins property. If* $|x| \neq |y|$*, then* $|x + y| = \max\{|x|, |y|\}$.

In addition, $\mathbb{Q}_p$ is totally disconnected, complete with respect to $|\cdot|$, and locally compact. Because $\mathbb{Q}_p$ is complete and locally compact, it has also has the property that every closed and bounded subset is compact.

1.3. **Ring of p-adic integers.** Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$ and is called the ring of p-adic integers. Similarly, let $p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| < 1\}$. Then $p\mathbb{Z}_p$ is the unique maximal ideal of $\mathbb{Z}_p$. Because $p\mathbb{Z}_p$ is a maximal ideal, $\mathbb{Z}_p/p\mathbb{Z}_p$ is a field which we call the residue field of $\mathbb{Z}_p$. In addition, $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$.

Define the reduction map $\bar{\cdot} : \mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p$ as the natural map that takes p-adic integers to their respective residue class in $\mathbb{Z}_p/p\mathbb{Z}_p$.

1.4. **Hensel's Lemma.** Hensel's Lemma, or sometimes called the p-adic version of Newton's method, gives conditions for when polynomials with p-adic integer coefficients have roots and is stated as follows:

If $f(x)$ is a polynomial in $\mathbb{Z}_p[X]$, and there exists a p-adic integer $b \in \mathbb{Z}_p$ such that $|f(b)| \leq \frac{1}{p}$ and $|f'(b)| = 1$, then there exists a unique root $\beta \in \mathbb{Z}_p$ such that $f(\beta) = 0$, and $|\beta - b| \leq \frac{1}{p}$.

There is a stronger version of Hensel's lemma that gives more general conditions for when polynomials have roots and is sometimes useful. The stronger version is stated as follows:

If $f(x)$ is a polynomial in $\mathbb{Z}_p[X]$, and there exists a p-adic integer $b \in \mathbb{Z}_p$ such that $|f(x)| \leq \frac{1}{p^n}$ and $k = v_p(f'(b)) < \frac{n}{2}$, then there exists a unique root $\beta \in \mathbb{Z}_p$ such that $f(\beta) = 0$, $|\beta - b| \leq \frac{1}{p^{n-k}}$, and $v_p(f'(\beta)) = v_p(f'(x)) = k$.

1.5. **Vector space $\mathbb{Q}_p^n$.** Analogously to the real case, $\mathbb{Q}_p^n$ can be given vector space structure over $\mathbb{Q}_p$ with component wise scalar multiplication. For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Q}_p^n$, the sup-norm $\|\cdot\|_p : \mathbb{Q}_p^n \to \mathbb{R}^+$ is defined by $\|\alpha\|_p = \max\{|\alpha_1|, |\alpha_2|, ..., |\alpha_n|\}$. From now on we will refer only to the p-adic sup-norm, so we will omit the p and write $\|\cdot\|_p$ as $\|\cdot\|$.

1.6. **The Hénon Map.** The Hénon map over $\mathbb{Q}_p^2$ is the function $\phi : \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ defined by:

$$\phi(x, y) = (a + by - x^2, x)$$

For some $a, b \in \mathbb{Q}_p$, and $b \neq 0$. The function ϕ is invertible with inverse:

$$\phi^{-1}(x, y) = (y, \frac{-a + x + y^2}{b})$$

The Hénon map is a polynomial automorphism, meaning it is a polynomial bijection with a polynomial inverse.

1.7. **The filled Julia set.** The forward filled Julia set of ϕ is the set:

$$J^+(\phi) = \{(x, y) \in \mathbb{Q}_p^2 | \ \{\phi^n(x, y)\} \text{ is a bounded with respect to} | \cdot |_p\}$$

The backwards filled Julia set of ϕ is the set:

$$J^-(\phi) = \{(x, y) \in \mathbb{Q}_p^2 | \ \{\phi^{-n}(x, y)\} \text{ is a bounded with respect to} | \cdot |_p\}$$

The filled Julia set of ϕ is the intersection of the forward and backward filled Julia sets:

$$J(\phi) = J^+(\phi) \cap J^-(\phi)$$

2. Generalities

For the remainder of this paper, we standardize the following conventions:

- p is an odd prime.
- In general, when we introduce a point indexed by 0, $\alpha_0 = (x_0, y_0)$, we take $\alpha_k = (x_k, y_k)$ to denote $\phi^k(\alpha_0) = \phi^k(x_0, y_0)$ for $k \in \mathbb{Z}$. We take $\phi^0 : \mathbb{Q}_p \to \mathbb{Q}_p$ to be the identity function $\phi^0(x, y) = (x, y)$.
- We let $D_r(x_0) \subset \mathbb{Q}_p$ denote the *closed* ball of radius r around the point $x_0 \in \mathbb{Q}_p$, that is
  $$D_r(x_0) = \{x \in \mathbb{Q}_p \mid |x - x_0|] \leq r\}.$$
  We let $B_r(x_0, y_0) \subset \mathbb{Q}_p^2$ denote the *closed* ball of radius r around the point $(x_0, y_0) \in \mathbb{Q}_p^2$, that is
  $$B_r(x_0, y_0) = \{(x, y) \mid \|(x, y) - (x_0, y_0)\| \leq r\}.$$
  We let $D_{r_1,r_2}(\alpha) = B_{r_1}(\pi_x(\alpha)) \times B_{r_2}(\pi_y(\alpha)) \subseteq \mathbb{Q}_p^2$ refer to the *closed* polydisc of radius r_1 in the first coordinate and radius r_2 in the second, that is
  $$D_{r_1,r_2}(x_0, y_0) = D_{r_1}(x_0) \times D_{r_2}(y_0)$$
  .

- For some ball $\mathcal{B} = I \times I \subset \mathbb{Q}_p^2$ where $I = D_R(z) \subset \mathbb{Q}_p$, function $f : I \to I$, and $r \leq R$ we denote the *"vertical tube of radius r around f through $\mathcal{B}$"* as

$$T_r^V(f(t)) = T_r^V(f) = \{(f(t)+\theta, t) \mid t \in I, |\theta| \leq r\} = \bigsqcup_{t \in I} D_{r,0}(f(t), t)$$

For a point $t_0 \in I$, we shall refer to the polydisc $D_{r,0}(f(t), t)$ as that tube's *"cross-section at t_0."* We use a mirrored definition for T_r^H, the horizontal tube through $\mathcal{B}$, simply switching the first and second coordinates. As before, we assume that $D_{r,0}(f(t), t)$ is the closed polydisc.

The first of our basic results comes from [5], the proof of which is a routine calculation:

Proposition 2.1 (Devaney-Nitecki). *$\phi_{a,b}(x, y)$ is topologically conjugate to $\phi^{-1}_{\frac{a}{b^2},\frac{1}{b}}$ via the linear map $f(x, y) = (-by, -bx)$, or more precisely*

$$f^{-1} \circ \phi_{a,b} \circ f = \phi^{-1}_{\frac{a}{b^2},\frac{1}{b}}$$

In addition, $J(\phi_{a/b^2,1/b}) = f^{-1}(J(\phi_{a,b}))$.

Lemma 2.2. *Let $T > 0$. ϕ is Lipschitz continuous when restricted to $B_T(\mathbf{0})$ with Lipschitz constant $C_T^+(a, b) = C_T^+ = \max\{|b|, T, 1\}$. More precisely, for all (x, y), $(x', y') \in B_T(\mathbf{0})$,*

$$\|\phi(x, y) - \phi(x', y')\| \leq C_T^+ \|(x, y) - (x', y')\|$$

Proof. Assume (x_0, y_0), $(x_0', y_0') \in B_T(\mathbf{0})$, and let $\delta_x = |x - x'|$, $\delta_y = |y - y'|$. By our choice of norm $\|(x_0, y_0) - (x_0', y_0')\| = \max\{\delta_x, \delta_y\}$. Let $(x_1, y_1) = \phi(x_0, y_0)$; $(x_1', y_1') = \phi(x_0, y_0)$ and write $\|(x_1, y_1) - (x_1', y_1')\| = \|(b(y_0 - y_0') - (x_0^2, {x_0'}^2), x_0 - x_0')\| = \max\{|(b(y_0 - y_0') - (x_0^2 - {x_0'}^2)|, \delta_x\}$. By the strong triangle inequality, $|(b(y_0 - y_0') - (x_0^2 - {x_0'}^2)| \leq \max\{|b|\delta_y, |x_0 + x_0'|\delta_x\}$. As $|x_0|, |x_0'| \leq T$, the strong triangle inequality implies $|x_0 + x_0'| \leq T$. Putting all this together, we now have $\|(x_1, y_1) - (x_1', y_1')\| \leq \max\{|b|\delta_y, T\delta_x, \delta_x\} \leq (\max\{|b|, T, 1\})(\max\{\delta_x, \delta_y\}) = (\max\{|b|, T, 1\}) \|(x_0, y_0) - (x_0', y_0')\|$. □

Lemma 2.3. *We let A,B,C be metric spaces with respective distance metrics d_a, d_b, d_c. For $f : B \to C$, $g : A \to B$ where f, g are Lipschitz continuous with respective Lipschitz constants c_f, c_g, $h = (f \circ g) : A \to C$ is Lipschitz continuous with Lipschitz constant $c_f c_g$.*

Proof. By hypothesis, $d_c(f(x), f(y)) \leq c_f d_b(x, y)$ and $d_b(g(x), g(y)) \leq c_g d_a(x, y)$. Letting $w = g(x), z = g(y)$, we have that $d_c(h(x), h(y)) = d_c(f(w), f(z)) \leq c_f d_b(w, z) = c_f d_b(g(x), g(y)) \leq c_f c_g d_a(x, y)$. □

Corollary 2.4. *Let $T > 0$ once more. ϕ^{-1} is Lipschitz continuous on $B_T(\mathbf{0})$ with Lipschitz constant* $\max\{|\frac{1}{b}|, \frac{T}{|b|}, 1\}$.

Proof. By proposition 2.1, we have that $\phi^{-1}_{a,b} = g^{-1} \circ \phi_{\frac{a}{b^2},\frac{1}{b}} \circ g$ where $g(x, y) = (-\frac{y}{b}, -\frac{x}{b})$. We note that the image of $B_T(\mathbf{0})$ under g is $B_{\frac{T}{|b|}}(\mathbf{0})$, so by lemma 2.2, $\phi_{\frac{a}{b^2}}$ has Lipschitz constant $\max\{|\frac{1}{b}|, \frac{T}{|b|}, 1\}$. g and g^{-1} have Lipschitz constants $\frac{1}{|b|}$, $|b|$ respectively, so by Lemma 2.3, we conclude that $(g^{-1} \circ \phi_{\frac{a}{b^2},\frac{1}{b}} \circ g) = \phi^{-1}_{a,b}$ has Lipschitz constant $\max\{|\frac{1}{b}|, \frac{T}{|b|}, 1\}$ □

Though Bedford and Smillie's paper [2] works in the space $\mathbb{C}^2$ rather than $\mathbb{Q}_p^2$, we can apply similar methods to those they use on pages 56-57 to obtain filtration properties of ϕ.

Proposition 2.5 (Adapted from Bedford-Smillie)**.** *We let* $R = \max\{|a|^{1/2}, |b|, 1\}$*. We partition* $\mathbb{Q}_p^2$ *into three sets* S*,* S_+ *and* S_- *where* $S = B_R(\mathbf{0})$*,* $S_+ = \{(x,y) \in \mathbb{Q}_p^2 \mid |x| \geq |y|;\ |x| > R\}$*, and* $S_- = \{(x,y) \in \mathbb{Q}_p^2 \mid |x| \leq |y|;\ |x| > R\}$*. We also let* $I = D_R(0) \subset \mathbb{Q}_p$ *and note that* $S = I \times I$*. The following filtration properties then hold:*

1) $\phi(S_+) \subset S_+$*;* $\phi^{-1}(S_-) \subset S_-$
2) For all $(x,y) \in S_\pm$*,* $\lim_{n\to\infty} \|\phi^{\pm n}(x,y)\| = \infty$
3) $\phi(S_-) \cap S_+ \neq \emptyset$ *and* $\phi^{-1}(S_+) \cap S_- \neq \emptyset$
4) $\phi(S) \cap S_- = \emptyset$ *and* $\phi^{-1}(S) \cap S_+ = \emptyset$
5) $\phi(S) \cap S_+ \neq \emptyset$ *if and only if* $\max\{|a|, |b|\} > 1$
6) $\phi^{-1}(S) \cap S_- \neq \emptyset$ *if and only if* $|b| < \max\{|a|^{1/2}, 1\}$
7) $\phi(S) \cap S = \emptyset$ *if and only if* $|a| > \max\{|b|^2, 1\}$ and *a is not a square in* $\mathbb{Q}_p$

Proof. We show Claims 1) and 2) only for S_-; the proof for S_+ can be acquired either through similar methods or an argument invoking proposition 2.1. We suppose $(x_0, y_0) \in S_-$ and then have that $(x_{-1}, y_{-1}) = (y_0, \frac{1}{b}(x_0 + y_0^2 - a))$. We note $|y_0^2| > |y_0| \geq |x_0|$ and $|y_0^2| > |a|$ by assumption, so $|y_{-1}| = |1/b||x_0 + y_0^2 - a| = |1/b||y_0^2| > |y_0| = |x_{-1}|$. Thus, $(x_{-1}, y_{-1}) \in S_-$ and $\|(x_{-1}, y_{-1})\| > \|(x_0, y_0)\|$, so inductively, $\|(x_{-n}, y_{-n})\| \to \infty$.

To see that Claim 3) holds, we need only select $(x_0, y_0) \in S_-$ such that $|y_0| \geq |x_0| > |by_0|^{1/2} > |a|^{1/2}$. We note that such a selection is indeed possible, as we can always select y_0 such that $|y_0| > |b|^2$. We then have that $|x_1| = |a + by_0 - x_0^2| \leq \max\{|a|, |by_0|, |x_0^2|\}$. We have by assumption that $|x_0^2| > |a|$, and as $|x_0| > |by_0|^{1/2}$, we have that $|x_0| > |by_0|$. By a strongest wins argument, we then have that $|x_1| = |x_0|^2 > |y_1| = |x_0|$, so $(x_1, y_1) \in S_+$. We've thus identified points $(x_0, y_0) \in S_- \cap \phi^{-1}(S_+)$ and $(x_1, y_1) \in S_+ \cap \phi(S_-)$.

Claim 4) is a simple corollary of Claim 1)– if $\phi(S) \cap S_- \neq \emptyset$, we then have that $S \cap \phi^{-1}(S_-) \neq \emptyset$, which is a contradiction as it has been shown that $\phi^{-1}(S_-) \subset S_-$. An identical argument shows $\phi^{-1}(S) \cap S_+ = \emptyset$.

To see the forward implication of claim 5), we show the contrapositive: If $\max\{|a|, |b|\} \leq 1$, we then have that $S = \mathbb{Z}_p^2$, and ϕ has $\mathbb{Z}_p$ coefficients, so $\phi(S) \subset S$. To show the other direction, we split into two cases. If $|a| \geq |b|$ with $|a| > 1$, we then have that $|a| > R$, so $\phi(0,0) = (a, 0) \in S_+$. If $|b| > \max\{|a|, 1\}$, we then have that $(b, 0) \in S$ and $|b^2| > R$, so $\phi(b, 0) = (b^2, b) \in S_+$, thus establishing the two-way implication.

To see the forward implication of claim 6), we again prove the contrapositive: if $|b| \geq \max\{|a|^{1/2}, 1\}$, then $R = |b|$ and recalling that $\phi^{-1}(x,y) = (y, \frac{-a}{b} + \frac{x}{b} + \frac{y^2}{b})$, we see that as $|y|, |x|, |a| \leq |b|$,

$$\|\phi(x,y)\| \leq \max\{|y|, \left|\frac{-a}{b}\right|, \left|\frac{x}{b}\right|, \left|\frac{y^2}{b}\right|\} \leq \max\{|b|, \left|\frac{-b^2}{b}\right|, \left|\frac{b}{b}\right|, \left|\frac{b^2}{b}\right|\} \leq |b| = R$$

so $\phi^{-1}(S) \subset S$. To show the other direction of Claim 6), we suppose $|b| < \max\{|a|^{1/2}, 1\} = R$ and then have that $|\frac{a}{b}| > \frac{|a|}{|a|^{1/2}} = |a|^{1/2}$, and as $|a| > |b|$, $|\frac{a}{b}| > 1$. Therefore $\phi^{-1}(0,0) = (0, \frac{a}{b}) \in S_-$, wrapping up our proof of claim 6).

Claim 7) will be a corollary of our later work. However, we can see now that $\phi(S) \cap S = \emptyset \implies |a| > \max\{|b|^2, 1\}$, as claim 3) implies that for any such ϕ, we must have that $\phi(S) \subset S_+$ and $\phi^{-1}(S) \subset S_-$, and claims 4 and 5 then combine to imply $|a| > \max\{|b|^2, 1\}$. □

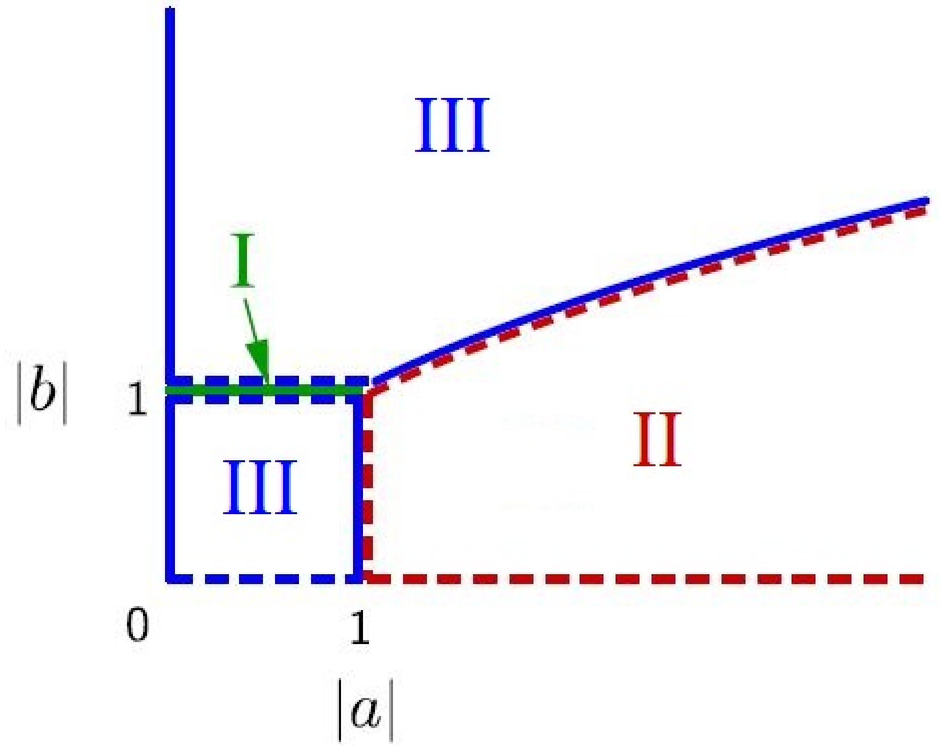

FIGURE 1. Regions I, II, and III

We note several consequences of proposition 2.5, the first two of which follow immediately from claims 1) and 2)

Corollary 2.6. *$J_+(\phi) \subseteq S_- \cup S$ and $J_-(\phi) \subseteq S_+ \cup S$.*

Corollary 2.7. *$J(\phi) \subseteq S$*

The filtration properties of ϕ motivate our partition of the parameter space into three regions.

Definition 2.8. As shown in figure 2.8, we partition our parameter space $\mathbb{Q}_p \times (\mathbb{Q}_p \setminus \{0\})$ into three regions. Let

- Region I$= \{(a, b) \mid |a| \leq 1,\ |b| = 1\}$
- Region II$= \{(a, b) \mid |a| \leq \max\{|b|^2, 1\},\ |b| \neq 1\}$
- Region III$= \{(a, b) \mid |a| > \max\{|b|^2, 1\}\}$

In particular, we see that when (a, b) is in Region III, both claim 5 and 6 hold; when (a, b) is in Region II one and only one of the two do; and when (a, b) is in Region I, neither does.

Corollary 2.9. *In Region III, one of $J_+(\phi)$ and $J_-(\phi)$ are unbounded.*

Proof. Without loss of generality, assume $|b| < 1$; the case $|b| > 1$ is identical with the indices $+$ and $-$ swapped. When $|b| < 1$, $|a| \leq 1$, claim 5) of proposition 2.5 holds, but claim 4) does not. Thus, points S_- are mapped into S by ϕ, but $\phi(S) \subsetneq S$–in other words, points are mapped into S, but never out. This implies that

$$J_+(\phi) = \bigcap_{k \geq 0} \phi^{-k}(S).$$

As $\phi^{-1}(S) \cap S_- \neq \emptyset$, we then have by claim 2) of proposition 2.5 that $J_+(\phi)$ is unbounded, as all points in S_- tend towards infinity upon iteration by ϕ. □

We shall investigate the (quite distinct!) dynamics for each region of the parameter space after getting a few more preliminary statements out of the way.

Proposition 2.10. *$J^+(\phi)$, $J^-(\phi)$, and $J(\phi)$ are closed. In addition, $J(\phi)$ is compact.*

Proof. Suppose $\alpha \notin J^+(\phi)$. Then by Proposition 2.5, $\phi^N(\alpha) \in S^+$ for some $N \geq 0$. Because S^+ is open, there exists some $0 < \epsilon < 1$ such that $B_\epsilon(\phi^N(\alpha)) \subset S^+$. We want to choose a $\delta > 0$ such that $\beta \in B_\delta(\alpha)$ implies $\phi^N(\beta) \in B_\epsilon(\phi^N(\alpha))$, then by proposition 2.5 we will know that $\beta \notin J(\phi)$. Note that we can choose $T > 1$ such that $\{\alpha, \phi(\alpha), \phi^2(\alpha), ..., \phi^N(\alpha)\} \subset B_T(\mathbf{0})$. Because ϕ is Lipschitz continuous on $B_T(\mathbf{0})$, there exists $C > 1$ such that for all $z, z' \in B_T(\mathbf{0})$, $\|\phi(z) - \phi(z')\| \leq C\|z - z'\|$.Choose $\delta = \frac{\epsilon}{C^N}$. Then $\|\phi^N(\alpha) - \phi^N(\beta)\| \leq C^N\|\alpha - \beta\| < C^N\delta = \epsilon$. Therefore $J^+(\phi)^c$ is open, so $J^+(\phi)$ is closed. An identical argument with the appropriate filtration for ϕ^{-1} shows that $J^-(\phi)$ is also closed. In addition, by corollary 2.7 $J(\phi)$ is bounded. As $\mathbb{Q}_p^2$ has the Heine-Borel property, it follows that $J(\phi)$ is compact. □

Lemma 2.11. *If $A = r^2$ such that $r \in \mathbb{Q}_p$, then when $|B| < |A|$ there exists $r' \in \mathbb{Q}_p$ such that $A + B = (r')^2$. Furthermore, we can choose r' such that $|r - r'| < |\frac{B}{r}|$.*

Proof. We let $A^* = p^{-v_p(A)}A$, $B^* = p^{-v_p(A)}B$ and $r^* = p^{-v_p(r)}r$. As $v_p(A) = 2v_p(r)$, we are still assured $(r^*)^2 = A^*$. As $A^* \in \mathbb{Z}_p^\times$ and $|B^*| < |A^*|$, we now have that $A^* + B^* \in \mathbb{Z}_p$, so the polynomial $F(z) = z^2 - (A^* + B^*) \in \mathbb{Z}_p[z]$, and furthermore has good reduction. We apply the reduction map to obtain $\overline{F(z)} = z^2 - \overline{A^*}$. As $A^* = (r^*)^2$, we have that $\overline{F(r^*)} = 0$ and $\overline{F'(r^*)} = 2\overline{r^*} \neq 0$. We then have by Hensel's Lemma that there exists $\zeta \in \mathbb{Z}_p$ such that $F(\zeta) = 0$, implying $\zeta^2 = A^* + B^*$. Furthermore, by the statement of Hensel's Lemma given by [11], as $F(r^*) \equiv 0 \mod B^*\mathbb{Z}_p$ and $|F'(r^*)| = 1$, we can conclude that $\zeta \equiv r^* \mod B^*\mathbb{Z}_p$. Letting $r' = p^{v_p(r)}\zeta$, we now have $(r')^2 = A + B$, and as $|\zeta - r^*| \leq |B^*|$, we have $|r - r'| \leq |p^{v_p(r)}B^*| = |\frac{B}{r}|$. Finally, we note that as $|\frac{B}{r}| < |\frac{A}{r}| = |r|$, we have that the neighborhoods containing the two roots $B_{|\frac{B}{r}|}(\pm r)$ as stated in the proposition are distinct. □

Remark 2.1. As we saw in the proof of the above lemma, we have that one possible value of $\sqrt{A + B}$ is in $B_{|\frac{B}{r}|}(\pm r)$–in general, when approaching square roots of these forms where A is a constant and B is an indeterminate such that $|A| > |B|$, we shall take the positive square root to be the one closest to the root of A we declare to be its "canonical" square root. Further, as a corollary via the strongest wins property, we have that $|\sqrt{A + B}| = |r|$.

Proposition 2.12. *ϕ has period one points if and only if $(b-1)^2+4a$ is a square in $\mathbb{Q}_p$. ϕ has points of minimal period two if and only if $-3(b-1)^2+4a$ is a square in $\mathbb{Q}_p$. When those conditions are met, ϕ has fixed points:*

$$\left(\frac{(b-1)\pm\sqrt{4a+(b-1)^2}}{2}, \frac{(b-1)\pm\sqrt{4a+(b-1)^2}}{2}\right)$$

and minimal period two points:

$$\left(\frac{(1-b)\pm\sqrt{4a-3(b-1)^2}}{2}, \frac{(1-b)\mp\sqrt{4a-3(b-1)^2}}{2}\right)$$

Proof. This follows from simple calculations based on the fact that ϕ has fixed points if and only if $\phi(x,y)=(x,y)$, and period two points if and only if $\phi^2(x,y)=(x,y)$. □

3. Region I: Good Reduction

Definition 3.1. In the case where $|a|\leq 1$ and $|b|=1$, we say that phi has good reduction. The motivation for this definition is that in the region $|a|\leq 1$, $|b|=1$, both ϕ and ϕ^{-1} reduces to a Henon map over $\mathbb{F}_p^2$.

Proposition 3.2. *The following are equivalent:*

(1) ϕ has good reduction.
(2) $J(\phi)=\mathbb{Z}_p^2$.
(3) $\phi(\mathbb{Z}_p^2)=\mathbb{Z}_p^2$.
(4) $J(\phi)$ is a polydisc.

Proof. Suppose ϕ has good reduction. Suppose $|x|\leq 1$ and $|y|\leq 1$. Then $\|\phi(x,y)\| = \max\{|a+by-x^2|,|x|\}$. Note that $|a+by-x^2|\leq\max\{|a|,|by|,|x^2|\}\leq 1$, and $|x|\leq 1$. So $\|\phi(x,y)\|\leq 1$. Iterating, we have $\|\phi^n(x,y)\|\leq 1$ for all $n\geq 1$, so $(x,y)\in J^+(\phi)$. Similarly, $\|\phi^{-1}(x,y)\| = \max\{|y|,|-\frac{a}{b}+\frac{1}{b}x+\frac{1}{b}y^2|\}\leq\max\{|y|,|\frac{a}{b}|,|\frac{1}{b}x|,|\frac{1}{b}y^2|\}\leq 1$. Iterating the argument, we have $\|\phi^{-n}(x,y)\|\leq 1$. So $(x,y)\in J^-(\phi)$. Therefore, $(x,y)\in J^-(\phi)\cap J^+(\phi)=J(\phi)$.

Suppose $|x|>1$ and $|x|\geq|y|$. Then $\|\phi(x,y)\|=\max\{|a+by-x^2|,|x|\}$. However, $|a+by-x^2|=|x^2|=|x|^2>|x|$. So $\|\phi(x,y)\|=|x|^2$. Moreover, $\|\phi^n(x,y)\|=|x|^{2^n}\to\infty$. So $(x,y)\notin J^+(\phi)$.

Similarly, suppose $|y|>1$ and $|y|\geq|x|$. Then $\|\phi^{-1}(x,y)\|=\max\{|y|,|-\frac{a}{b}+\frac{1}{b}x+\frac{1}{b}y^2|\}=|y|^2$. Moreover, $\|\phi^{-n}(x,y)\|=|y|^{2^n}\to\infty$. So $(x,y)\notin J^-(\phi)$. Therefore $J(\phi)=\mathbb{Z}_p^2$.

Suppose $J(\phi)=\mathbb{Z}_p^2$. Then because $J(\phi)$ is invariant under ϕ, then $\phi(\mathbb{Z}_p^2)=\mathbb{Z}_p^2$.

Suppose $\phi(\mathbb{Z}_p^2)=\mathbb{Z}_p^2$. Then because $(0,0)\in\mathbb{Z}_p^2$ and $\|\phi(0,0)\|=\|(a,0)\|=|a|$, then it must be true that $|a|\leq 1$. Also, if $|b|>1$, and $|y|=1$, then by strongest wins, $|a+by-x^2|=|by|=|b|>1$, contradiction. So $|b|\leq 1$. Note that $\phi^{-1}(\mathbb{Z}_p^2)=\mathbb{Z}_p^2$. So if $(x,y)=(a+1,0)\in\mathbb{Z}_p^2$, then $\|\phi^{-1}(a+1,0)\|=|\frac{1}{b}|\leq 1$. Therefore, $|b|=1$, so ϕ has good reduction.

The proof that statements 1 through 3 are equivalent to the statement that $J(\phi)$ is a polydisc follows after further study of regions II and III. □

Proposition 3.3. *If $|a| < 1$ and $|b-1| > \frac{1}{p}$, then ϕ has fixed points. In addition, if -3 is a quadratic residue $\mod p$, then ϕ has minimal period two points.*

Proof. Consider the functions $g(z) = z^2 - (4a + (b-1)^2)$ and $f(z) = z^2 - (4a + 3(b-1)^2)$ and apply Hensel's lemma. □

4. Region II: The Horseshoe

Theorem 4.1. *We assume $|a|^{1/2} > \max\{|b|, 1\}$. If $a = \gamma^2$ for some $\gamma \in \mathbb{Q}_p$, then $J(\phi)$ is a topological horseshoe, with the action of ϕ upon it conjugate to the 2-shift. Otherwise, $J(\phi)$ is empty.*

As much of the "usual machinery" assumes connectedness, it is inaccessible to us. As a consequence, the proof of 4.1 is lengthy, with many moving parts to examine. We break it up over the following sections:

(1) **Preliminaries:** In which we prove that $J(\phi) = \emptyset$ when a is not a square in $\mathbb{Q}_p$ and establish some basic facts and necessary terminology regarding $J(\phi)$ –most importantly that $J(\phi)$ is contained withing the union of two disjoint subsets of S we name A_+ and A_-.
(2) **Finite and Infinite Trajectory Pre-images** In which we seek to obtain a description of the points of S that follow a given trajectory through the sets A_+, A_- for finitely many iterations. We then use that description to characterize the set of points that follow a given infinite trajectory.
(3) **Extending our results to ϕ^{-1}** In which we use a conjugacy argument to find a characterization of the points whose preimages follow a given trajectory through A_+ and A_-
(4) **Proof the Main Theorem**

4.1. Preliminaries.

We begin by recalling that by Corollary 2.7, we have that $J(\phi) \subset S = B_{|a|^{1/2}}(\mathbf{0})$. In fact, we can say something stronger.

Lemma 4.2. *For (a, b) in Region II, $J(\phi) \subseteq H = \left\{(x, y) \in \mathbb{Q}_p^2;\ |x| = |y| = |a|^{1/2}\right\}$*

Proof. First we will prove the proposition for the case $|a|^{1/2} > |b| \geq 1$. Suppose $\|(x, y)\| > |a|^{1/2}$. Then $\|(x, y)\| > |b|$ and $\|(x, y))\| > |\frac{1}{b}|$. So by Proposition 2.5, $(x, y) \notin J(\phi)$. Suppose $|x| < |a|^{1/2}$ and $|y| \leq |a|^{1/2}$. Then $\|\phi(x, y)\| = \max\{|a + by - x^2|, |x|\}$. Since $|x|^2 < |a|$ and $|by| \leq |b||a|^{1/2} < |a|^{1/2}|a|^{1/2} = |a|$, then by strongest wins $|a + by - x^2| = |a|$. So $\|\phi(x, y)\| = |a| > |a|^{1/2}$. So after one iteration of ϕ we are in our previous case, so $(x, y) \notin J(\phi)$. Similarly, suppose $|x| \leq |a|^{1/2}$ and $|y| < |a|^{1/2}$. Then $\|\phi^{-1}(x, y)\| = \max\{|y|, |\frac{1}{b}||-a+x+y^2|\}$. Since $|y|^2 < |a|$ and $|x| \leq |a|^{\frac{1}{2}} < |a|$, then by strongest wins $\max\{|y|, |\frac{1}{b}||-a + x + y^2|\} = \max\{|y|, |\frac{a}{b}|\}$. However, $|y| < |a|^{1/2} = \frac{|a|}{|a|^{1/2}} < |\frac{a}{b}|$. So $\|\phi^{-1}(x, y)\| = |\frac{a}{b}| > |a|^{1/2}$, so $(x, y) \notin J(\phi)$.

Now when conjugating ϕ with $f(x, y) = (-by, -bx)$, we know that $\phi_{a,b} \sim \phi^{-1}_{\frac{a}{b^2}, \frac{1}{b}}$. Note that $|a|^{1/2} > |b| \geq 1$ if and only if $|\frac{a}{b^2}| > 1 \geq |\frac{1}{b}|$, and $J(\phi^{-1}_{\frac{a}{b^2}, \frac{1}{b}}) = J(f^{-1} \circ \phi_{a,b} \circ f) = f^{-1}(J(\phi_{a,b}))$.

Therefore, $J(\phi_{\frac{a}{b^2},\frac{1}{b}}) \subset f^{-1}(H) = \{(x,y) \in \mathbb{Q}_p^2;\ |x| = |y| = |\frac{a}{b^2}|^{1/2}\}$. Substituting $a^* = \frac{a}{b^2}$ and $b^* = \frac{1}{b}$, we get $J(\phi_{a^*,b^*}) \subseteq \left\{(x,y) \in \mathbb{Q}_p^2;\ |x| = |y| = |a^*|^{1/2}\right\}$ where $|a^*|^{\frac{1}{2}} > 1 \geq |b^*|$. □

Proposition 4.3. *For (a,b) in Region II, $J(\phi) \neq \emptyset$ if and only if $a = \gamma^2$ for some $\gamma \in \mathbb{Q}_p$.*

Proof. Suppose $|a|^{\frac{1}{2}} > \max\{|b|, 1\}$ and $(x_0, y_0) \in J(\phi)$. We then have that $|x_k| = |y_k| = |a|^{1/2}$ for all $k \in \mathbb{Z}$ by Lemma 4.2. We let

$$f(z) = z^2 - 1 + \frac{by_0 - x_1}{x_0^2}$$

We note that $|\frac{by_0 - x_1}{x_0^2}| = \frac{1}{|a|}|by_0 - x_1| \leq \frac{1}{|a|}\max\{|by_0|, |x_1|\} = \frac{1}{|a|}\max\{|b||a|^{\frac{1}{2}}, |a|^{\frac{1}{2}}\}$. If $|b| \leq 1$, then $\frac{1}{|a|}\max\{|b||a|^{\frac{1}{2}}, |a|^{\frac{1}{2}}\} = \frac{|a|^{1/2}}{|a|} = |a|^{-1/2} < 1$. If $|b| > 1$, then $\frac{1}{|a|}\max\{|b||a|^{\frac{1}{2}}, |a|^{\frac{1}{2}}\} = \frac{|b|}{|a|^{1/2}} < \frac{|a|^{1/2}}{|a|^{1/2}} = 1$, so $|\frac{by_0 - x_1}{x_0^2}| < 1$. Thus, $f \in \mathbb{Z}_p[z]$, so we may attempt to apply Hensel's lemma. By our observation above, $\frac{by_0 - x_1}{x_0^2} \equiv 0 \mod p$. Thus, $\overline{f(1)} = 1^2 - 1 = 0$ and $\overline{f'(1)} = 2 \neq 0$, so Hensel's lemma ensures that there exists a root $r \in \mathbb{Q}_p$ such that

$$f(r) = r^2 - 1 + \frac{by_0 - x_1}{x_0^2} = 0.$$

Noting that $by_0 - x_1 = x_0^2 - a$, this simplifies to $(rx_0)^2 = a$. Letting $\gamma = rx_0 \in \mathbb{Q}_p$, this proves the proposition. □

We have now completely characterized the dynamics when a is not a square. Henceforth, we shall assume $a = \gamma^2$ where $\gamma \in \mathbb{Q}_p$.

Proposition 4.4. $\phi^{-1}(S) \cap (S) = A_+ \sqcup A_-$ *where* $A_\pm = \{(x,y) \in \mathbb{Q}_p^2 \mid |\sqrt{a + by} \mp x| \leq 1\}$

Proof. We can rephrase our characterization of the set $\phi^{-1}(S) \cap S$ as $\phi^{-1}(S) \cap S = \{(x,y) \in S \mid \phi(x,y) \in S\} = \{(x,y) \in \mathbb{Q}_p \mid \|(x,y)\|, \|\phi(x,y)\| \leq |\gamma|\}$. We assume $(x,y) \in S$ and write $\phi(x,y) = (a + by - x^2, x)$. We recall that $|x| \leq |\gamma|$ and thus have that $(x,y) \in \phi^{-1}(S) \cap (S)$ if and only if $|a + by - x^2| \leq |\gamma|$. By Lemma 2.11, $a + by$ is a square in $\mathbb{Q}_p$ as $|by| \leq |b\gamma| < |\gamma^2| = |a|$. Keeping our conventions of Lemma 2.11, we let the root of $a + by$ near γ be written as the positive square root. We now have that $(x,y) \in \phi^{-1}(S) \cap (S)$ if and only if $|\sqrt{a + by} - x||\sqrt{a + by} + x| \leq |\gamma|$. We claim that one of $|\sqrt{a + by} - x|$ and $|\sqrt{a + by} + x|$ has absolute value $|\gamma|$. To see this, we note that as $|\sqrt{a + by}| = |\gamma|$, if $|\sqrt{a + by} \pm x| \leq |\gamma|$, by the strongest wins property, we must have that $|x| = \gamma$. We can then write that $|\sqrt{a + by} \mp x| = |\left(\sqrt{a + by} \pm x\right) \mp 2x|$. By the strongest wins property (and because $p \neq 2$), we then have that $|\sqrt{a + by} \mp x| = |2x| = |\gamma|$ as desired. Without loss of generality, we then assume that $|\sqrt{a + by} \mp x| = \gamma$, and have that $(x,y) \in \phi(S) \cap S$ if and only if

$$|\sqrt{a + by} \pm x| \leq \frac{|\gamma|}{|\sqrt{a + by} \mp x|} = 1,$$

establishing that

$$\begin{aligned}\phi(S)\cap S &= \{(x,y)\in S \mid |\sqrt{a+by}\pm x|\leq 1\}\\ &= \{(x,y)\in S \mid |\sqrt{a+by}-x|\leq 1\}\sqcup\{(x,y)\in S \mid |\sqrt{a+by}+x|\leq 1\}\end{aligned}$$

We let $\ell_0^{\pm}(t) = \pm\sqrt{a+bt}$ and let $A_+ = \{(x,y)\in S \mid |\sqrt{a+by}-x|\leq 1\} = T_1^V(\ell_0^+)$ and $A_- = \{(x,y)\in S \mid |\sqrt{a+by}+x|\leq 1\} = T_1^V(\ell_0^-)$ □

Proposition 4.5. *$\phi(S)\cap S = B_+\sqcup B_-$ where $B_\pm = T_{|b|}^H(\omega_0^\pm)$ and $\omega_0^\pm(t) = \pm\sqrt{a-t}$.*

The proof of proposition 4.5 is much the same as that of proposition 4.4 and is thus omitted. The difference in the width of $B_\pm$ from that of $A_\pm$ comes from the b^{-1} factor in our operative inequality $|\frac{-a+x+y^2}{b}|\leq|\gamma|$, or equivalently, $|-a+x+y^2| = |b\gamma|$. We also remark that as we define the positive square root function to return the value closer to γ (in this case), we necessarily have that $\phi(A_\pm) = B_\pm$, as $\phi(\phi^{-1}(S)\cap S) = \phi(S)\cap S$ and $\pi_x(A_\pm) = \pi_y(\phi(A_\pm))$.

Corollary 4.6. *$\ell_0^\pm : I\to I$ has Lipschitz constant $|\frac{b}{\gamma}|$*

Proof. We let $t,t'\in I$ be arbitrary and write $|\ell_0^\pm(t)-\ell_0^\pm(t')| = |\sqrt{a+bt}-\sqrt{(a+bt)-b(t-t')}|$. By lemma 2.11, we have that $|\ell_0^\pm(t)-\ell_0^\pm(t')|\leq|\frac{b}{\sqrt{a+bt}}||t-t'| = |\frac{b}{\gamma}||t-t'|$. □

Corollary 4.7. *When $|b|\leq 1$, $|a|>1$, $A_\pm = T_1^H(\gamma) = D_{1,|\gamma|}(\gamma,0)$*

Proof. By lemma 2.11, we have that $|\sqrt{a+by}-|\gamma||\leq|\frac{by}{\gamma}|\leq 1$. Thus, for each cross-section at vertical coordinate t_0, we have that $(\gamma,t_0)\in D_{1,0}(\sqrt{a+bt_0},t_0)$ and thus $D_{1,0}(\sqrt{a+bt_0},t_0) = D_{1,0}(\gamma,t_0)$ □

Corollary 4.8. *When $|b|>1$, A_+ and A_- can each be partitioned into the disjoint union of $|b| = p^w$ polydiscs of polyradius $(1,|\gamma|/|b|)$*

Proof. By corollary 4.6, if $|y-y'|\leq|\gamma|/|b|$, then $|\ell_0^\pm(y)-\ell_0^(y')|\leq 1$. As all points in a p-adic disc are the center of that disc, we then have that the horizontal cross-sections of A_+ at vertical coordinates y,y' project in the x-coordinate to the same subset of $I\subset\mathbb{Q}_p$. By an elementary congruence argument, I can be partitioned into p^w discs $\mathcal{D}_0,\mathcal{D}_1,\ldots,\mathcal{D}_{p^w-1}$ of radius $|\gamma|/|b|$. Letting y_i^* be an arbitrary point in $\mathcal{D}_i$, we now have that

$$A_\pm = \bigsqcup_{k=0}^{p^w-1}\overline{D}_{1,|\gamma|/|b|}\left(\ell_0^\pm(y_i^*),y_i^*\right)$$

□

4.2. **Finite and Infinite Trajectory Pre-images.**

Definition 4.9. Let $X = \{s = \ldots s_{-2}s_{-1}.s_0s_1\ldots \mid s_i\in\{+,-\}\}$ be the space of two-sided sequences in two symbols, in our case letting those symbols be $+$ and $-$. X is here equipped with one of many canonical choices of distance metrics, our choice being for $w,s\in X$

$$d(s,w) = e^{-\min\{|m| \mid s_m\neq w_m\}}$$

In keeping with canon, X is naturally equipped with the self-homeomorphism $\sigma : X \to X$ where $\sigma(s)_k = s_{k-1}$.

Definition 4.10. For $s \in X$, we denote the *length-m pre-image of forward trajectory s* and the *infinite pre-image of forward trajectory s* respectively as

$$V_m^s = \bigcap_{k=0}^{m} \phi^{-k}(A_{s_k}), \quad V^s = \bigcap_{k=0}^{\infty} \phi^{-k}(A_{s_k}).$$

Similarly, we denote the *length-m pre-image of backward trajectory s* and the *infinite pre-image of backward trajectory s* respectively as

$$H_m^s = \bigcap_{k=-(m+1)}^{-1} \phi^{-k}(A_{s_k}), \quad H^s = \bigcap_{k=-\infty}^{-1} \phi^{-k}(A_{s_k}).$$

We proceed to our main project of the section by utilizing a logical trick Gouvêa [6] links spiritually to Saint Anselm of Canterbury: assuming the existence of an object, determining the properties such an object would have, and then using those properties to prove the existence of such an object. In particular, we wish to show that V^s is non-empty for all $s \in X$, and is moreover a vertical tube of radius 0.

Lemma 4.11. *(1)* $V_m^s \subset V_{m-1}^s$
(2) $\phi(V_m^s) \subset V_{m-1}^{\sigma(s)}$

Proof. The proofs of each statement consists simply of recalling definitions. The first statement is obvious once the definitions of V_m^s and V_{m-1}^s are written out:

$$V_m^s = \bigcap_{k=0}^{m} \phi^{-k}(A_{s_k}) \subset \bigcap_{k=0}^{m-1} \phi^{-k}(A_{s_k}) = V_{m-1}^s$$

The second statement follows with only slightly more difficulty

$$\begin{aligned}
\phi\left(V_m^s\right) &= \phi\left(\bigcap_{k=0}^{m} \phi^{-k}(A_{s_k})\right) \\
&= \bigcap_{k=0}^{m} \phi^{1-k}(A_{s_k}) \\
&= \phi(A_{s_0}) \cap \left(\bigcap_{k=0}^{m-1} \phi^{-k}(A_{\sigma(s)_k})\right) \\
&= \phi(A_{s_0}) \cap V_{m-1}^{\sigma(s)} \subset V_{m-1}^{\sigma(s)}
\end{aligned}$$

□

Corollary 4.12. *For all* $s \in X$*,* $m \in \mathbb{N}$*,* $(x, y) \in V_m^s$*, we have* $|x| = |\gamma|$*.*

Definition 4.13. We let $c_m = c_m(a, b) = |\frac{1}{\gamma^m}|$, and $d_m = d_m(a, b) = \frac{1}{|\gamma|^{m-1}|b|}$.

Proposition 4.14. *For all* $m \geq 0$*, if* $(x_0, y_0) \in V_m^s$*, then* $D_{c_m,d_m}(x_0, y_0) \subseteq V_m^s$

Proof. For the case $m \geq 1$, we break Proposition 4.14 up into three lemmas, then show that those lemmas imply the proposition.

Lemma 4.15. *Proposition 4.14 holds for the special case $m = 0$.*

Proof. $V_0^s = A_{s_0}$, $C_0 = 1$ and $d_0 = |\frac{\gamma}{b}|$. By Corollaries 4.7 and 4.8 (and recalling that any point in a p-adic ball can be written to be at its center), we have that $D_{1,|\frac{\gamma}{b}|}(x_0, y_0) \subset A_{s_0}$ for all $(x_0, y_0) \in A_{s_0}$, establishing our claim for the special case. □

Lemma 4.16. $B_{c_m}(x_0, y_0) \subset V_m^s$ *for all* $(x_0, y_0) \subset V_m^s$

Proof. We proceed by strong induction on m, letting lemma 4.15 serve as our basis. We assume for our inductive hypothesis that $B_{c_m}(x_0, y_0) \subset V_m^s$ for all $m \leq (n-1)$. By proposition 2.2, we have that ϕ has Lipschitz constant $|\gamma|$ on S. Thus for any $(x_0', y_0') \in B_{c_n}(x_0, y_0)$, we have that $\|\phi(x_0, y_0) - \phi(x_0', y_0')\| \leq \gamma c_n = c_{n-1}$. By our inductive hypothesis and lemma 4.11, we then have that $\phi(x_0, y_0) \in V_{n-1}^{\sigma}(s)$, and as $B_{c_n}(x_0, y_0) \subset A_{s_0}$, we have that $(x_0', y_0') \in \phi^{-1}(V_{n-1}^{\sigma}(s)) \cap A_{s_0} = V_n^s$. □

Lemma 4.17. $D_{0,d_m}(x_0, y_0) \subset V_m^s$ *for all* $(x_0, y_0) \subset V_m^s$

Proof. We let $(x_0', y_0') = (x_0, y_0') \in D_{0,d_m}(x_0, y_0)$ and let $\phi(x_0, y_0) = (x_1, y_1)$ and $\phi(x_0', y_0') = (x_1', y_1')$. We write

$$\begin{aligned}
\|(x_1, y_1) - (x_1', y_1')\| &= \|\big(b(y_0 - y_0') - (x_0^2 - {x_0'}^2), x_0 - x_0'\big)\| \\
&= \|(b(y_0 - y_0'), 0)\| = |b||y_0 - y_0'| \\
&\leq |b| d_m = c_{m-1}
\end{aligned}$$

We then have by Lemma 4.16 that $(x_1', y_1') \in V_{m-1}^{\sigma(s)}$, and as $D_{0,d_m}(x_0, y_0) \subset A_{s_0}$, we conclude that $(x_0', y_0') \in \phi^{-1}(V_{m-1}^{\sigma}(s)) \cap A_{s_0} = V_m^s$ □

To wrap up the proof of proposition 4.14, we simply note that for all $(x_0', y_0') \in D_{c_m,d_m}(x_0, y_0)$, we have that $(x_0', y_0') \in D_{0,d_m}(x_0', y_0)$, and $(x_0', y_0) \in B_{c_m}(x_0, y_0)$. By lemmas 4.16 and 4.17, we can conclude that $(x_0', y_0') \in V_m^s$. □

We have now assembled the machinery we need to attack non-emptiness of V_m^s and V^s.

Proposition 4.18. *When $a = \gamma^2$, $|\gamma| > \max\{|b|, 1\}$, for all $n \in \mathbb{N}$, $s \in X$ $V_n^s = T_{|\gamma|^{-n}}^V(\ell_n^s(t))$ where $\ell_n^s : I \to I$ is a continuous function.*

Proof. We use induct on n. For the purposes of our induction, as we must call on V_{n-1}^s, we let $V_{-1}^s = S = T_{|\gamma|}^V(0)$, which fulfills that $\phi(V_0^s) \subset V_{-1}^s$ and that V_{-1}^s is a tube of width $\frac{1}{|\gamma|^{-1}} = |\gamma|$ as needed. Along with that we have that $V_0^s = A_{s_0} = T_{|\gamma|^0}^V(\ell_0^{s_0})$, which establishes a basis.

For the inductive step, we assume for all $m \leq n \geq 0, s \in X$ that $V_m^s = T_{|\gamma|^{-m}}^V(\ell_m^s)$. Though there may be many choices for ℓ_m^s, we assume it is a fixed continuous function once named. By Lemma 4.11, we have that $V_m^s = T_{|\gamma|^{-m}}^V(\ell_m^s) \subset T_{|\gamma|^{-(m-1)}}^V(\ell_{m-1}^s) = V_{m-1}^s$. Thus, for all

$t \in I$, $|\ell^s_m(t) - \ell^s_{m-1}(t)| \leq |\gamma|^{-(m-1)} = p^{-(m-1)v}$ where $v = -v_p(\gamma)$. We can then write

$$\ell^s_m(t) = \sum_{k=0}^{m} p^{(k-1)v} \psi^s_m(t)$$

where $\psi^s_k : I \to \mathbb{Z}_p$ is a continuous function defined by

$$\psi^s_k(t) = \frac{\ell^s_k(t) - \ell^s_{k-1}(t)}{p^{(m-1)v}}$$

Fixing s, we wish to find $\ell^s_{n+1} : I \to I$ such that $V^s_{n+1} = T^V_{|\gamma|^{-(n+1)}}(\ell^s_{n+1})$ (and as a consequence, is nonempty). By Lemma 4.11, we have that $\phi(\ell^s_n(t), t) = (a + bt - \ell^s_n(t)^2, \ell^s_n(t)) \in V^{\sigma(s)}_{n-1} = T^V_{|\gamma|^{-(n-1)}}(\ell^{\sigma(s)}_{n-1})$ for all $t \in I$–in particular we have that $|\ell^{\sigma(s)}_{n-1}(\ell^s_n(t)) - (a + bt - \ell^s_n(t)^2)| \leq |\gamma|^{-(n-1)} = p^{-(n-1)v}$. We let $r^s_n(t) : I \to \mathbb{Z}_p$ be defined as

$$r^s_n(t) = \frac{\ell^{\sigma(s)}_{n-1}(\ell^s_n(t)) - (a + bt - \ell^s_n(t)^2)}{p^{(n-1)v}}$$

so now $\phi(\ell^s_n(t), t) = \left(\ell^{\sigma(s)}_{n-1}(t) + p^{(n-1)v} r^s_n(t), \ell^s_n(t)\right)$.

We claim that

$$h(t) = \frac{-\psi^{\sigma(s)}_n(\ell^s_n(t)) + r^s_n(t)}{2p^v \ell^s_n(t)}$$

fulfills our desired conditions for ψ^s_{n+1}, that is $V^s_{n+1} = T^V_{|\gamma|^{-(n+1)}}(\ell^s_n(t) + p^{nv} h(t)) = T^*$. We have by construction that $\psi^{\sigma(s)}_n$and r^s_n have image in $\mathbb{Z}_p$, and by corollary 4.12 $|2p^v \ell^s_n(t)| = 1$, so we have that $\mathrm{Im}(h) \subseteq \mathbb{Z}_p$ as desired.

To show that $T^* \subset V^s_{n+1}$, we need only show that $\phi\left(\{(\ell^s_n(t) + p^{nv} h(t), t) \mid t \in I\}\right) \subset V^{\sigma(s)}_n$, as lemma 4.16 gives us that $(x, y) \in V^s_{n+1}$ implies $B_{|\gamma|^{-(n+1)}}(x, y) \subseteq V^s_{n+1}$. We let $t_0 \in I$ be arbitrary, and let $\alpha_0 = (x_0, y_0) = (\ell^s_n(t_0) + p^{nv} h(t_0), t_0)$. We let $\phi^k(\alpha_0) = \alpha_k = (x_k, y_k)$ and write

$$\begin{aligned}
\alpha_1 &= \left(a + bt_0 - (\ell^s_n(t_0) + p^{nv} h(t_0))^2, \ell^s_n(t_0) + p^{nv} h(t_0)\right) \\
&= \left((a + bt_0 - \ell^s_n(t_0)^2) - 2p^{nv} \ell^s_n(t_0) h(t_0) - p^{2nv} h(t_0)^2, \ell^s_n(t_0) + p^{nv} h(t_0)\right) \\
&= \left(\ell^{\sigma(s)}_{n-1}(\ell^s_n(t_0)) + p^{(n-1)v} r^s_n(t_0) - 2p^{nv} \ell^s_n(t_0) h(t_0) - p^{2nv} h(t_0)^2, \ell^s_n(t_0) + p^{nv} h(t_0)\right)
\end{aligned}$$

From our form for h, we note $2p^{nv} \ell^s_n(t_0) h(t_0) = p^{(n-1)v}\left(-\psi^{\sigma(s)}_n(\ell^s_n(t_0)) + r^s_n(t_0)\right)$. As the r^s_n terms cancel and $\ell^{\sigma(s)}_{n-1} + \psi^{\sigma(s)}_n = \ell^{\sigma(s)}_n$, we now have

$$\alpha_1 = \left(\ell^{\sigma(s)}_n(\ell^s_n(t_0)) + p^{2nv} h(t_0)^2, \ell^s_n(t_0) + p^{nv} h(t_0)\right).$$

By lemma 4.17, we need only that $|x_1 - \ell^{\sigma(s)}_n(\ell^s_n(t_0))| \leq |\gamma|^n$, as $|\ell^s_n(t_0) - y_1| \leq |\gamma|^{-n} < |\gamma|^{-(n-1)}|b|^{-1} = d_n$ since $|b| < |\gamma|$ by assumption. This condition clearly holds however, as $|\left(\ell^{\sigma(s)}_n(\ell^s_n(t_0)) + p^{2nv} h(t_0)^2\right) - \ell^{\sigma(s)}_n(\ell^s_n(t_0))| = |p^{2nv} h(t_0)^2| \leq p^{-2nv} < |\gamma|^n$. Thus, $\phi(\alpha) \in V^{\sigma(s)}_n$ and $\alpha \in V^s_n \subset A_{s_0}$ by assumption, so we have that $\alpha \in \phi^{-1}(V^{\sigma(s)}_n) \cap A_{s_0} = V^s_{n+1}$ as desired.

To show $T^* \supset V_{n+1}^s$, we recall that $V_{n+1}^s \subset V_n^s$ by Lemma 4.11 and consider an arbitrary element $\beta \in V_n^s \setminus T^*$. We seek to show that $\phi(\beta) \notin V_n^{\sigma(s)}$. We keep that $\alpha_0 = (x_0, y_0) = (\ell_n^s(t_0) + p^{nv}h(t_0), t_0)$ and use the form $\beta = (\ell_n^s(t_0) + p^{nv}h(t_0) + \eta, t_0)$ where $p^{-nv} \geq |\eta| > p^{(n+1)v}$. We then write

$$\begin{aligned}\phi(\beta) &= \big((a + bt_0 - (\ell_n^s(t_0) + p^{nv}h(t_0))^2) - 2\eta(\ell_n^s(t_0) + p^{nv}h(t_0)) - \eta^2, \ell_n^s(t_0) + p^{nv}h(t_0) + \eta\big) \\ &= \big(\ell_n^{\sigma(s)}(\ell_n^s(t_0)) + p^{2nv}h(t_0)^2 - 2\eta(\ell_n^s(t_0) + p^{nv}h(t_0)) - \eta^2, \ell_n^s(t_0) + p^{nv}h(t_0) + \eta\big) \\ &= \phi(\alpha_0) + \big(-2\eta(\ell_n^s(t_0) + p^{nv}h(t_0)) - \eta^2, \eta\big) = \phi(\alpha_0) + (\theta, \eta)\end{aligned}$$

As $|\eta| \leq p^{-nv} < d_n$, we have by Corollary 4.14 and inductive hypothesis that $\phi(\beta) \in V_n^{\sigma(s)}$ if and only if $|\theta| \leq |\gamma|^{-n}$. However, as $|\ell_n^s(t_0) + p^{nv}h(t_0)| = |\gamma|$, we have $|-2\eta(\ell_n^s(t_0) + p^{nv}h(t_0))| = |\gamma||\eta| > |\gamma|^{-n}$, and as $|\gamma\eta| > |\eta|^2$, we have by a strongest wins argument that $|\theta| > |\gamma|^{-n}$. Thus we have shown our function h is indeed a valid choice for ψ_{n+1}^s, and $T^* = V_{n+1}^s$. As such, we can write $\ell_{n+1}^s(t) = \ell_n^s(t) + p^{nv}h(t)$ and have that $V_{n+1}^s = T^V_{|\gamma|^{-(n+1)}}(\ell_{n+1}^s)$, as desired. ☐

Corollary 4.19. *V^s is non-empty for all s, and in particular*

$$V^s = \{\ell^s(t), t \mid t \in I\} = T_0^V(\ell^s)$$

where $\ell^s : I \to I$ is defined pointwise by

$$\ell^s(t) = \lim_{m\to\infty} \ell_m^s(t).$$

Furthermore, ℓ^s is Lipschitz continuous with Lipschitz constant $|pb|$.

Proof. We first show that $\lim_{m\to\infty} \ell_m^s(t)$ exists for all s, t. By iterating corollary 4.11, we see that $\{\ell_m^s(t)\}_{m\geq 0}$ is Cauchy, as for all $m \geq n$, each $\ell_m^s(t) \in D_{|\gamma|^{-n}}(\ell_n^s(t))$. Thus, $\{\ell_m^s\}_{\geq 0}$ converges uniformly to a continuous function $\ell^s(t)$, with uniformity coming from each tube V_m^s being of constant width which decreases monotonically.

To see that ℓ^s is Lipschitz continuous with Lipschitz constant $|pb|$, we split into two cases. If $|y - y'| \geq pd_0 = |\frac{\gamma}{pb}|$, we then have that since the image of ℓ is in I, $|\ell^s(y) - \ell^s(y')| \leq |\gamma| \leq |pb||y - y'|$. In the other case that $|y - y'| \leq d_0$, we let k be such that $d_{k+1} < |y - y'| \leq d_k$. Since $(\ell^s(y), y) \in V^s \subset V_k^s$, by lemma 4.14 we have that $(\ell^s(y'), y) \in V_k^s$, so $|\ell^s(y') - \ell^s(y)| \leq c_k = |b|d_{k+1} \leq |pb||y - y'|$, as desired. ☐

4.3. **Extending our results to ϕ^{-1}.** We recall that by 2.1 $f^{-1} \circ \phi_{a,b} \circ f = \phi^{-1}_{\frac{a}{b^2}, \frac{1}{b}}$ where $f(x, y) = (-by, -bx)$.

Notation 4.20. For the remainder of this subsection, we shall use the following conventions $a^* = \frac{a}{b^2}$, $b^* = \frac{1}{b}$. If an object has a $*$ as a subscript or a superscript, it is to be interpreted as with respect to a^* and b^*.

Definition 4.21. We let $g : I \to I^*$ be defined as $g(t) = \frac{-t}{b}$.

Lemma 4.22. *We let $\rho^* : I^* \to I^*$ be a continuous function and let $\mu : I \to I$ be defined as $\mu = (\rho^*)^g = g^{-1} \circ \rho^* \circ g$. Then*

$$f\left(T^V_{p^{-k}}(\rho^*(t))\right) = T^H_{p^{-k}|b|}(\mu)$$

Proof. We write $T^V_\epsilon(\rho^*) = \{(\rho^*(u) + p^k\theta, u) \mid u \in I^*, \theta \in \mathbb{Z}_p\}$. We apply f and perform the following calculations:

$$\begin{aligned} f(T^V_\epsilon(\rho^*)) &= \{f(\rho^*(u) + p^k\theta, u) \mid u \in I^*, \theta \in \mathbb{Z}_p\}\} \\ &= \{(-bu, -b\rho^*(u) - bp^k\theta) \mid u \in I^*, \theta \in \mathbb{Z}_p\} \end{aligned}$$

We make substitutions $-bu = t$ and $-bp^{-v_p(b)}\theta = \eta$. We then have that

$$\begin{aligned} f(T^V_\epsilon(\rho^*)) &= \{(t, -b\rho^*(\frac{-t}{b}) + p^{k+v_p(b)}\eta) \mid t \in I, \eta \in \mathbb{Z}_p\} \\ &= \{(t, \mu(t) + p^{k+v_p(b)}\eta) \mid t \in I, \eta \in \mathbb{Z}_p\} \\ &= T^H_{p^{-k}|b|}(\mu) \end{aligned}$$

□

Remark 4.1. While written for positive radius p^{-k} for simplicity and clarity, Lemma 4.22 in fact applies identically to describing $f(T^V_0(\rho^*))$

Lemma 4.23. $f(A^*_\mp) = B_\pm = \phi(A_\pm)$

Proof. We recall from 4.4 that $A^*_\mp = T^V_1(\ell^{*\mp}_0)$ where $\ell^{*\mp}_0(t) = \mp\sqrt{a^* + b^*t}$. We then conjugate $\ell^{*\mp}_0$ by g to yield

$$\begin{aligned} (\ell^{*\mp}_0)^g &= \pm b\sqrt{a^* + b^*(\frac{-t}{b})} \\ &= \pm b\sqrt{\frac{a}{b^2} - \frac{t}{b^2}} \\ &= \pm\sqrt{a - t} = \omega^\pm_0(t) \end{aligned}$$

By lemma 4.22, we can then conclude that $f(A^*_\mp) = T^H_{|b|}(\omega^\pm_0(t))$. Recalling proposition 4.5, we now have that $f(A^*_\mp) = B_\pm$. □

Definition 4.24. We let $\tilde{\cdot} : \{+, -\} \to \{+, -\}$ be defined as $\tilde{\mp} = \pm$.

Corollary 4.25. *Fixing s, we let $w \in X$ be defined by $w_i = \tilde{s}_{-(i+1)}$. We let $\omega^s_m : I \to I$ be defined as $(\ell^{*w}_m)^g$. For all $m \in \mathbb{N}$, $H^s_m = f(V^{*w}_m) = T^H_{|b\gamma^{-m}|}(\omega^s_m)$.*

Proof. Equipped with the previous two lemmas, our work reduces to recalling definitions and applying our newfound technical results. We write:

$$\begin{aligned} V^{*w}_m &= \bigcap_{k=0}^m (\phi^*)^{-k}(A^*_{w_k}) \\ &= \bigcap_{k=0}^m (f^{-1} \circ \phi^k \circ f)(A^*_{w_k}) \end{aligned}$$

Applying f to both sides and using lemma 4.23 to make substitution $\phi(A_{\tilde{w}_k}) = f(A^*_{w_k})$, we now have

$$f(V^{*w}_m) = \bigcap_{k=0}^m \phi^{k+1}(A_{\tilde{w}_k})$$

We now reindex by the substitution $k \mapsto j$ where $j = -(k+1)$.

$$f(V^{*w}_{\ m}) = \bigcap_{j=-(m+1)}^{-1} \phi^{-j}(A_{\tilde{w}_{-(j+1)}})$$

Finally we recall that by definition $\tilde{w}_{-(j+1)} = s_j$ and yield

$$f(V^{*w}_{\ m}) = \bigcap_{j=-(m+1)}^{-1} \phi^{-j}(A_{s_j}) = H^s_m$$

By lemma 4.22, we have that $f(V^{*w}_{\ m}) = T^H_{|b\gamma^{-m}|}(\omega^s_m)$. We can thus conclude that $H^s_m = T^H_{|b\gamma^{-m}|}(\omega^s_m)$.

□

By extending our methods for corollary 4.25 (effectively replacing each m with an ∞ and taking limits where appropriate), we arrive at a virtually identical result characterizing H^s.

Corollary 4.26. $H^s = T^H_0(\omega^s)$ *where* $\omega^s = (\ell^w_*)^g$.

Corollary 4.27. ω^s *is Lipschitz continuous with Lipschitz constant* $|pb^*|$

Proof. Recall from 4.19 that $|\ell^w_*(t) - \ell^w_*(t')| \leq |pb^*||t - t'|$, for $t, t' \in S^*$. By lemma 2.11, as g g^{-1} have reciprocal Lipschitz constants, we have that conjugation by linear maps preserves Lipschitz constant, so ω^s has Lipschitz constant $|pb^*| = |\frac{p}{b}|$ □

4.4. **Proof of the Main Theorem.**

Lemma 4.28. *Each* $H^s \bigcap V^s$ *consists of precisely one point* α_s.

Proof. Recall the definitions of H^s and V^s:

$$V^s = \{(\ell^s(t), t) \mid t \in I\} \qquad\qquad H^s = \{(u, \omega^s(u)) \mid u \in I\}$$

We then have that $V^s \cap H^s = \{(u,t) \mid u = \ell^s(t);\ t = \omega^s(u)\}$. Consider the two-variable system of equations $u = \ell^s(t);\ t = \omega^s(u)$. By making the appropriate substitutions, one sees that (u_0, t_0) is a solution if and only if $t_0 = \omega^s(\ell^s(t_0))$, $u_0 = \ell^s(\omega^s(u_0))$. Let $G_1 : I \to I$ be the composition $G = \omega^s \circ \ell^s$, and $H : I \to I$ the composition $H = \ell^s \circ \omega^s$. As corollaries 4.19 and 4.27 inform us, ℓ^s has Lipschitz constant $|pb|$ and ω^s has Lipschitz constant $|\frac{p}{b}|$. By lemma 2.3, G and H both have Lipschitz constant $|p^2| < 1$, implying that both functions are contractions of I. By the Banach fixed-point theorem, as I is a complete metric space, we have that G and H have unique fixed points t_0 and u_0 respectively. Note as well that as $t_0 = G(t_0)$, we can write $\ell^s(t_0) = \ell^s(G(t_0)) = \ell^s(\omega^s(\ell^s(t_0))) = H(\ell^s(t_0))$, implying that $\ell^s(t_0)$ is a fixed point of H. As u_0 is the *unique* fixed point of H, we then have that $\ell^s(t_0) = u_0$. By an identical argument, we also have $\omega^s(u_0) = t_0$, thus establishing that $(u_0, t_0) \in V^s \cap H^s$ so H^s is nonempty. To confirm uniqueness, we assume $(u_0', t_0') \in V^s \cap H^s$. By the arguments above, we must have that t_0' is a fixed point of G, and as t_0 is the unique fixed point of G, we have $t_0' = t_0$, and as $(u_0', t_0) \in V^s$, we must have $u_0' = \ell^s(t_0) = u_0$. We thus name $(u_0, t_0) = \alpha^s$ with our proof complete. □

We have now finally established the last of the technical details necessary to finish off our proof of Theorem 4.1 once and for all. To recall the statement of Theorem 4.1, we seek to establish a topological conjugacy to the action of σ on X. Let the conjugacy map $\Psi : X \to J(\phi)$ be defined by $\Psi(s) = \alpha_s$, keeping our definition of α_s from Lemma 4.28. That Ψ is a bijection is an obvious consequence of Lemma 4.28, as each point in $J(\phi)$ must follow some infinite forward and backward trajectory through the sets A_+ and A_- by proposition 4.4 To see that $\Psi \circ \sigma = \phi \circ \Psi$, write

$$\begin{aligned}
\phi(\alpha^s) &= \phi\left(\bigcap_{k=-\infty}^{\infty} \phi^{-k}(A_{s_k})\right) \\
&= \bigcap_{k=-\infty}^{\infty} \phi^{-(k-1)}(A_{s_k}) \\
&= \bigcap_{k=-\infty}^{\infty} \phi^{-k}(A_{s_{k+1}}) \\
&= \bigcap_{k=-\infty}^{\infty} \phi^{-k}(A_{\sigma(s)_k}) = \alpha_{\sigma(s)}
\end{aligned}$$

Finally, to see that Ψ is indeed bicontinuous, we recall a few facts from point-set topology:

(1) If A and B are both compact metric spaces and $F : A \to B$ is bijective and continuous in the forward direction, then F^{-1} is continuous as well and F is a homeomorphism between A and B.
(2) In a metric space A, a sequence a_n converges to a limit $L \in A$ if and only if every subsequence a_{n_k} has a further subsequence $a_{n_{k_m}}$ converging to L.
(3) A function $F : A \to B$ between metric spaces is continuous if and only if whenever a sequence a_n converges to $L \in A$, $F(a_n)$ converges to $F(L) \in B$.
(4) In a compact set, every sequence has a convergent subsequence.

As X is well-known to be compact and $J(\phi)$ is compact by Proposition 2.10, fact (1) implies that just the one-directional continuity of Ψ is sufficient to prove bicontinuity. Let s be arbitrary, $\{s^n\}_{n\geq 1} \subset X$ be a sequence converging to s. By fact (3) and the arbitrarity of our choice of s, Ψ is continuous if and only if $\{\Psi(s^n)\}_{n\geq 1} \subset J(\phi)$ converges to $\Psi(s) = \alpha^s$. Consider an arbitrary subsequence $\{\Psi(s^{n_k})\}_{k\geq 1}$. By fact (4), as $J(\phi)$ is compact, there exists a further subsequence $\{\Psi(s^{n_{k_m}}\}_{k\geq 1}$ converging to some $L \in J(\phi)$. However, as $\{s^{n_{k_m}}\}$ converges to s, we have that for any ϵ, there exists an M such that for all $m > M$, $d(s^{n_{k_m}}, s) \leq \epsilon$–in other words, for all N, there exists an M such that for all $m > M$, $s_i^{n_{k_m}} = s_i$ for all $0 \leq i \leq N$. Thus, $\Psi(s^{n_{k_m}}) \in V_N^s$, so it follows that $L \in V^s$. An identical argument can be made to establish that $L \in H^s$. As α^s is the unique element of the intersection $V^s \cap H^s$, the sequence $\{\Psi(s^{n_{k_m}})\}_{m\geq 1}$ converges to $\alpha^s = \Psi(s)$, thus confirming continuity. At long last, this concludes the proof of our main theorem.

5. Region III: One-directional Reduction

5.1. General Results.

We will now consider the region in the parameter space where $|b| \neq 1$, $|a|^{1/2} \leq \max\{|b|, 1\}$

Proposition 5.1. *If $|a|^{1/2} < \max\{|b|, 1\}$, then $J(\phi) \neq \emptyset$. In particular, $J(\phi)$ contains both fixed points.*

Proof. First, suppose $|a| < 1$ and $|b| < 1$. Consider the function $g(z) = z^2 - ((b-1)^2 + 4a) = z^2 - b^2 + 2b - 1 + 4a$. Then $\overline{g(z)} = \overline{z^2} - 1$. So $\overline{g(1)} = 0$ and $\overline{g'(1)} = 2 \neq 0$, so by Hensel's lemma there exists r such that $g(r) = 0$. Therefore $(b-1)^2 + 4a$ is a square in $\mathbb{Q}_p$, so there both periodic points of ϕ exist. Now because ϕ_{a^*,b^*} for $|a^*|^{1/2} < |b^*|$ and $|b^*| > 1$ is topologically conjugate to $\phi_{a,b}^{-1}$ where $|a| < 1$ and $|b| < 1$, that must mean that ϕ_{a^*,b^*} admits fixed points as well. □

Proposition 5.2. *Suppose $p \neq 3$. If $|a|^{1/2} < \max\{|b|, 1\}$, then ϕ has minimal period two points if and only if $p \equiv 1 \mod 3$.*

Proof. Suppose $|a| < 1$ and $|b| < 1$. We know that ϕ has period two points if and only if $h(z) = z^2 - (-3(b+1)^2 + 4a)$ has roots. However, by Hensel's lemma h has roots if and only if $\overline{h(z)} = z^2 + 3$ has roots in $\mathbb{Z}/p\mathbb{Z}$, which has roots if and only if -3 is a quadratic residue mod p. But by quadratic reciprocity, -3 is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1 \mod 3$. By conjugation with h, we get the desired result for the case $|a|^{1/2} < |b|$ and $|b| > 1$.

□

Proposition 5.3. *Suppose $|a|^{1/2} = \max\{1, |b|\}$ and $|b| \neq 1$. Then ϕ has fixed points if and only if $1 + 4\bar{a}$ is a quadratic residue $\mod p$. ϕ has period two points if and only if $-3 + 4\bar{a}$ is a quadratic residue $\mod p$.*

Proof. The proof follows similarly to the previous two propositions. Suppose $|b| < 1$. By Hensel's lemma, ϕ has periodic points if and only if $\overline{g(z)} = z^2 - (1 + 4\overline{a})$ has roots in $\mathbb{Z}/p\mathbb{Z}$, and period two points if and only if the reduction of $h(z) = z^2 - (-3 + 4a)$ has roots in $\mathbb{Z}/p\mathbb{Z}$.

□

Proposition 5.4. *If $|a| \leq 1$ and $|b| < 1$ then $J(\phi) \subset \mathbb{Z}_p^2$, and $\mathbb{Z}_p^2$ is also the smallest polydisc that contains $J(\phi)$.*

Proof. From our bounds on the filled Julia set, we know that in the case where $|b| > 1$ and $|b| \geq |a|^{1/2}$, then $J(\phi) \subset S = \{(x, y) \in \mathbb{Q}_p : |x| \leq |b|, \text{and } |y| \leq |b|\}$. However, through conjugation with the map $f(x, y) = (by, bx)$, we know that $\phi_{a,b} \sim \phi_{\frac{a}{b^2}, \frac{1}{b}}$, which corresponds exactly to the case $|a| \leq 1, |b| < 1$. Therefore, $J(\phi_{\frac{a}{b^2}, \frac{1}{b}}) = f^{-1}(J(\phi_{a,b}))$. So, $J(\phi_{\frac{a}{b^2}, \frac{1}{b}}) \subset f^{-1}(S) = \mathbb{Z}_p^2$. To prove that $\mathbb{Z}_p^2$ is the smallest polydisc containing $J(\phi)$, suppose Q is a polydisc such that $J(\phi) \subset Q$. We know that ϕ admits two fixed points, namely $(\frac{1-b+\sqrt{(b-1)^2+4a}}{-2}, \frac{1-b+\sqrt{(b-1)^2+4a}}{-2})$ and $(\frac{1-b-\sqrt{(b-1)^2+4a}}{-2}, \frac{1-b-\sqrt{(b-1)^2+4a}}{-2})$.The absolute value of the difference of each of these co-ordinates is $|(b-1)^2 + 4a|^{\frac{1}{2}} = 1$ by strongest wins. Because the absolute value of the difference of each co-ordinate of two points in the filled Julia set can be 1, this must mean that $\mathbb{Z}_p^2 \subset Q$. So because Q was arbitrary, that must mean that $\mathbb{Z}_p^2$ is the smallest polydisc containing $J(\phi)$.

$\square$

Proposition 5.5. *If $|a|^{1/2} \leq |b|$ and $|b| > 1$, then $S = \{(x, y) \in \mathbb{Q}_p : |x| \leq |b|$, and $|y| \leq |b|\}$ is the smallest polydisc that contains $J(\phi)$ in this region.*

Proof. Because $\mathbb{Z}_p^2$ is the smallest polydisc for the case $|a| \leq 1$ and $|b| < 1$, then by conjugacy $f(\mathbb{Z}_p^2) = S$ must be the smallest polydisc in the case $|a|^{1/2} \leq |b|$ and $|b| > 1$. $\square$

Proposition 5.6. *If $|a| < 1$ and $|b| \leq 1$, then $\phi^n(\mathbb{Z}_p^2) \subset \mathbb{Z}_p^2$, for $n \geq 1$. In addition, $\phi(\mathbb{Z}_p^2) \cap \mathbb{Z}_p^2 = \{(x, y) \in \mathbb{Q}_p^2 : |-a + x + y^2| \leq |b|\}$. Similarly, if $|a|^{1/2} \leq |b|$ and $|b| > 1$, then $\phi^{-n}(S) \subset S$ for $n \geq 1$, and $\phi^{-1}(S) \cap S = \{(x, y) \in \mathbb{Q}_p^2 : |a + by + x^2| \leq |b|\}$.*

Proof. Suppose $|a| < 1$ and $|b| \leq 1$. Suppose $(x, y) \in \mathbb{Z}_p^2$. Then $\|\phi(x, y)\| = \max\{|a + by - x^2|, |x|\} \leq 1$. Iterating the argument, we have $\|\phi^n(x, y)\| \leq 1$, so $\phi^n(\mathbb{Z}_p^2) \subseteq \mathbb{Z}_p^2$ However, since ϕ does not have good reduction, then $\phi(\mathbb{Z}_p^2) \subset \mathbb{Z}_p^2$. Therefore, $\phi^n(\mathbb{Z}_p^2) \subsetneq \mathbb{Z}_p^2$.

Suppose $(x, y) \in \mathbb{Z}_p^2$ and $\phi^{-1}(x, y) \in \mathbb{Z}_p^2$. Then $(x, y) \in \phi(\mathbb{Z}_p^2) \cap \mathbb{Z}_p^2$, so $\|(x, y)\| \leq 1$ and $\|\phi^{-1}(x, y)\| = \|(y, \frac{-a+x+y^2}{b})\| \leq 1$. Therefore, $|-a + x + y^2| \leq |b|$. The proof for part two of the proposition works identically.

$\square$

Proposition 5.7. *If $|a|, |b| < 0$, then ϕ admits an attracting fixed point $\alpha = (\alpha, \alpha)$ with basin of attraction $(p\mathbb{Z}_p)^2$.*

Proof. To show that ϕ admits a fixed point $\alpha \in (p\mathbb{Z}_p)^2$, we recall 2.12, which stated that the formula for fixed points of ϕ can be written as (q, q), where $q = \frac{b-1}{2} \pm \frac{\sqrt{(b-1)^2+4a}}{2}$, and note that by 5.1, the term under the square root is indeed a square in $\mathbb{Q}_p$. We note that

$$\overline{q} = \overline{\left(\frac{b-1}{2} \pm \frac{\sqrt{(b-1)^2 + 4a}}{2}\right)} = \frac{1}{2} \pm \frac{1}{2}.$$

We let $\alpha = (\alpha, \alpha)$ be such a fixed point.
We let $x, y \in (p\mathbb{Z}_p)^2$ be written $(x, y) = (\alpha + \theta, \alpha + \eta)$ with $\theta, \eta \in p\mathbb{Z}_p$. A straightforward calculation shows that

$$\phi^2(x, y) - \alpha = \left(-2\alpha(b\eta - 2\alpha\theta - \theta^2) - (b\eta - 2\alpha\theta - \theta^2)^2 + b\theta, b\eta - 2\alpha\theta - \theta^2\right)$$

Recalling that $|\alpha|, |b|, |\theta|, |\eta| < 1$ by assumption, inspection reveals that for any monomial term w on either side of the polynomial in θ, η above that $|w| < \max\{|\theta|, |\eta|\}$, as one or both of θ, η is a factor of each term, with each coefficient having absolute value below 1. By the ultrametric inequality, we conclude that $\|\phi^2(x, y) - \alpha\| < \|(x, y) - \alpha\|$, proving the proposition. $\square$

5.2. **The Attractor.** We assume without loss of generality that $|b| < 1$, $|a| \leq |b^2|$, as the remainder of region III conjugates to that case via Lemma 2.1.

Notation 5.8. When relevant, we let $\mathbb{Z}_p/p^k\mathbb{Z}_p$ refer to the *ring* of integers modulo p^k. We let $i_k : \mathbb{Z}_p/p^k\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p$ be defined such that $i_k(z)$ is the unique integer such that $i_k(z) \equiv z \mod p^k\mathbb{Z}_p$ and $0 \leq i_k(z) < p^k$. We let $\bar{\cdot}_k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$ be the reduction map modulo p^k. As it does not induce any ambiguity, we refer to the counterparts for multiple dimensions the same way.

Proposition 5.9. *As defined in Milnor's article* [9]*, ϕ admits a trapped attracting set $\mathcal{A} \subset \mathbb{Z}_p^2$, which here does* not *imply any notion of indecomposability. Furthermore, $J(\phi) = \mathcal{A}$*

Proof. We have by Proposition 2.5 that $J(\phi) \subset \mathbb{Z}_p^2$ and as $\phi^{-1}(\mathbb{Z}_p^2) \not\subseteq \mathbb{Z}_p^2$, we have that $\phi(\mathbb{Z}_p^2)$ is a proper subset of $\mathbb{Z}_p^2$. By [9], as $\mathbb{Z}_p^2$ is a compact set, we have that $\mathcal{A} = \bigcap_{k\in\mathbb{N}} \phi^k(\mathbb{Z}_p)$ is a non-empty ϕ-invariant set, and as $\mathcal{A} \subset \overline{B}(\mathbf{0}, 1) = \mathbb{Z}_p$, we have that it is bounded and therefore $\mathcal{A} \subset J(\phi)$. Further, we have for all $\beta \in \mathbb{Z}_p \setminus \mathcal{A}$, there exists $m_\beta \in \mathbb{N}$ such that $\phi^{m_\beta}(\beta) \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, and thus $\beta \notin J(\phi)$, so $J(\phi) = \mathcal{A}$. □

Conjecture 5.10. *For some $(a, b) \in \mathbb{Z}_p \times p\mathbb{Z}_p \subset$ Region III, $\mathcal{A}$ admits a bounded nonperiodic orbit.*

Remark 5.1. While there is no small amount of discussion among dynamicists regarding the proper definition of strange attractor in the Euclidean case (see for example Ruelle [13]), at present there seems to be very little written about strange attractors in the p-adic or more general non-Archimedean context. In light of that, we shall tentatively call any attractor that admits a bounded nonperiodic orbit "strange." In the absence of more concrete results regarding our attracting set for the time being, we will not worry ourselves too much regarding decomposability and may in fact slightly abuse terminology by referring to an attracting set as an attractor when it in fact may be decomposable–such ambiguities can hopefully be cleared up through further investigation.

To discuss our attracting set, we introduce a notion of Khrennikov and Anashin [1]. While their work on the subject deals with functions on $\mathbb{Q}_p$, we shall find that it adapts quite readily to the two-variable case.

Definition 5.11 (Anashin & Khrennikov)**.** A set of k balls of radius $\epsilon = 1/p^m$ in $\mathbb{Q}_p^2$

$$\{B_\epsilon(\alpha_1), B_\epsilon(\alpha_2), \ldots, B_\epsilon(\alpha_k)\}$$

is called a *fuzzy cycle* of order m and length k if $\phi(B_\epsilon(\alpha_i)) \subseteq B_\epsilon(\alpha_{i+1 \mod k})$. We refer to the balls $B_\epsilon(\beta)$ where $\phi^n(B_\epsilon(\beta)) \subset B_\epsilon(\alpha_i)$ but $\beta \neq \alpha_i$ for all $i \leq k$ as *fuzzily pre-periodic* to the fuzzy cycle in question.

Remark 5.2. In order to understand the relevance of definition 5.11, we recall a fact from ring theory

Fact 5.12. *Consider a polynomial $\psi \in \mathcal{R}[x]$ where $\mathcal{R}$ is a ring with ideal $\mathcal{I}$. Let $\cdot^* : \mathcal{R} \to \mathcal{R}/\mathcal{I}$ be the natural reduction map and $\phi^* : \mathcal{R}/\mathcal{I} \to \mathcal{R}/\mathcal{I}$ be the polynomial obtained by applying $\cdot^*$ to the coefficients of ϕ. Then for all $z \in \mathcal{R}$, $\phi^*(z^*) = (\phi(z))^*$.*

From this, we have that $\alpha \equiv \beta \mod p^k\mathbb{Z}_p$ implies $\phi(\alpha) \equiv \phi(\beta) \mod p^k\mathbb{Z}_p$–equivalently, for all $\beta \in B_{p^{-k}}(\alpha)$, $\phi(\beta) \in B_{p^{-k}}(\phi(\alpha))$. Thus, any cycle of $\overline{\phi}_k$ corresponds to a fuzzy cycle of order k of ϕ.

Corollary 5.13.

$$\mathcal{A} = \{\alpha \in \mathbb{Z}_p \mid \overline{\alpha}_k \in \operatorname{Per}(\overline{\phi}_k) \ \forall\ k \in \mathbb{N}\},$$

that is $\mathcal{A}$ consists of the points that project to $\overline{\phi}_k$-periodic points by each reduction map $\bar{\cdot}_k$.

Proof. We let M_k be the length of the longest preperiodic "tail" of any $\overline{\phi}_k$-orbit in $\left(\mathbb{Z}_p/p^k\mathbb{Z}_p\right)^2$, that is

$$M_k = \min\left\{m \in \mathbb{N} \mid \overline{\phi}_k^m(\zeta) \in \operatorname{Per}(\overline{\phi}_k) \ \forall\ \zeta \in \left(\mathbb{Z}_p/p^k\mathbb{Z}_p\right)^2\right\}.$$

As all orbits of a finite dynamical system are either periodic or strictly pre-periodic, we can be assured that such a minimum exists. We then have that

$$\mathcal{A} \subset \bigcap_{k=0}^{M_k} \phi^k(\mathbb{Z}_p^2) \subset \bigsqcup_{\zeta \in \operatorname{Per}(\overline{\phi}_k)} B_{p^{-k}}\left(i_k(\zeta)\right).$$

This establishes that $\mathcal{A} \subseteq \{\alpha \in \mathbb{Z}_p \mid \overline{\alpha}_k \in \operatorname{Per}(\overline{\phi}_k) \ \forall\ k \in \mathbb{N}\}$. Furthermore, for all $\zeta \in \operatorname{Per}(\overline{\phi}_k)$, we have that $\phi^M(\mathbb{Z}_p^2) \cap B_{p^{-k}}\left(i_k(\zeta)\right) \neq \emptyset$, as there always exists some $\beta \in \left(\mathbb{Z}_p/p^k\mathbb{Z}_p\right)^2$ such that $\overline{\phi}_k^M(\beta) = \zeta$. Thus, for any $\alpha \in \{\alpha \in \mathbb{Z}_p \mid \overline{\alpha}_k \in \operatorname{Per}(\overline{\phi}_k) \ \forall\ k \in \mathbb{N}\}$, we have that $\mathcal{B}_k(\alpha) = B_{p^{-k}}(\alpha) \cap \mathcal{A} \neq \emptyset$. As such, we have a family of nested nonempty subsets of $\mathbb{Z}_p^2$, $\{\mathcal{B}_k(\alpha) \mid k \in \mathbb{N}\}$. By the Cantor intersection theorem, as $\mathbb{Z}_p^2$ is a compact metric space, we can write

$$\bigcap_{k \geq 0} \mathcal{B}_k(\alpha) \neq \emptyset$$

However, we note that $\bigcap_{k\geq 0} \mathcal{B}_k(\alpha) \subset \bigcap_{k\geq 0} B_{p^{-k}}(\alpha) = \{\alpha\}$. Thus $\alpha \in \mathcal{A}$, so we have that $\mathcal{A} \supseteq \{\alpha \in \mathbb{Z}_p \mid \overline{\alpha}_k \in \operatorname{Per}(\overline{\phi}_k) \ \forall\ k \in \mathbb{N}\}$. This establishes a two-way containment and thus the result. ☐

We finish this section by introducing two sequences of statistics, further results regarding which may lead to a better understanding of the p-adic Hénon attractor.

Definition 5.14. Let $\mathcal{G}_m(a,b) = \mathcal{G}_m$ be the directed graph with vertex set $V = (\mathbb{Z}_p/p^m\mathbb{Z}_p)^2$ and edge set

$$E = \{(\alpha, \beta) \in (\mathbb{Z}_p/p^m\mathbb{Z}_p)^2 \times (\mathbb{Z}_p/p^m\mathbb{Z}_p)^2 \mid \overline{\phi}_m(\alpha) = \beta\}$$

Let $C_m(a,b) = \#\{\text{connected components of } \mathcal{G}_m\}$. Let $\mathcal{C}(a,b) = \mathcal{C}$ be the sequence $\{C_m(a,b)\}_{m\geq 1}$.

Proposition 5.15. *If the monotone increasing sequence $\mathcal{C}(a,b)$ converges to some $\kappa(a,b) \in \mathbb{N}$, then $\mathcal{A}$ can be decomposed into $\kappa(a,b)$ indecomposable attractors*

Proof. To see that $\mathcal{C}$ is non-decreasing, one must only note that the directed graph $\mathcal{G}_m$ can be acquired from $\mathcal{G}_{m+1}$ via taking the quotient graph obtained from partitioning V into equivalence classes by residue class $\mod \mathbb{Z}_p/p^m\mathbb{Z}_p$–this process obviously cannot result in a new connected component arising when moving from $\mathcal{G}_{m+1}$ to $\mathcal{G}_m$.

As follows from our proof of corollary 5.13, for all $\beta \in (\mathbb{Z}_p/p^m\mathbb{Z}_p)^2$ such that β is preperiodic to a length-k cycle $\mathcal{O} = \{\alpha_1, \ldots, \alpha_k\} \subset (\mathbb{Z}_p/p^m\mathbb{Z}_p)^2$, the ball $B_{p^{-m}}(i_m(\beta))$ is eventually mapped to some $B_{p^{-m}}(\alpha_i)$. If the preimage of $\mathcal{O}$ and its preperiodic tails under the quotienting process described above is contained within one connected component of $\mathcal{G}_n$ for all $n > m$, because all connected components of $\mathcal{G}_n$ are the union of a cycle and its preperiodic tails, we can conclude that the union of the balls "fuzzily-preperiodic" to the fuzzy cycle corresponding to $\mathcal{O}$ or contained within it form a basin of attraction to an indecomposable attractor. If $\mathcal{C}$ is indeed convergent to κ, there then exists some $N \in \mathbb{N}$ such that $C_n = \kappa$ for all $n > N$, so each of the κ connected components of $\mathcal{G}_n$ correspond to one indecomposable attractor. □

Definition 5.16. Let $P_+^m(a,b) = \{\max\{k \mid \mathrm{Per}_k^*(\overline{\phi}_m) \neq \emptyset\}\}_{m\geq 1}$, that is the length of the longest cycle of $(\mathbb{Z}_p/p^m\mathbb{Z}_p)^2$ under the action of $\overline{\phi}_m$. Define the sequence $\mathcal{P}_+(a,b) = \mathcal{P}_+$ as $\mathcal{P}_+ = \{P_+^m(a,b)\}_{m\geq 1}$ is taken as a *sequence*. As established by our previous remarks, the mth entry of $\mathcal{P}_+$ is the maximum length of fuzzy cycles of order m admitted by ϕ.

Proposition 5.17. *If the monotone increasing sequence $\mathcal{P}_+$ is bounded, then $\mathcal{A}$ is a collection of attracting cycles. If $\mathcal{P}_+$ is unbounded, then $\mathcal{A}$ admits a bounded nonperiodic orbit.*

Proof. $\mathcal{P}_+$ is monotone increasing by a similar argument to that used to establish that $\mathcal{C}$ is monotone increasing. Suppose $\mathcal{P}_+$ is unbounded. We present an algorithm to find a point $\beta \in \mathbb{Z}_p^2$ such that the orbit of β orbit is bounded and nonperiodic. We determine β through its canonical p-adic representation digit-by-digit. We suppose that β is chosen mod $p^{k-1}\mathbb{Z}_p^2$ such that there are fuzzy cycles of unbounded length components of which project by the reduction map $\bar{\cdot}_{k-1}$ to β. To establish a basis, we note that our process works for the $k = 1$ case in precisely the same way as for the $k > 1$ case.
As the length of fuzzy cycles of arbitrary order m is unbounded, we have by the pigeonhole principle that for at least one residue class ζ_k of $\left(\mathbb{Z}_p/p^k\mathbb{Z}_p\right)^2$, fuzzy cycles of unbounded length contain balls that project to ζ_k under $\bar{\cdot}_m$. We choose the $\nu_k \in \left(\mathbb{Z}_p/p^k\mathbb{Z}_p\right)^2$ fulfuilling that property of maximal minimal period under $\overline{\phi}_k$ and let $\beta \equiv \nu_k \mod p^k\mathbb{Z}_p^2$. Repeating this process for infinitely many k, we construct a β such that $\overline{\beta}_k$ has unbounded minimal period under $\overline{\phi}_k$ as k increases. Thus, $\beta \in \mathcal{A}$ by corollary 5.13, but for all $m \in \mathbb{N}$, β cannot be periodic with length at most m, as there exists some k such that $\overline{\beta}_k$ is periodic under $\overline{\phi}_k$ with minimal period exceeding m. Thus β fulfills our desired conditions.

On the other hand, if $\mathcal{P}$ is bounded, it is necessarily eventually constant. In that case, for any point in $\alpha \in \mathcal{A}$, there exists integers k, N such that for all $n > N$, $\overline{\alpha}_n$ has minimal period k–thus, α is of period k. □

We end this section by remarking on the plausibility of our conjecture. In particular, points exactly like β from the proof of the final proposition necessarily exist in the case of good reduction (Region I), as in that case, all points of $\mathbb{Z}_p^2$ are periodic under every reduction map, but only countably many are periodic. More study is required to settle Conjecture 5.10 as well as to answer the question raised by Proposition 5.15–it would most certainly be a surprising result if it is indeed the case that for some choice of a, b, $\mathcal{A}$ cannot be broken into a collection of indecomposable attractors.

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