REU 1991

Counting Domino Tilings of Mutilated Grids

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August 19, 1991

Introduction

In this paper I will discuss a method of counting the number of ways of tiling a mutilated grid with dominos. A domino is a 1 x 2 or 2 x 1 rectangle, and a mutilated grid is a rectangular grid from which some squares have been removed. A tiling of a grid is simply an assignment of dominos to pairs of adjacent squares so that each square is covered by one domino and each domino covers two squares. Here is a tiling of a 4 x 3 grid:

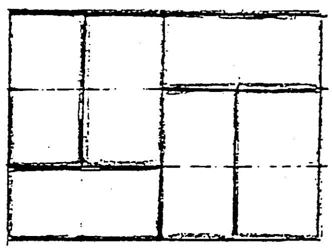

Figure 1

Not all regions can be tiled by dominos; grids on which there is an odd number of squares, for example, can never be tiled, since each domino covers two squares. Also, if a rectangular grid is colored as in a checkerboard (see Figure 2), it becomes clear that if two squares of the same color are removed from the grid, the remaining squares cannot be tiled, since each domino must cover one square of each color, and there is no longer an equal number of squares of each color.

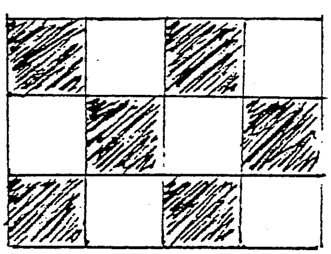

Figure 2

There are, however, many ways of removing squares from a grid so that the mutilated grid can still be tiled. In this paper I will study the effect of removing two squares from the boundary of a rectangular grid.

The Pfaffian

Kasteleyn (1961) has shown that the number of ways of tiling an m x n grid with dominos can always be counted by evaluating the Pfaffian of an mn x mn matrix. A Pfaffian is evaluated over the upper triangular components of an n x n matrix $A=[a_{ij}]$ (n even). Its formula is given by:

$$Pf(A) = \sum_{\sigma \in P} sgn(\sigma)\, a_{k_1 k_2} a_{k_3 k_4} \ldots a_{k_{n-1} k_n}.$$

where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ k_1 & k_2 & k_3 & k_4 & \ldots & k_n \end{pmatrix}$

and P is the set of all permutations σ of {1,2, . . . , n} which satisfy

$k_1 < k_2$; $k_3 < k_4$; . . . ; $k_{n-1} < k_n$

and

$k_1 < k_3 < k_5 < \ldots < k_{n-1}$.

In other words, a term of the Pfaffian can by computed by forming ordered pairs of the numbers from 1 to n so that each number appears in exactly one pair and the smaller element in each pair is the first component of the pair. The term of the Pfaffian is then

$sgn(\sigma)\, a_{k_1 k_2} \ldots a_{k_{n-1} k_n}$

where $\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ k_1 & k_2 & k_3 & \ldots & k_n \end{pmatrix}$.

and $(k_1,k_2), \ldots, (k_{n-1},k_n)$ are the ordered pairs placed in increasing order according to their first component. It has been proved that if M is a skew-symmetrix matrix, then $(DetM)^{1/2} = Pf(M)$.

Application of the Pfaffian to Counting Domino Tilings

There are many ways to label the squares of an m x n grid. In this paper I will use two, which I will call the "dictionary" labeling and the "Cartesian" labeling. The first assigns each square a number, the second a pair of numbers, as shown in Figure 3.

$(n-1)m+1$	$(n-1)m+2$	$(n-1)m+3$		nm
⋮	⋮	⋮		⋮
$m+1$	$m+2$	$m+3$	. . .	$2m$
1	2	3	. . .	m

Dictionary labeling

$(1,n)$	$(2,n)$	$(3,n)$		(m,n)
⋮				
$(1,2)$	$(2,2)$	$(3,2)$	. . .	$(m,2)$
$(1,1)$	$(2,1)$	$(3,1)$	. . .	$(m,1)$

Cartesian labeling

Figure 3

Consider the matrix $A=[a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if the squares numbered } i \text{ and } j \text{ (dictionary labeling) on the grid can be covered by a domino} \\ 0 & \text{otherwise.} \end{cases}$$

For example, with the 2 x 2 grid (see Figure 4)

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} .$$

3	4
1	2

Figure 4

The matrix A will clearly have mn rows and columns since the grid has mn squares. A term of Pf(A) is of the form

$$\operatorname{sgn}\sigma\, a_{k_1 k_2} a_{k_3 k_4} \cdots a_{k_{mn-1} k_{mn}}.$$

This will be zero unless $a_{k_i k_{i+1}} = 1$ for all (odd) i; that is, each pair (k_i, k_{i+1}), where i is odd, must correspond to a pair of squares which may be tiled by the same domino. Then, since every number from 1 to mn appears in exactly one ordered pair in the Pfaffian term, each square has been paired with exactly one adjacent square, and a tiling of the grid has been found. Since every term of the Pfaffian has a different permutation, each non-zero term must correspond to a different tiling of the grid. Also, every possible tiling corresponds to a non-zero term of the Pfaffian; the permutation can be found by pairing elements that have been covered by the same domino, and the appropriate terms of A will all be non-zero by construction.

Thus, there is a one-to-one correspondence between tilings of the mxn grid and non-zero terms of the Pfaffian of A. The way A is constructed, however, the Pfaffian does not count the tilings; the sgnσ in the terms of the Pfaffian makes some of the terms -1 instead of 1. This can be avoided, as Kasteleyn has shown, by changing appropriate elements of A to -1.

The signs of the elements can be determined by associating with each domino a direction, as shown in Figure 5. In this figure, the points represent the squares which are to be covered, and the bonds represent the oriented dominos.

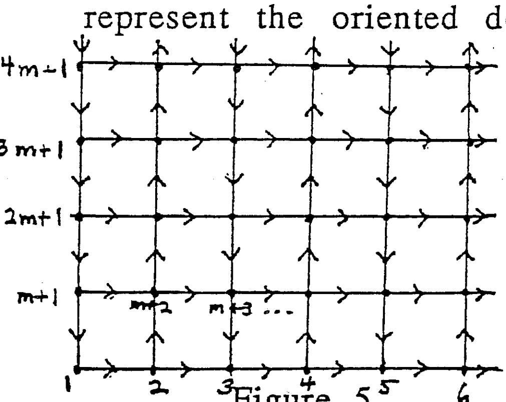

Figure 5

Using this orientation, an mn x mn matrix D can be constructed in the following manner:

$$D_{ij} = \begin{cases} 1 & \text{if the bond between i and j runs from i to j} \\ -1 & \text{if the bond runs from j to i} \\ 0 & \text{if there is no bond between i and j.} \end{cases}$$

Kasteleyn (1963) has proved the following theorem:

If the bonds of a planar graph G are oriented so that the number of arrows pointing clockwise around each mesh is odd, which is always possible, and if the elements of D are assigned as described above, then Pf(D) will equal the number of domino tilings which completely cover G.

In terms of the Cartesian labeling of the grid, the matrix D that has been constructed can be described by:

$D(r,s;\ r+1,s) = -D(r+1,s;\ r,s) = 1$

$D(r,s;\ r,s+1) = -D(r,s+1;\ r,s) = (-1)^r$

$D(r,s;\ r',s') = 0$ otherwise.

D is skew-symmetric, so $(\text{Det } D)^{1/2} = \text{Pf}(D)$, which is the number of ways of tiling the grid.

D is not the only matrix that can be constructed to count the number of tilings. Another matrix, D_o, can be used to count the number of tilings of a square grid, and will be used in this paper. D_o satisfies the following:

$D_o(r,s;\ r+1,s) = -D_o(r+1,s;\ r,s) = 1$

$D_o(r,s;\ r,s+1) = -D_o(r,s+1;\ r,s) = i$

$D_o(r,s;r',s') = 0$ otherwise.

D_o can be obtained directly from D by multiplying each row of D that corresponds to an even r by i, and each column that corresponds to an odd r by $-i$. In the case of a square (n x n) lattice, this multiplies Det(D) by $(i)^{n/2}(-i)^{n/2} = (-i^2)^{n/2} = 1$, since n is even.

Here is D_o for a 2 x 2 grid (see Figure 6):

$$\begin{bmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & i \\ -i & 0 & 0 & 1 \\ 0 & -i & -1 & 0 \end{bmatrix}$$

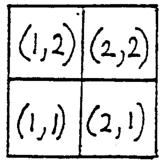

Figure 6

$\mathrm{Det}(D_o) = 4 = 2^2$, and there are two ways of tiling the 2 x 2 grid.

Matrices for Mutilated Grids

Kasteleyn's theorem enables us to extend the matrix counting method to mutilated grids; we will study grids in which two squares have been removed from the boundary of the grid. A directed graph can be constructed which is identical to that of the unmutilated grid except that the two squares removed from the grid are connected by a bond, and all other bonds between them and other squares are removed (see Figure 7).

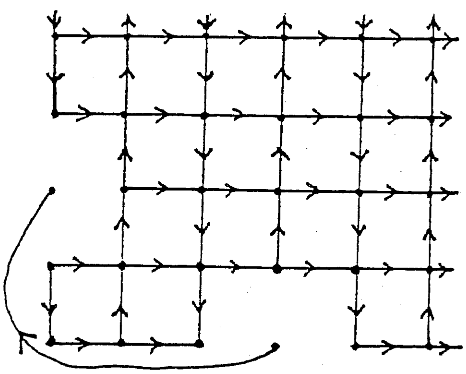

Figure 7

This grid satisfies the hypothesis of Kasteleyn's theorem, so if the i^{th} and j^{th} squares of the grid are removed, a matrix F can be constructed whose Pfaffian is the number of tilings, and which is identical to D except that $F_{ij} = -F_{ji} = 1$ $(i<j)$ and the remaining elements of the i^{th} and j^{th} rows and columns are all zero.

It is more difficult to construct F_o, the matrix obtained from D_o counting the ways of tiling the mutilated grid; F_o can be gotten directly from F by multiplying the rows of even r by i and the columns of odd r by $-i$, but the resulting matrix will depend on the r values of the squares being removed. In the case where two adjacent corners are removed (i=(1,1) and j=(n,1)), $(F_o)_{ij} = -(F_o)_{ji} = 1$, the remaining elements of the i^{th} and j^{th} rows and columns are all zero, and the rest of F_o is identical to D_o. For example, in the 2 x 2 grid if the two bottom corners are removed (see Figure 8),

$$F_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

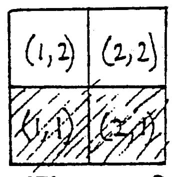

Figure 8

$Det(F_o) = 1$, and there is clearly only one way of tiling the remaining area.

If only two squares are removed from a grid, the matrix F_o will differ from D_o in only a few elements. Fisher and Stephenson have developed a theory showing how the determinant of a matrix which differs in only a few elements from another matrix can be calculated from the second matrix.

Let $A = Det(D_o)$ and

$$B = Det(F_o) = Det(D_o + E)$$

where D_o, F_o, and E are antisymmetric and most elements of E are zero. Assume $A \neq 0$ (since otherwise there would be no tilings of the original matrix).

Let $G = D_o^{-1}$, which has been named the Green's function matrix.

Then,

$B = \mathrm{Det}(D_O + E) = \mathrm{Det}(D_O\,(I + GE)) = \mathrm{Det}(D_O)\,\mathrm{Det}(I + GE) = A\,\mathrm{Det}(I + GE)$

where I is the identity matrix of the appropriate size.

For every column of E that is identically zero, the corresponding column of GE will be identically zero. When Det(I + GE) is expanded by cofactors by one of these columns, say the i^{th} column, it can be seen that the determinant is unchanged if the i^{th} row and column are crossed out. In this way, I + GE can be reduced to a much smaller matrix with the same determinant, and it can easily be verified that

$$\mathrm{Det}(I + GE) = \mathrm{Det}(I' + ge)$$

where g and e are the matrices formed from G and E by crossing out the rows and columns in which E is identically zero, and I' is the identity matrix of the appropriate size.

Thus, $B/A = \mathrm{Det}(I' + ge)$. Since $A^{1/2}$ is the number of ways of tiling the mutilated grid and $B^{1/2}$ is the number of ways of tiling the original grid,

$$\mathrm{Det}(I' + ge) = \left(\frac{\text{the number of ways of tiling the mutilated grid}}{\text{the number of ways of tiling the original grid}}\right)^2 .$$

Counting Tilings Using Reduced Matrices

When two squares are removed from a grid, relatively few elements of D_O are changed. The only elements which may change are in the rows and columns of the removed squares and the squares adjacent to them. For example, if square number three is removed, the entry in the second row, third column changes from 1 to 0, since there can no longer be a bond between squares two and three. If two squares are removed from an edge of the grid, only eight squares are affected, so eight rows and columns of D_0 are changed, and the reduced g and e will be 8 x 8 matrices. In the case where two adjacent corners are removed, only six squares are affected (see Figure 9), so the reduced g and e are 6 x 6 matrices.

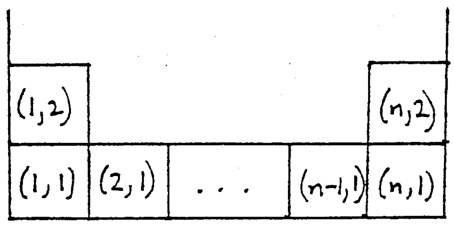

Figure 9

Here is the reduced matrix e for the removal of two corners:

	(1,1)	(2,1)	(n-1,1)	(n,1)	(1,2)	(n,2)
(1,1)	0	-1	0	-1	$-i$	0
(2,1)	1	0	0	0	0	0
(n-1,1)	0	0	0	-1	0	0
(n,1)	1	0	1	0	0	$-i$
(1,2)	i	0	0	0	0	0
(n,2)	0	0	0	i	0	0

The corresponding g would be found by removing from the Green's function matrix all the rows and columns except the 1^{st}, 2^{nd}, $(n-1)^{th}$, n^{th}, $(n+1)^{th}$, and $(2n)^{th}$. Similar e and g matrices can be found for the various ways of removing two squares from the edges of the grid. For example, here is the matrix e for the removal of the i^{th} and j^{th} squares (see Figure 10) on one edge of the grid (assuming that i and j have different parities, as they must have if any tilings of the mutilated grid are to be possible; this corresponds with removing two squares of different colors):

	(i-1,1)	(i,1)	(i+1,1)	(j-1,1)	(j,1)	(j+1,1)	(i,2)	(j,2)
(i-1,1)	0	-1	0	0	0	0	0	0
(i,1)	1	0	-1	0	-1	0	$-i$	0
(i+1,1)	0	1	0	0	0	0	0	0
(j-1,1)	0	0	0	0	-1	0	0	0
(j,1)	0	1	0	1	0	-1	0	$-i$
(j+1,1)	0	0	0	0	1	0	0	0
(i,2)	0	i	0	0	0	0	0	0
(j,2)	0	0	0	0	i	0	0	0

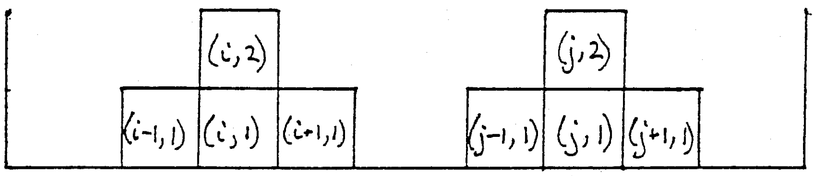

Figure 10

Thus, the ratio of the number of ways of tiling the mutilated grid to the number of ways of tiling the unmutilated grid can be calculated in terms of appropriate Green's function elements. Fisher and Stephenson have found a formula for the Green's function. If i=(r,s) and j=(r',s') are two squares in an n x n grid, the Green's function for i and j is given by:

$$G_{ij} = G(r,s;\, r',s') = \frac{2 i^{\,r'-r+s'-s+1}}{(n+1)^2} \times$$

$$\sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\sin(r\theta_p)\sin(r'\theta_p)\sin(s\phi_q)\sin(s'\phi_q)(-\cos(\theta_p) + i\cos(\phi_q))}{\cos^2\theta_p + \cos^2\phi_q}$$

where $\theta_p = \frac{p\pi}{n+1}$ and $\phi_q = \frac{q\pi}{n+1}$.

It has been shown that if $t=r'-r$ and $u=s'-s$ are of the same parity then $G(r,s;\ r',s') = 0$. If t is odd and u is even then $G(r,s;\ r',s')$ is real, and if t is even and u is odd then $G(r,s;\ r',s')$ is pure imaginary.

Since the reduced matrix e for a particular mutilation (e.g. removal of two corners, removal of two squares from the same edge) remains invariant over the grids of different sizes, the ratio of the number of tilings of the mutilated grid to the number of tilings of the unmutilated grid can be computed entirely in terms of Green's function values. Table 1 shows numerical approximations of the ratios ($= \mathrm{Det}(I' + ge)^{1/2}$) for the removal of corners from various sized grids.

Table 1: ratio of number of tilings of mutilated grid to number of tilings of unmutilated grid when two adjacent corners are removed from an n x n grid.

n	Ratio
2	0.5
4	0.333333
6	0.241379
8	0.187569
10	0.153082
12	0.129264
14	0.111862
16	0.0985963
18	0.0881489
20	0.0797073
22	0.0727439

It can be seen from these results that as n gets large, the number of ways of tiling the mutilated grid becomes substantially smaller than the number of ways of tiling the unmutilated grid, and that the ratio decreases as n increases.

These figures make it seem extremely likely that whenever two corners are removed from a square grid, the number of ways of tiling the grid are reduced. To prove this would be equivalent to showing that $\mathrm{Det}(I' + ge) < 1$ for all n. One possible way of doing this is by studying the eigenvalues of $I' + ge$. If they are all between -1 and 1, with some strictly between -1 and 1, the result will follow.

Suppose $(I' + ge)v = \lambda v$.

Then $\quad I'v + (ge)v = \lambda v.$

$$(ge)v = \lambda v - I'v = \lambda v - v = (\lambda - 1)v.$$

Thus, λ is an eigenvalue of $I' + ge$ if and only if $\lambda - 1$ is an eigenvalue of ge. So it is sufficient to show that the eigenvalues of ge are between -2 and 0, with some strictly between -2 and 0.

Wittemeyer's Theorem, obtained from E. Bodewig's Matrix Calculus, states that if $AA^T = A^TA$ and $BB^T = B^TB$, then

$$|\lambda^{AB}|_{max} \leq |\lambda^{A}|_{max}\, |\lambda^{B}|_{max},$$

where λ^M denotes an eigenvalue of the matrix M. Since g and e are skew-symmetric, they satisfy the hypothesis of the theorem, so finding a bound for the eigenvalues of ge can be reduced to finding a bound for the eigenvalues of g and e. For the removal of two corners, the eigenvalues of e are $\{0,0,0,0,i,-i\}$, which are bounded by1, so it is necessary to show that the eigenvalues of g are bounded above by 2. Frobenius' theorem, which also may be found in Bodewig's book, states that

$$|\lambda|_{max} \leq \max_i \sum_k |a_{ik}|$$

Thus, a bound can be found for the eigenvalues of g by finding bounds for the Green's function elements that make up g. One possible way of doing this is by approximating the Green's function elements by integrals. If it can be shown that the eigenvalues of g are bounded by 2, it will only remain to show that the eigenvalues of ge are not positive and not all 0 or -2.

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