PREPERIODIC POINTS AND LINEARIZATION IN p-ADIC DYNAMICS

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Abstract. In this project, we study properties of maps in one and multiple variables over $\mathbb{Q}_p$. First, we study the roots of unity in $\mathbb{Q}_p$ for prime $p \geq 2$. Specifically, we prove the existence of $(p-1)$-st roots of unity for odd primes p and that there are no other roots of unity in $\mathbb{Q}_p$. We prove similar results for $p = 2$. We then study linearization and isolated periodic points of polynomial maps. Finally, we study multivariable maps and, more specifically, Hénon maps.

1. Introduction

In this section, we introduce relevant terms and notations that will be used throughout this document. We will study dynamical systems $T : X \to X$ over a space X and make frequent use of the notation

$$T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ iterations}}.$$

Definition 1.1. Let S be a set, $T : S \to S$ be any mapping.

(1) A point $a \in S$ is called a *fixed point* of T is $T(a) = a$. The set of all fixed points of T is denoted $\mathrm{Fix}(T)$
(2) Let $m \geq 1$. A point a is called m-periodic if $T^m(a) = a$, and is called *periodic* if it is m-periodic for some $m \geq 1$. The set of m-periodic points is denoted $\mathrm{Per}_m(T)$, and the set of periodic points is denoted $\mathrm{Per}(T)$.
(3) A point $a \in S$ is said to be *preperiodic* if $T^m(a) = T^{m+k}(a)$ for some $m \geq 0$ and $k \geq 1$. Equivalently, a is preperiodic if $T^k(a)$ is periodic for some $m \geq 0$. Clearly periodic points are preperiodic (set $m = 0$). A preperiodic point $a \in S$ is said to be *strictly preperiodic* if $m \geq 1$, that is a is preperioidic but not periodic. The set of preperiodic points is denoted $\mathrm{PrePer}(T)$.

Definition 1.2. A dynamical system $T : X \to X$ is a conjugate of dynamical system $S : Y \to Y$ if there exists a bijective function $f : X \to Y$ such that $f \circ T = S \circ f$, that is f makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

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An advantage of conjugacy is that it preserves preperiodic points.

Theorem 1.3. *Let $T : X \to X$ be conjugate to $S : Y \to Y$ via $f : X \to Y$. A point $\alpha \in X$ is T-preperiodic if and only if $f(\alpha) \in Y$ is S-preperiodic.*

Proof. Since T is conjugate to S, we have

$$S = f \circ T \circ f^{-1}.$$

This implies

$$S^n = f \circ T^n \circ f^{-1} \text{ and therefore } S^n \circ f = f \circ T^n. \tag{1}$$

We prove if α is a preperiodic point of T, then $f(\alpha)$ is preperiodic point of S. Assume α is T-preperiodic. So $T^n(\alpha) = T^{n+k}(\alpha)$ for some $n, k \in \mathbb{N}$. Evaluating both sides with the function f yields the following.

$$f(T^n(\alpha)) = f(T^{n+k}(\alpha))$$

By conjugacy of T and S, we see that $f(\alpha)$ is S-preperiodic.

$$S^n(f(\alpha)) = S^{n+k}(f(\alpha))$$

Similarly, it follows by conjugacy that if we start with $S^n(f(\alpha)) = S^{n+k}(f(\alpha))$, then we obtain $f(T^n(\alpha)) = f(T^{n+k}(\alpha))$. Taking the inverse of f on both sides yields the desired equation, $T^n(\alpha) = T^{n+k}(\alpha)$, i.e. α is T-preperiodic. ♡

Definition 1.4. The **p-adic absolute value** is defined on the rational numbers $\alpha \in \mathbb{Q}$ by

$$|\alpha|_p = \begin{cases} \left|p^k \frac{a}{b}\right|_p = p^{-k} & \alpha \neq 0 \\ 0 & \alpha = 0 \end{cases} \qquad \text{for } p \text{ prime and } p \nmid a, b$$

This absolute value satisfies:

- $|\alpha|_p = 0$ if and only if $\alpha = 0$.
- $|\alpha\beta|_p = |\alpha|_p|\beta|_p$.
- $|\alpha + \beta|_p \le \max\{|\alpha|_p, |\beta|_p\}$ with equality whenever $|\alpha|_p \neq |\beta|_p$.

The final property above is known as the **ultrametric inequality** or the **strong triangle inequality**.

The set of p-adic numbers $\mathbb{Q}_p$ is the completion of the set of rational numbers $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$ [3]. A p-adic number $\alpha \in \mathbb{Q}_p$ is a number described as a series of the form

$$\alpha = \sum_{n \ge n_0}^{\infty} a_n p^n \qquad n_0 \in \mathbb{Z},\ a_n \in \{0, 1, \ldots, p-1\}$$

which converges under the p-adic absolute value $|\cdot|_p$.

Definition 1.5. A fixed point α of a polynomial map $T \in \mathbb{Q}_p[x]$ is an *attracting fixed point* if $|T'(\alpha)| < 1$. The set B of points b for which $T^n(b) \to \alpha$ as $n \to \infty$ is called the *basin of attraction* of α.

2. Roots of Unity in $\mathbb{Q}_p$ via Dynamical Systems

We divide this section into two parts. The first part proves the existence of $(p-1)$-st roots of unity in $\mathbb{Q}_p$ for odd prime p. In addition, we prove that there are no other roots of unity except those $(p-1) - st$ roots of unity. In the second part, we prove that 1 and -1 are the only roots of unity in $\mathbb{Q}_2$.

2.1. **Odd Prime p.** Let p be an odd prime, $p \neq 2$.

Theorem 2.1. *For each $1 \leq k \leq p-1$ there exists $(p-1)$-st roots of unity ζ_k in the disk $D_{1/p}(k)$.*

Proof. Let $f(x) = x^{p-1} - 1$ and let $1 \leq k \leq p-1$. Since $p \nmid k$, by Fermat's Little Theorem, $k^{p-1} \equiv 1 \pmod p$. Therefore, we have

$$|\, f(k)\,|_p \leq 1/p. \tag{2}$$

In addition, $f'(k) = (p-1)k^{p-2}$. Since $p \nmid p-1$ and $p \nmid k$, we have

$$|\, f'(k)\,|_p = 1. \tag{3}$$

Since $k \in \mathbb{Z}_p$ along with (2) and (3), Hensel's Lemma [3] applies. Therefore, there exists $\zeta_k \in \mathbb{Z}_p$ such that $f(\zeta_k) = 0$, i.e. ζ_k is $(p-1)$-st roots of unity, and $|\, k - \zeta_k\,|_p \leq 1/p$. Therefore, we have $\zeta_k \in D_{1/p}(k)$. ♡

Next, we take a dynamical systems approach in proving that there are no other roots of unity outside the ones constructed in Theorem 2.1. We first establish equivalency between roots of unity in arbitrary fields and preperiodic points with respect to powering maps.

Proposition 2.2. *Let K be a field and let $m \geq 2$. An element $\alpha \in K$ is a preperiodic point with respect to $T(x) = x^m$ if and only if $\alpha = 0$ or α is a root of unity.*

Proof. Let $T^n(x)$ denote the composition of T with itself n times.

$$T^n(x) = x^{m^n} \tag{4}$$

We first prove if α is preperiodic, then it is either 0 or a root of unity. By definition, α is preperiodic if there exists $n \geq 0$ such that $T^n(\alpha)$ is periodic. Therefore, we have

$$T^n(\alpha) = T^{n+k}(\alpha). \qquad\qquad k \in \mathbb{N}$$

Substituting (4), we have

$$x^{m^n} - x^{m^{(n+k)}} = 0$$
$$x^{m^n}(1 - x^{m^k}) = 0.$$

Since K is a field, its an integral domain. Therefore, there are no zero divisors, i.e. either $x^{m^n} = 0$ or $x^{m^k} = 1$. Therefore, either $x = 0$ or x is a root of unity.

Now we prove implication in the opposite direction. Let $\alpha = 0$. It follows that $T^n(\alpha) = 0$. Therefore, it is preperiodic. Now we look at the second case where α is the n-th root of unity. By definition, $\alpha^n = 1$. Therefore, there are only finitely distinct elements of the form α^k for $k \geq 0$. In other words, the set

$$S = \{\alpha^k \in K \mid k \geq 0\}$$

is finite.

Consider the following sequence

$$\alpha, T(\alpha), \ldots, T^n(\alpha), \ldots \qquad\qquad T^i(\alpha) \in S.$$

Since every element in the sequence is an element of S, by the pigeonhole principle, we must have $T^i(\alpha) = T^j(\alpha)$ for some $i \neq j$. Therefore, α is T-preperiodic. ♡

Since a preperiodic point with respect to T is also a root of unity, we can show that there are no other roots of unity if there are no other preperiodic points. We do this by defining a conjugacy that takes a dynamical system T in the form of $T(x) = x^m$ to another dynamical system S that is linear. However, we must first prove that T is a dynamical system over the disk of radius $1/p$.

Proposition 2.3. *Let $1 \leq k \leq p-1$ and define $T(x) = x^p$. Then $T(D_{1/p}(k)) \subseteq D_{1/p}(k)$.*

Proof. Assume $\alpha \in D_{1/p}(k)$. Therefore, $\alpha \equiv k \pmod p$.

Since $T(\alpha) = \alpha^p$, we have

$$T(\alpha) \equiv k^p \pmod p.$$

Since $k \in \mathbb{Z}$, by Fermat's Little Theorem, $k^p \equiv k \pmod p$. Therefore, we obtain

$$T(\alpha) \equiv k \pmod p.$$

This means $|\, T(\alpha) - k \,|_p \leq 1/p$ and so $T(\alpha) \in D_{1/p}(k)$. ♡

Notice the p-adic absolute for any root of unity ζ must be 1. By definition, $\zeta^m = 1$ for some $m \in \mathbb{N}$. Taking the p-adic absolute value on both sides yield $|\, \zeta \,|_p = 1$. This implies $\zeta \in \mathbb{Z}_p^\times$.

With respect to our goal of determining that there are no other preperiodic points inside the disk except the ones we mentioned in 2.1, based on Theorem 1.3, we can try to obtain conjugacy.

$$\begin{tikzcd}
D_{1/p}(k) \arrow[r, "T(x) = x^p"] \arrow[d, "f"'] & D_{1/p}(k) \arrow[d, "f"] \\
D_{1/p}(0) \arrow[r, "S(x) = px"] & D_{1/p}(0)
\end{tikzcd}$$

FIGURE 1. Conjugacy between T and S for odd p.

We will show that a conjugacy between T and S in figure 1 above is possible through the bijective function $f(x) = \log_p(x/\zeta_k)$ where $\log_p : D_{1/p}(1) \to \mathbb{Q}_p$ is the p-adic logarithm [3].

Definition 2.4. The *p-adic logarithm* is defined by the power series

$$\log_p(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \qquad x \in D_{1/p}(1) = \{x \in \mathbb{Z}_p : |x-1|_p < 1\} = 1 + p\mathbb{Z}_p$$

Theorem 2.5. *If $a, b \in 1 + p\mathbb{Z}_p$, then $\log_p(ab) = \log_p(a) + \log_p(b)$*

Proof. The proof follows from Gouvea [3]. ♡

Definition 2.6. Let $D = \{x \in \mathbb{Z}_p : |x|_p < p^{-1/(p-1)}\}$. The *p-adic exponential*, denoted $\exp_p : D \to \mathbb{Q}_p$ is defined by the power series

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Theorem 2.7. *Let*

$$U = 1 + p\mathbb{Z}_p \qquad U_1 = \begin{cases} U & p \neq 2 \\ 1 + 4\mathbb{Z}_2 & p = 2 \end{cases} \qquad W = \begin{cases} p\mathbb{Z}_p & p \neq 2 \\ 4\mathbb{Z}_2 & p = 2 \end{cases}$$

be considered as an additive group. Then the p-adic logarithm defines an isomorphism of groups

$$\log_p : U_1 \to W$$

with inverse $\exp_p$.

Proof. The proof follows from Gouvea [3]. ♡

We must first ensure that when x is in the disk $D_{1/p}(k)$, then $y = x/\zeta_k$ is contained in the domain of $\log_p(y)$. This ensures the following:

- The function $f(x) = \log_p(x/\zeta_k)$ is well-defined over the domain that we want $D_{1/p}(k)$.
- Conjugacy requires f has an inverse f^{-1}. Since preperiodic point of $f(\alpha) \in S$ is a preperiodic point $\alpha \in T$, it is important that the range of f^{-1} covers the entire disk $D_{1/p}(k)$ so that we can be ensured if there exists any preperiodic point $f(\alpha)$ of S, $f^{-1}(f(\alpha)) = \alpha$ must lie within the disk $D_{1/p}(k)$.

Lemma 2.8. *The disk $D_{1/p}(k)$ is contained in the domain of $f(x) = \log_p(x/\zeta_k)$ where $0 < k < p$ and $\zeta_k \in D_{1/p}(k)$ is a root of unity.*

Proof. We want to prove the following.

$$x, \zeta_k \in D_{1/p}(k) \implies \frac{x}{\zeta_k} \in D_{1/p}(1)$$

where $0 < k < p$ and ζ_k is a root of unity.

Since x, ζ_k is in the closed disk $D_{1/p}(k)$, by definition, we have the following.

$$|\, x - \zeta_k \,|_p \;\le\; \frac{1}{p} \iff x \equiv \zeta_k \equiv k \pmod p \tag{5}$$

From (5), we have the following.

$$\left| \frac{x}{\zeta_k} - 1 \right|_p = \left| \frac{x - \zeta_k}{\zeta_k} \right|_p = |\, x - \zeta_k \,|_p \;\le\; \frac{1}{p} < 1$$

Therefore, x/ζ_k is in the domain of $\log_p(y)$. In other words, the disk of $D_{1/p}(k)$ is contained in the domain of $f(x) = \log_p(x/\zeta_k)$. ♡

Our last verification of the conjugacy between T and S in figure 1 above is that f must be bijective. A function is bijective in a domain D if and only if it has an inverse in D.

Lemma 2.9. *The function $f : D_{1/p}(k) \to D_{1/p}(0)$ where $f(x) = \log_p\left(\frac{x}{\zeta_k}\right)$ is bijective with inverse $f^{-1}(x) = \zeta_k \exp_p(x)$.*

Proof. The proof follows from Proposition 4.5.9 in Gouvea [3]. From Gouvea, $\log_p$ defines an isomorphisms of groups $D_{1/p}(1) \to D_{1/p}(0)$ with inverse $\exp_p$. By Lemma 2.8, x/ζ_k sends $D_{1/p}(k)$ to $D_{1/p}(1)$. Therefore, $f : D_{1/p}(k) \to D_{1/p}(0)$ is bijective.

Since the inverse of $\log_p()$ is $\exp_p()$, rough algebra of f^{-1} yields the following.

$$f((f^{-1}(x)) = \log_p\left(\frac{f^{-1}(x)}{\zeta_k}\right) = x \qquad \text{implies} \qquad f^{-1}(x) = \zeta_k \exp_p(x)$$

Similarly, we have the following.

$$f^{-1}(f(x)) = \zeta_k \exp_p\left(\log_p\left(\frac{x}{\zeta_k}\right)\right) = x$$

Therefore, $f^{-1}(x) = \zeta_k \exp_p(x)$. ♡

Finally, we prove that the $\log_p()$ function defines a conjugacy between T and S as follows.

Theorem 2.10. *The function $g(x) = \log_p(x/\zeta_k)$ where ζ_k is a root of unity defines a conjugacy between*

$$T : D_{1/p}(k) \to D_{1/p}(k) \text{ and } S : D_{1/p}(0) \to D_{1/p}(0)$$
$$\text{where } T(x) = x^p \text{ and } S(x) = px \text{ for } 0 < k < p$$

Proof. Given $T(x) = x^p$ defined on a disk $D_{1/p}(k)$, we have the following.

$$f(T(x)) = \log_p\left(\frac{x^p}{\zeta_k}\right)$$

From Theorem 2.1, ζ_k is a $(p-1)$-st root of unity. Therefore, $\zeta_k^{p-1} = 1$. It follows that $\zeta_k^p = \zeta_k$. Therefore, we have the following.

$$f(T(x)) = \log_p\left(\frac{x^p}{\zeta_k^p}\right) = p\,\log_p\left(\frac{x}{\zeta_k}\right) = S(f(x)) \qquad \text{where } S(x) = px \tag{6}$$

♡

Theorem 2.11. *$\mathbb{Q}_p$ contains no roots of unity except the $(p-1)$-st roots of unity for odd prime p.*

Proof. By definition, a preperiodic point α of S must satisfy $S^m(\alpha) = S^n(\alpha)$ for $m \neq n$.

$$\begin{aligned} p^m\alpha &= p^n\alpha \\ p^m(\alpha - p^{n-m}\alpha) &= 0 \qquad\qquad \text{for } n > m \end{aligned}$$

Since $p \neq 0$, it must be that α is zero. Therefore, there is only one preperiodic point, i.e. $\alpha = 0$, in some subset V of the disk $D_{1/p}(0)$. Since there is only one preperiodic point in S, by Theorem 1.3, there is only one preperiodic point in the disk $D_{1/p}(k)$. Therefore, by Proposition 2.2, there is only one $(p-1)$-st root of unity ζ_k in each of the disk $D_{1/p}(k)$ per theorem 2.1 for $0 < k < p$. ♡

2.2. **For $p = 2$.** We know that any root of unity ζ has p-adic absolute value $|\zeta|_p = 1$. For $\mathbb{Q}_2$, this implies there are no roots of unity in $D_{1/2}(0)$ since all elements in $D_{1/2}(0)$ by definition have p-adic absolute value less than $1/2$. Therefore, any roots of unity must be in the disk $D_{1/2}(1)$. We know that $D_{1/2}(1)$ consists of only disks $D_{1/4}(1)$ and $D_{1/4}(-1)$ where for every element $x \in D_{1/4}(1)$, its negative $-x$ is in $D_{1/4}(-1)$ and vice versa. In addition, we know $-\alpha$ is a root of unity if and only if α is a root of unity. Therefore, it suffices to find all roots of unity in the disc $D_{1/4}(1)$.

We can apply the same strategy of using conjugacy to determine the existence of roots of unity. We use the following conjugacy.

$$\begin{tikzcd} D_{1/4}(1) \arrow[r, "T(x) = x^2"] \arrow[d, "f"'] & D_{1/4}(1) \arrow[d, "f"] \\ D_{1/4}(0) \arrow[r, "S(x) = 2x"] & D_{1/4}(0) \end{tikzcd}$$

Figure 2. Conjugacy between T and S for $p = 2$.

Theorem 2.12. *Let $T(x) = x^2$. Then $T(D_{1/4}(1)) \subseteq D_{1/4}(1)$.*

Proof. Assume $x \in D_{1/4}(1)$. By definition, $x \equiv 1 \pmod 4$. Therefore, $x^2 \equiv 1 \pmod 4$. So $T(x) \equiv 1 \pmod 4$. This means, $T(x)$ is in the disk $D_{1/4}(1)$. ♡

Theorem 2.13. *The only roots of unity in* $\mathbb{Q}_2$ *are* 1 *and* -1.

Proof. We know $\log_2$ defines an isomorphism from $D_{1/4}(1)$ to $D_{1/4}(0)$ from [3]. Therefore, the conjugacy between T and S follows from $f(x) = \log_2(x)$ where $\log_2$ is the 2-adic logarithm.

$$f(T(x)) = \log_2(x^2) = 2\log_2(x) = S(f(x))$$

The proof for Theorem 2.11 shows that $S(x) = 2x$ has 0 as the only preperiodic point in $D_{1/4}(0)$. Therefore, by conjugacy, there is only one preperiodic point $\alpha \in D_{1/4}(1)$ where $f(\alpha) = \log_2(\alpha) = 0$. Therefore, $\alpha = 1$. By proposition 2.2, it follows that $\alpha = 1$ is a root of unity in $\mathbb{Q}_2$.

Since $\alpha = 1$ is a root of unity in $D_{1/4}(1)$, we know $-\alpha = -1$ is also a root of unity in $D_{1/4}(-1)$. In addition, since there is only one root of unity in $D_{1/4}(1)$, there is also only one root of unity in $D_{1/4}(-1)$. Therefore, there are only 1 and -1 as roots of unity in $\mathbb{Q}_2$. ♡

3. Preperiodic Points in One Dimension

In this section, we will explore linearization of maps $f : \mathbb{Q}_p \to \mathbb{Q}_p$ and what information we can infer regarding periodic points. We will make use of the theorems of Lindahl [4]. We also explore the effect of perturbation on the dynamics of polynomial maps and power series. Note that $B_r(x)$ denotes the *closed ball* of radius r surrounding x. Since we are often working with the same prime p throughout a proof, we let $|x|$ denote $|x|_p$ whenever the prime in reference is clear.

3.1. **An Introductory Example.** We begin this section with an example. Consider the function $f(x) = x^2 - 6$. Note that this map is not an automorphism of $\mathbb{Q}_p$. This map has fixed points $x = 3$ and $x = -2$. The fixed point $x = 3$ is attracting, as we can easily see by letting $y = 3 + 3^k\frac{a}{b}$ with $3 \nmid a, b$, $b \neq 0$, $k > 0$. We then have

$$\begin{aligned} f(y) &= (3 + 3^k\frac{a}{b})^2 - 6 \\ &= 3 + 2 \cdot 3^{k+1}\frac{a}{b} + 3^{2k}\frac{a^2}{b^2} \\ &= 3 + 3^{k+1}\frac{2ab + 3^{k-1}a^2}{b^2}, \end{aligned}$$

so $|f(y) - f(x)| \leq |y - x|$ for all $y \in B_{1/3}(0)$, with equality if and only if $a = 0$ or $a = -2$. Thus $x = 3$ and $x = -3$ are the only preperiodic points in this ball.

Next, we will show that the point $x = -2$ is the only remaining preperiodic point of f. First, note that for $|x| > 1$, we have $|f(x)| = |x|^2 > |x|$, so we need only look in the unit ball. As we have already considered the case $x \in B_{1/3}(0)$, we need to consider the cases $x \in B_{1/3}(1)$ and $x \in B_{1/3}(2)$. We may reduce this if we note, for $x \in B_{1/3}(2)$, we have $f(x) = (2+3^k\frac{a}{b})^2 - 6 = -2+3^k(\frac{2ab+3^ka^2}{b^2}) \in B_{1/3}(1)$. Thus we need only consider $x \in B_{1/3}(1)$ to classify all preperiodic points of f.

First, note $S(x) = f(x-2) + 2 = -4x + x^2$ is conjugate to f, so periodic points $x \in B_{1/3}(0)$ of S correspond to periodic points $y \in B_{1/3}(0)$. We may then use lemma 4.1 of Lindahl [4] (with $m = 2, k = 7, s = 0$) to write (with $\lambda_1 = f'(-2) = -4$)

$$\overline{r}(-4) = \left[3^{\frac{1}{2}}\left(\prod_{n=1}^{6}|1 - (-4)^n|\right)^{-1}\right]^{-\frac{1}{6}} = \frac{1}{3}^{\frac{3}{4}}$$

So that S is analytically conjugate to $g(x) = -4x$ in the open disk $D_{1/3^{\frac{3}{4}}}(0)$ in $\mathbb{C}_p$, or equivalently (in $\mathbb{Q}_p$) the closed ball $B_{1/3}(0)$. Thus S has no preperiodic points besides $x = 0$ in $B_{1/3}(0)$, so f has no preperiodic points besides $x = -2$ in $B_{1/3}(1)$. Thus f has no preperiodic points except for $x = 3$ and $x = -2$.

3.2. **Preperiodic Points of Perturbed Maps.** Inspired by the beginning of the above example, we begin with the following lemma:

Lemma 3.1. *Suppose $T : \mathbb{Q}_p \to \mathbb{Q}_p$ is a polynomial with coefficients in $\mathbb{Z}_p$ such that $T(\alpha) = \alpha$ and $|T'(\alpha)| < 1$ for some $\alpha \in \mathbb{Z}_p$. Then α is an attracting fixed point with basin of attraction $B_{1/p}(\alpha)$.*

Proof. First, let $T(x) = \alpha + T'(\alpha)(x-\alpha) + \sum_{n=2}^{\infty} a_n(x-\alpha)^n$ with $|a_n| \leq 1$. Then, we can write any point in $B_{1/p}(\alpha)$ in the form $\alpha + p^k c$ with $k \geq 1$ and $|c| = 1$. We begin with $T(x)$:

$$\begin{aligned} S(x) &= \alpha + \lambda(x-\alpha) + \sum_{n=2}^{\infty} a_n(x-\alpha)^n \\ S(\alpha + p^k c) &= \alpha + \lambda(p^k c) + \sum_{n=2}^{\infty} a_n(p^k c)^n \\ &= \alpha + \lambda p^k c + p^{2k} c^2 \sum_{n=2}^{\infty} a_n (p^k c)^{n-2} \end{aligned}$$

Then, continuing from above,

$$\begin{aligned} |S(\alpha + p^k c) - \alpha| &= \left| \lambda p^k c + p^{2k} c^2 \sum_{n=2}^{d} a_n (p^k c)^{n-2} \right| \\ &\leq \max \left\{ \left| \lambda p^k c \right|, \left| p^{2k} c^2 \sum_{n=2}^{d} a_n (p^k c)^{n-2} \right| \right\} \\ &\leq \max \left\{ \left| \lambda p^k \right|, \left| p^{2k} \right| \right\} \\ &\leq p^{-(k+1)}, \end{aligned}$$

which concludes our proof. ♡

Following similar logic to the example we began with, we will prove the following theorem:

Theorem 3.2. *Let $T : \mathbb{Q}_3 \to \mathbb{Q}_3$ be of the form $T(x) = x^2 + c$, where $|c| \leq \frac{1}{3}$. By Hensel's lemma, T has a unique fixed point $\alpha_0 \in B_{1/3}(0)$ and a unique fixed point in $\alpha_1 \in B_{1/3}(1)$. Let $\lambda = T'(\alpha_1)$. Then if $|1 - \lambda^2| = \frac{1}{3}$, T has exactly two preperiodic points in $\mathbb{Q}_3 - B_{1/3}(0)$ and finitely many in $B_{1/3}(0)$.*

Proof. First we see that, letting $g(x) = T(x) - x$, $g(0) = c, g'(0) = -1$ and $g(1) = c, g'(1) = 1$, so by Hensel's lemma T has unique fixed points α_1 and α_0 as described above. Next, similarly to the example above, note that $T(2 + 3k) = 4 + 12k + 9k^2 + c \in B_{1/3}(1)$ for all $k \in \mathbb{Z}_3$, so we may again consider only elements of $B_{1/3}(i)$ for $i = 0, 1$. Next, we will show the fixed point α_0 is attracting.

Let $\alpha_0 \in B_{1/3}(0)$ be a fixed point of T. Then, given $x \in B_{1/3}(0) \setminus \{\alpha_0\}$, we have $x = \alpha_0 + 3^k a$ for $|a| = 1$, $k \in \mathbb{Z}^+$, so

$$\begin{aligned}
T(x) - \alpha_0 &= (\alpha_0 + 3^k a)^2 + c - \alpha_0 \\
&= \alpha_0^2 + 2 \cdot 3^k a \alpha_0 + 3^{2k} a^2 + c - \alpha_0 \\
&= \left(\alpha_0^2 + c - \alpha_0\right) + 3^{k+1}\left(2a\frac{\alpha_0}{3} + 3^{k-1}a^2\right) \\
&= 3^{k+1}\left(2a\frac{\alpha_0}{3} + 3^{k-1}a^2\right).
\end{aligned}$$

Observing that $2a\frac{\alpha_0}{3} + 3^{k-1}a^2 \in \mathbb{Z}_3$, we have shown that α_0 is an attracting fixed point, and therefore, since points which eventually map to α_0 must be contained in $\mathbb{Z}_p$, and each point can have at most 2 elements in its preimage under a quadratic map, T has finitely many preperiodic points in $B_{1/3}(0) \setminus \{\pm\alpha_0\}$.

Now, consider the equation $T(x) - x = x^2 - x + c = 0$. This has solutions of the form $x^{\pm} = \frac{1 \pm \sqrt{1-4c}}{2}$, with x^+ and x^- corresponding to the choice of sign. By the Hensel's lemma argument above, we know x^+ and x^- are both elements of $\mathbb{Z}_3$, and furthermore we know that one is an element of $B_{1/3}(1)$, and the other of $B_{1/3}(0)$. We now split our argument into two cases:

(1) Suppose $x^- \in B_{1/3}(1)$. Then, let $\lambda = T'(x^-) = 1 - \sqrt{1-4c} \in B_{1/3}(2)$. We then see that T is topologically conjugate to the map $S(x) = T(x + x^-) - x^- = \lambda x + x^2$, which has a fixed point at $x = 0$. We will use the theorems of Lindahl [4] we used previously to determine the radius on which S is analytically conjugate to its linear part. First, we find m:

$$\begin{aligned}
|1-\lambda| &= |\sqrt{1-4c}| = 1 \\
|1-\lambda^2| &= |1 - 1 + 2\sqrt{1-4c} - 1 + 4c| \\
&\le \max\left\{|4c|, |1 - 2\sqrt{1-4c}|\right\} \le \frac{1}{3}.
\end{aligned}$$

To justify the final inequality note that $1 - 2\sqrt{1-4c} = 2\lambda - 1$ is the difference of two elements of $B_{1/3}(1)$ and is therefore an element of $B_{1/3}(0)$. Thus $m = 2$, which tells us $s = 0$ as in our example. We then let $k = 3$, so $\frac{k-1}{mp^s} = 1$ is a nonnegative integer power of 3 and thus, by lemma 4.1 of Lindahl, a lower bound for the radius of analytic conjugacy of S to its linear part can be found by

$$\begin{aligned}
\overline{r}(\lambda) &= \left[3^{\frac{1}{2}}\left(|1-\lambda||1-\lambda^2|\right)^{-1}\right]^{-\frac{1}{2}} \\
&= 3^{-\frac{1}{4}}|1-\lambda^2|^{\frac{1}{2}} \\
&\begin{cases} = \frac{1}{3}^{\frac{3}{4}} > \frac{1}{3} & |1-\lambda^2| = \frac{1}{3} \\ \le \frac{1}{3}^{\frac{5}{4}} < \frac{1}{3} & |1-\lambda^2| < \frac{1}{3} \end{cases}.
\end{aligned}$$

By a similar argument to the example, we have shown T has no preperiodic points in $B_{1/3}(1) \setminus \{\alpha_1\}$ and thus T has exactly two preperiodic points.

(2) We will use a similar argument to the above in the second case. Suppose $x^+ \in B_{1/3}(1)$. Then, let $\lambda = T'(x^+) = 1 + \sqrt{1-4c} \in B_{1/3}(2)$. We then see that T is topologically conjugate to the map $S(x) = T(x + x^+) - x^+ = \lambda x + x^2$, which has a fixed point at $x = 0$.

We will use the theorems of Lindahl [4] we used previously to determine the radius on which S is analytically conjugate to its linear part. First, we find m:

$$\begin{aligned}|1-\lambda| &= |\sqrt{1-4c}| = 1\\ |1-\lambda^2| &= |1-1-2\sqrt{1-4c}-1+4c|\\ &\le \max\left\{|4c|, |1+2\sqrt{1-4c}|\right\} \le \frac{1}{3}.\end{aligned}$$

To justify the final inequality note that $1+2\sqrt{1-4c} = 2\lambda - 1$ is the difference of two elements of $B_{1/3}(1)$. The remainder of the argument follows similarly to the above.

If we note that $T^{-1}(\alpha_1)$ has exactly one element $-\alpha_1$ in $B_{1/3}(2)$, we have now shown the theorem to be true. ♡

We continue with a generalization of the above, which does not quite cover each ball of radius $\frac{1}{p}$. We begin with a second theorem.

Theorem 3.3. *Let $T : \mathbb{Q}_p \to \mathbb{Q}_p$ for any odd prime p such that $T(x) = x^2 + c$, where $|c|_p < 1$. If $|1-\lambda^m| > p^{-m}$, where m is the smallest integer such that $|1-\lambda^m| < 1$, then the set*

$$\operatorname{PrePer}(f) \cap \left(\bigcup_{\ell\in L} B_{\frac{1}{p}}(\ell)\right)$$

is finite. In this case, L is the set

$$L = \left\{\ell \in \{1,2,\ldots,p-1\} : \ell^{(2^k)} \equiv 1 \text{ mod } p \text{ for some } k \in \mathbb{Z}\right\}.$$

Proof. From Hensel's lemma, using the same argument we used above, f has a fixed point $\alpha_0 \in B_{1/p}(0)$ and $\alpha_1 \in B_{1/p}(1)$. These take form similar to $x^{\pm}$ above. Now, note that the eigenvalue $f'(\alpha_1) = \lambda \in B_{1/p}(2)$. To use the linearization theorem of Lindahl [4], we find m to be the smallest integer such that $|1-\lambda^m| < 1$, or equivalently $2^m \equiv 1 \bmod p$. Because $\lambda \in \mathbb{Q}_p$, $|1-\lambda^m|_p < 1 \Rightarrow |1-\lambda^m|_p \le \frac{1}{p}$. Then, we know Lindahl's parameter $s = 0$. Then, let $k = m+1$, so that [4]

$$\begin{aligned}\overline{r}(\lambda) &= \left[p^{\frac{1}{p-1}}|1-\lambda^m|^{-1}\right]^{-\frac{1}{m}}\\ &= p^{-\frac{1}{m(p-1)}}|1-\lambda^m|^{\frac{1}{m}}.\end{aligned}$$

Letting $|1-\lambda^m| = p^{-q}$, we continue

$$\begin{aligned}\overline{r}(\lambda) > \frac{1}{p} &\Leftrightarrow \frac{1}{m(p-1)} + \frac{q}{m} < 1\\ &\Leftrightarrow \frac{1+q(p-1)}{m(p-1)} < 1\\ &\Leftrightarrow q < m - \frac{1}{p-1}\\ &\Leftrightarrow q < m \text{ because } q, m \in \mathbb{Z}.\end{aligned}$$

The remainder of the theorem follows similarly to above, by induction.

♡

We proceed with a general result about preperiodic points of functions $T : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $T(x) = x^k + c$ with k being a power of p and $|c|_p < 1$.

Theorem 3.4. *Let T be as above with p an odd prime. Then* $\mathrm{PrePer}(T)$ *contains exactly $p-1$ points outside of $B_{1/p}(0)$, all of which are fixed points. It contains finitely many preperiodic points inside $B_{1/p}(0)$.*

Before we prove the above theorem, we will prove the following lemma:

Lemma 3.5. *Let T be described as above. Then* $\mathrm{Per}(T)$ *contains exactly p points, all of which are fixed points.*

Proof. Similarly to above, define $g_n(x) = T^n(x) - x$. Then, because $c \equiv 0 \bmod p$, we have $g_n(x) \equiv x^{k^n} - x \equiv 0 \bmod p$ for all $n \in \mathbb{Z}^+$, $x \in \{0, 1, \ldots, p-1\}$. Furthermore, we see that

$$g_n'(x) = T'(T^{n-1}(x))(T^{n-1})'(x) - 1. \tag{7}$$

Then, if we note that $T'(x) = kx^{k-1} \equiv 0 \bmod p$ for all $x \in \mathbb{Z}_p$, and that all terms of (7) are surely in $\mathbb{Z}_p$, we see that $g_n'(x) \equiv -1 \bmod p$, so by Hensel's lemma, for all $n \in \mathbb{Z}^+$, T^n has a unique periodic point of period n in each ball $B_{1/p}(i)$, $i \in \{0, 1, \ldots, p-1\}$. Because a fixed point has period n for all $n \in \mathbb{Z}^+$, we have shown the theorem to be true. ♡

We proceed with the proof of theorem 3.4:

Proof. For the sake of notation, we write $T(x) = x^{p^r} + c$. Then, let $k \geq 1$, $|a| = 1$, and let α be a fixed point of f outside of $B_{1/p}(0)$. We then have

$$\begin{aligned} T(\alpha + p^k a) - \alpha &= (\alpha + p^k a)^{p^r} + c - \alpha \\ &= \sum_{i=0}^{p^r-1} \binom{p^r}{i} \alpha^i (p^k a)^{p^r-i} + 0 \\ &= \alpha^{p^r-1} p^{k+r} a + \binom{p^r}{p^r-2} \alpha^{p^r-2} p^{2k} a^2 + \sum_{i=1}^{p^r-3} \binom{p^r}{i} \alpha^i (p^k a)^{p^r-i}, \end{aligned}$$

so that

$$\begin{aligned} |T(\alpha + p^k a) - \alpha| &= |\alpha^{p^r-1} p^{k+r} a + \binom{p^r-1}{p^r-2} \frac{\alpha^{p^r-2}}{p^r-2} p^{2k+r} a^2 + \sum_{i=1}^{p^r-3} \binom{p^r}{i} \alpha^i (p^k a)^{p^r-i}| \\ &= |\alpha^{p^r-1} p^{k+r} a| = p^{-(k+r)}, \end{aligned}$$

thus the point α is attracting with no preperiodic points in $B_{1/p}(\alpha)$. Next, we observe that the fixed point $\alpha \in B_{1/p}(0)$ is an attracting fixed point and thus by a similar argument to that in theorem 3.2, there are finitely many preperiodic points in $B_{1/p}(0)$. Alongside lemma 3.5, this concludes our proof. ♡

We continue to explore the concept of small perturbations of maps with the following theorems. First, consider a map

$$T(x) = \alpha + \lambda(x - \alpha) + \sum_{n=2}^{\infty} a_n (x - \alpha)^2$$

whos power series converges on $\mathbb{Z}_p$, with $a_n \in \mathbb{Z}_p$ and λ not a root of unity. Consider also a perturbation

$$S(x) = T(x) + E(x), \ E(x) = \sum_{k=0}^{\infty} e_k x^k$$

whos additional power series converges on $\mathbb{Z}_p$ with $|e_k| \leq \epsilon$ for all $k \in \mathbb{Z}_{\geq 0}$. We may use a power series extension of Hensel's lemma [2] to observe that S has a fixed point β such that $|\beta-\alpha| < |1-\lambda|$. We then state the theorem:

Theorem 3.6. *Let T and S be as above. Then, $\beta \to \alpha$ as $\epsilon \to 0$. More precisely, $|\beta - \alpha| \leq \frac{\epsilon}{|1-\lambda|}$. Furthermore, if $\lambda = T'(\alpha)$ and $\mu = S'(\beta)$, then $\mu \to \lambda$ as $\epsilon \to 0$. More precisely, $|\mu - \lambda| \leq \frac{\epsilon}{|1-\lambda|}$.*

Proof. Consider the following:

$$\begin{aligned} S(\beta) - T(\beta) &= \beta - \alpha - \lambda(\beta - \alpha) - \sum_{n=2}^{\infty} a_n(\beta - \alpha)^n \\ E(\beta) &= (1 - \lambda)(\beta - \alpha) - (\beta - \alpha)^2 \sum_{n=2}^{\infty} a_n(\beta - \alpha)^{n-2}. \end{aligned}$$

Using the fact that $|\alpha - \beta| < |1 - \lambda|$, we then have

$$|1 - \lambda||\beta - \alpha| = |E(\beta)| \leq \epsilon$$

by the strong triangle inequality. Thus $|\beta - \alpha| < \frac{\epsilon}{|1-\lambda|}$. This concludes the proof of the first statement in the theorem. We will then prove the second statement:

$$\begin{aligned} |\lambda - \mu| &= |T'(\alpha) - S'(\beta)| \\ &\leq \max\{|T'(\alpha) - S'(\alpha)|, |S'(\alpha) - S'(\beta)|\} \\ &\leq \max\left\{\epsilon, \frac{\epsilon}{|1 - \lambda|}\right\} = \frac{\epsilon}{|1 - \lambda|}, \end{aligned}$$

which concludes the proof. ♡

4. Multivariable Maps

We begin this section with the following lemma:

Lemma 4.1. *Let $T : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ be a map of formal power series,*

$$T(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \qquad \text{for } f_i(x_1, \ldots, x_n) \in \mathbb{Z}_p[[x_1, \ldots, x_n]]$$

and let α be a fixed point of T in $\mathbb{Z}_p$. If $||J_T(\alpha)|| < 1$, then α is an attracting fixed point with basin of attraction containing $B_{1/p}(\alpha)$.

Proof. Let

$$f_i(X_1, \ldots, X_n) = \alpha + J_T(\alpha)(X - \alpha) + \sum_{\mathbf{k} \in \mathbb{N}^n \,:\, |\mathbf{k}|_{\ell^1} \geq 2} c_{i,k}(X - \alpha)^k$$

with $||J_T(\alpha)|| < 1$. Then for $x \in B_{1/p}(\alpha)$, we can write $x = \alpha + p^m h$ with $||h|| = 1$.

$$\begin{aligned}
|f_i(x) - \alpha|_p &= \left| \alpha + J_T(\alpha)(x-\alpha) + \sum_{\mathbf{k} \in \mathbb{N}^n \,:\, |\mathbf{k}|_{\ell^1} \geq 2} c_{i,k}(x-\alpha)^{\mathbf{k}} - \alpha \right| \\
&= \left| J_T(\alpha)(p^m h) + \sum_{\mathbf{k} \in \mathbb{N}^n \,:\, |\mathbf{k}|_{\ell^1} \geq 2} c_{i,k} p^{m|\mathbf{k}|} h^k \right| \\
&\leq \max\left\{ ||J_T(\alpha)|| p^{-m}, |c_{i,k}|_p p^{-m|\mathbf{k}|} ||h||^{|\mathbf{k}|} \right\} \\
&\leq p^{-(m+1)},
\end{aligned}$$

hence

$$||T(x) - \alpha|| \leq p^{-(m+1)},$$

thus if $\mathbf{x} \in B_{1/p^k}(\alpha)$, then $T(\mathbf{x}) \in B_{1/p^{k+1}}(\alpha)$, so $T^{(n)}(\mathbf{x}) \in B_{1/p^{k+n}}(\alpha)$, therefore $B_{1/p}(\alpha)$ is contained in the basin of attraction of α as $T^{(n)}(\mathbf{x}) \to \alpha$ as $n \to \infty$. ♡

We proceed with an overview of relevant results which describe multivariate maps.

Definition 4.2. Let $T : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$, and write $T = (T_1, \ldots, T_m)$ where $T_i : \mathbb{Q}_p^n \to \mathbb{Q}_p$. If the partial derivatives $\frac{\partial T_i}{\partial x_j}$ (defined as difference quotients in the usual way) exist and are continuous at some point $\mathbf{a} \in \mathbb{Q}_p^n$, we say that T is *differentiable at* $\mathbf{a}$. If T is said to be differentiable if it is differentiable at all points $\mathbf{a} \in \mathbb{Q}_p^n$. Furthermore, if T is differentiable at $\mathbf{a}$, we define the *Jacobian* of T to be

$$J_T(\mathbf{a}) = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_m}{\partial x_1} & \cdots & \frac{\partial T_m}{\partial x_n} \end{bmatrix}$$

where all derivatives are evaluated at $\mathbf{a}$.

Proposition 4.3. *Chain Rule: If $T : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ and $S : \mathbb{Q}_p^m \to \mathbb{Q}_p^l$, then*

$$J_{S \circ T} = J_S(T) J_T, \text{ i.e. for all } a \in \mathbb{Q}_p^n,\ J_{S \circ T}(a) = J_S(T(a)) J_T(a).$$

Lemma 4.4. *Let T^m denote $T \circ \cdots \circ T$ composed with itself m times. Then*

$$J_{T^m}(a) = J_T(T^{m-1}(a)) J_T(T^{m-2}(a)) \cdots J_T(T(a)) J_T(a).$$

Lemma 4.5. *(Keith Conrad) Let $T : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ be a polynomial map with coefficients in $\mathbb{Z}_p$. Then if $a \in \mathbb{Z}_p^n$ satisfies*

$$||T(a)|| < |\det J_T(a)|^2,$$

there is a unique $\alpha \in \mathbb{Z}_p^n$ such that $f(\alpha) = 0$ and $||\alpha - a|| < |\det J_T(a)|$. More precisely,

(1) $||\alpha - a|| = ||J_T(a)^{-1} T(a)|| \leq ||T(a)||/|\det J_T(a)| < |\det J_T(a)|$.
(2) $|\det J_T(\alpha)| = |\det_T(a)|$.

In particular, if $||T(a)|| < 1$ (e.g. $T(\mathbb{Z}_p^n) \subset \mathbb{Z}_p^n$) and $|\det J_T(a)| = 1$, then there is a unique $\alpha \in \mathbb{Z}_p^n$ such that $T(\alpha) = 0$ and $||\alpha - a|| < 1$. The solution may be obtained by "Newton's method", that is with the recurrence

$$\alpha_{n+1} = \alpha_n - (J_T(\alpha_n))^{-1} T(\alpha_n)$$

with $\alpha_1 = a$.

We study the case of so-called *Hénon maps* with good reduction (meaning T and T^{-1} have coefficients in $\mathbb{Z}_p$). Hénon maps are maps of the form $T : \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ via $(x, y) \mapsto (A + By - \phi(x), x)$. Indeed,

$$J_T((a, b)) = \begin{bmatrix} -\phi'(a) & 1 \\ B & 0 \end{bmatrix}$$

which has determinant $-B$. If T is invertible, the inverse is

$$(y, B^{-1}(-A - x + \phi(y)))$$

which has coefficients in $\mathbb{Z}_p$ if and only if $B^{-1} \in \mathbb{Z}_p$, i.e. $|B| = 1$, so $|\det J_T((a, b))| = 1$. The only hitch is that $||T(a, b)||$ could equal 1; indeed this is very likely the case.

Let's compute the eigenvalues for $J_T((a, b))$. First, $\chi_{J_T}(X) = X^2 + \phi'(a)X - B$, so the eigenvalues are

$$\lambda = \frac{-\phi'(a) \pm \sqrt{\phi'(a)^2 + 4B^2}}{2} = (-\phi'(a)/2) \pm \sqrt{(\phi'(a)/2)^2 + B^2}.$$

We would extend the norm to $K/\mathbb{Q}_p$ where $K = \mathbb{Q}_p(\sqrt{(\phi'(a)/2)^2 + B^2})$, thus

$$|\lambda|_K = |N_{K/\mathbb{Q}_p}(\lambda)|_p^{1/|K:\mathbb{Q}_p|} = |-B|_p^{1/d} = 1$$

where $|K : \mathbb{Q}_p|$ denotes the degree of the field extension $K/\mathbb{Q}_p$ which is 1 or 2 depending on whether or not λ is a square in $\mathbb{Q}_p$ (though this is not relevant since $|B|_p = 1$). Both eigenvalues have norm 1.

5. Hénon Maps

In this section, we will briefly explore Hénon maps, mostly through examples. We will show results including repeated eigenvalues and irrational periodic points. The definition provided in the previous section pertains to *generalized Hénon maps.* A degree 2 Hénon map is a map $T : \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ of the form

$$T_{A,B} = (A + By - x^2, x)$$

with $B \neq 0$. Note that this map can be defined by the ordered pair A, B. A Hénon map has inverse

$$T_{A,B}^{-1} = (y, \frac{1}{B}(x + y^2 - A)).$$

We say a Hénon map has *good reduction* if both $T_{A,B}$ and $T_{A,B}^{-1}$ have coefficients in $\mathbb{Z}_p$. That is, if $A \in \mathbb{Z}_p$ and $|B| = 1$.

5.1. **An Interesting Function?** Consider the function $T(x, y) = (2 - y - x^2, x)$ from $\mathbb{Q}_3^2$ to $\mathbb{Q}_3^2$. Using sage, we found the following 3-cycles of T:

$$(-1, -1) \to (2, -1) \to (-1, 2) \to (-1, -1) \tag{8}$$

$$(1, 1) \to (0, 1) \to (1, 0) \to (1, 1) \tag{9}$$

The cycle (1) is contained entirely in $B_{\frac{1}{3}}\big((-1, -1)\big)$, while the points in the cycle (2) are each distance 1 from each other. If we consider the function

$$H_n(x, y) = T^n(x, y) - (x, y)$$

which may not be an automorphism, we can spot a difference in the Jacobian of H. Note that a root of H_3 is a fixed point of T^3. First, for points (a, b) in the cycle (1),

$$|\det(J_{H_3})|_3 = \frac{1}{9}$$

while for points (a, b) in the cycle (2),

$$|\det(J_{H_3})|_3 = 1.$$

This suggests a relationship between $\det(J_H)$ and the potential closeness of periodic points of T, which is consistent with a multivariable Hensel's lemma [2]. For example, using the multivariate Hensel's lemma we could prove that any 3-cycle of T must be at least distance 1 from the points in (2), and at least distance $\frac{1}{81}$ from the points in (1). We then continue to observe fixed points for different cycles to look for patterns in the Jacobian:

- For the Hénon map defined by $(A, B) = (1, 1)$, we have fixed points or points of period 2 at $(\pm 1, \pm 1)$, and $|\det(J_{H_2})| = 1$.
- For $(A, B) = (1, -1)$, the 3-cycle $\{(1, 0), (0, 1), (0, 0)\}$ produces $|\det(J_{H_3})| = 0$, while the 4-cycle $\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$ produces $|\det(J_{H_4})| = 1$.
- For $(A, B) = (4, -1)$, the 2-cycle $\{(2, 0), (-2, 2), (0, -2), (0, 0)\}$ produces $|\det(J_{H_4})| = 0$.

Each of these points is a root of H_n for some n, and the multivariate Hensel's lemma in theorem 3.3 of [2] provides us with a bound on how nearby a different T-periodic point of the same period can be (in the cases where $|\det(J_H)| \neq 0$). That is, we can prove the uniqueness of these periodic points within balls of radius $|\det(J_H)|$ for those with nonzero determinant. We also may want to observe more points of varying periods and various cycles or search for a reason justifying these conjectures, as well as look further into those with determinant 0.

5.2. **Proving the existence of a periodic point using Hensel's lemma.** Consider the Hénon map $T(x, y) = (-10 + 2y - x^2, x)$. Using SageMath, we found that the point $(1, 2)$ has period 5 mod 3^6. Through further exploration, we found more accurate descriptions of the periodic point near $(1, 2)$, as $(1, 2)$ itself is not periodic. We then observe the following 5-cycle mod 3 of $T(x, y)$:

$$(1, 2) \to (2, 1) \to (0, 2) \to (0, 0) \to (2, 0) \to (1, 2). \tag{10}$$

We proceed with the following proposition:

Proposition 5.1. *For each point $\vec{a} \in \{(1, 2), (2, 1), (0, 2), (0, 0), (2, 0)\}$, there is a unique $\vec{\alpha}$ in the closed ball $B_{\frac{1}{3}}(\vec{a})$ such that $\vec{\alpha}$ is T-periodic with period 5.*

Proof. First, note that this is equivalent to having a unique root of H_5, with H_5 defined as above. We then apply the multivariable Hensel's lemma provided in [2]. Given each of these points, we calculate the value $||H_5(\vec{a})||$ as well as $|\det(J_{H_5})|$:

$\vec{a}$	$\|\|H_5(\vec{a})\|\|$	$\|\det(J_{H_5})\|$
(1,2)	1/729	1
(2,1)	1/9	1
(0,2)	1/3	1
(0,0)	1/3	1
(2,0)	1/3	1

Because each of these points $\vec{a}$ satisfies $||H_5(\vec{a})|| < |\det(J_{H_5})|^2$, we know that, for each $\vec{a}$ listed in (3), there is exactly one root of H_5, and therefore exactly one T-periodic point of period 5, in each closed ball $B_{\frac{1}{3}}(\vec{a})$ (equivalently in each open ball $B_1(\vec{a})$). ♡

We now move on to a proposition regarding the rationality of these points.

Proposition 5.2. *The periodic point $\vec{\alpha} \in B_{\frac{1}{3}}(0, 0)$ described above is irrational. That is, $\vec{\alpha} \notin \mathbb{Q}^2$.*

Proof. We will begin by supposing $\vec{\alpha} = (x, y) \in \mathbb{Q}^2$ has period 5 with respect to T and is within one of the balls $B_{\frac{1}{3}}(\vec{a})$ described above.

First, note that if any of the points in the 5-cycle $\{T^n(x, y)\}_{n=0}^4$ is rational, they must all be. Then without loss of generality we may assume that $(x, y) \in B_{\frac{1}{3}}(0, 0)$. Because T has good reduction (as defined in [1]), we know from [1] that the filled Julia set $F_p(T) = \mathbb{Z}_p^2$ for each odd prime p. Thus our point (x, y) must take on the form $(\frac{3A}{2^m}, \frac{3B}{2^n})$ for some $A, B \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$. We will now provide bounds on $F_2(T)$ and $F_\infty(T)$, using a proof modelled after the proof of proposition 7 from [1].

First, define the sets

$$S^+ = \left\{(x, y) \in \mathbb{Q}_2^2 \;:\; ||(x, y)|| > 1, |x| \geq |y|\right\}$$
$$S^- = \left\{(x, y) \in \mathbb{Q}_2^2 \;:\; ||(x, y)|| > 1, |x| \leq |y|\right\}.$$

Suppose $(x, y) \in S^+$. Then we have that $|x|^2 > |y| > |2y|$ and $|x|^2 > 1 > |-10|$. From this, let $T(x, y) = (x', y')$. Then

$$|x'| = |-10 + 2y - x^2| = |x^2| > |x| = |y'|,$$

so $||(x', y')|| = ||x||^2$ and $(x', y') \in S^+$. By induction, $||T^n(x, y)|| = ||x||^{2n} \to \infty$, so $(x, y) \notin F(T)$. Next, suppose $(x, y) \in S^-$. This tells us that $|y^2| > |x|$ and $|y^2| > |10|$ and $|\frac{1}{2}y^2| = 2|y| > |y|$. Let $T^{-1}(x, y) = (y, \frac{1}{2}(x + 10 + y^2)) = (x', y')$. We then have

$$|y'| = |\frac{1}{2}(x + 10 + y^2)| = 2|y|^2 > |y| = |x'|$$

so $||T^{-1}(x, y)|| = 2|y^2|$ and by induction $||T^{-n}(x, y)|| = 2^n|y|^{2n} \to \infty$, so $(x, y) \notin F_2(T)$. We have now shown that $F_2(T) \subseteq \mathbb{Z}_2^2$ (it may be a proper subset, but that isn't important for this proof). This tells us that our T-periodic point of period 5 must take the form $(3A, 3B)$ with A and B integers. To find a bound on A and B, we will consider the filled Julia set $F_\infty(T)$ in the complex plane $\mathbb{C}^2$.

First, given $(x, y) \in \mathbb{C}^2$ suppose $|x| \geq |y| > 10$ where $|\cdot|$ is now the standard absolute value in $\mathbb{C}$. Then if we let $T(x, y) = (x', y')$ we have that

$$\begin{aligned}
|x'| = |-10 + 2y - x^2| &\geq |x^2| - |-10 + 2y| \\
&\geq |x|^2 - |2y| - 10 \\
&\geq |x|(|x| - 2) - 10 \\
&\geq 8|x| - 10 \\
&\geq 7|x| > |x| = |y'|.
\end{aligned}$$

If we notice that $||x, y|| = \sqrt{|x|^2 + |y|^2} \leq \sqrt{2}|x|$, we can then see that

$$||T(x, y)|| > |x'| \geq 7|x| \geq \frac{7}{\sqrt{2}}||(x, y)|| > 3||(x, y)||$$

and, because $|x'| > |y'| > 10$, we then have by induction $||T^n(x,y)|| > 3^n||(x,y)|| \to \infty$, so $(x,y) \notin F_\infty(T)$. Now suppose $|y| \geq |x| > 10$. Then, letting $T^{-1}(x,y) = (x',y')$, we see that

$$\begin{aligned} |y'| = \frac{1}{2}|x + 10 + y^2| &\geq \frac{1}{2}\left(|y|^2 - |x| - 10\right) \\ &\geq \frac{1}{2}\left(|y|(10-1) - 10\right) \\ &> \frac{1}{2}(8|y|) = 4|y| > |y| = |x'|. \end{aligned}$$

We then see that $||T(x,y)|| > |y'| \geq 4|y| \geq 2\sqrt{2}||(x,y)|| > 2||(x,y)||$. Because $|y'| > |x'| > 10$, we have by induction that $||T^{-n}(x,y)|| > 2^n||(x,y)|| \to \infty$ and therefore $F_\infty(T) \subseteq B_{10}(0,0)$. Note that this is a naïve upper bound and the ball containing $F_\infty(T)$ is likely much smaller.

From the above, we know that, if our periodic point $(x,y) \in B_{\frac{1}{3}}(0,0)$ in $\mathbb{Q}_3^2$ is rational, we must have $(x,y) = (3A, 3B)$ for $A, B \in \{0, \pm 1, \pm 2, \pm 3\}$. By checking each of these points for periodicity in sage, we find that none of them are periodic with period 5. Thus the periodic point $(x,y) \in B_{\frac{1}{3}}(0,0)$ is contained in $\mathbb{Q}_3^2 \setminus \mathbb{Q}^2$. ♡

Corollary 5.3. *None of the periodic points $\vec{\alpha}$ described in the propositions above are rational.*

5.3. **Repeated Eigenvalues.** We now want to investigate Hénon maps alongside the eigenvalues of their Jacobian matrices. We begin with a proposition:

Proposition 5.4. *Suppose a Hénon map T has a fixed point (x,x) such that the Jacobian matrix $J_T(x,x)$ has a repeated eigenvalue λ. If T has good reduction, then $|\lambda| = |x| = 1$.*

Proof. Let $T(x,y) = (a + by - x^2, x)$. Then $J_T = \left(\begin{smallmatrix} -2x & b \\ 1 & 0 \end{smallmatrix}\right)$, so we have characteristic polynomial

$$\begin{aligned} p_{J_T}(\lambda) &= \lambda^2 + 2x\lambda - b = 0 \\ &\Rightarrow \lambda = -x \pm \sqrt{x^2 + b}. \end{aligned}$$

To have repeated eigenvalues, we must have $-b$ be a square with $x^2 = -b$. Because T has good reduction, we then have $|x|^2 = |b| = 1$ so that $|x| = 1$, and $|\lambda| = |-x| = |x| = 1$. ♡

Corollary 5.5. *Choice of repeated eigenvalue λ (or equivalently choice of point x) above determines a unique map.*

5.4. **Nearby Fixed Points.** Consider the set of Hénon maps $T : \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ such that $T(x,y) = ((p^k+1)y - x^2, x)$. These maps all have good reduction, and they have fixed points at the origin $(0,0)$ as well as (p^k, p^k). These two fixed points both lie in $B_{1/p^k}(0,0)$.

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