A WEAK TYPE ESTIMATE FOR BASES OF RECTANGLES IN $\mathbb{R}^3$

MARY AMBROSINO AND PATRIK NABELEK

Advisor: Nick Michalowski
Oregon State University

Abstract. In this paper we will attempt to extract a property of the basis of three-dimensional rectangles with dimensions $(t, 1/t, s)$ that will guarantee the associated maximal operator will have weak type $L(\log^+ L)$. We base our property off of ones introduced by Stokolos. Two of our three attempts at proofs are based off Cordoba's proof that a basis with dimensions $(t, s, \phi(s,t))$ has weak type $L(\log^+ L)$. Ultimately, we were unable to provide a proof that guarantees weak type $L(\log^+ L)$ for a basis with our property.

1. Introduction

First, let us define a *basis* of rectangles $\mathcal{B}$ in $\mathbb{R}^n$ to be a collection of n-dimensional rectangles that are translation invariant. By translation invariant we mean given a rectangle in the basis and any arbitrary vector, the rectangle translated by that vector is still in the basis. We define two rectangles from the basis to be *comparable*, denoted $R \sim R'$, if there exists a translation such that one is completely contained in the other. In the opposite case we call them *incomparable*, denoted $R \nsim R'$.

Associated with a basis of rectangles, we have a *maximal operator*,

$$M_{\mathcal{B}} f(x) = \sup_{\mathcal{B} \ni R \ni x} \frac{1}{\mu(R)} \int_R |f(t)| dt,$$

where μ is the *Lebesgue measure*.

We say that a basis has the *weak type (1, 1) estimate*, or equivalently the basis differentiates L^1, when

$$\mu(\{x : M_{\mathcal{B}} f > \alpha\}) \leq C \int_{\mathbb{R}^n} \frac{|f|}{\alpha} dt, \quad \alpha > 0$$

for some universal constant $C < \infty$.

We say that a basis has the *weak type* $L(\log^+ L)^k$ *estimate*, or equivalently the basis differentiates $L(\log^+ L)^k$, when

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$$\mu(\{x : M_{\mathcal{B}}f > \alpha\}) \leq C \int_{\mathbb{R}^n} \left(\frac{|f|}{\alpha} \left(1 + \log^+ \frac{f}{\alpha} \right)^k \right) dt, \quad \alpha > 0$$

for some universal constant $C < \infty$.

An *open dyadic interval* is an interval of the form $\big(j2^k, (j+1)2^k\big)$, $j, k \in \mathbb{Z}$. It should be noted that if two open dyadic intervals intersect, then one must be fully contained in the other. We define a *dyadic rectangle* in $\mathbb{R}^n$ to be the Cartesian product of n open dyadic intervals.

The worst weak type estimate for a basis of rectangles in $\mathbb{R}^n$ is weak type $L(\log^+ L)^{n-1}$. If $\mathcal{B}^*$ is the basis of dyadic rectangles contained in a basis $\mathcal{B}$, then $\mathcal{B}$ has a weak type estimate no worse than $\mathcal{B}^*$. It has been shown that one can achieve a better estimate for the weak type estimate by examining properties of the basis of rectangles. The weak type of the maximal operator associated with a basis depends on the covering properties of that basis. Covering properties for a basis involve taking a finite collection of rectangles, $\{R_\alpha\}$, and carefully selecting a subfamily whose measure is comparable to the original collection, but whose elements have as little overlap as possible. In $\mathbb{R}$, bases can only be weak type (1, 1). The proof [6] involves the Vitali Covering Lemma in one-dimension, which states: Given a collection of intervals in $\mathbb{R}$, $\{I_\alpha\}$, there exists a sub collection, $\{I_j\}$, such that the elements of $\{I_j\}$ are disjoint, and $\mu(\cup I_\alpha) \leq C\mu(\cup I_j)$ for some universal constant C.

However, in $\mathbb{R}^2$, bases can be either weak type (1, 1), or weak type $L \log^+ L$. Examples of bases in $\mathbb{R}^2$ with weak type (1, 1) are the collection of rectangles with side lengths (t, t), and $(t, \phi(t))$, where $\phi(t)$ is a function non-decreasing in t. Examples of a basis in $\mathbb{R}^2$ with weak type $L \log^+ L$ are the families of rectangles with side lengths $(t, 1/t)$ and (s, t).

In general, Zygmund [9] showed that the basis of rectangles constructed by the Cartesian product of k-dimensional cubes and $n - k$ one-dimensional intervals is of weak type $L(\log^+ L)^{n-k}$. As more degrees of freedom are added to the basis, the weak type estimate becomes worse. Conversely, Zygmund conjectured that given a basis in $\mathbb{R}^n$ with side lengths as functions of k independent variables (for $k < n$), the basis would behave like the basis of all k-dimensional rectangles. However, this conjecture is generally false when applied to higher dimensions, with counterexamples provided by Soria [5].

Since the three-dimensional case is more complex than the two-dimensional case, it is worth examining families of rectangles in $\mathbb{R}^3$. Cordoba [1] proved that families of rectangles with side lengths $(t, s, \phi(s, t))$ with $\phi(s, t)$ non-decreasing in respect to both variables has weak type $L \log^+ L$. His proof involves the following covering lemma, and it is known that a basis satisfying this lemma will be weak type $L \log^+ L$:

Covering Lemma 1. Let $\mathcal{B}$ be a family of dyadic rectangles in $\mathbb{R}^3$ satisfying the following monotonicity property: if R_1, $R_2 \in \mathcal{B}$ and the horizontal dimensions of R_1 are both strictly smaller than the corresponding dimensions of R_2, then the vertical dimension of R_1 must be not bigger than the vertical dimension of R_2. Under these circumstances the family $\mathcal{B}$

satisfies the exponential type covering property, that is: Given $\{R_\alpha\} \subset \mathcal{B}$ one can select a subfamily $\{R_j\} \subset \{R_\alpha\}$ such that

(1) $\mu(\cup R_\alpha) \leq C\mu(\cup R_j)$

(2) $\int_{\cup R_j} \exp(\sum \chi_{R_j}(x))dx \leq C\mu(\cup R_j)$

for some universal constant $C < \infty$.

Soria [5] noticed that for any basis of rectangles $\mathcal{B}$ where if R and R' are two elements of $\mathcal{B}$ and $\ell_1(R) > \ell_1(R')$, then either $\ell_2(R) \geq \ell_2(R')$ or $\ell_3(R) \geq \ell_3(R')$, also has weak type $L\log^+ L$, where ℓ_1, ℓ_2, and ℓ_3 are the side lengths of the rectangles parallel to the different coordinate axes. This is because Cordoba's proof also applies to this $\mathcal{B}$. We will call this property Soria's property.

An interesting example in $\mathbb{R}^3$, and one that was used as the foundation of our research, is the family of rectangles with side lengths $(t, 1/t, s)$. It is known that this basis is weak type $L\log^+ L$, even though there is one more degree of freedom than the corresponding basis in $\mathbb{R}^2$. The original goal of our research was to generalize a property of this basis to encompass more families of rectangles, and prove that bases with this property also differentiate $L\log^+ L$.

The first property we tried was based on properties defined by Stokolos [**?**]. We present his work then introduce a property of our own. He first classified bases by determining which of the following two properties it has:

(w) $$\exists k > 1 \;\; \forall R_1, \ldots, R_k \in \mathcal{B} \; s.t. \; \exists i \neq j, \quad R_i \sim R_j;$$

(s) $$\forall k > 1 \;\; \exists R_1, \ldots, R_k \in \mathcal{B} \;\; s.t. \; \forall i \neq j, \quad R_i \nsim R_j.$$

In any dimension, a basis $\mathcal{B}$ either has property (w) or (s). A basis $\mathcal{B}$ has weak type (1,1) if and only if it has property (w) [7]. In $\mathbb{R}^2$, a basis with property (s) has weak type $L\log^+ L$. If a basis of rectangles in $\mathbb{R}^3$ has property (s), then it can either have weak type $L\log^+ L$ or $L(\log^+ L)^2$. He then introduces a new property, (is), which says

(is) $$\forall k > 1 \;\; \exists R_1, \ldots, R_k \in \mathcal{B} \;\; s.t. \; \forall i \neq j, \quad (R_i \nsim R_j) \;\&\; (R_i \cap R_j \in \mathcal{B}_0),$$

and proves that bases with this property have weak type $L(\log^+ L)^2$. The family of rectangles $(t, 1/t, s)$ behaves differently than this, so we introduced a new property

(ns) $$\forall k > 1 \;\; \exists R_1, \ldots, R_k \in \mathcal{B} \;\; s.t. \; \forall i \neq j, \quad (R_i \nsim R_j) \;\&\; (R_i \cap R_j \notin \mathcal{B}_0)$$

and wished to prove that bases satisfying this property have weak type $L\log^+ L$. However we decided to try to prove first that if a basis satisfies the following property,

$$\text{(P)} \qquad \text{if } R_1, R_2 \in \mathcal{B}, \text{ then } R_1 \cap R_2 \notin \mathcal{B}_0,$$

then it is weak type $L\log^+ L$. We would then like to be able to apply a similar argument for (ns). But we were unable to prove (P) implied weak type $L\log^+ L$.

2. Attempts At Proofs

2.1. **The First Attempt.** Let $\mathcal{B}$ be a basis which satisfies property (P). Note that any two rectangles from this basis are incomparable; otherwise the intersection of the two would be the smaller rectangle, which is an element of the basis. This implies that if one side of R_i is larger than the corresponding side of R_j, then R_i must have another side smaller than the corresponding R_j in at least one of the other two directions. Our idea was to split this basis into finite sub-bases, each of which satisfied Soria's property. We began by taking a collection of rectangles, $\{R_\alpha\}$, and selecting an arbitrary first rectangle, R_1. We then sorted the rest of the rectangles into families by comparing them to this rectangle. Take $R_2 \in \{R_\alpha\}$.

If $\ell_x(R_1) > \ell_x(R_2)$ and $\ell_y(R_1) \geq \ell_y(R_2)$, then this implies $\ell_z(R_1) \leq \ell_z(R_2)$, place R_2 in $\mathcal{B}_1$.
If $\ell_x(R_1) > \ell_x(R_2)$, $\ell_y(R_1) \leq \ell_y(R_2)$, and $\ell_z(R_1) \geq \ell_z(R_2)$, place R_2 in $\mathcal{B}_2$.
If $\ell_x(R_1) > \ell_x(R_2)$, $\ell_y(R_1) \leq \ell_y(R_2)$, and $\ell_z(R_1) \leq \ell_z(R_2)$, place R_2 in $\mathcal{B}_3$.
If $\ell_x(R_1) < \ell_x(R_2)$ and $\ell_y(R_1) \leq \ell_y(R_2)$, then this implies $\ell_z(R_1) \geq \ell_z(R_2)$, place R_2 in $\mathcal{B}_4$.
If $\ell_x(R_1) < \ell_x(R_2)$, $\ell_y(R_1) \geq \ell_y(R_2)$, and $\ell_z(R_1) \leq \ell_z(R_2)$, place R_2 in $\mathcal{B}_5$.
If $\ell_x(R_1) < \ell_x(R_2)$, $\ell_y(R_1) \geq \ell_y(R_2)$, and $\ell_z(R_1) \geq \ell_z(R_2)$, place R_2 in $\mathcal{B}_6$.

Of course, there are many ways to sort the rectangles, but the same problem arose in every method we tried. While each of $\mathcal{B}_1, \ldots, \mathcal{B}_6$ has Soria's property when comparing only the first two rectangles, as soon as another rectangle is compared, these sub-bases do not necessarily have Soria's property. For example, say R_2 and R_3 were both selected to be an element of $\mathcal{B}_1$. Even though each satisfy Soria's property when compared to R_1, there is no guarantee that the property will still hold true when compared to each other. We tried several other ways of sorting the rectangles into a finite number of bases which satisfied Soria's property, but in every attempt we were unable to ensure that the bases which resulted would still have his property.

2.2. **The Second Attempt.** Since trying to manipulate our basis to coincide with Soria's property didn't work, we took a step back to examine Cordoba's original proof off of which Soria's property was based. Here is an outline of his proof of **Covering Lemma 1**.

Proof. He begins by choosing R_1 to be an element of $\{R_\alpha\}$ with the biggest vertical side. Assuming the rectangles $R_1, R_2, \ldots, R_{j-1}$ have been chosen, let R_j be an element in $\{R_\alpha\}$ with the largest vertical side such that

$$\frac{1}{\mu(R_j)} \int_{R_j} \exp\left(\sum_{k=1}^{j-1} \chi_{R_k}(x)\right) dx \leq C$$

for some constant $1 < C < \infty$. The subfamily $\{R_j\}_{j=1,\ldots,M}$ satisfies the second part of his lemma. He then uses the fact that if R was a rectangle not chosen, then the elements of $\{R_j\}$ with a bigger vertical side than R have another side larger as well. This allows him to conclude after a complex argument,

$$\mu(\cup R_\alpha) \leq C\mu(\cup R_j).$$

□

It should also be noted that in his proof Cordoba uses the following inequality,

$$\sum_{j=1}^{n} \mu(R_j) \leq \left(1 - \frac{C}{e}\right)^{-1} \mu(\cup_{j=1}^{n} R_j).$$

Here is a proof of the inequality, which is not supplied in his original paper.

Proof. The following holds by properties of the Lebesgue measure of a set,

$$\mu(R_j) = \mu\left(R_j \setminus (\cup_{k=1}^{j-1} R_k)\right) + \mu\left(R_j \cap (\cup_{k=1}^{j-1} R_k)\right).$$

Rearranging the equation gives

$$\mu\left(R_j \setminus (\cup_{k=1}^{j-1} R_k)\right) = \mu(R_j) - \mu\left(R_j \cap (\cup_{k=1}^{j-1} R_k)\right).$$

The selection criterion

$$\int_{R_n} \exp\left(\sum_{j=1}^{n-1} \chi_{R_j}(x)\right) dx \leq C\mu(R_n)$$

implies

$$e\,\mu\left(R_j \cap (\cup_{k=1}^{j-1} R_k)\right) \leq \int_{R_j \cap \left(\cup_{k=1}^{j-1} R_k\right)} \exp\left(\sum_{k=1}^{j-1} \chi_{R_k}(x)\right) dx \leq C\mu(R_j).$$

Dividing by e gives

$$\mu\left(R_j \cap (\cup_{k=1}^{j-1} R_k)\right) \leq \frac{C}{e}\mu(R_j).$$

Substitution yields

$$\mu\left(R_j \setminus (\cup_{k=1}^{j-1} R_k)\right) = \mu(R_j) - \mu\left(R_j \cap (\cup_{k=1}^{j-1} R_k)\right) \geq \left(1 - \frac{C}{e}\right)\mu(R_j). \tag{1}$$

By iteratively splitting up $\mu(\cup_{j=1}^{n} R_j)$, the following equalities are true:

$$\begin{aligned}
\mu(\cup_{j=1}^{n} R_j) &= \mu(R_1) + \mu\left(\cup_{j=1}^{n} R_j \setminus R_1\right) \\
&= \mu(R_1) + \mu(R_2 \setminus R_1) + \mu\left(\cup_{j=1}^{n} R_j \setminus (R_1 \cup R_2)\right) \\
&\vdots \\
&= \sum_{j=1}^{n} \mu\left(R_j \setminus \cup_{k=1}^{j-1} R_k\right).
\end{aligned}$$

Therefore, because of (1), we get

$$\sum_{j=1}^{n} \mu(R_j) \leq \left(1 - \frac{C}{e}\right)^{-1} \mu(\cup_{j=1}^{n} R_j).$$

□

We modified his selection criterion to attempt to prove the following conjecture, but our argument has mistakes which we will explain after it is presented.

Conjecture 1. Let $\mathcal{B}$ be a family of dyadic rectangles in $\mathbb{R}^3$ such that

$$\forall R_i,\ R_j \in \mathcal{B},\ i \neq j,\ R_i \cap R_j \notin \mathcal{B}.$$

Then the family $\mathcal{B}$ has the exponential-type covering property, that is:

Given $\{R_\alpha\} \subset \mathcal{B}$ we can select a subfamily $\{R_j\} \subset \{R_\alpha\}$ such that,

(1) $\mu(\cup\{R_\alpha\}) \leq C\mu(\cup\{R_j\})$

(2) $\int_{\cup R_j} \exp\left(\sum \chi_{R_j}(x)\right)dx \ \leq C\mu(\cup\{R_j\})$

for some universal constant $1 \leq C < \infty$.

Argument. We will begin by attempting to prove the second part of the conjecture.

Order the elements of $\{R_\alpha\}$ first by their length in the y-direction:

$\{R^y\} = R_1^y, R_2^y, \ldots, R_N^y$ where R_1^y is the element of $\{R_\alpha\}$ with the largest side in the y-direction.

And again by their length in the z-direction:

$\{R^z\} = R_1^z, R_2^z, \ldots, R_N^z$ where R_1^z is the element of $\{R_\alpha\}$ with the largest side in the z-direction.

Create a sequence $\{R_j\}_{j=1,\ldots,n}$ by alternating selection in sequential order from $\{R^y\}$ and $\{R^z\}$, and when a rectangle is chosen from one, remove that rectangle from the other set. If a rectangle is not selected from one set, remove it from the other as well. Let R_1 be R_1^y.

Assuming we have chosen $R_1, \ldots, R_{j-1}$, let R_j be an element remaining in $\{R^z\}$ or $\{R^y\}$ with the largest z or y side satisfying:

$$\frac{1}{\mu(R_j)} \int_{R_j} \exp\left(2 \sum_{R \in \{S_j^y\}} \chi_R(x)\right) dx \leq C$$

and also

$$\frac{1}{\mu(R_j)}\int_{R_j}\exp\left(2\sum_{R\in\{S_j^z\}}\chi_R(x)\right)dx\le C$$

where $\{S_j^y\}$ is the set of rectangles selected from $\{R^y\}$ with a second side larger than R_j and $\{S_j^z\}$ is the set of rectangles selected from $\{R^z\}$ with a second side larger than R_j.

Let $\{R_j^y\}_{j=1,\ldots,l}$ be the elements of $\{R_j\}_{j=1,\ldots,n}$ selected from $\{R^y\}$, and $\{R_j^z\}_{j=1,\ldots,m}$ be the elements of $\{R_j\}_{j=1,\ldots,n}$ selected from $\{R^z\}$.

The subfamily $\{R_j^y\}_{j=1,\ldots,l}$ satisfies

$$\begin{aligned}\int\limits_{\bigcup_{j=1}^{l}R_j^y}\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}(x)\right)dx &= \int\limits_{\bigcup_{j=1}^{l-1}R_j^y}\exp\left(2\sum_{j=1}^{l-1}\chi_{R_j^y}(x)\right)dx+e^2\int\limits_{R_l^y}\exp\left(2\sum_{j=1}^{l-1}\chi_{R_j^y}(x)\right)dx\\ \text{(X)}\qquad &\le \int\limits_{\bigcup_{j=1}^{l-1}R_j^y}\exp\left(2\sum_{j=1}^{l-1}\chi_{R_j^y}(x)\right)dx+e^2C\mu(R_l^y)\\ &\le e^2C\sum_{j=1}^{l}\mu(R_j^y)\le C'\mu\left(\bigcup_{j=1}^{l}R_j^y\right),\end{aligned}$$

$$\text{(2)}\qquad \int\limits_{\bigcup_{j=1}^{l}R_j^y}\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}(x)\right)dx\le C'\mu\left(\bigcup_{j=1}^{n}R_j\right).$$

The subfamily $\{R_j^z\}_{j=1,\ldots,m}$ satisfies

$$\text{(3)}\qquad \int\limits_{\bigcup_{j=1}^{m}R_j^z}\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}(x)\right)dx\le C'\mu\left(\bigcup_{j=1}^{n}R_j\right)$$

by a similar argument.

Since $\{R_j^z\}_{j=1,\ldots,m}$ and $\{R_j^y\}_{j=1,\ldots,l}$ have no elements in common, we can say that

$$\int\limits_{\bigcup_{j=1}^{n} R_j} \exp\left(\sum_{j=1}^{n} \chi_{R_j}(x)\right) dx \leq \int\limits_{\bigcup_{j=1}^{l} R_j^y} \exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}(x)\right) dx + \int\limits_{\bigcup_{j=1}^{m} R_j^z} \exp\left(2\sum_{j=1}^{m} \chi_{R_j^z}(x)\right) dx \leq C'\mu\left(\bigcup_{j=1}^{n} R_j\right).$$

Now we will begin the argument towards the first part of the conjecture.

Suppose $\exists R_\beta \in \{R_\alpha\} \setminus \{R_j\}$. This implies that either

$$\frac{1}{\mu(R_\beta)} \int_{R_\beta} \exp\left(2 \sum_{R \in \{S_\beta^y\}} \chi_R(x)\right) dx \ > C \tag{4}$$

or

$$\frac{1}{\mu(R_\beta)} \int_{R_\beta} \exp\left(2 \sum_{R \in \{S_\beta^z\}} \chi_R(x)\right) dx \ > C \tag{5}$$

where $\{S_\beta^y\}$ is the set of rectangles selected from $\{R^y\}$ that appear before R_β with a second side larger than R_β and $\{S_\beta^z\}$ is the set of rectangles selected from $\{R^z\}$ that appear before R_β with a second side larger than R_β.

In case (4), if R_β was not selected, then it must have been true that those rectangles in $\{S_\beta^y\}$ had one or both of side lengths in the x- and z-direction larger than R_β. However, if R_β were smaller in both the x- and z-directions, this would imply that $R_\beta \sim R$, for all rectangles in $\{S_\beta^y\}$. If those rectangles in $\{S_\beta^y\}$ had only one side length in the x- or z-direction larger than R_β, then let $R_{k_1}^y, \ldots, R_{k_r}^y$ be the elements of $\{S_\beta^y\}$ with x-dimension larger than that of R_β, and $R_{l_1}^y, \ldots, R_{l_s}^y$ be the elements of $\{S_\beta^y\}$ with z-dimension larger than that of R_β. By splitting the sum of the characteristic functions of the elements of $\{S_\beta^y\}$ over the two sequences and integrating the series expansion of the exponential function, we have that

$$\frac{1}{\mu(R_\beta)} \int_{R_\beta} \exp\left(2 \sum_{R \in \{S_\beta^y\}} \chi_R(x)\right) dx \ = \sum_{r,\, s=0}^{\infty} \frac{2^r}{r!} \frac{2^s}{s!} \sum_{\substack{k_1,\ldots,k_r \\ l_1,\ldots,l_s}} \frac{\mu(R_{k_1}^y \cap \cdots \cap R_{k_r}^y \cap R_{l_1}^y \cap \cdots \cap R_{l_s}^y \cap R_\beta)}{\mu(R_\beta)}.$$

If $\mu(R_{k_1}^y \cap \cdots \cap R_{k_r}^y \cap R_{l_1}^y \cap \cdots \cap R_{l_s}^y \cap R_\beta) \neq 0$, then the intersection must be of the form shown in Figure 1, where the block corresponding to ε is the intersection of the $R_{k_i}^y$'s with R_β and the block corresponding to δ is the intersection of the $R_{l_i}^y$'s with R_β Let 2^k be the x-dimension of R_β and 2^l be the z-dimension of R_β.

Assuming that $\varepsilon \neq 0$ and $\delta \neq 0$, we have

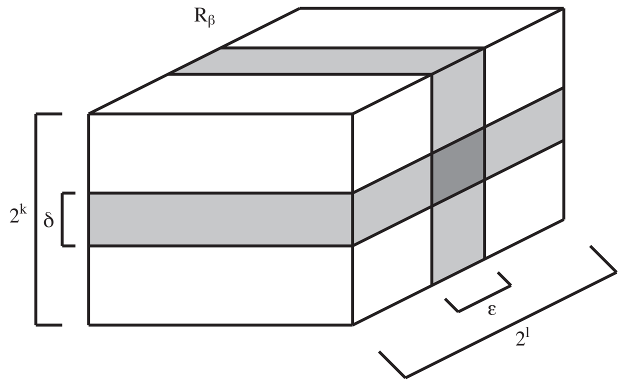

FIGURE 1. The intersections through R_β

$$\frac{\mu(R^y_{k_1}\cap\cdots\cap R^y_{k_r}\cap R^y_{l_1}\cap\cdots\cap R^y_{l_s}\cap R_\beta)}{\mu(R_\beta)}=\frac{\varepsilon\times\delta}{2^k\times 2^l}.$$

Let $Q=(x_0,y_0,z_0)\in R_\beta$,

$$I^1_q=\{(x,y_0,z_0)\in R_\beta\} \text{ and } \quad I^3_q=\{(x_0,y_0,z)\in R_\beta\}.$$

Then

$$\frac{|R^y_{k_1}\cap\cdots\cap R^y_{k_r}\cap I^1_q|}{|I^1_q|}\geq\frac{\varepsilon}{2^k},$$

$$\frac{|R^y_{l_1}\cap\cdots\cap R^y_{l_s}\cap I^3_q|}{|I^3_q|}\geq\frac{\delta}{2^l}.$$

$$\begin{aligned} C &\leq \sum_{r,\,s=0}^{\infty}\frac{2^r}{r!}\frac{2^s}{s!}\sum_{\substack{k_1,\dots,k_r\\ l_1,\dots,l_s}}\frac{|R^y_{k_1}\cap\cdots\cap R^y_{k_r}\cap I^1_q|\times|R^y_{l_1}\cap\cdots\cap R^y_{l_s}\cap I^3_q|}{|I^1_q|\times|I^3_q|}\\ &\leq \left(\frac{1}{|I^1_q|}\int_{I^1_q}\exp\left(2\sum_{R\in\{S^y_\beta\}}\chi_R\right)dx\right)\times\left(\frac{1}{|I^3_q|}\int_{I^3_q}\exp\left(2\sum_{R\in\{S^y_\beta\}}\chi_R\right)dx\right)\\ &\leq \left(\frac{1}{|I^1_q|}\int_{I^1_q}\exp\left(2\sum_{j=1}^{l}\chi_{R^y_j}\right)dx\right)\times\left(\frac{1}{|I^3_q|}\int_{I^3_q}\exp\left(2\sum_{j=1}^{l}\chi_{R^y_j}\right)dx\right). \end{aligned}$$

If M_x, M_z denote the extensions to $\mathbb{R}^3$ of the one-dimensional Hardy-Littlewood maximal function in the x- and z-directions respectively, we have

$$R_\beta \subseteq \left\{ x \in \cup\{R_\alpha\} : \; M_x \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \times M_z \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \geq C \right\},$$

and therefore

(6)

$$R_\beta \subseteq \left\{ x \in \cup\{R_\alpha\} : \; M_x \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \geq \sqrt{C} \right\} \cup \left\{ x \in \cup\{R_\alpha\} : \; M_z \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \geq \sqrt{C} \right\}.$$

Similarly, in case (5), if R_β was not selected, and if M_x, M_y denote the extensions to $\mathbb{R}^3$ of the one-dimensional Hardy-Littlewood maximal function in the x- and y-directions respectively, we have that

(7)

$$R_\beta \subseteq \left\{ x \in \cup\{R_\alpha\} : \; M_x \left(\exp\left(2\sum_{j=1}^{m} \chi_{R_j^z}\right)\right) \geq \sqrt{C} \right\} \cup \left\{ x \in \cup\{R_\alpha\} : \; M_y \left(\exp\left(2\sum_{j=1}^{m} \chi_{R_j^z}\right)\right) \geq \sqrt{C} \right\}.$$

Combining (6) and (7) yields

$$\begin{aligned} \cup\{R_\alpha\} \setminus \{R_j\} \;\; \subseteq \;\; & \{x \in \cup\{R_\alpha\} : \; M_x \left(\exp\left(2\sum_{j=1}^{m} \chi_{R_j^z}\right)\right) \geq \sqrt{C}\} \\ & \cup \{x \in \cup\{R_\alpha\} : \; M_y \left(\exp\left(2\sum_{j=1}^{m} \chi_{R_j^z}\right)\right) \geq \sqrt{C}\} \\ & \cup \{x \in \cup\{R_\alpha\} : \; M_x \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \geq \sqrt{C}\} \\ & \cup \{x \in \cup\{R_\alpha\} : \; M_z \left(\exp\left(2\sum_{j=1}^{l} \chi_{R_j^y}\right)\right) \geq \sqrt{C}\}. \end{aligned}$$

Then

$$\begin{aligned}
\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right) \;&\leq\; \mu\left(\left\{x\in\cup\{R_\alpha\}:\; M_x\left(\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}\right)\right)\geq\sqrt{C}\right\}\right)\\
&\quad+\mu\left(\left\{x\in\cup\{R_\alpha\}:\; M_y\left(\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}\right)\right)\geq\sqrt{C}\right\}\right)\\
&\quad+\mu\left(\left\{x\in\cup\{R_\alpha\}:\; M_x\left(\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}\right)\right)\geq\sqrt{C}\right\}\right)\\
&\quad+\mu\left(\left\{x\in\cup\{R_\alpha\}:\; M_z\left(\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}\right)\right)\geq\sqrt{C}\right\}\right).
\end{aligned}$$

Since the one-dimensional Hardy-Littlewood maximal functions are weak-type (1,1), we have that

$$\begin{aligned}
\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right) \;&\leq\; \frac{2C_1}{\sqrt{C}}\left(\int_{\cup\{R_\alpha\}}\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}\right)dx \;+\; \int_{\cup\{R_\alpha\}}\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}\right)dx\right)\\
&\leq\; \frac{2C_1}{\sqrt{C}}\left(\int_{\cup\{R_j^y\}}\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}\right)dx+\int_{\cup\{R_\alpha\}\setminus\{R_j^y\}}\exp\left(2\sum_{j=1}^{l}\chi_{R_j^y}\right)dx\right.\\
&\qquad\left.+\int_{\cup\{R_j^z\}}\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}\right)dx+\int_{\cup\{R_\alpha\}\setminus\{R_j^z\}}\exp\left(2\sum_{j=1}^{m}\chi_{R_j^z}\right)dx\right).
\end{aligned}$$

By inequalities (2) and (3),

$$\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right)\leq\frac{2C_1}{\sqrt{C}}\left(C'\mu(\cup\{R_j\})\;+\;\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right)+C'\mu(\cup\{R_j\})+\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right)\right).$$

Fix C so that $\frac{4C_1}{\sqrt{C}}<\frac{1}{2}$.

Taking $\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right)$ to the left-hand side, we have that

$$\mu\left(\cup\{R_\alpha\}\setminus\{R_j\}\right)\leq C'\mu(\cup\{R_j\}).$$

Since

$$\mu(\cup\{R_\alpha\})\leq\mu(\cup\{R_j\})+\mu(\cup\{R_\alpha\}\setminus\{R_j\})$$

we can say that

$$\begin{aligned}
\mu(\cup\{R_\alpha\}) \;&\leq\; \mu(\cup\{R_j\})+C'\mu(\cup\{R_j\})\\
&\leq\; C'\mu(\cup\{R_j\}).
\end{aligned}$$

This ends the argument.

The fatal mistake lies in line (X). The inequality is not true, because the sum taken in the selection criteria was only over those rectangles selected from $\{R_y\}$ before R_j with a second side larger than R_j, rather than the entire set of rectangles previously selected from $\{R_y\}$. Therefore, using our selection criteria, we are not able to get an upper bound for

$$\int_{R_l^y} \exp\left(2\sum_{j=1}^{l-1} \chi_{R_j^y}(x)\right) dx.$$

2.3. **The Third Attempt.** After failing to fix our argument, we decided to modify our approach. Instead of dividing $\{R_j\}$ we decided to create one sequence and use the following selection criterion:
Let R_1 be the element of $\{R_\alpha\}$ with the largest volume. Assuming $R_1, \ldots, R_{j-1}$ have been chosen, let R_j be an element remaining in $\{R_\alpha\}$ with the largest volume such that

$$\frac{1}{\mu(R_j)} \int_{R_j} \exp\left(\sum_{k=1}^{j-1} \chi_{R_k}(x)\right) dx \leq C. \tag{8}$$

This selection criterion implies

$$\int_{\bigcup_{j=1}^{n} R_j} \exp\left(\sum_{j=1}^{n} R_j(x)\right) dx \leq C'\mu\left(\bigcup_{j=1}^{n} R_j\right),$$

by an argument similar to the one presented in our second attempt. However, in this case, the argument is valid because each element of $\{R_j\}$ is compared to all previously selected elements of $\{R_j\}$.

Instead our argument broke down when trying to show the first part of the conjecture. Suppose there is $R_\beta \in \{R_\alpha\} \setminus \{R_j\}$. Let $\{R_k\}_{k=1,\ldots,m}$ be the elements of $\{R_j\}$ with larger volume than R_β, then

$$\frac{1}{\mu(R_\beta)} \int_{R_\beta} \exp\left(\sum_{k=1}^{m} \chi_{R_k}(x)\right) dx > C. \tag{9}$$

Elements of $\{R_k\}_{k=1,\ldots,m}$ must have at least one side smaller than R_β, otherwise some elements of $\{R_k\}_{k=1,\ldots,m}$ would be comparable to R_β.

If $\ell_x(R) < \ell_x(R_\beta)$, put $R \in \{R_k\}_{k=1,\ldots,m}$ into the subsequence $\{R_{k_i}\}_{i=1,\ldots,r}$.
If $\ell_y(R) < \ell_y(R_\beta)$, put $R \in \{R_k\}_{k=1,\ldots,m}$ into the subsequence $\{R_{e_i}\}_{i=1,\ldots,s}$.
If $\ell_z(R) < \ell_z(R_\beta)$, put $R \in \{R_k\}_{k=1,\ldots,m}$ into the subsequence $\{R_{f_i}\}_{i=1,\ldots,t}$.

Expanding the exponential and integrating gives us

$$\begin{aligned} C &< \frac{1}{\mu(R_\beta)} \int_{R_\beta} \exp\left(\sum_{k=1}^{m} \chi_{R_k}(x)\right) dx \\ &< \sum_{r,s,t=0}^{\infty} \frac{1}{r!}\frac{1}{s!}\frac{1}{t!} \sum_{\substack{k_1 \ldots k_r \\ e_1 \ldots e_s \\ f_1 \ldots f_t}} \frac{\mu(R_{k_1} \cap \cdots \cap R_{k_r} \cap R_{e_1} \cap \cdots \cap R_{e_s} \cap R_{f_1} \cap \cdots \cap R_{f_t} \cap R_\beta)}{\mu(R_\beta)}. \end{aligned}$$

If $\mu(R_{k_1} \cap \cdots \cap R_{k_r} \cap R_{e_1} \cap \cdots \cap R_{e_s} \cap R_{f_1} \cap \cdots \cap R_{f_t} \cap R_\beta) \neq 0$, then the intersection must be of the form shown in Figure 2, where the block corresponding to ε is the intersection of the R_{k_i}'s with R_β, the block corresponding to δ is the intersection of the R_{e_i}'s with R_β, and the block corresponding to γ is the intersection of the R_{f_i}'s with R_β.

Let 2^k be the x-dimension of R_β, 2^l be the y-dimension of R_β, and 2^m be the z-dimension of R_β.

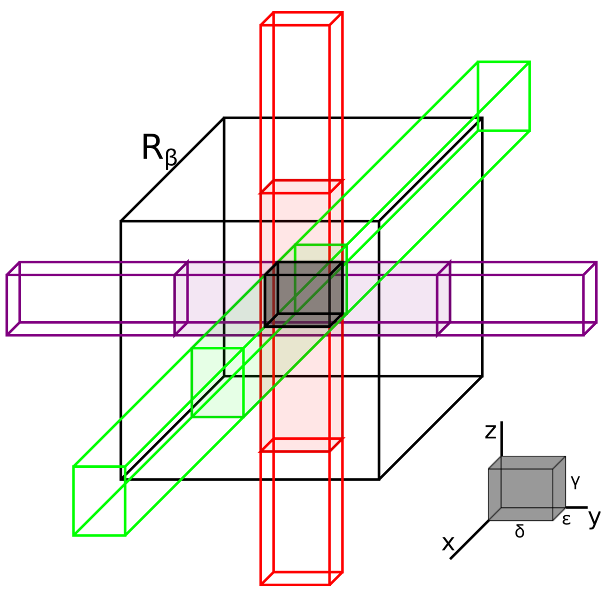

FIGURE 2. The intersections through R_β

Therefore,

$$\frac{\mu(R_{k_1} \cap \cdots \cap R_{f_t} \cap R_\beta)}{\mu(R_\beta)} = \frac{\varepsilon \times \delta \times \gamma}{2^k \times 2^l \times 2^m}. \tag{10}$$

Let $P = (x_0, y_0, z_0) \in R_\beta$, and consider

$$\begin{aligned} I_p^1 &= \{(x, y_0, z_0) \in R_\beta\}, \\ I_p^2 &= \{(x_0, y, z_0) \in R_\beta\}, \\ I_p^3 &= \{(x_0, y_0, z) \in R_\beta\}. \end{aligned}$$

From here we would like to say

$$\frac{|R_{k_1} \cap \cdots \cap R_{k_r} \cap I_p^1|}{|I_p^1|} = \frac{\varepsilon}{2^k},$$
$$\frac{|R_{e_1} \cap \cdots \cap R_{e_s} \cap I_p^2|}{|I_p^2|} = \frac{\delta}{2^l},$$
$$\frac{|R_{f_1} \cap \cdots \cap R_{f_t} \cap I_p^3|}{|I_p^3|} = \frac{\gamma}{2^m}.$$

However, we are unable to ensure that I_p^1, I_p^2, and I_p^3 intersect with the ε-, δ- and γ-blocks respectively, which would make it possible for one of $|R_{k_1} \cap \cdots \cap R_{k_r} \cap I_p^1|$, $|R_{e_1} \cap \cdots \cap R_{e_s} \cap I_p^2|$, or $|R_{f_1} \cap \cdots \cap R_{f_t} \cap I_p^3|$ to be zero. Thus, we can not guarantee lower bounds on the extensions to $\mathbb{R}^3$ of the one-dimensional Hardy-Littlewood maximal functions as we could in our second attempt.

3. Conclusion

In conclusion, we were unable to verify that any basis with property (P) is weak type $L(\log^+ L)$. We thought the best method to prove this would be to modify Cordoba's proof, but we were unable to apply it to our property. Before we continue with more attempts to prove bases with our property are weak type $L(\log^+ L)$, it might be better to compute some examples and see if any contradict our conjecture. As a potential counterexample, we could compute the value of the maximal function of the characteristic function of a cube associated with the basis of all rectangles with side lengths $(t, s, 1/st)$. If we were able to show that

$$\mu(\{x : M_{\mathcal{B}} f > \alpha\}) > C \int_{\mathbb{R}^3} \left(\frac{|f|}{\alpha} \left(1 + \log^+ \frac{f}{\alpha} \right) \right) dt,$$

then we would know that neither (ns) nor (P) guarantee that a basis will have weak type $L(\log^+ L)$. In this case, it would be necessary to extract a different property from our original basis $(t, 1/t, s)$.

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The College of New Jersey

E-mail address: `ambrosi2@tcnj.edu`

Oregon State University

E-mail address: `nabelekp@onid.oregonstate.edu`