

# ETA-QUOTIENTS OF PRIME OR SEMIPRIME LEVEL AND ELLIPTIC CURVES

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ABSTRACT. In this paper, we investigate questions related to eta-quotients. We prove that eta-quotients that satisfy a condition of Newman are always modular forms if  $N$  is coprime to 6, and study the case of all odd levels. We also prove that all eta-quotients which are modular forms for  $\Gamma_1(N)$  are also modular forms for  $\Gamma_0(N)$ . We found a condition for the existence of an eta-quotient as a modular form for prime levels, and generalized this for semiprime levels as well, which has improved upon recent work of Arnold-Roksandich, James, and Keaton. In addition, we investigate representing modular forms associated to elliptic curves in terms of linear combinations of eta-quotients, and provide some new examples.

## 1. INTRODUCTION

**1.1. Background: Modular Forms.** We will first provide some definitions, notation, and basic facts regarding modular forms.

**Definition 1.1.** The set denoted  $SL_2(\mathbb{Z})$ , is the multiplicative group of  $2 \times 2$  matrices with determinant 1. We call this the modular group.

We specifically turn our attention to certain subgroups of  $SL_2(\mathbb{Z})$ , namely

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N}, a, d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : b, c \equiv 0 \pmod{N}, a, d \equiv 1 \pmod{N} \right\}.$$

These are examples of congruence subgroups, which are subgroups of  $SL_2(\mathbb{Z})$  which contain  $\Gamma(N)$  for some  $N \in \mathbb{N}$ . We denote an arbitrary congruence subgroup as  $\Gamma_N$ , where  $N$  is the least positive integer for which  $\Gamma(N) \subset \Gamma_N$ . We call  $N$  the level of  $\Gamma_N$ .

The modular group acts on the upper-half plane,  $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ , in the following manner:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

This group action can be generalized to the extended upper-half plane,  $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ .

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*Date:* August 17, 2018.

This work was done during the Summer 2018 REU program in Mathematics at Oregon State University, supported by grant DMS-1359173.

**Definition 1.2.** Let  $\Gamma_N$  be a congruence subgroup of level  $N$ . We define an equivalence relation on  $\mathbb{H}^*$  by  $\tau_1 \sim \tau_2$  if and only if there exists  $M \in \Gamma_N$  where  $M\tau_1 = \tau_2$ . The equivalence classes in  $\mathbb{Q} \cup \{i\infty\}$  are known as the cusps of  $\Gamma_N$ .

In particular, Martin [6] gives the following complete set of representatives for the cusps of  $\Gamma_0(N)$ :

$$(1) \quad C_N = \left\{ \frac{a_c}{c} \in \mathbb{Q} : c \mid N, 1 \leq a_c \leq N, \gcd(a_c, N) = 1 \text{ and } a_c \equiv a'_c \pmod{\gcd\left(c, \frac{N}{c}\right)} \text{ iff } a_c = a'_c \right\}.$$

In the case where  $N$  is squarefree,  $\gcd(c, N/c) = 1$  for all  $c$  which divide  $N$ . Therefore, in this case we have the following as a complete set of representatives:

$$(2) \quad C_N = \left\{ \frac{1}{d} \in \mathbb{Q} : d \mid N \right\}.$$

**Definition 1.3.** A Dirichlet character  $\chi$  modulo  $N$  is a group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}.$$

With this in mind we define our main objects of interest.

**Definition 1.4.** Let  $N \in \mathbb{N}$  and  $f : \mathbb{H}^* \rightarrow \mathbb{C}$ ,  $\Gamma_N$  a congruence subgroup of level  $N$ , and  $\chi$  a Dirichlet character modulo  $N$ . Then  $f$  is a modular form of weight  $k$  and character  $\chi$  for  $\Gamma_N$  if and only if:

- (1)  $f$  is holomorphic on all of  $\mathbb{H}$ ,
- (2)  $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau\right) = \chi(d)(c\tau + d)^k f(\tau)$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N$ ,
- (3)  $f$  is holomorphic at each cusp of  $\Gamma_N$ .

If, however,  $f$  is meromorphic at the cusps, then  $f$  is called a weakly holomorphic modular form. Additionally, if  $f$  is holomorphic and vanishes at all of the cusps,  $f$  is called a cusp form.

We determine holomorphicity by calculating the *orders of vanishing* at the cusps; that is, a value assigned to each cusp that determines how quickly the function tends to zero. If the order of vanishing is negative at a cusp, the function is not holomorphic at the cusp; if the order of vanishing is zero, the function is holomorphic but nonvanishing; if the order of vanishing is positive, the function is holomorphic and vanishes.

For a given congruence subgroup  $\Gamma_N$ , denote by  $M_k^!(\Gamma_N, \chi)$ ,  $M_k(\Gamma_N, \chi)$ , and  $S_k(\Gamma_N, \chi)$  the sets of weakly holomorphic modular forms, modular forms, and cusp forms of weight  $k$ , and character  $\chi$  for the group  $\Gamma_N$ . If the character is omitted the trivial character is assumed. These sets all form vector spaces over  $\mathbb{C}$ . In the case of  $M_k(\Gamma_N, \chi)$  and  $S_k(\Gamma_N, \chi)$  they are finite-dimensional subspaces of  $M_k^!(\Gamma_N, \chi)$ . We typically only need to concern ourselves with characters in the case of  $\Gamma_0(N)$  due to the following identity.

**Proposition 1.5** ([8], Eqn. 1.7). *We have that*

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N), \chi)$$

In section 5, we will explore  $q$ -expansions of  $\eta$ -quotients, and will be using the following theorem:

**Theorem 1.6** ([7], 6.4.7). *Let  $\Gamma$  be a congruence subgroup and  $f \in M_k(\Gamma)$ . Let  $r_1, \dots, r_t$  be the cusps of  $\Gamma$ . If*

$$\sum_{i=1}^t v_{r_i}(f) > \frac{k[\mathrm{SL}_2(\mathbb{Z}) : \{\pm I\}\Gamma]}{12},$$

then  $f = 0$ .

**1.2. Background: Eta-Quotients.** Richard Dedekind defined the function  $\eta : \mathbb{H} \rightarrow \mathbb{C}$  as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where throughout this paper  $q := e^{2\pi i\tau}$  unless explicitly stated otherwise. We will often denote such an  $\eta$ -quotient  $f(\tau)$  as  $\eta_g(\tau)$  where  $g$  is the formal product

$$g := \prod_{0 < \delta | N} \delta^{r_\delta}.$$

This function has shed light on many different branches of mathematics; for example, Ramanujan's  $\Delta$ -function is defined as  $\Delta(\tau) = \eta^{24}(\tau)$ . However, we are interested in  $\eta$ -quotients, which are functions of form

$$f(\tau) = \prod_{0 < \delta | N} \eta(\delta\tau)^{r_\delta},$$

where each  $r_\delta$  is an integer and  $N \in \mathbb{N}$ .

The following theorem, which is a variant of a theorem of Gordon and Hughes in [3], it provides two sufficient criteria to determine the modularity of an  $\eta$ -quotient.

**Theorem 1.7** (Theorem 1.64, [8]). *Let  $\eta_g$  be the eta quotient given by*

$$\eta_g(\tau) = \prod_{\delta | N} \eta^{r_\delta}(\delta\tau).$$

If  $\eta_g$  satisfies

$$(3) \quad \sum_{0 < \delta | N} \delta r_\delta \equiv 0 \pmod{24},$$

$$(4) \quad \sum_{0 < \delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then for  $k = \frac{1}{2} \sum_{0 < \delta | N} r_\delta$  and  $\chi(d) = \left(\frac{-1}{d}\right)^s$  where  $s = \sum_{0 < \delta | N} \delta r_\delta$ , we have  $\eta_g \in M_k^!(\Gamma_0(N), \chi)$ , i.e for all  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$\eta_g(M\tau) = \chi(d)(c\tau + d)^k \eta_g(\tau).$$

The next theorem is also originally due to Gordon and Hughes, it calculates the orders of vanishing at each of the cusps of our  $\eta$ -quotient.

**Theorem 1.8** (Theorem 1.65, [8]). *Let  $c$ ,  $d$ , and  $N$  be positive integers with  $d \mid N$  and  $\gcd(c, d) = 1$ . Then if  $f(\tau)$  is an  $\eta$ -quotient satisfying the conditions given in Theorem 1.7 for  $N$ , then the order of vanishing for  $f(\tau)$  at the cusp  $\frac{c}{d}$  is*

$$\frac{N}{24} \sum_{0 < \delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d \delta}.$$

This allows us to discern whether an  $\eta$ -quotient is holomorphic or weakly holomorphic, depending on whether or not the formula yields non negative orders of vanishing at each cusp, which we will define in Section 2. Note also that if all orders of vanishing are strictly positive, our  $\eta$ -quotient is a cusp form.

**1.3. Background: Elliptic Curves.** In Section 5 of this paper, we will explore connections between elliptic curves and linear combinations of  $\eta$ -quotients. We must, however, recall what elliptic curves are.

**Definition 1.9.** An **elliptic curve**  $E$  over a field  $K$ , denoted  $E/K$ , is given by an equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x,$$

where each  $a_i \in K$ . If the characteristic of  $K$  is not 2 or 3, through a proper change of variables, we may reduce this equation to

$$E : y^2 = x^3 + Ax + B$$

with  $A, B \in K$ . This simplified equation is known as the **reduced Weierstrass form** of  $E$ .

Now, consider an elliptic curve  $E/\mathbb{Q}$ . Each such elliptic curve has an invariant integer associated with it known as the conductor. In short, the conductor is a number associated to an elliptic curve that is determined by the how the elliptic curve reduces modulo  $p$ . Elliptic curves have attached functions of the form

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}, \quad s \in \mathbb{C},$$

known as *L-functions*. From here, we can discuss the relationship to modular forms.

**Theorem 1.10.** [Modularity Theorem] *Every elliptic curve  $E$  over  $\mathbb{Q}$  with conductor  $N$  has an L-function*

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

such that the Fourier series

$$f(\tau) = \sum_{n=1}^{\infty} a_E(n) q^n, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}$$

represents a level  $N$  cusp form of weight 2.

This theorem allows us to ask when a modular form associated to an elliptic curve can be written as a linear combination of  $\eta$ -quotients.

**1.4. Results.** Our first theorem was motivated by the ubiquitous use of Theorem 1.7 in the literature on  $\eta$ -quotients. Our goal was to focus on the necessity and or sharpness of Conditions (3) and (4). The following theorem demonstrates the necessity of Conditions (3) and (4) in Theorem 1.7 in the case where  $N$  is coprime to 6, and we later discuss some partial results in the case of general odd levels.

**Theorem 1.11.** *Let  $N \in \mathbb{N}$  be such that  $\gcd(N, 6) = 1$  and let  $\eta_g$  be the eta quotient given by  $\eta_g(\tau) = \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$ . Then  $\eta_g(\tau) \in M_k^!(\Gamma_0(N), \chi)$  for some even weight  $k$  and character  $\chi$  if and only if*

$$\begin{aligned} \sum_{0 < \delta|N} \delta r_\delta &\equiv 0 \pmod{24}, \\ \sum_{0 < \delta|N} \frac{N}{\delta} r_\delta &\equiv 0 \pmod{24}. \end{aligned}$$

*In particular,  $\eta_g$  must satisfy all conditions of weight and character provided by Theorem 1.7.*

Theorem 1.11 allows us to prove that all weakly holomorphic modular  $\eta$ -quotients of weight  $k$  and character  $\chi$  for  $\Gamma_N$ , where  $N$  is coprime to 6 and character  $\chi$  must be elements of  $M_k^!(\Gamma_0(N), \chi)$ .

**Theorem 1.12.** *Fix  $N \in \mathbb{N}$  such that  $\gcd(N, 6) = 1$ ,  $\Gamma_N$  a level  $N$  congruence subgroup, and let  $\chi$  be a character modulo  $N$ . If  $\eta_g \in M_k^!(\Gamma_N)$ , then  $\eta_g \in M_k^!(\Gamma_0(N), \chi)$ .*

The following result shows the necessary and sufficient conditions for the existence of eta quotients in  $M_k^!(\Gamma_1(p))$ .

**Theorem 1.13.** *Let  $p \geq 5$  be a prime number, set  $h = \frac{1}{2} \gcd(p-1, 24)$ , and let  $k$  be an even integer. Then there exists  $f = \eta^{r_1}(\tau) \eta^{r_p}(p\tau)$  in  $M_k(\Gamma_1(p))$  if and only if both the following are satisfied:*

- (1)  $h \mid k$ , and
- (2) *It is not the case that  $p \neq 5$ ,  $p \equiv 5 \pmod{24}$ , and  $k = 2$ .*

In our attempt to generalize the methods used by Arnold-Roksandich, James, and Keaton in [1] to semiprime levels, we provide our own proof of Theorem 1.2 in [1] in a manner that easily generalizes to semiprime levels. The following theorem deals with the existence of  $\eta$ -quotients of semiprime levels, in very much the same spirit as Theorem 1.13.

**Theorem 1.14.** *Let  $p, q \geq 5$  be distinct primes,  $N = pq$ , and  $k$  an even integer. Then, there exists  $f(\tau) = \prod_{\delta|N} \eta^{r_\delta}(\delta\tau) \in M_k(\Gamma_1(pq))$  if and only if both the following are satisfied:*

- (1)  $h = \frac{1}{2} \gcd(24, p-1, q-1)$  divides  $k$ , and
- (2) *it is not the specific case wherein  $(p, q) \pmod{24} \in \{(1, 5)(5, 5)\}$ ,  $p, q \neq 5$ , and  $k = 2$ .*

Using our results in Section 4, we consider spaces of cusp forms of semiprime level. We have been able to successfully find an  $\eta$ -quotient basis of  $S_2(\Gamma_0(35))$ , which is provided in the following theorem.

**Theorem 1.15.** *The following linear combination of  $\eta$ -quotients is equal to the modular form corresponding to the elliptic curves with conductor  $N = 35$ :*

Conductor $N$	$\eta$ - quotient $f(z)$
35	$\eta(\tau)^2\eta(35\tau)^2 + \eta(5\tau)^2\eta(7\tau)^2$

The following sections will focus on the proofs of the aforementioned theorems. Section 2 will focus on Theorems 1.11 and 1.12, Section 3 will focus on Theorems 1.13, Section 4 will focus on Theorem 1.14, and Section 5 will focus on Theorem 1.15.

## 2. A CONVERSE TO THEOREM 1.7

This section of the paper deals with the following conjecture we have posed:

**Conjecture 2.1.** *Let  $\eta_g$  be the eta quotient given by  $\eta_g(\tau) = \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$ . Then  $\eta_g(\tau) \in M_k^!(\Gamma_0(N), \chi)$  for some weight  $k$  and character  $\chi$  if and only if*

$$\sum_{0 < \delta|N} \delta r_\delta \equiv 0 \pmod{24},$$

$$\sum_{0 < \delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

This conjecture arises from our study of the following theorem of Hughes and Gordon as well as Theorem 1.7 which we have already stated.

**Theorem 2.2** ([3], Theorem 3). *If  $\eta_g$  is an  $\eta$ -quotient given by  $\eta_g = \prod_{0 < \delta|N} \eta^{r_\delta}(\delta\tau)$ , then if*

$$(5) \quad \frac{1}{24} \sum_{0 < \delta|N} \delta r_\delta = \frac{c_1}{e_1},$$

$$(6) \quad \frac{1}{24} \sum_{0 < \delta|N} \frac{N}{\delta} r_\delta = \frac{c_2}{e_2},$$

*are such that  $\frac{c_1}{e_1}$  and  $\frac{c_2}{e_2}$  are given in lowest terms, then  $\eta_g(\tau) \in M_k^!(\Gamma_0(Ne_1e_2), \chi)$ , where  $k = \frac{1}{2} \sum_{0 < \delta|N} r_\delta$ ,  $\chi = \left(\frac{(-1)^{ks}}{d}\right)$ , and  $s = \prod_{0 < \delta|N} \delta^{r_\delta}$ .*

However, the theorem does not prove  $\eta_g$  is not a modular form at a lower level, nor does it discuss the necessity of either condition. We also note that Theorem 1.7 is the special case of Theorem 2.2 when  $e_1 = e_2 = 1$ .

In our attempt to prove Conjecture 2.1 we prove the case when  $\gcd(N, 6) = 1$ . We also consider the case of all odd levels, but our proof techniques do not generalize in some cases. In order to prove this, we will first show that Conditions (3) and (4) of Theorem 1.7 are equivalent when  $\gcd(N, 6) = 1$ . We will then prove that there is no  $\eta$ -quotient  $\eta_g$  of the form given in Theorem 1.11 which satisfies  $\eta_g \in M_k^!(\Gamma_0(N), \chi)$  but does not satisfy Condition (3).

The following lemma, which is a general result, shows that Conditions (3) and (4) of Theorem 1.7 are equivalent.

**Lemma 2.3.** *If  $N$  is an integer such that  $\gcd(N, 6) = 1$  then the following are equivalent:*

$$\begin{aligned} \sum_{0 < \delta | N} \delta r_\delta &\equiv 0 \pmod{24}, \\ \sum_{0 < \delta | N} \frac{N}{\delta} r_\delta &\equiv 0 \pmod{24}, \end{aligned}$$

where each  $r_\delta$  is an integer.

*Proof.* Suppose that  $N \in \mathbb{N}$  is such that  $\gcd(N, 6) = 1$ . Observe that all divisors  $\delta$  of  $N$  will be congruent to units in  $\mathbb{Z}/24\mathbb{Z}$ . Furthermore, all units are their own inverse modulo 24. Thus all divisors of  $N$  are their own inverse modulo 24.

Now suppose that  $\sum_{0 < \delta | N} \delta r_\delta \equiv 0 \pmod{24}$ . Using the above observation that  $\delta^2 \equiv 1 \pmod{24}$  for all divisors  $\delta$  of  $N$ , we obtain:

$$\begin{aligned} 0 &\equiv \sum_{0 < \delta | N} \delta r_\delta \\ &\equiv N \sum_{0 < \delta | N} \delta^{-1} r_\delta \\ &\equiv \sum_{0 < \delta | N} \frac{N}{\delta} r_\delta \pmod{24} \end{aligned}$$

as desired.

Conversely, suppose  $\sum_{0 < \delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$ . Observe that  $N$  can be factored out from this congruence and that for all divisors  $\delta$  of  $N$ ,  $\delta^2 \equiv 1 \pmod{24}$ . We see that

$$\sum_{0 < \delta | N} \frac{N}{\delta} r_\delta \equiv N \sum_{0 < \delta | N} \delta r_\delta \pmod{24}.$$

Since  $\gcd(N, 6) = 1$  this implies that  $N$  is a unit in  $\mathbb{Z}/N\mathbb{Z}$ , therefore

$$\sum_{0 < \delta | N} \delta r_\delta \equiv 0 \pmod{24},$$

which concludes the proof.  $\square$

We know it is sufficient to prove that only one of Conditions (3) and (4) given in Theorem 1.11 must hold. In order to show that Condition (3) must hold for a weakly holomorphic  $\eta$ -quotient as given in Theorem 1.11, we will use the the following lemma.

**Lemma 2.4.** *Suppose  $N \in \mathbb{N}$  is such that  $\gcd(N, 6) = 1$ , and  $\eta_g$  is an  $\eta$ -quotient given by  $\eta_g(\tau) = \prod_{0 < \delta | N} \eta^{r_\delta}(\delta\tau)$  such that  $\eta_g \in M_{k_g}^!(\Gamma_N, \chi_g)$  and*

$$(7) \quad \sum_{0 < \delta | N} \delta r_\delta \equiv t \pmod{24}.$$

Then for every  $\eta$ -quotient  $\eta_f$  given by  $\eta_f(\tau) = \prod_{0 < \alpha|N} \eta^{r_\alpha}(\alpha\tau)$  such that

$$(8) \quad \sum_{0 < \alpha|N} \alpha r_\alpha \equiv -t \pmod{24},$$

we must have that  $\eta_f \in M_{k_f}^!(\Gamma_0(N) \cap \Gamma_N, \chi_f)$ , where  $k_f = \frac{1}{2} \left( \sum_{0 < \delta|N} r_\delta + \sum_{0 < \alpha|N} r_\alpha \right) - k_g$ .

*Proof.* Suppose  $N \in \mathbb{N}$  is such that  $\gcd(N, 6) = 1$ , and  $\eta_g$  and  $\eta_f$  are defined as above, and  $\eta_g \in M_{k_g}^!(\Gamma_N, \chi_g)$ . We also assume that

$$\begin{aligned} \sum_{0 < \delta|N} \delta r_\delta &\equiv t, \\ \sum_{0 < \alpha|N} \alpha r_\alpha &\equiv -t. \end{aligned}$$

Observe that the product  $\eta_g \eta_f$  satisfies

$$\sum_{0 < \delta|N} \delta r_\delta + \sum_{0 < \alpha|N} \alpha r_\alpha \equiv t - t \equiv 0 \pmod{24}$$

and thus by Lemma 2.3 the product  $\eta_g \eta_f$  satisfies Conditions (3) and (4) of Theorem 1.7. In particular  $\eta_g \eta_f \in M_{k_{fg}}^!(\Gamma_0(N), \chi_{fg})$ , where

$$k_{fg} = \frac{1}{2} \left( \sum_{0 < \delta|N} r_\delta + \sum_{0 < \alpha|N} r_\alpha \right)$$

and

$$\chi_{fg}(d) = \left( \frac{(-1)^{k_{fg}} s_{fg}}{d} \right),$$

where

$$s_{fg} = \prod_{0 < \delta|N} \delta^{r_\delta} \prod_{0 < \alpha|N} \alpha^{r_\alpha}.$$

Note that that  $\eta_f = \frac{\eta_g \eta_f}{\eta_g}$  now satisfies for every  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N \cap \Gamma_0(N)$  :

$$\begin{aligned} \eta_f(M\tau) &= \frac{(\eta_f \eta_g)(M\tau)}{\eta_g(M\tau)} \\ &= \frac{\chi_{fg}(d)(c\tau + d)^{k'} \eta_f(\tau) \eta_g(\tau)}{\chi_g(d)(c\tau + d)^k \eta_g(\tau)} \\ &= \chi_f(d)(c\tau + d)^{k_{fg} - k_g} \eta_f(\tau). \end{aligned}$$

In particular,  $\eta_f \in M_{k' - k}^!(\Gamma_N, \chi_f)$ . □

We will want to use the contraposition of this lemma, that is, we want to consider  $\eta_f$  which satisfy  $\sum_{0 < \alpha|N} \alpha r_\alpha \equiv -t \pmod{24}$  that are not elements of  $M_k^!(\Gamma_N, \chi)$  when  $\gcd(N, 6) = 1$ . If for each  $t \in \{\pm 1, \dots, \pm 12\}$  we can find such an  $\eta_f$  then it will immediately imply that no such  $\eta_g$  satisfying the Condition (7) given in Lemma 2.4 ever existed, except in the case when  $t = 0$ . The most natural choices for  $\eta_f$  are exactly the  $t^{\text{th}}$  powers of the  $\eta$ -function. We now prove that none of these is a modular form on a level  $N$  coprime to 6.

**Lemma 2.5.** *If  $t \in \{\pm 1, \dots, \pm 12\}$  then  $\eta^t(\tau)$  is not a weakly holomorphic modular form for any level  $N$  congruence subgroup  $\Gamma_N$  where  $\gcd(N, 6) = 1$ .*

*Proof.* Let  $N$  be an integer such that  $\gcd(N, 6) = 1$ , and  $\Gamma_N$  be a level  $N$  congruence subgroup, and let  $t \in \{\pm 1, \dots, \pm 12\}$ .

Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$  then  $\eta(\tau)$  transforms as [4]

$$\eta(M\tau) = v(M)(c\tau + d)^{\frac{1}{2}}\eta(\tau),$$

where

$$v(M) := \begin{cases} \left(\frac{d}{|c|}\right) e^{\frac{i\pi}{12}((a+d)c + bd(c^2-1) - 3c)} & \text{if } c \equiv 1 \pmod{2} \\ \left(\frac{c}{d}\right) e^{\frac{i\pi}{12}((a+d)c + bd(c^2-1) + 3d - 3 - 3cd)} & \text{if } d \equiv 1 \pmod{2}. \end{cases}$$

We note that  $V(I) = 1$ .

It suffices to show that  $v^t(M)$  does not depend on  $d$  modulo  $N$  for any choice of  $t$  above. This is sufficient as  $v^t(M)$  must evaluate to a character depending on  $d$ . Since we're guaranteed that  $v(I) = 1$  for the identity matrix we know that if  $d = 1$  we must have  $v(M) = 1$  for any matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $d \equiv 1 \pmod{N}$ . So we consider the matrix  $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma(N) \subset \Gamma_N$ . We note  $c = N$  is odd so  $v(M)$  evaluates in the case  $c \equiv 1 \pmod{2}$  above. We also can also see that since  $d = 1$  that the generalized Legendre symbol in this expression is trivial. So we need only consider the evaluation of

$$\frac{i\pi}{12}((a+d)c + bd(c^2-1) - 3c).$$

It is sufficient to show that this does not evaluate in the form  $2k\pi i$  for  $k \in \mathbb{Z}$ , for any choice of  $N$  with  $\gcd(N, 6) = 1$  and  $t \in \{\pm 1, \dots, \pm 12\}$ . Given our choice of  $M$ , this reduces to

$$(9) \quad \frac{t\pi i}{12}(1 + 1 - 3)N = \frac{-tN\pi i}{12},$$

where  $N$  is an odd number not divisible by 3, which implies for all choices of  $t$  this will not evaluate in the form  $2k\pi i$ . Therefore this is not a well defined function in  $d$  as both 1 and the value given in 9 above, which will not be equal for fixed  $d$ .  $\square$

We note that the argument used in the proof of Lemma 2.5 will extend to all odd levels in every case except when  $t = \pm 8$ .

We now prove Theorem 1.11.

*Proof.* (Theorem 1.11)

Let  $N$  be an integer coprime to 6. And suppose  $\eta_g(\tau) \in M_k^1(\Gamma_0(N), \chi)$  for some weight  $k$ , character  $\chi$ , is such that, for  $t \not\equiv 0 \pmod{24}$ ,  $\eta_g$  satisfies:

$$\sum_{0 < \delta | N} \delta r_\delta \equiv -t \pmod{24}$$

Using Lemma 2.5, we see that for all  $t \in \{\pm 1, \dots, \pm 12\}$  that  $\eta^t(\tau)$  satisfies

$$\sum_{0 < \delta | N} \delta r_\delta \equiv t \pmod{24}.$$

Namely  $\eta^t(\tau)$  satisfies the conditions needed for  $\eta_f$  in Lemma 2.4 but is not a weakly holomorphic modular form of level  $N$  which is a contradiction. Therefore no such  $\eta_g$  can satisfy  $\eta_g \in M_k^!(\Gamma_N, \chi)$  and  $\sum_{0 < \delta | N} \delta r_\delta \equiv -t \pmod{24}$ , for any congruence classes of  $t$  modulo 24 except for 0. This immediately implies that  $\eta_g \in M_k^!(\Gamma_N, \chi)$  must satisfy the Condition (3) in Theorem 1.7 and by Lemma 2.3 it must satisfy Condition (4).

The converse of this theorem is exactly given by Theorem 1.7.  $\square$

We use Theorem 1.11 to immediately prove an even stronger result, Theorem 1.12.

*Proof.* (Theorem 1.12)

This follows from the proof of Lemmas 2.5 and 2.4, as we did not require  $\eta_g$  to be an  $\eta$ -quotient on  $\Gamma_0(N)$  but  $\Gamma_N$ . Therefore it must be the case that  $\eta_g$  satisfies Conditions (3) and (4) of Theorem 1.7, so it is also a weakly holomorphic modular form of weight  $k$  and character  $\chi$  for  $\Gamma_0(N)$ .  $\square$

This theorem immediately lends itself to a variant of Theorem 1.11.

**Corollary 2.6.** *If  $\eta_g \in M_k^!(\Gamma_N, \chi)$  where  $\gcd(N, 6) = 1$  is given by  $\eta_g = \prod_{0 < \delta | N} \eta^{r_\delta}(\delta\tau)$ , then  $\eta_g$  must satisfy:*

$$\sum_{0 \leq \delta | N} \delta r_\delta \equiv 0 \pmod{24},$$

$$\sum_{0 \leq \delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

$k = \frac{1}{2} \sum_{0 < \delta | N} r_\delta$ , and  $\chi(d) = \left(\frac{-1}{d}\right)^{k_s}$  for  $s = \prod_{0 < \delta | N} \delta^{r_\delta}$ .

*Proof.* This is a direct result of Theorems 1.11 and 1.12.  $\square$

**2.1. The Odd Case.** In our attempt to extend Theorem 1.11 to all odd cases, we have noted that there is an issue in proving Lemma 2.5 in the case where  $j = 8$ . This is not to say, however, that the lemma does not hold, only that the proof method used does not account for this case. Furthermore, for lemma 2.4 to hold, we would need to show, for  $n = \text{ord}_3(N)$ :

$$\sum_{\substack{3^n | \delta \\ \frac{N}{\delta} \equiv 1 \pmod{3}}} r_\delta \equiv \sum_{\substack{3^n | \delta \\ \frac{N}{\delta} \equiv 2 \pmod{3}}} r_\delta \pmod{3}$$

must hold for all  $\eta_g \in M_k^!(\Gamma(N), \chi)$ . Note that proving this would also prove that Theorem 2.2 is sharp in the case where  $N$  is odd. Combined with a variation of Lemma 2.5 for all odd cases, including  $j = 8$ , we would see that Conditions (5) and (6) are necessary and sufficient.

### 3. $\eta$ -QUOTIENTS OF PRIME LEVEL

The goal of this section is to prove Theorem 1.13. As discussed in the intro, the results of the prior section will allow us to determine all  $\eta$ -quotients in  $M_k^!(\Gamma_1(p))$  by looking only at the cusps of  $\Gamma_0(p)$ . By (2), there are two cusps of  $\Gamma_0(p)$ , at 1 and  $1/p$ . We first state the following useful result from a recent paper by Arnold-Roksandich, James and Keaton [1], but offer an alternative proof which we will generalize to squarefree levels in Section 4.

**Lemma 3.1** ([1], Theorem 1.2). *Let  $p \geq 5$  be a prime. There exists  $f(\tau) = \eta^{r_1}(\tau)\eta^{r_p}(p\tau)$  in  $M_k^1(\Gamma_1(p))$  if and only if  $h = \frac{1}{2} \gcd(p-1, 24) \mid k$ .*

*Proof.* Let  $v_1$  and  $v_p$  be the orders of vanishing at 1 and  $1/p$ , respectively. Using Theorem 1.8 we calculate

$$(10) \quad v_1 = \frac{1}{24}(pr_1 + r_p)$$

and

$$(11) \quad v_p = \frac{1}{24}(r_1 + pr_p).$$

We rewrite (10) as  $24v_1 = pr_1 + r_p$ . Since  $2k = r_1 + r_p$ , we may further rewrite the equation as  $24v_1 = 2k + (p-1)r_1$ , and then as

$$(12) \quad 24v_1 - (p-1)r_1 = 2k.$$

Equation (12) can be viewed as a linear Diophantine equation in variables  $v_1$  and  $r_1$ . In this light, the existence of a solution implies  $\gcd(p-1, 24) \mid 2k$ . As  $p \neq 2$ ,  $p-1$  is even and so  $2 \mid \gcd(24, p-1)$ . Therefore,  $h = \frac{1}{2} \gcd(p-1, 24)$  is an integer and  $h \mid k$ .

Conversely, if  $h \mid k$ , then (12) has integer solutions for  $v_1$  and  $r_1$ . We can then use equation (10) to solve for  $r_p$ . Moreover, as (10) has an integer solution, condition (3) of Theorem 1.7 is satisfied. As noted in 2.3 and as  $(p, 6) = 1$ , this is enough to imply that  $f(\tau) \in M_k^1(\Gamma_1(p))$ .  $\square$

Before progressing we require two additional lemmas. These lemmas will allow us to deal with certain cases in the proof of Theorem 1.13.

**Lemma 3.2.** *Let  $p$  be prime,  $h = \frac{1}{2} \gcd(p-1, 24)$ , and  $k$  be an even integer. If  $h \mid k$  and  $k(p+1)/12$  is odd, then  $(p-1)/h$  is odd.*

*Proof.* We first consider the case  $p = 2$ . In this case,  $h = 1/2$ , and so  $(p-1)/2h = 1$  is always odd, regardless of any conditions on  $k$ . We prove the case  $p \geq 3$  by contrapositive. By definition of  $h$ ,  $(p-1)/2h$  and  $12/h$  are relatively prime. Thus, it suffices to show that  $12/h$  is even. Let  $p$ ,  $h$ , and  $k$  be defined as in the statement of the lemma. Suppose  $h \mid k$  and  $12/h$  is odd. We aim to show that  $k(p+1)/12$  is even. As  $12/h$  is odd, it follows that  $4 \mid h$ . This implies  $4 \mid k$ . As  $p \neq 2$ ,  $2 \mid p+1$  and so  $8 \mid k(p+1)$ . Therefore  $k(p+1)/12$  is even.  $\square$

**Lemma 3.3.** *Let  $p \geq 5$  be prime,  $h = \frac{1}{2} \gcd(p-1, 24)$ , and  $k > 0$  be an even integer. If  $k(p+1)/12$  is even, then there exists an  $\eta$ -quotient in  $M_k(\Gamma_1(p))$ .*

*Proof.* Suppose  $p, h$ , and  $k$  satisfy the given hypotheses. Consider the following  $\eta$ -quotient:

$$f(\tau) = \eta^k(\tau)\eta^k(p\tau).$$

For this  $\eta$ -quotient, we have

$$(13) \quad \sum_{\delta \mid N} \delta r_\delta = \sum_{\delta \mid N} \frac{N}{\delta} r_\delta = k(p+1).$$

As  $k(p+1)/12$  was assumed to be even,  $24 \mid k(p+1)$ , and so (13) shows that  $f(\tau)$  satisfies the conditions of Theorem 1.7. By equations (10) and (11), we have

$$(14) \quad v_1 = v_p = \frac{k(p+1)}{24}.$$

This is a positive integer, and so  $f(\tau) \in M_k(\Gamma_1(p))$ .  $\square$

**Remark 3.4.** Note that we did not need the assumption that  $h \mid k$  in the previous lemma. This is simply as the condition that  $k(p+1)/12$  be even implies that  $h \mid k$ . To see this, observe that if  $k(p+1)/12 = 2n$ , then

$$2k = 24n - k(p-1),$$

and so  $\gcd(24, p-1) \mid 2k$ .

Finally, we will also need the following result of Rouse and Webb [10].

**Theorem 3.5** ([10], Theorem 2). *Suppose that  $f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k(\Gamma_0(N))$ . Then we have*

$$(15) \quad \sum_{\delta \mid N} |r_\delta| \leq 2k \prod_{p \mid N} \left( \frac{p+1}{p-1} \right)^{\min\{2, \text{ord}_p(N)\}}.$$

Although Theorem 3.5 specifies the space  $M_k(\Gamma_0(N))$ , the proof does not depend on any specific character, so in particular, the theorem holds for any  $\eta$ -quotient in  $M_k(\Gamma_1(N))$ . Also note that, as we will only be working with squarefree choices of  $N$ , the exponent in the product will always be 1. With these lemmas we now prove Theorem 1.13.

*Proof.* (Theorem 1.13)

Let  $p \geq 5$  be a prime,  $h = \frac{1}{2} \gcd(p-1, 24)$ , and  $k$  be an even integer. We first prove the converse by construction. Suppose that  $h \mid k$ . We proceed by cases on  $p$  and  $k$ . Recall that we want to construct a valid  $\eta$ -quotient for every case except for the case where  $p \neq 5$ ,  $p \equiv 5 \pmod{24}$ , and  $k = 2$ .

We first consider the case where  $p \not\equiv 5 \pmod{8}$ . By Lemma 3.3, it suffices to prove the following claim.

**Claim:** If  $p \not\equiv 5 \pmod{8}$ , then  $k(p+1)/12$  is even.

*Proof:* The condition that  $p \not\equiv 5 \pmod{8}$  can be further refined into the conditions that  $p \equiv 3 \pmod{4}$  or  $p \equiv 1 \pmod{8}$ . If  $p \equiv 3 \pmod{4}$ , then  $4 \mid p+1$ . As  $k$  is even,  $8 \mid k(p+1)$ , and so  $k(p+1)/12$  is even. We approach the case of  $p \equiv 1 \pmod{8}$  via the contrapositive. Suppose that  $k(p+1)/12$  is odd, we will show that  $p \not\equiv 1 \pmod{8}$ . As  $p+1$  is even, the only way  $k(p+1)/12$  can be odd is if  $4 \nmid k$ . As  $h \mid k$  it follows that  $4 \nmid h$ . This implies that  $8 \nmid p-1$ , as otherwise  $(p-1)/2h$  would be even, contradicting Lemma 3.2. Hence, if  $k(p+1)/12$  is odd then  $p$  is not congruent to 1 modulo 8.  $\blacksquare$

Turning our attention to the case that  $p \equiv 5 \pmod{8}$ , we see that  $p$  is congruent to either 5 or 13 modulo 24. We need only show the existence of an  $\eta$ -quotient in  $M_h(\Gamma_1(p))$ . This suffices as, if  $f(\tau)$  is such an  $\eta$ -quotient, then for any  $k$  such that  $h \mid k$ ,  $f^{k/h}(\tau) \in M_k(\Gamma_1(p))$ . Suppose that  $p \equiv 13 \pmod{24}$ . Then  $h = 6$ , and using Theorems 1.7 and 1.8 one can quickly check that

$$\eta^9(\tau)\eta^3(p\tau) \in M_6(\Gamma_1(p)).$$

Next we consider  $p = 5$ . We have  $h = 2$ , and

$$\frac{\eta^5(p\tau)}{\eta(\tau)} \in M_2(\Gamma_1(p)).$$

Finally, suppose  $p \neq 5$ ,  $p \equiv 5 \pmod{24}$ , and  $k > 2$ . As we are claiming that there does not exist an  $\eta$ -quotient when  $k = h = 2$ , our approach for the last two cases will not work. However, it is sufficient to show that there exist  $\eta$ -quotients  $f_4(\tau) \in M_4(\Gamma_1(p))$  and  $f_6(\tau) \in M_6(\Gamma_1(p))$ . For any even integer  $k > 6$ , either  $4 \mid k$  or  $4 \mid (k - 6)$ . In the first case,  $f_4^{k/4}(\tau) \in M_k(\Gamma_1(p))$  and in the second  $f_4^{(k-6)/4}(\tau)f_6(\tau) \in M_k(\Gamma_1(p))$ . We can again check that

$$f_4(\tau) := \eta^4(\tau)\eta^4(p\tau) \in M_4(\Gamma_1(p))$$

and

$$f_6(\tau) := \eta^9(\tau)\eta^3(p\tau) \in M_6(\Gamma_1(p)),$$

resolving this final case.

For the forwards direction, suppose that  $p, h$ , and  $k$  are as defined and  $f(\tau) = \eta^{r_1}(\tau)\eta^{r_p}(p\tau)$  belongs to  $M_k(\Gamma_1(p))$ . As  $M_k(\Gamma_1(p)) \subset M_k^1(\Gamma_1(p))$ , Lemma 3.1 implies that  $h \mid k$ . To complete the proof, we need only show that there is no  $\eta$ -quotient in  $M_2(\Gamma_1(p))$  when  $p \equiv 5 \pmod{24}$  and  $p \neq 5$ . To do this, we will use Theorem 3.5. In our current setting, the inequality (15) gives us

$$(16) \quad |r_1| + |r_p| \leq 4 \left( \frac{p+1}{p-1} \right).$$

This upper bound decreases as  $p$  increases, so the largest the bound could be in our current setting is when  $p = 29$ . Even in this worst case, the bound is less than 5 and so as the left-hand side in (16) is an integer, we have

$$(17) \quad |r_1| + |r_p| \leq 4.$$

Also, by Corollary 1.11 and the assumption that  $p \equiv 5 \pmod{24}$  we have

$$(18) \quad r_1 + 5r_p \equiv 0 \pmod{24}.$$

Lastly, recall from (10) and (11):

$$\begin{aligned} 24v_1 &= r_1 + pr_p \\ 24v_p &= pr_1 + r_p. \end{aligned}$$

As  $f(\tau) \in M_k(\Gamma_1(p))$ , neither  $v_1$  nor  $v_p$  is negative. This guarantees that  $r_1$  and  $r_p$  must both be positive. If, say,  $r_1$  were negative, then the bound in (17) and the fact that  $p$  is at least 29 would force  $v_p$  to be negative. Similarly, if  $r_p$  were negative then  $v_1$  would be as well. However, there exist no non-negative, not both zero integers  $r_1$  and  $r_p$  satisfying both (17) and (18), and so no such  $\eta$ -quotient  $f(\tau)$  can exist.  $\square$

## 4. ETA-QUOTIENTS OF SEMIPRIME LEVEL

Having found a criterion for existence of an  $\eta$ -quotient as a modular form for prime level, we wish to generalize this for composite numbers. Specifically, we shall focus on semiprimes heretofore. Throughout this section, we will let  $N = pq$ , where  $p, q$  are distinct prime integers greater than or equal to 5. Also note that these primes are interchangeable; that is, if we write  $(p, q) \equiv (1, 5) \pmod{24}$ , this case is identical to  $(p, q) \equiv (5, 1) \pmod{24}$ .

We shall first generalize the result of Lemma 3.1, but we need a preceding lemma to do so:

**Lemma 4.1.** *Let  $p, q \geq 5$  be distinct primes, and  $N = pq$ . Then,  $f(z) = \prod_{0 < \delta | N} \eta^{r_\delta}(\delta z)$  satisfies the conditions of Theorem 1.7, then all orders of vanishing are integral.*

*Proof.* Denote  $v_1, v_p, v_q, v_N$  as the orders of vanishing at each respective cusp. We first use theorem 1.8 and calculate all orders of vanishing at the cusps:

$$(19) \quad v_1 = \frac{1}{24}(Nr_1 + qr_p + pr_q + r_N)$$

$$(20) \quad v_p = \frac{1}{24}(qr_1 + Nr_p + r_q + pr_N)$$

$$(21) \quad v_q = \frac{1}{24}(pr_1 + r_p + Nr_q + qr_N)$$

$$(22) \quad v_N = \frac{1}{24}(r_1 + pr_p + qr_q + Nr_N)$$

Since  $f$  satisfies the conditions of Theorem 1.7,  $v_1$  and  $v_N$  are integral. All that remains is to show  $v_p$  and  $v_q$  are as well. We see that from equation 19 that  $v_p$  is an integer if and only if  $qr_1 + pqr_p + r_q + pr_N \equiv 0 \pmod{24}$ . Since  $p^{-1} \equiv p \pmod{24}$ , we have that

$$\begin{aligned} p(qr_1 + pqr_p + r_q + pr_N) &\equiv pqr_1 + qr_p + pr_q + r_N \pmod{24} \\ &\equiv v_1 \pmod{24} \\ &\equiv 0 \pmod{24}. \end{aligned}$$

Thus,  $v_p$  is integral. An equivalent argument can be made for  $v_q$ . □

We need Lemma 4.1 to affirm that we may use the principles of linear Diophantine equations, similar to the prime case of Lemma 3.1.

**Lemma 4.2.** *Let  $p, q \geq 5$  be distinct primes,  $N = pq$ , and  $h = \frac{1}{2} \gcd(24, p-1, q-1)$ . Then, there exists  $f(\tau) = \prod_{0 < \delta | N} \eta^{r_\delta}(\delta \tau) \in M_k^1(\Gamma_1(N))$  if and only if  $h \mid k$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $f \in M_k^1(\Gamma_0(N))$ , and hence equation 19 yields an integral value for  $v_1$ . We now multiply both sides of 19 by 24 to obtain  $24v_1 = Nr_1 + qr_p + pr_q + r_N$ . Next, since  $2k = r_1 + r_p + r_q + r_N$ , we may rewrite our equation to be

$$(23) \quad 24v_1 = 2k + (N-1)r_1 + (q-1)r_p + (p-1)r_q,$$

and then again to

$$(24) \quad 24v_1 - (N-1)r_1 - (q-1)r_p - (p-1)r_q = 2k.$$

We see that 24 is a linear Diophantine equation with coefficients 24,  $N-1$ ,  $q-1$ , and  $p-1$ . Since we already have our solution, we know that  $\gcd(24, N-1, q-1, p-1)$  divides  $2k$ , which of course means  $\frac{1}{2}\gcd(24, N-1, q-1, p-1)$  divides  $k$ . Note that since  $N-1 = p(q-1) + (p-1)$ , the  $N-1$  term is irrelevant, and so  $h = \frac{1}{2}\gcd(24, p-1, q-1)$  divides  $k$ .

( $\Leftarrow$ ) Suppose  $h \mid k$ . Note that equation 24 will have solutions if and only if  $2h \mid 2k$ , which we have already assumed. Hence, we conclude the existence of  $v_1, r_1, r_p$  and  $r_q$  which satisfy 24, and we may easily calculate the  $r_N$  necessary to satisfy the conditions of Theorem 1.7.  $\square$

Now, we define  $G_{p,q} := \gcd(p-1, q-1)$ . Observe that, in equation 24, every term is divisible by  $2h$ . Furthermore, note that, since  $2h = \gcd(24, G_{p,q})$ , we have that  $1 = \gcd(24/2h, G_{p,q}/2h)$ , meaning  $(24/2h) \in (\mathbb{Z}/(G_{p,q}/2h)\mathbb{Z})^\times$ . Thus, we can divide equation 24 by  $2h$ , mod out by  $G_{p,q}/2h$ , and multiply both sides by  $(24/2h)^{-1}$  to obtain

$$v_1 \equiv \left(\frac{2k}{2h}\right) \left(\frac{24}{2h}\right)^{-1} \left(\text{mod } \frac{G_{p,q}}{2h}\right).$$

We may solve for each order of vanishing in a similar fashion, and find that all four have the same congruence modulo  $G/2h$ . Thus, we conclude the following.

**Corollary 4.3.** *Let  $p, q \geq 5$  be distinct primes,  $N = pq$ ,  $h = \frac{1}{2}\gcd(24, p-1, q-1)$ , and  $G_{p,q} = \gcd(p-1, q-1)$ . If  $f(z) = \prod_{0 < \delta \mid N} \eta^{r_\delta}(\delta z)$  is a weakly holomorphic modular form with orders of vanishing  $v_1, v_p, v_q, v_N$ , then*

$$(25) \quad v_1 \equiv v_p \equiv v_q \equiv v_N \left(\text{mod } \frac{G_{p,q}}{2h}\right)$$

and

$$(26) \quad v_1 + v_p + v_q + v_N = \frac{k(p+1)(q+1)}{12}.$$

Equation 26 can easily be verified by adding 19, 20, 21, and 22 together and making a few substitutions..

We now will focus on what cases there can exist all non-negative orders of vanishing.

**Lemma 4.4.** *Let  $p, q \geq 5$  be prime,  $N = pq$ , and  $k$  be an even integer. If  $k(p+1)(q+1)/12$  is divisible by 4 and  $h = \frac{1}{2}\gcd(24, p-1, q-1)$  divides  $k$ , then there exists an  $\eta$ -quotient in  $M_k(\Gamma_1(N))$ .*

*Proof.* Recall equations 19, 20, 21, and 22. By our hypothesis, we may write  $k(p+1)(q+1)/12 = 4m$  for some  $m \in \mathbb{Z}$ . Then, if we let  $v_1 = v_p = v_q = v_N = m$ , we obtain  $r_1 = r_p = r_q = r_N = k/2$ . So, we obtain the  $\eta$ -quotient  $f(\tau) = \eta^{k/2}(\tau)\eta^{k/2}(p\tau)\eta^{k/2}(q\tau)\eta^{k/2}(N\tau)$ , which, since all orders of vanishing are positive, is a holomorphic modular form in  $M_k(\Gamma_1(N))$ .  $\square$

Lemma 4.4 guarantees  $\eta$ -quotients in all  $M_k(\Gamma_1(pq))$ , except when  $p+1 \equiv q+1 \equiv k \equiv 2 \pmod{4}$ , so we only need to focus on this case hereafter. We turn our attention specifically to  $k=2$  and  $M_2(\Gamma_1(pq))$ . This is because an  $\eta$ -quotient  $\eta_g \in M_2(\Gamma_1(pq))$  will satisfy  $h \mid k$  for all even  $k$  by

Lemma 4, and, since we are guaranteed an  $\eta$ -quotient  $\eta_f \in M_4(\Gamma_1(pq))$  by Lemma 4.4, the new  $\eta$ -quotient  $\eta_g \cdot \eta_f^n \in M_{2+4n}(\Gamma_1(pq))$ . As a result, there will be an  $\eta$ -quotient in  $M_k(\Gamma_1(pq))$  for all  $k \equiv 2 \pmod{4}$ , which, combined with the result of Lemma 4.4, will cover all possible values of  $k$ . In short, existence of an  $\eta$ -quotient of weight 2 will guarantee existence for every weight. In fact, if  $h \mid 2$ , an  $\eta$ -quotient of weight  $k_0 \equiv 2 \pmod{4}$  implies existence of an  $\eta$ -quotient of weight  $k > k_0$  of the same level.

With this in mind, we consider all  $(m, n) \in \mathbb{Z}/24\mathbb{Z}$  such that  $m, n \not\equiv 3 \pmod{4}$ , and where  $h \mid 2$ . This gives the set  $\{(1, 5), (5, 5), (5, 13), (5, 17)\}$ .

**Lemma 4.5.** *Let  $S = \{p : p \text{ prime}, p > 5, p \equiv 1 \text{ or } 5 \pmod{24}\}$ . If  $p, q \in S$ , then there does not exist an  $\eta$ -quotient in  $M_2(\Gamma_1(pq))$ .*

*Proof.* The smallest choices for  $p$  and  $q$  in this case is  $p = 29, q = 53$ . Even for this smallest case, Theorem 3.5 yields

$$\sum_{0 < \delta \mid N} |r_\delta| \leq 4.$$

Now, if any  $r_\delta$  were negative, since  $N > 3 \max\{p, q\}$ ,  $v_\delta$  would have to be negative (refer to equations 19, 20, 21 and 22). Hence, all exponents of our  $\eta$ -quotient must be positive. If both  $p$  and  $q$  are congruent to 1 modulo 24, then  $h = 12$ , so  $h \nmid k$ . Now, for  $(p, q) \equiv (1, 5) \pmod{24}$ , we may reduce modulo 24 equation 19:

$$24v_1 \equiv (1 \cdot 5)r_1 + 5r_p + r_q + r_N \equiv 5r_1 + 5r_p + r_q + r_N \equiv 0 \pmod{24}.$$

Since our exponents must abide by Theorem 2 of [10], the quantity  $5r_1 + 5r_p + r_q + r_N \leq 20$ , and hence cannot be congruent to zero modulo 24. As such, there are no  $\eta$ -quotients in  $M_2(\Gamma_1(pq))$ . In the case of  $(p, q) \equiv (5, 5) \pmod{24}$ , one can make a similar argument. □

**Remark 4.6.** If  $5 \mid N$ , i.e.  $p$  or  $q$  equals 5, there are automatically modular forms since there are modular forms in  $M_2(\Gamma_1(5))$  and  $M_2(\Gamma_1(5)) \subseteq M_2(\Gamma_1(5n))$  for any  $n \in \mathbb{N}$ .

In contrast, when  $(p, q) \equiv (5, 13) \pmod{24}$ , we obtain  $(5 \cdot 13)r_1 + 13r_p + 5r_q + r_N \equiv 0 \pmod{24}$ . This equation *does* have a solution, namely  $(r_1, r_p, r_q, r_N) = (0, 1, 2, 1)$ . Similarly, for  $(p, q) \equiv (5, 17) \pmod{24}$ , we have a valid solution of  $(r_1, r_p, r_q, r_N) = (1, 0, 2, 1)$ . We have now covered all possible cases and can now fully characterize the semiprimes that yield  $\eta$ -quotients of weight 2:

**Lemma 4.7.** *Let  $p, q \geq 5$  and  $N = pq$ . Then, there exists an  $\eta$ -quotient in  $M_2(\Gamma_1(N))$  if and only if  $h = \frac{1}{2} \gcd(24, p-1, q-1)$  divides  $k = 2$ , and one of the following holds:*

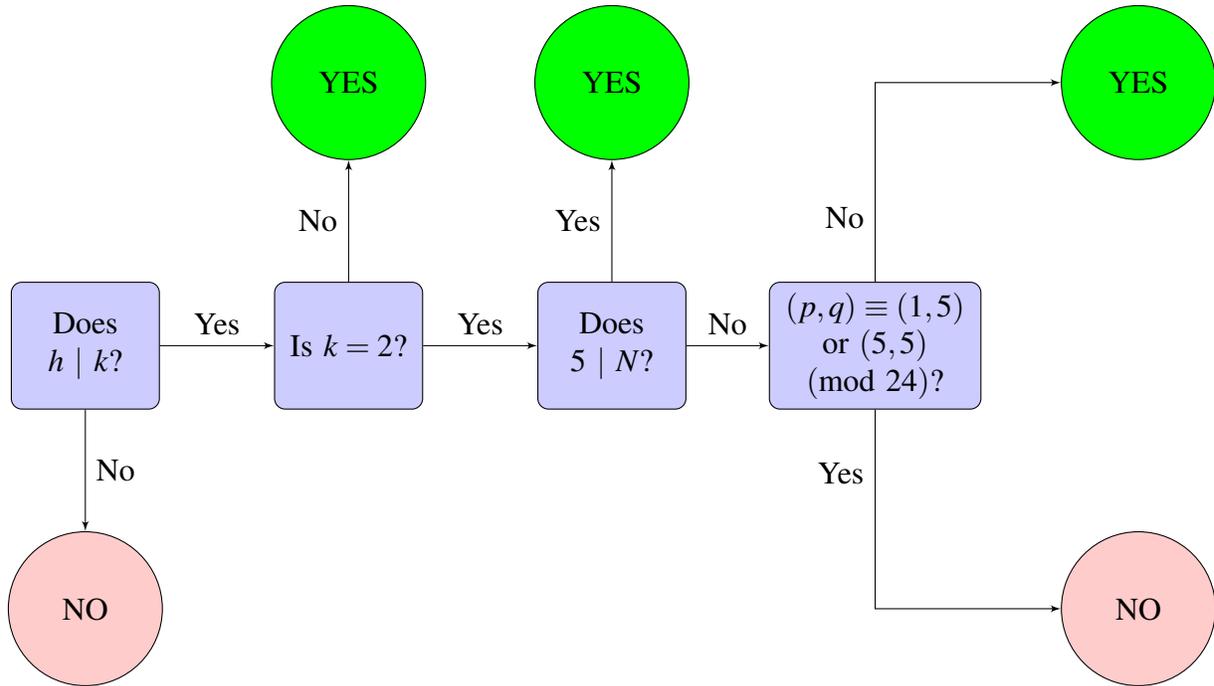
- (1) *At least one of  $p, q$  is congruent to 3 modulo 4*
- (2) *Either  $p = 5$  or  $q = 5$ .*
- (3)  *$(p, q)$  is equivalent to  $(5, 13)$  or  $(5, 17) \pmod{24}$*

We are now ready to prove Theorem 1.14:

*Proof.* For the cases of  $(p, q) \pmod{24} \in \{(1, 5), (5, 5)\}$ , we note that  $M_6(\Gamma_1(pq))$  contains  $\eta$ -quotients, namely  $\eta^9(\tau)\eta^3(N\tau)$  and  $\eta^9(\tau)\eta^3(p\tau)$ , respectively. Thus, we have that for  $(p, q) \pmod{24} \in \{(1, 5), (5, 5)\}$ , there are  $\eta$ -quotients of weight 6 or more.

Finally, we must consider the cases where  $h \nmid 2$ , those being  $(p, q) \pmod{24} \in \{(1, 1), (1, 13), (1, 17), (13, 13), (13, 17), (17, 17)\}$ . Note that if  $\eta_g \in M_k(\Gamma_1(N))$ , then  $\eta_g^n \in M_{nk}(\Gamma_1(N))$ . So, to show existence in for all weights  $k$  such that  $h \mid k$ , it suffices to show existence in  $M_h(\Gamma_1(N))$ . This also means that for  $h = 4$ , we know there will be  $\eta$ -quotients by 4.4, reducing our cases to  $(p, q) \pmod{24} \in \{(1, 13), (13, 13)\}$ , both of which have  $h = 6$ . For  $(p, q) \pmod{24} \equiv (1, 13)$ , one can verify that  $\eta^9(\tau)\eta^3(N\tau) \in M_6(\Gamma_1(N))$ , and for  $(13, 13)$ ,  $\eta^9(\tau)\eta^3(p\tau) \in M_6(\Gamma_1(N))$ . Therefore, in addition with Lemma 4.4 and 4.7, we obtain our desired result. □

We may visualize Theorem 1.14 with the following flowchart:



### 5. ELLIPTIC CURVES AND ETA-QUOTIENTS

In this section, we describe the process of creating a nice representation of the modular form associated to a specific elliptic curve. In particular, we demonstrate a method to represent a modular form attached to an elliptic curve as a linear combination of  $\eta$ -quotients.

Let  $E$  be an elliptic curve over  $\mathbb{Q}$  with conductor  $N$ . Specifically, here we consider values where  $N$  is a semiprime coprime to 6. Given the modularity theorem, Theorem 1.10, we know that the  $L$ -function of an elliptic curve at conductor  $N$  corresponds to a cusp form of weight 2 and level  $N$ . One way to prove that a representation of  $f(\tau)$  in terms of a linear combination of  $\eta$ -quotients exists, is to construct a basis of  $S_2(\Gamma_0(N))$  consisting of  $\eta$ -quotients. Using a similar method to that used by Pathakjee, RosnBrick, and Yoong [9], we calculate the dimension of the space of cusp forms of weight 2 and level  $N$ ,  $S_2(\Gamma_0(N))$ . First we find a basis of  $S_2(\Gamma_0(N))$  consisting of  $\eta$ -quotients, if possible. We then use linear algebra to write the Fourier coefficients of the modular

form attached to an elliptic curve of conductor  $N$  in terms of this basis, utilizing the Sturm bound in Theorem 1.6 for modular forms.

The linear combinations of eta-quotients in Theorem 1.15 were generated by the following algorithm:

- (1) Choose a level  $N$  such that  $N$  is a semiprime, such that  $N$  is coprime to 6.
- (2) Compute  $\dim_{\mathbb{C}} S_2(\Gamma_0(N))$ .
- (3) Generate  $\eta$ -quotients which are elements of  $S_2(\Gamma_0(N))$ .
- (4) Attempt to construct a basis for  $S_2(\Gamma_0(N))$  using these  $\eta$ -quotients.
- (5) Write the modular form attached to an elliptic curve of conductor  $N$  in terms of this basis.
- (6) Once a basis of  $\eta$ -quotients has been generated for  $S_2(\Gamma_0(N))$ , it is simple to express  $f(\tau)$  in terms of the basis.

After choosing a level  $N$ , such that  $N$  is also the conductor of an elliptic curve  $E$ , we compute  $\dim_{\mathbb{C}} S_2(\Gamma_0(N))$  using SageMath [11]. With Theorem 1.11, we are able to check the conditions in which an  $\eta$ -quotient is an element of  $S_2(\Gamma_0(N))$ . If  $f(\tau)$  is an  $\eta$ -quotient which vanishes at each cusp of  $\Gamma_0(N)$ , such that the conditions in Theorem 1.7 is satisfied, then  $f(\tau) \in S_2(\Gamma_0(N))$ . In our algorithm, we programmed a function "vanish", Appendix 2, to calculate the order of vanishing using Theorem 1.8. If this calculation returns a positive order of vanishing for an  $\eta$ -quotient at a cusp, then that  $\eta$ -quotient satisfies Theorem 1.7, and that the  $r_{\delta}$  of the  $\eta$ -quotient sums to 4. We also wrote a function "check", Appendix 3, to check that the  $\eta$ -quotient satisfies Theorem 1.7, and that the  $r_{\delta}$  of the  $\eta$ -quotient sums to 4.

In order to calculate the possible  $r_{\delta}$ 's, for an  $\eta$ -quotient  $\eta(\tau)^{r_1} \eta(p\tau)^{r_p} \eta(q\tau)^{r_q} \eta(N\tau)^{r_N}$ , we need to know how many ways there are to partition

$$(27) \quad v_1 + v_p + v_q + v_N = \frac{k(p+1)(q+1)}{12},$$

using Theorem 4.3. We then use the possible order of vanishings to find possible  $r_{\delta}$ 's, and then programmed another function "compute", Appendix 4, to set these coefficients equal to  $d$  (the integers of a partition), and then solve the linear system of four equations with four unknowns (the four unknowns being the  $r_{\delta}$ 's). Once the linear system of equations has been solved, the program returns the possible integral values of the  $r_{\delta}$ 's.

Once we are armed with the  $r_{\delta}$ 's, we may begin checking if our  $\eta$ -quotients satisfy Theorem 1.7, and are linearly independent. If after this, we are left with a number of  $\eta$ -quotients that is greater than or equal to the dimension of the space  $S_2(\Gamma_0(N))$ , then we have a basis for  $S_2(\Gamma_0(N))$ , and can begin trying to write these  $\eta$ -quotients as a linear combination of the modular form associated with the elliptic curve  $E$  of conductor  $N$ . We do this by looking at the  $q$ -expansion of the modular form linked to  $E$  and comparing it to the  $q$ -expansion's of the  $\eta$ -quotients that form a basis for the space  $S_2(\Gamma_0(N))$ .

We utilized this method to prove Theorem 1.15 for  $N = 35$ .

**Theorem.** The following linear combination of  $\eta$ -quotients is equal to the modular form corresponding to the elliptic curves with conductor  $N = 35$ :

Conductor $N$	$\eta$ - quotient $f(z)$
35	$\eta(\tau)^2 \eta(35\tau)^2 + \eta(5\tau)^2 \eta(7\tau)^2$

*Proof.* Fix  $k = 2$ , and chose  $N = 35$ , for which  $p = 5$  and  $q = 7$ . Then, using SageMath [11] and checking LMFDB [5], we compute that  $\dim_{\mathbb{C}}S_2(\Gamma_0(35)) = 3$ . Searching elliptic curves over  $\mathbb{Q}$  with conductor  $N = 35$ , we find three curves; however, the three curves are in the same isogeny class (with LMFDB label 35.a), and thus correspond to the same modular form. The  $q$ -expansion of  $f(\tau)$  begins with:

$$f(\tau) := q + q^3 - 2q^4 - q^5 + q^7 - 2q^9 - 3q^{11} - 2q^{12} + 5q^{13} - q^{15} + 4q^{16} + 3q^{17} + 2q^{19} + O(q^{20}).$$

Using (27), we calculate that

$$\frac{2(5+1)(7+1)}{12} = \frac{2(6)(8)}{12} = 8.$$

Using the function "findNumParts" in our algorithm, Appendix 6, we computed all of the possible ways to partition 8 into 4 integers, and all of the rearrangements of these partitions. Running these partitions through another function called "listMatrices", Appendix 7, we were able to find three  $\eta$ -quotients that span  $\dim_{\mathbb{C}}S_2(\Gamma_0(35))$ , and their corresponding  $q$ -expansions:

$\eta$ - quotient	$q$ - expansion
$\eta(\tau)\eta(5\tau)\eta(7\tau)\eta(35\tau)$	$q^2 - q^3 - q^4 + q^8 + q^9 + q^{10} + q^{11} - 2q^{12} + q^{13} - q^{14} - q^{15} - 3q^{16} - q^{17} + q^{18} + 2q^{19} + O(q^{20})$
$\eta(\tau)^2\eta(35\tau)^2$	$q^3 - q^4 - q^5 + 2q^6 + q^7 + 2q^8 - 2q^9 - 2q^{11} - 2q^{12} + q^{13} + 2q^{16} + 3q^{17} - 2q^{18} + 2q^{19} + O(q^{20})$
$\eta(5\tau)^2\eta(7\tau)^2$	$q - 2q^6 - 2q^8 - q^{11} + 4q^{13} - q^{15} + 2q^{16} + 2q^{18} + O(q^{20})$

Given these results, we were able to write the  $q$ -expansion of the modular form linked to  $N = 35$  as a linear combination of the  $\eta$ -quotients. We produced:  $\eta(\tau)^2\eta(35\tau)^2 + \eta(5\tau)^2\eta(7\tau)^2$ , which can easily be shown to match  $f(\tau)$  up to the Sturm bound, Theorem 1.6. □

For  $N = 35$ , we were lucky because we had enough linearly independent  $\eta$ -quotients to span  $\dim_{\mathbb{C}}S_2(\Gamma_0(35))$ . If this were not the case, then we would have to take an approach similar to an example of Rouse and Webb described in [10].

**Example 5.1.**  $M_2(\Gamma_0(22))$  has dimension 5, but contains only four linearly independent  $\eta$ -quotients:

$$\frac{\eta(2\tau)^4\eta(22\tau)^4}{\eta(\tau)^2\eta(11\tau)^2}, \eta(\tau)^2\eta(11\tau)^2, \eta(2\tau)^2\eta(22\tau)^2, \text{ and } \frac{\eta(\tau)^4\eta(11\tau)^4}{\eta(2\tau)^2\eta(22\tau)^2}.$$

A fifth basis element is  $f(\tau) = q^4 + q^6 + q^8 + \dots$  cannot be expressed as a linear combination of the  $\eta$ -quotients from above.

However, if  $g(\tau) = \frac{\eta(22\tau)^{22}\eta(\tau)}{\eta(2\tau)^{22}\eta(11\tau)^{11}}$ , then  $f(\tau)g(\tau) \in M_{12}(\Gamma_0(22))$  and every holomorphic modular form in  $M_{12}(\Gamma_0(22))$  is a linear combination of (holomorphic)  $\eta$ -quotients.

Then dividing the linear combination of  $\eta$ -quotients equivalent to  $f(\tau)g(\tau)$  by  $g(\tau)$  gives a way to write  $f(\tau)$  as a linear combination of  $\eta$ -quotients.

**Remark 5.2.** The method in Example 5.1 generates poles, but only at cusps. Thus, the  $\eta$ -quotients produced are weakly holomorphic, rather than holomorphic.

For any  $N$ , if there are not enough  $\eta$ -quotients in  $S_2(\Gamma_0(N))$  to span  $\dim_{\mathbb{C}} S_2(\Gamma_0(N))$ , then the same process can be repeated for  $S_4(\Gamma_0(N))$ , and so on. We come across this case when we look at  $N = 55$ .

Fix  $k = 2$  and  $N = 55$ , for which  $p = 5$  and  $q = 11$ . Then, using SageMath [11] and checking LMFDB [5], we compute that  $\dim_{\mathbb{C}} S_2(\Gamma_0(55)) = 5$ . Searching elliptic curves over  $\mathbb{Q}$  with conductor  $N = 55$ , we find four curves; however, the four curves are in the same isogeny class (with LMFDB label 55.a), and thus correspond to the same modular form. The  $q$ -expansion of  $f(\tau)$  begins with:

$$f(\tau) := q + q^2 - q^4 + q^5 - 3q^8 - 3q^9 + q^{10} - q^{11} + 2q^{13} - q^{16} + 6q^{17} - 3q^{18} - 4q^{19} + O(q^{20}).$$

Using (27), we calculate that

$$\frac{2(5+1)(11+1)}{12} = \frac{2(6)(12)}{12} = 12.$$

Using the function "findNumParts" in our algorithm, Appendix 6, we computed all of the possible ways to partition 12 into 4 integers, and all of the rearrangements of these partitions. Running these partitions through another function called "listMatrices", Appendix 7, we were able to find three linearly independent  $\eta$ -quotients that span  $\dim_{\mathbb{C}} S_2(\Gamma_0(55))$ , and their corresponding  $q$ -expansions:

$\eta$ - quotient	$q$ - expansion
$\eta(\tau)\eta(5\tau)\eta(11\tau)\eta(55\tau)$	$q^3 - q^4 - q^5 + q^9 + q^{10} + q^{11} + q^{12} - 2q^{13} + q^{14} - q^{15} - q^{16} - 3q^{17} - q^{18} + q^{19} + O(q^{20})$
$\eta(\tau)^2\eta(11\tau)^2$	$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + O(q^{20})$
$\eta(5\tau)^2\eta(55\tau)^2$	$q^5 - 2q^{10} - q^{15} + O(q^{20})$

Since the number of  $\eta$ -quotients produced is less than the size of  $\dim_{\mathbb{C}} S_2(\Gamma_0(55))$ , we must reach out to a higher weight in order to produce the two  $\eta$ -quotients missing from the basis. Let  $h_1$  and  $h_2$  be the two missing  $\eta$ -quotients. We use SageMath [11] to compute the  $q$ -expansions for  $h_1$  and  $h_2$ . We write  $f(\tau)$  in terms of the  $q$ -expansions of the three  $\eta$ -quotients,  $h_1$ , and  $h_2$ .

To write  $h_1$  and  $h_2$  in terms of  $\eta$ -quotients, we must search a higher weight. We were not able to generate a basis of linearly independent  $\eta$ -quotients for  $S_4(\Gamma_0(55))$  with  $k = 4$ , because the  $\dim_{\mathbb{C}} S_4(\Gamma_0(55)) = 16$  and there are only nine linearly independent  $\eta$ -quotients generated. We were not able to generate a basis of linearly independent  $\eta$ -quotients for  $S_6(\Gamma_0(55))$  with  $k = 6$ , because the  $\dim_{\mathbb{C}} S_6(\Gamma_0(55)) = 28$  and there are only 27 linearly independent  $\eta$ -quotients generated; however, we were able to generate a basis of linearly independent  $\eta$ -quotients for  $S_8(\Gamma_0(55))$  with  $k = 8$ . The  $\dim_{\mathbb{C}} S_8(\Gamma_0(55)) = 40$  and we are able to produce the following 40 linearly independent  $\eta$ -quotients:

Basis of $S_8(\Gamma_0(55))$	
$\eta$ - quotient	$q$ - expansion
$\eta(\tau)^4\eta(5\tau)^4\eta(11\tau)^4\eta(55\tau)^4$	$q^{12} - 4q^{13} + 2q^{14} + 8q^{15} - 5q^{16} - 8q^{17} + 6q^{18} + \dots$
$\eta(\tau)^5\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^3$	$q^{10} - 5q^{11} + 5q^{12} + 10q^{13} - 15q^{14} - 9q^{15} + 10q^{16} + \dots$
$\eta(\tau)^3\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)^5$	$q^{14} - 3q^{15} + 5q^{17} - 5q^{19} + 8q^{20} - 25q^{21} + 14q^{23} + \dots$
$\eta(\tau)^2\eta(5\tau)^6\eta(11\tau)^2\eta(55\tau)^6$	$q^{16} - 2q^{17} - q^{18} + 2q^{19} + q^{20} - 4q^{21} + 10q^{22} + \dots$

$\eta(\tau)^6\eta(5\tau)^2\eta(11\tau)^6\eta(55\tau)^2$	$q^8 - 6q^9 + 9q^{10} + 10q^{11} - 30q^{12} - 2q^{13} + 23q^{14} + \dots$
$\eta(\tau)^7\eta(5\tau)\eta(11\tau)\eta(55\tau)^7$	$q^{17} - 7q^{18} + 14q^{19} + 7q^{20} - 49q^{21} + 20q^{22} + 42q^{23} + \dots$
$\eta(\tau)^6\eta(5\tau)^6\eta(11\tau)^2\eta(55\tau)^2$	$q^7 - 6q^8 + 9q^9 + 10q^{10} - 30q^{11} - 6q^{12} + 47q^{13} + \dots$
$\eta(\tau)^7\eta(5\tau)\eta(11\tau)^7\eta(55\tau)$	$q^6 - 7q^7 + 14q^8 + 7q^9 - 49q^{10} + 20q^{11} + 42q^{12} + \dots$
$\eta(\tau)\eta(5\tau)^7\eta(11\tau)\eta(55\tau)^7$	$q^{18} - q^{19} - q^{20} - 6q^{23} + 7q^{24} + 8q^{25} + 7q^{28} + \dots$
$\eta(\tau)^8\eta(11\tau)^2\eta(55\tau)^6$	$q^{15} - 8q^{16} + 20q^{17} - 70q^{19} + 64q^{20} + 56q^{21} - 125q^{23} + \dots$
$\eta(\tau)^7\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)$	$q^5 - 7q^6 + 14q^7 + 7q^8 - 49q^9 + 16q^{10} + 70q^{11} + \dots$
$\eta(\tau)\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^7$	$q^{19} - q^{20} - q^{21} - 2q^{24} + 3q^{25} + 4q^{26} - 3q^{29} + \dots$
$\eta(\tau)^5\eta(5\tau)^7\eta(11\tau)\eta(55\tau)^3$	$q^9 - 5q^{10} + 5q^{11} + 10q^{12} - 15q^{13} - 13q^{14} + 30q^{15} + \dots$
$\eta(5\tau)^8\eta(55\tau)^8$	$q^{20} - 5q^{25} + 5q^{30} + 10q^{35} - 15q^{40} - 6q^{45} + \dots$
$\eta(\tau)^8\eta(11\tau)^8$	$q^4 - 8q^5 + 20q^6 - 70q^8 + 64q^9 + 56q^{10} - 125q^{12} + \dots$
$\eta(\tau)^9\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^5$	$q^{13} - 9q^{14} + 27q^{15} - 12q^{16} - 90q^{17} + 136q^{18} + 45q^{19} + \dots$
$\eta(\tau)^5\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^9$	$q^{21} - 5q^{22} + 5q^{23} + 10q^{24} - 15q^{25} - 9q^{26} + 10q^{27} + \dots$
$\eta(\tau)^{-1}\eta(5\tau)^9\eta(11\tau)^5\eta(55\tau)^3$	$q^{11} + q^{12} + 2q^{13} + 3q^{14} + 5q^{15} - 2q^{16} + 2q^{17} + \dots$
$\eta(\tau)^3\eta(5\tau)^5\eta(11\tau)^9\eta(55\tau)^{-1}$	$q^3 - 3q^4 + 5q^6 - 5q^8 + 8q^9 - 25q^{11} + 14q^{13} + \dots$
$\eta(\tau)^3\eta(5\tau)^9\eta(11\tau)^5\eta(55\tau)^{-1}$	$q^2 - 3q^3 + 5q^5 - 9q^7 + 20q^8 - 45q^{10} + 36q^{12} + \dots$
$\eta(\tau)^5\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^9$	$q^{22} - 5q^{23} + 5q^{24} + 10q^{25} - 15q^{26} - 5q^{27} - 10q^{28} + \dots$
$\eta(\tau)^9\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^{-1}$	$q - 9q^2 + 27q^3 - 12q^4 - 90q^5 + 132q^6 + 81q^7 + \dots$
$\eta(\tau)^{-1}\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)^9$	$q^{23} + q^{24} + 2q^{25} + 3q^{26} + 5q^{27} + 2q^{28} + 6q^{29} + \dots$
$\eta(\tau)^4\eta(11\tau)^2\eta(55\tau)^{10}$	$q^{24} - 4q^{25} + 2q^{26} + 8q^{27} - 5q^{28} - 4q^{29} - 10q^{30} + \dots$
$\eta(\tau)^3\eta(5\tau)\eta(11\tau)\eta(55\tau)^{11}$	$q^{26} - 3q^{27} + q^{29} - q^{31} - 4q^{32} - 5q^{34} + 8q^{36} + \dots$
$\eta(\tau)^2\eta(5\tau)^2\eta(55\tau)^{12}$	$q^{28} - 2q^{29} - q^{30} + 2q^{31} + q^{32} + 2q^{34} + 2q^{35} + \dots$
$\eta(\tau)\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^{13}$	$q^{30} - q^{31} - q^{32} - 2q^{35} + 3q^{36} + 4q^{37} - 3q^{40} + \dots$
$\eta(\tau)^6\eta(5\tau)^{-2}\eta(11\tau)^{-2}\eta(55\tau)^{14}$	$q^{31} - 6q^{32} + 9q^{33} + 10q^{34} - 30q^{35} + 2q^{36} - q^{37} + \dots$
$\eta(5\tau)^4\eta(11\tau)^{-2}\eta(55\tau)^{14}$	$q^{32} - 4q^{37} + 2q^{42} + 2q^{43} + 8q^{47} - 8q^{48} + \dots$
$\eta(11\tau)^2\eta(55\tau)^{14}$	$q^{33} - 2q^{44} + \dots$
$\eta(\tau)\eta(5\tau)^{-1}\eta(11\tau)^{-3}\eta(55\tau)^{19}$	$q^{42} - q^{43} - q^{44} + 2q^{47} - q^{48} + \dots$
$\eta(\tau)\eta(5\tau)^7\eta(11\tau)^7\eta(55\tau)$	$q^7 - q^8 - q^9 - 6q^{12} + 7q^{13} + 8q^{14} + 7q^{17} + \dots$
$\eta(\tau)^2\eta(5\tau)^2\eta(11\tau)^6\eta(55\tau)^6$	$q^{17} - 2q^{18} - q^{19} + 2q^{20} + q^{21} + 2q^{23} + 2q^{24} + \dots$
$\eta(\tau)^3\eta(5\tau)\eta(11\tau)^7\eta(55\tau)^5$	$q^{15} - 3q^{16} + 5q^{18} - q^{20} - 4q^{21} - 5q^{23} + 8q^{24} + \dots$
$\eta(\tau)^2\eta(5\tau)^6\eta(11\tau)^8$	$q^5 - 2q^6 - q^7 + 2q^8 + q^9 - 4q^{10} + 10q^{11} + \dots$
$\eta(\tau)^6\eta(5\tau)^2\eta(55\tau)^8$	$q^{19} - 6q^{20} + 9q^{21} + 10q^{22} - 30q^{23} - 2q^{24} + 23q^{25} + \dots$
$\eta(5\tau)^8\eta(11\tau)^6\eta(55\tau)^2$	$q^9 - 8q^{14} + 20q^{19} - 6q^{20} + 48q^{25} - 70q^{29} - 120q^{30} + \dots$
$\eta(\tau)^4\eta(11\tau)^8\eta(55\tau)^4$	$q^{13} - 4q^{14} + 2q^{15} + 8q^{16} - 5q^{17} - 4q^{18} - 10q^{19} + \dots$
$\eta(\tau)^8\eta(5\tau)^4\eta(11\tau)^4$	$q^3 - 8q^4 + 20q^5 - 70q^7 + 60q^8 + 88q^9 - 80q^{10} + \dots$
$\eta(\tau)^3\eta(5\tau)^9\eta(11\tau)^{-1}\eta(55\tau)^5$	$q^{21} - 4q^{26} + 2q^{31} - 4q^{32} + 8q^{36} + 16q^{37} - 5q^{41} + \dots$

Since we have computed the  $\eta$ -quotients that span  $S_6(\Gamma_0(55))$ , we let  $a \in S_6(\Gamma_0(55))$ , such that  $a$  is an arbitrary  $\eta$ -quotient that was generated. We then compute  $ah_1$  and  $ah_2$ , and know that

$ah_1, ah_2 \in S_8(\Gamma_0(55))$ . We use the basis of 40  $\eta$ -quotients stated in the table above to write  $ah_1$  and  $ah_2$ , such that  $ah_1$  and  $ah_2$  is written as a linear combination of the  $\eta$ -quotients from the basis of  $S_8(\Gamma_0(55))$ .

Once this is done, we divide the linear combination of  $\eta$ -quotients that equal  $ah_1$  and  $ah_2$  by  $a$ , and are left with a linear combination of  $\eta$ -quotients that equal  $h_1$  and  $h_2$ . Since  $h_1$  and  $h_2$  are in terms of  $\eta$ -quotients, we add them to the other three  $\eta$ -quotients to generate a complete basis for  $S_2(\Gamma_0(55))$ . Now that we have a complete basis for  $S_2(\Gamma_0(55))$ , it is simple to express  $f(\tau)$  in terms of the basis.

## 6. L-FUNCTION'S AND HIGHER WEIGHT

Recent work of Farmer, Koutsoliotas, and Lemurell [2] demonstrates how it is possible to find the Dirichlet coefficients and functional equation parameters of the L-function of an object without having access to the object it is associated with. Through a series of computer calculations, Farmer, Koutsoliotas, and Lemurell [2] are able to find the Dirichlet coefficients and functional equation of the L-function. Their procedure assumes that the Hasse-Weil L-function of the variety satisfies its conjectured functional equation, with no assumption of an associated automorphic object or Galois representation. In this Section, we will explain the method used in [2] and try to apply it to a weight 4 cusp form.

### 6.1. Background Definitions.

**Definition 6.1.** Dirichlet Series:

$$L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s}, \quad A_n \in \mathbb{Z}$$

with  $A_n \ll n^{\frac{1}{2}+\varepsilon}$ .

**Definition 6.2.** Functional Equation:

$$\Lambda(s) := N^{s/2} \Gamma_{\mathbb{C}} \left( s + \frac{1}{2} \right)^2 L(s),$$

originally defined by the Dirichlet series for  $s = \sigma + it$  with  $\sigma > 1$ , continues to an entire function which satisfies the functional equation

$$\Lambda(s) = \varepsilon \Lambda(1-s),$$

where  $\varepsilon = \pm 1$ . We call the positive integer  $N$  the *conductor* of the L-function, and  $\varepsilon$  the *sign*.

**Definition 6.3.** Euler Product:

$$L(s) = \prod_p F_p(p^{-s})^{-1},$$

with  $F_p(z) = G_p(z/\sqrt{p})$  where  $G_p(z) \in \mathbb{Z}[z]$ . Furthermore, if  $p \nmid N$  then

$$G_p(z) = 1 - A_p z + (A_p^2 - A_{p^2}) z^2 - A_p z^3 + z^4$$

and the roots of  $F_p(z)$  lie on  $|z| = 1$ , and if  $p|n$  then  $F_p$  has degree  $< 4$  and each root of  $F_p(z)$  lies on  $|z| = p^{m/2}$  for some  $m \in \{0, 1, 2, 3\}$ .

The restrictions on  $F_p$  imply that the Dirichlet coefficients satisfy the bound  $|A_n|/\sqrt{n} \leq d_4(n)$ , where  $d_4(n) = \sum_{abcd=1} 1$  is the 4-fold divisor function; in particular,  $d_4(n) = 4$  if  $p$  is prime.

These definitions are from [2].

**6.2. Weight 4 Cusp Forms.** The five weight 4 cusp forms that we have chosen to look at are listed below.

Weight 4 Cusp Forms	
Level	$\eta$ - product
5	$\eta(z)^4\eta(5z)^4$
6	$\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2$
8	$\eta(2z)^4\eta(4z)^4$
9	$\eta(3z)^8$
16	$\frac{\eta(4z)^{16}}{\eta(2z)^4\eta(8z)^4}$ ( $\eta$ - quotient)

**6.3. Method.** For each  $\eta$ -products (and quotients) associated L-series, we have a conductor  $N$ , and a sign of the functional equation  $\epsilon$ , and we have to determine if there is an L-function that satisfies the functional equation and the properties described in Section 6.1. Since we don't know the Dirichlet coefficients of the L-function, we treat them as unknowns and form a system of equations by evaluating the L-function through using the functional equation.

The Dirichlet coefficients are multiplicative and satisfy a recursion for the prime powers that arise from the shape of the local factors in the Euler product Definition 6.3. Thus, it is possible to construct a nonlinear system of equations and solve directly for the coefficients. Since the coefficients are arithmetically normalized, we know that they are rational integers. Since the coefficients are rational integers, we know that there are finitely many choices for the local factor at each prime  $p$ .

The weight function  $g(s) = e^{ibs+cs^2}$ , is used with  $b$  chosen to balance the size of the terms in the functional equation to minimize the loss in precision of the calculation. Farmer, Koutsoliotas, and Lemurell [2] state that they find it more convenient to use the Hardy Z-function instead of using the L-function itself. This is not a problem because the Z-function associated to an L-function is defined by the properties:  $Z(\frac{1}{2} + it)$  is a smooth function which is real if  $t$  is real, and  $|Z(\frac{1}{2} + it)| = |L(\frac{1}{2} + it)|$ .

**Example 6.4.** Take the first  $\eta$  - product in the table from section 3.

We have  $\eta(z)^4\eta(5z)^4$  has level  $(N) = 5$ , sign  $(\epsilon) = +1$ , weight  $(k) = 4$ , and degree  $(d) = 2$ . We would begin by choosing  $s = \frac{1}{2} + 2i$  and  $g(z) = 1$  to obtain an equation for  $Z(\frac{1}{2} + 2i)$ . Then we would choose  $s = \frac{1}{2} + 2i$  and  $g(z) = e^z$  to obtain another equation for  $Z(\frac{1}{2} + 2i)$ . Since both equations yield  $Z(\frac{1}{2} + 2i)$ , we can subtract one from the other to get another equation that equals 0. Using this method of choosing different pairs of test functions or evaluating  $Z(s)$  at different points, we can create more equations, all that have conductor,  $N$ , and sign,  $\epsilon$ , in the functional equation. We can then solve this system subject to the constraints imposed by the Euler product.

Farmer, Koutsoliotas, and Lemurell [2] constructed a list of possible local factors for each small prime  $p$ .

Number of possible local factors $F_p$		
prime, $p$	good	bad
2	35	26
3	63	32
5	129	38
7	207	44

If  $p \nmid N$ , then there are good possible local factors at  $p$  in the Euler product. Likewise, if  $p|N$ , then there are bad possible local factors at  $p$  in the Euler product.

Since the nonlinear system of equations has infinitely many unknowns, Farmer, Koutsoliotas, and Lemurell [2] truncate it and track the error due to the omitted terms and solve it by testing all of the possible values of the remaining coefficients. This data is organized as a breadth-first search of a tree. In order to truncate the system of equation, we must bound the contribution of the entries which have been eliminated. The bound consists of estimating the size of the Dirichlet coefficients and estimating the size of the terms appearing in the equations.

The method used in [2] has been described, except for the "searching the tree." This part seems a bit out of reach in terms of what may be plausible for us during a short summer program. Thus, we decided not to pursue it.

## 7. APPENDIX

This section will outline the methods that are a part of the algorithm used in Section 6.

- (1) The method "E" was used to calculate the  $q$ -expansions for the eta-quotients we found.
- (2) The method "vanish" returns the coefficients of each  $r_1, r_p, r_q, \text{ and } r_N$  for a certain  $v_d$ , level  $N$ , and sets it equal to  $n$  (which is essentially  $d$ ). Basically computes the right hand side of the vanishing cusps equation, and puts it into a matrix with  $d$  on the right (which is the left hand side of the vanishing cusps equation).
- (3) The method "check" checks the conditions from Theorem 1.7. It takes in a list of the exponents ( $r_\delta$ 's) for each eta-quotient, and checks to see if the following conditions are all satisfied. If they are not satisfied, then the program returns False.
- (4) The method "compute" sets up our system of equations by using the "vanish" method to set the coefficients = to each  $d$ , and put them into a matrix. The method then puts the matrix in reduce row echelon form, and returns the reduced matrix.
- (5) The method "fourParts" returns how many distinct ways  $n$  can be partitioned into 4 parts.
- (6) The method "findNumParts" counts the number of distinct permutations of each partition of  $n$  (from the "fourParts" method), and stores each distinct permutation in a set, and then converts the set to a list of lists (which consists of each distinct partition).
- (7) The method "listMatrices" takes in a list of all distinct partitions generated by the "find-NumParts" method, using the "fourParts" method, and creates matrices for each one using the "compute" method, and checks the last column entry of each matrix to see if it is integral. If every entry in the last column of the matrix is integral, the matrix is then added to a list of new matrices (this list consists of the matrices that create the exponents for the eta-quotients). The method takes the last column of each of the matrices (this is the column that has our exponent solutions to  $r_1, r_p, r_q, \text{ and } r_N$ ), and stores them in a list, and returns a list of lists of exponents values.

- (8) The method “etasQ” creates a list of all the  $q$  expansions of the eta-quotients we find. This would then store the coefficients of each  $q$ -expansion in a matrix and see if there is a linear combination of them that equals the  $q$ -expansion from the modular form that is attached to our  $N$ .
- (9) The method “qFinder” calculates the  $q$  term that needs to be in front of each eta-quotient.
- (10) The method “linInd” calculates the minimum number of eta-quotients that much be linearly independent using the “qFinder” method and counting the distinct number of exponents.

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