A p-ADIC PERRON-FROBENIUS THEOREM

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Abstract. We prove a result for square matrices over the p-adic numbers akin to the Perron-Frobenius Theorem for square matrices over the real numbers. In particular, we show that if a square $n \times n$ matrix A has all entries p-adically close to 1, then this matrix will possess a unique maximal eigenvalue λ_0 such that: (a) λ_0 is a p-adic integer, (b) λ_0 has an algebraic multiplicity of one, and (c) there exists an eigenvector associated to λ_0 with all entries p-adically close to 1. Furthermore, we show that iteration of A/λ_0 converges to a projection operator onto the eigenspace of λ_0.

1. Introduction

The classical Perron-Frobenius Theorem gives us information about square matrices with all entries given by positive real numbers.

Theorem 1.1 (Perron-Frobenius, Theorem 1 in Chapter 16 of [3]). *Let A be a square matrix with all entries given by positive real numbers. Then there exists an eigenvalue λ_0 of A of multiplicity one such that λ_0 is a positive real number and $|\lambda| < \lambda_0$ for all other complex eigenvalues λ of A. Furthermore, there exists an eigenvector $\mathbf{v}$ of A with eigenvalue λ_0 such that all components of $\mathbf{v}$ are positive real numbers.*

A useful application of Theorem 1.1 is the following.

Theorem 1.2 (Theorem 3 in Chapter 16 of [3]). *Let A be a square $n \times n$ matrix with all entries given by positive real numbers such that, for each column of A, the sum of the entries in that column is 1. Then the maximal eigenvalue λ_0 of A guaranteed by Theorem 1.1 is equal to 1, and for any vector $\mathbf{x} \in \mathbb{R}^n$ with all entries nonnegative, the sequence $(A^k\mathbf{x})$ converges to an eigenvector of λ_0.*

The dynamical implications of Theorem 1.2 make it useful for studying many real-world phenomena. Matrices in the form of the matrix A from Theorem 1.2 are used to model changes in atomic nuclei and populations in ecological systems [3, p. 241], and Theorem 1.2 itself forms the basis of Google's search strategy [3, p. 242].

Our first result over the p-adic numbers is analogous to Theorem 1.1. This result applies to any p-adic $n \times n$ matrix A, where: (a) all entries of A are in $1 + p\mathbb{Z}_p$, and (b) $p \nmid n$. For a demonstration of why these conditions cannot be relaxed, see Example 4.3.

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Theorem 1.3. *Let p be a prime and let n be a positive integer such that $p \nmid n$. Let A be a square $n \times n$ matrix with all entries in $1 + p\mathbb{Z}_p$; that is, let*

$$A = \begin{bmatrix} 1+pa_{11} & 1+pa_{12} & \dots & 1+pa_{1n} \\ 1+pa_{21} & 1+pa_{22} & \dots & 1+pa_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1+pa_{n1} & 1+pa_{n2} & \dots & 1+pa_{nn} \end{bmatrix}, \tag{1}$$

where each $a_{ij} \in \mathbb{Z}_p$. Then there exists an eigenvalue λ_0 of A of multiplicity one such that $\lambda_0 \in \mathbb{Z}_p$, $\lambda_0 \equiv n \pmod{p\mathbb{Z}_p}$, and $|\lambda|_p < |\lambda_0|_p$ for all other eigenvalues λ of A in $\mathbb{C}_p$. In addition, there exists an eigenvector $\mathbf{v}$ of A with eigenvalue λ_0 such that all components of $\mathbf{v}$ are elements of $1 + p\mathbb{Z}_p$.

Our next result states that we can relax the condition of Theorem 1.3 that $p \nmid n$ as long as we strengthen the other condition according to the extent to which $p \mid n$. Specifically, given a prime p and positive integer n, we show that the conclusion of Theorem 1.3 about the existence of a strictly maximal eigenvalue λ_0 of A still holds if we require that all entries of A be close enough to 1.

Theorem 1.4. *Let p be a prime and let n, ℓ be positive integers. Let A be a square $n \times n$ matrix with all entries in $1 + p^\ell\mathbb{Z}_p$; that is, let*

$$A = \begin{bmatrix} 1+p^\ell a_{11} & 1+p^\ell a_{12} & \dots & 1+p^\ell a_{1n} \\ 1+p^\ell a_{21} & 1+p^\ell a_{22} & \dots & 1+p^\ell a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1+p^\ell a_{n1} & 1+p^\ell a_{n2} & \dots & 1+p^\ell a_{nn} \end{bmatrix}, \tag{2}$$

where each $a_{ij} \in \mathbb{Z}_p$. Suppose $\ell > 2(n-1)\nu_p(n)$. Then there exists an eigenvalue λ_0 of A of multiplicity one such that $\lambda_0 \in \mathbb{Z}_p$, $|\lambda_0 - n|_p \le p^{-\ell}/|n^{n-1}|_p$, and $|\lambda|_p < |\lambda_0|_p$ for all other eigenvalues λ of A in $\mathbb{C}_p$.

Our final result analyzes the forward orbits of vectors with p-adic components under iteration by p-adic square matrices with a strictly dominant eigenvalue of multiplicity one. We produce a statement for matrices over the p-adic numbers analogous to that of Theorem 1.2 for matrices over the real numbers.

Theorem 1.5. *Let A be a square $n \times n$ matrix with all entries in $\mathbb{Q}_p$. Suppose that A has an eigenvalue λ_0 of multiplicity one such that $|\lambda|_p < |\lambda_0|_p$ for all other eigenvalues λ of A in $\mathbb{C}_p$. Then iteration of A/λ_0 converges to a projection operator onto the eigenspace of λ_0.*

2. Background on the p-adic Numbers

We begin with some remarks on the p-adic numbers $\mathbb{Q}_p$ and their complete, algebraically closed extension $\mathbb{C}_p$. Unless otherwise specified, we will take p to be any prime number. We refer the reader to [2] for more details on $\mathbb{Q}_p$ and $\mathbb{C}_p$.

2.1. Construction of $\mathbb{Q}_p$.

Definition 2.1. The *p-adic valuation* on $\mathbb{Z}$ is the function $\nu_p\colon \mathbb{Z}\setminus\{0\}\to\mathbb{Z}$ defined by

$$\nu_p(n) = \begin{cases} \max\{\, k\in\mathbb{N} : p^k \mid n \,\} & \text{if } p \mid n \\ 0 & \text{if } p \nmid n. \end{cases} \tag{3}$$

We extend ν_p to $\mathbb{Q}$ as follows: we define $\nu_p\colon \mathbb{Q}\to\mathbb{Z}\cup\{\infty\}$ by

$$\nu_p(x) = \begin{cases} \nu_p(a)-\nu_p(b) & \text{if } x = a/b \in \mathbb{Q}^\times \\ \infty & \text{if } x = 0. \end{cases} \tag{4}$$

Definition 2.2. The *p-adic absolute value* on $\mathbb{Q}$ is the function $|\cdot|_p\colon \mathbb{Q}\to\mathbb{R}$ defined by

$$|x|_p = p^{-\nu_p(x)}, \tag{5}$$

where we define $p^{-\infty} = 0$.

We recall that a field K is *complete* if every Cauchy sequence in K converges in K. It is known that $\mathbb{Q}$ is not complete with respect to any of its nontrivial absolute values [2, Lemma 3.2.3].

Definition 2.3. The *field $\mathbb{Q}_p$ of p-adic numbers* is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value.

2.2. Properties of $\mathbb{Q}_p$.

There are several useful properties of $\mathbb{Q}_p$ that we will keep in mind throughout our work. The first of these appears as part of Theorem 3.2.13 in [2].

Theorem 2.4. *Fix a prime p and let $a, b\in\mathbb{Q}_p$. Then*

$$|a+b|_p \le \max\{\, |a|_p, |b|_p \,\}. \tag{6}$$

Theorem 2.5 (Proposition 2.3.3 in [2]). *Fix a prime p and let $a, b\in\mathbb{Q}_p$ with $|a|_p \neq |b|_p$. Then*

$$|a+b|_p = \max\{\, |a|_p, |b|_p \,\}. \tag{7}$$

It is a consequence of Theorem 2.4 that $\mathbb{Z}_p = \{\, x\in\mathbb{Q}_p : |x|_p \le 1 \,\}$ is a subring of $\mathbb{Q}_p$; we call this the *ring $\mathbb{Z}_p$ of p-adic integers*. It follows similarly that $p\mathbb{Z}_p = \{\, x\in\mathbb{Q}_p : |x|_p < 1 \,\}$ is the unique maximal ideal of $\mathbb{Z}_p$. The quotient field $\mathbb{Z}_p/p\mathbb{Z}_p$ is called the *residue field of* $\mathbb{Z}_p$; it is an immediate consequence of [2, Corollary 3.3.6] that $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$.

The following results allow us to deduce information about the roots of a polynomial $f(X)\in\mathbb{Z}_p[X]$ from the behavior of the reductions of $f(X)$ and $f'(X)$ modulo $p\mathbb{Z}_p$ ($f'(X)$ being the *formal derivative* of $f(X)$).

Theorem 2.6 (Hensel's Lemma, Theorem 3.4.1 in [2]). *Let $f(X)$ be a polynomial whose coefficients are in $\mathbb{Z}_p$ and suppose that there exists a p-adic integer $\alpha_1\in\mathbb{Z}_p$ such that*

$$f(\alpha_1) \equiv 0 \pmod{p\mathbb{Z}_p} \tag{8}$$

and

$$f'(\alpha_1) \not\equiv 0 \pmod{p\mathbb{Z}_p}. \tag{9}$$

Then there exists a unique p-adic integer $\alpha\in\mathbb{Z}_p$ such that $f(\alpha)=0$ and $\alpha\equiv\alpha_1 \pmod{p\mathbb{Z}_p}$.

Theorem 2.7 (Strong Hensel's Lemma, Lemma 3.1 and Corollary 1 in [1])**.** *Let $f(X) \in \mathbb{Z}_p[X]$ be a polynomial whose coefficients are in $\mathbb{Z}_p$ and suppose that there exists a p-adic integer $\alpha_1 \in \mathbb{Z}_p$ such that*

$$|f(\alpha_1)|_p < |f'(\alpha_1)|_p^2. \tag{10}$$

Then there exists a unique p-adic integer $\alpha \in \mathbb{Z}_p$ for which $f(\alpha) = 0$ and

$$|\alpha - \alpha_1|_p \leq \frac{|f(\alpha_1)|_p}{|f'(\alpha_1)|_p}. \tag{11}$$

We conclude this subsection with a theorem that gives us sufficient conditions for a polynomial in $\mathbb{Z}_p[x]$ to be irreducible over $\mathbb{Q}_p$.

Theorem 2.8 (Eisenstein Irreducibility Criterion, Proposition 5.3.11 in [2])**.** *Let*

$$f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}_p[x] \tag{12}$$

be a polynomial satisfying the conditions

i) $|a_n|_p = 1$,

ii) $|a_i|_p < 1$ *for* $0 \leq i < n$, *and*

iii) $|a_0|_p = 1/p$.

Then $f(X)$ is irreducible over $\mathbb{Q}_p$.

2.3. **The Field $\mathbb{C}_p$ and Newton Polygons.** While the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete [2, Theorem 5.7.4], the completion of $\overline{\mathbb{Q}}_p$ is algebraically closed [2, Proposition 5.7.8]; we call this field $\mathbb{C}_p$.

Definition 2.9. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}_p[X]$ be a polynomial. The *Newton polygon of f* is the lower convex hull of the set of points

$$S = \{\, (i, \nu_p(a_i)) : 1 \leq i \leq n \,\}. \tag{13}$$

By a *slope* of the Newton polygon, we will mean the slope of a line segment appearing in the polygon. Given a slope of the polygon, by the *length* of the slope, we will mean the length of the projection of the corresponding segment on the x-axis. It is clear that since the Newton polygon is the lower convex hull of a set of points in $\mathbb{R}^2$, the slopes form a nondecreasing sequence if we pick the corresponding segments from left to right.

Theorem 2.10 (Theorem 6.4.7 in [2])**.** *Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}_p[X]$ be a polynomial, and let $m_1, m_2, \ldots, m_r$ be the slopes of its Newton polygon (in increasing order). Let $i_1, i_2, \ldots, i_r$ be the corresponding lengths. Then, for each k, $1 \leq k \leq r$, $f(X)$ has exactly i_k roots (counting multiplicities) of absolute value p^{m_k}.*

3. Background on Linear Algebra

We now review necessary background material on linear algebra. We will take A to be a square $n \times n$ with entries from a field F. We refer the reader to [3] and [4] for additional details on linear algebra.

3.1. **Calculation of Determinants.** A useful formula for computing determinants is provided in [3, Chapter 5, Equation 16]. This formula expresses the determinant of A as the sum of signed products of entries from distinct rows and columns in A. We will quote this formula below for convenience.

Lemma 3.1 (Determinant Formula)**.** *Let S_n denote the symmetric group on $\{1, \ldots, n\}$, and let* sgn *denote the sign function of permutations in S_n. Then*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{14}$$

3.2. **Calculations Involving Jordan Normal Forms.** We briefly review how to compute powers of matrices given in Jordan normal form. Let A be a matrix in Jordan normal form:

$$J = \begin{bmatrix} J_{m1} & 0 & \ldots & 0 \\ 0 & J_{m_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_{m_j} \end{bmatrix}, \tag{15}$$

where, for $1 \leq i \leq j$, J_{m_i} is the $m_i \times m_i$ Jordan block associated to the eigenvalue λ_i of J:

$$J_{m_i} = \begin{bmatrix} \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & \lambda_i & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix}. \tag{16}$$

Different Jordan blocks may share the same eigenvalue, but different eigenvalues may not share the same Jordan bock. A useful computational property of Jordan normal forms, as noted in Section 5.5 of [4], is that we can easily compute powers of A:

$$J^n = \begin{bmatrix} J_{m1}^n & 0 & \ldots & 0 \\ 0 & J_{m_2}^n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_{m_j}^n \end{bmatrix}. \tag{17}$$

Thus, in order to compute J^n for a positive integer n, we only have to compute the power $J_{m_i}^n$ of each Jordan block J_{m_i}. By induction, we will show that, for any positive integer n,

$$J_{m_i}^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \ldots & \binom{n}{m_i-2}\lambda_i^{n-(m_i-2)} & \binom{n}{m_i-1}\lambda_i^{n-(m_i-1)} \\ 0 & \lambda_i^n & \ldots & \binom{n}{m_i-3}\lambda_i^{n-(m_i-3)} & \binom{n}{m_i-2}\lambda_i^{n-(m_i-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} \\ 0 & 0 & \ldots & 0 & \lambda_i^n \end{bmatrix}. \tag{18}$$

If $a < b$, we say $\binom{a}{b} = 0$, as is standard. In the case $n = 1$, the formula (18) agrees with the Jordan block J_{m_i}. Suppose inductively that the formula (18) holds for some integer $n = k \geq 1$. That is,

$$J_{m_i}^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \dots & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} & \binom{k}{m_i-1}\lambda_i^{k-(m_i-1)} \\ 0 & \lambda_i^k & \dots & \binom{n}{m_i-3}\lambda_i^{k-(m_i-3)} & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & 0 & \dots & 0 & \lambda_i^k \end{bmatrix}. \tag{19}$$

We compute $J_{m_i}^{k+1}$:

$$J_{m_i}^{k+1} = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \dots & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} & \binom{k}{m_i-1}\lambda_i^{k-(m_i-1)} \\ 0 & \lambda_i^k & \dots & \binom{k}{m_i-3}\lambda_i^{k-(m_i-3)} & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & 0 & \dots & 0 & \lambda_i^k \end{bmatrix} \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} \lambda_i^{k+1} & \lambda_i^k + \binom{k}{1}\lambda_i^k & \dots & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} + \binom{k}{m_i-1}\lambda_i^{k-(m_i-2)} \\ 0 & \lambda_i^{k+1} & \dots & \binom{k}{m_i-3}\lambda_i^{k-(m_i-3)} + \binom{k}{m_i-2}\lambda_i^{k-(m_i-3)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_i^k + \binom{k}{1}\lambda_i^k \\ 0 & 0 & \dots & \lambda_i^{k+1} \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} \lambda_i^{k+1} & \binom{k+1}{1}\lambda_i^{(k+1)-1} & \dots & \binom{k+1}{m_i-2}\lambda_i^{(k+1)-(m_i-2)} & \binom{k+1}{m_i-1}\lambda_i^{(k+1)-(m_i-1)} \\ 0 & \lambda_i^{k+1} & \dots & \binom{k+1}{m_i-3}\lambda_i^{(k+1)-(m_i-3)} & \binom{k+1}{m_i-2}\lambda_i^{(k+1)-(m_i-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i^{k+1} & \binom{k+1}{1}\lambda_i^{(k+1)-1} \\ 0 & 0 & \dots & 0 & \lambda_i^{k+1} \end{bmatrix}. \tag{22}$$

The final step follows by Pascal's identity for binomial coefficients:

$$\binom{a+1}{b+1} = \binom{a}{b} + \binom{a}{b+1} \tag{23}$$

for nonnegative integers a and b. Hence, the formula (18) holds for $n = k + 1$. By the Principle of Mathematical Induction, this formula holds for all positive integers n.

4. Proofs of Main Results

4.1. **Proof of Theorem 1.3.** In this subsection, we establish Theorem 1.3. Our argument will make use of the Newton polygon associated to the characteristic polynomial $f(X)$ of A. First we establish several lemmas.

Lemma 4.1. *Let p be a prime and let n be a positive integer. Let A be a square $n \times n$ matrix with all entries in $1 + p\mathbb{Z}_p$; that is, let*

$$A = \begin{bmatrix} 1 + pa_{11} & 1 + pa_{12} & \dots & 1 + pa_{1n} \\ 1 + pa_{21} & 1 + pa_{22} & \dots & 1 + pa_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 + pa_{n1} & 1 + pa_{n2} & \dots & 1 + pa_{nn} \end{bmatrix}. \tag{24}$$

Then $|\det(A)|_p \le p^{1-n}$.

Proof. Using Lemma 3.1, we calculate:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (1 + a_{i,\sigma(i)} p) \tag{25}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{I \subseteq \{1,\dots,n\}} \prod_{i \in I} a_{i,\sigma(i)} p \tag{26}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{k=0}^{n} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = k}} \prod_{i \in I} a_{i,\sigma(i)} p \tag{27}$$

$$= \sum_{k=0}^{n} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = k}} p^k S(I), \tag{28}$$

where for each $I \subseteq \{1, \dots, n\}$ we define

$$S(I) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in I} a_{i,\sigma(i)}. \tag{29}$$

Now suppose that $0 \le k \le n - 2$. Let $I \subseteq \{1, \dots, n\}$ with $|I| = k$. Since $|I| \le n - 2$, there exists a transposition $\epsilon \in S_n$ that fixes the elements of I. Then since the map $\sigma \mapsto \sigma\epsilon$ is a bijection from S_n to itself, we can write

$$S(I) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma\epsilon) \prod_{i \in I} a_{i,\sigma\epsilon(i)} \tag{30}$$

$$= \operatorname{sgn}(\epsilon) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in I} a_{i,\sigma(i)} \tag{31}$$

$$= -S(I), \tag{32}$$

where in line (31) we use the fact that $a_{i,\sigma\epsilon(i)} = a_{i,\sigma(i)}$ for all $\sigma \in S_n$ and for all $i \in I$. The statement $S(I) = -S(I)$ implies that $S(I) = 0$, so line (28) becomes

$$\det(A) = \sum_{k=n-1}^{n} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} p^k S(I). \tag{33}$$

We see from Theorem 2.4 that for any subset I of $\{1, \ldots, n\}$, $|S(I)|_p \leq 1$. Taking the p-adic absolute value of both sides of equation (33) and again applying Theorem 2.4, we find that

$$|\det(A)|_p \leq \left| \sum_{k=n-1}^{n} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} p^k \right|_p \leq \max\{\, |p^{n-1}|_p, |p^n|_p \,\} = p^{1-n}. \tag{34}$$

$\square$

Lemma 4.2. *Let the matrix A be as in the statement of Lemma 4.1 Let the characteristic polynomial $f(X)$ of A be given as:*

$$f(X) = X^n + C_{n-1}X^{n-1} + \cdots + C_1 X + C_0. \tag{35}$$

Then $|C_i|_p \leq p^{1-n+i}$ for $0 \leq i \leq n-1$, and if additionally $p \nmid n$, $|C_{n-1}|_p = 1$.

Proof. First we establish some notations. Given positive integers i and j, let $\delta_{i,j}$ represent the Kronecker delta function of i and j. Given a permutation $\sigma \in S_n$, let $\mathrm{Fix}(\sigma)$ denote the set of all elements of $\{1, \ldots, n\}$ that are fixed by σ. Given a subset I of $\{1, \ldots, n\}$, let I^c represent the set $\{1, \ldots, n\} \setminus I$.

We apply Lemma 3.1 to $A - X\boldsymbol{I}$ to find the characteristic polynomial $f(X)$ of A:

$$\begin{align}
f(X) &= (-1)^n \det(A - X\boldsymbol{I}) \tag{36}\\
&= (-1)^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (1 + pa_{i,\sigma(i)} - \delta_{i,\sigma(i)}X) \tag{37}\\
&= (-1)^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{I \subseteq \{1,\ldots,n\}} \left(\prod_{i \in I^c} \big(1 + pa_{i,\sigma(i)}\big) \right) \left(\prod_{i \in I} \big(-\delta_{i,\sigma(i)}X\big) \right) \tag{38}\\
&= (-1)^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{I \subseteq \mathrm{Fix}(\sigma)} \left(\prod_{i \in I^c} \big(1 + pa_{i,\sigma(i)}\big) \right) (-1)^{|I|} X^{|I|} \tag{39}\\
&= (-1)^n \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} X^{|I|} \left(\sum_{\substack{\sigma \in S_n \\ I \subseteq \mathrm{Fix}(\sigma)}} \mathrm{sgn}(\sigma) \prod_{i \in I^c} \big(1 + pa_{i,\sigma(i)}\big) \right). \tag{40}
\end{align}$$

Note that if we let A_I denote the $(n - |I|) \times (n - |I|)$ matrix formed by removing the ith row and column from A for each $i \in I$, then line (40) becomes

$$f(X) = (-1)^n \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} X^{|I|} \det(A_I). \tag{41}$$

This is because, given $I \subseteq \{ 1, \ldots, n \}$, applying Lemma 3.1 to A_I tells us that

$$\det(A_I) = \sum_{\sigma \in \mathrm{Sym}(I^c)} \mathrm{sgn}(\sigma) \prod_{i \in I^c} (1 + pa_{i,\sigma(i)}) \tag{42}$$

$$= \sum_{\substack{\sigma \in S_n \\ I \subseteq \mathrm{Fix}(\sigma)}} \mathrm{sgn}(\sigma) \prod_{i \in I^c} \big(1 + pa_{i,\sigma(i)}\big). \tag{43}$$

It follows that for $0 \leq i \leq n-1$, the coefficient C_i can be calculated by adding and subtracting the determinants of $(n-i) \times (n-i)$ matrices with entries in $1 + p\mathbb{Z}_p$; Lemma 4.1 implies that each of these determinants has a p-adic absolute value of at most $p^{1-(n-i)} = p^{1-n+i}$, and Theorem 2.4 then implies that

$$|C_i|_p \leq p^{1-n+i} \tag{44}$$

Suppose, additionally, that $p \nmid n$. We note by line (41) that

$$|C_{n-1}|_p = \left| \sum_{|I|=n-1} \det(A_I) \right|_p \tag{45}$$

$$= |\mathrm{tr}(A)|_p \tag{46}$$

$$= |(1 + a_{11}p) + \cdots + (1 + a_{nn}p)|_p \tag{47}$$

$$= |n + (a_{11} + \cdots + a_{nn})p|_p. \tag{48}$$

It follows from Theorem 2.4 that $|a_{11} + \cdots + a_{nn}|_p \leq 1$, and so $|(a_{11} + \cdots + a_{nn})p|_p \leq p^{-1}$. But $|n|_p = 1$ since $p \nmid n$, so by Theorem 2.5, line (48) becomes

$$|C_{n-1}|_p = \max \{ |n|_p, |(a_{11} + \cdots + a_{nn})p|_p \} = |n|_p = 1. \tag{49}$$

□

Proof of Theorem 1.3. Let the matrix A be as in the statement of Theorem 1.3. Suppose that $p \nmid n$. Our first goal is to show the existence of a strictly maximal eigenvalue λ_0 of A of multiplicity one. We reach this goal by an analysis of the characteristic polynomial $f(X)$ of A.

We first construct a partial Newton polygon for $f(X)$. Referring to Lemma 4.2, we find that $\nu_p(C_i) \geq n - (i+1)$ for $0 \leq i \leq n-2$, and $\nu_p(C_{n-1}) = 0 = \nu_p(1)$; see Figure 1.

The slope of the Newton polygon from $(n-2, \nu_p(C_{n-2}))$ to $(n-1, 0)$ must be at most $0 - 1 = -1$. The fact that the Newton polygon is the lower convex hull of a set of points in $\mathbb{R}^2$ implies that the slopes are nonincreasing from right to left, so we conclude that all slopes to the left of $(n-1, 0)$ must be at most -1. Applying Theorem 2.10 to this Newton polygon, we find that there are $n-1$ roots of $f(X)$ (in $\mathbb{C}_p$, counting multiplicities) of absolute value at most p^{-1}, and there is one root λ_0 with

$$|\lambda_0|_p = p^0 = 1. \tag{50}$$

It follows that λ_0 is a strictly maximal eigenvalue of A of multiplicity one.

Our second goal is to show that $\lambda_0 \in \mathbb{Z}_p$. This will require an application of Theorem 2.6, Hensel's lemma. In particular, we will show that $\lambda_0 \equiv n \pmod{p\mathbb{Z}_p}$. Recall we showed in

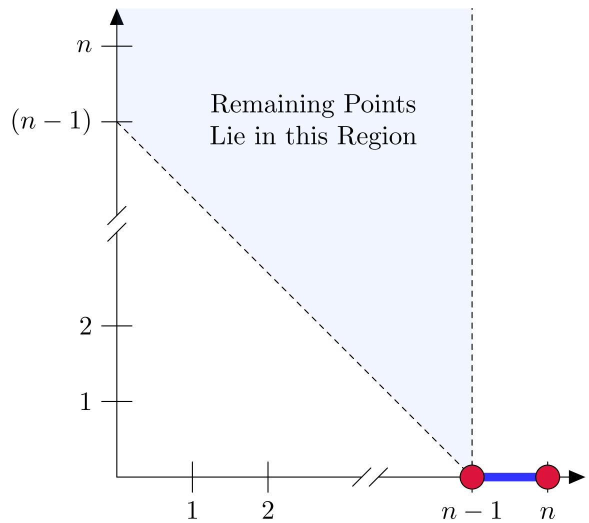

FIGURE 1. Newton Polygon for $f(X)$ in the proof of Theorem 1.3.

Lemma 4.2 that if we write the characteristic polynomial $f(X)$ of A as

$$f(X) = X^n + C_{n-1}X^{n-1} + \cdots + C_1 X + C_0, \tag{51}$$

then we know that $|C_i|_p \leq p^{1-n+i}$ for $0 \leq i \leq n-1$, and $|C_{n-1}|_p = 1$ since $p \nmid n$. Thus, $f(X) \in \mathbb{Z}_p[X]$, so $f(n)$ and $f'(n)$ are in $\mathbb{Z}_p$. Note that

$$C_{n-1} = -\operatorname{tr}(A) = -n - (a_{11} + \cdots + a_{nn})p \equiv -n \pmod{p\mathbb{Z}_p}. \tag{52}$$

Reducing $f(n)$ and $f'(n)$ modulo $p\mathbb{Z}_p$, we find that

$$\begin{aligned} f(n) &\equiv n^n - nn^{n-1} && \pmod{p\mathbb{Z}_p} && (53)\\ &\equiv 0 && \pmod{p\mathbb{Z}_p}, && (54) \end{aligned}$$

since $|C_i|_p \leq p^{-1}$ for $0 \leq i \leq n-2$, whereas

$$\begin{aligned} f'(n) &\equiv nn^{n-1} - (n^2 - n)n^{n-2} && \pmod{p\mathbb{Z}_p} && (55)\\ &\equiv n^n - n^n + n^{n-1} && \pmod{p\mathbb{Z}_p} && (56)\\ &\equiv n^{n-1} \not\equiv 0 && \pmod{p\mathbb{Z}_p}. && (57) \end{aligned}$$

We now apply Theorem 2.6 to conclude that there exists a unique p-adic integer $\alpha \in \mathbb{Z}_p$ such that $f(\alpha) = 0$ and $\alpha \equiv n \pmod{p\mathbb{Z}_p}$. Writing $\alpha = n + x$ for some $x \in p\mathbb{Z}_p$, it follows from Theorem 2.5 that $|\alpha|_p = |n|_p = 1$. But we saw in equation (50) that λ_0 is the unique root of $f(X)$ of p-adic absolute value 1, so it follows that $\lambda_0 = \alpha \in \mathbb{Z}_p$. We now have that

$$\lambda_0 \equiv n \pmod{p\mathbb{Z}_p}. \tag{58}$$

What remains is to show that there exists an eigenvector of A with eigenvalue λ_0 such that all of its components are elements of $1 + p\mathbb{Z}_p$. We can solve a linear system of equations in $\mathbb{Q}_p$ to find a nonzero eigenvector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Q}_p^n$ of λ_0. If we pick a component v_i of $\mathbf{v}$ such that $|v_i|_p = \max\{\,|v_1|_p, \ldots, |v_n|_p\,\}$, then it follows that

$$\mathbf{u} = \frac{\mathbf{v}}{v_i} \tag{59}$$

is an eigenvector of λ_0 contained in $\mathbb{Z}_p^n$. Since $u_i = 1$, we see that

$$(A\mathbf{u})_i = \lambda_0 u_i = \lambda_0 \equiv n \pmod{p\mathbb{Z}_p}. \tag{60}$$

The matrix A is equivalent to the matrix of all 1's modulo $p\mathbb{Z}_p$, so for all j with $1 \le j \le n$, calculating $(A\mathbf{u})_j$ as the scalar product of the jth row of A with $\mathbf{u}$ shows that

$$(A\mathbf{u})_j \equiv u_1 + \cdots + u_n \pmod{p\mathbb{Z}_p}. \tag{61}$$

Thus, all of the components of $(A\mathbf{u})_j$ are equivalent to each other modulo $p\mathbb{Z}_p$, and by equation (60), they are all equivalent to n modulo $p\mathbb{Z}_p$. But since $A\mathbf{u} = \lambda_0\mathbf{u}$, we also know that

$$(A\mathbf{u})_j = \lambda_0 u_j \equiv n u_j \pmod{p\mathbb{Z}_p} \tag{62}$$

for all j with $1 \le j \le n$. Then we have the system of equations:

$$n \equiv n u_1 \pmod{p\mathbb{Z}_p} \tag{63}$$
$$n \equiv n u_2 \pmod{p\mathbb{Z}_p} \tag{64}$$
$$\vdots$$
$$n \equiv n u_n \pmod{p\mathbb{Z}_p}. \tag{65}$$

Since $n \not\equiv 0 \pmod{p\mathbb{Z}_p}$, we can divide each line above through by n to see that every component u_j of $\mathbf{u}$ is equivalent to 1 modulo $p\mathbb{Z}_p$ and is hence an element of $1 + p\mathbb{Z}_p$. $\square$

Example 4.3. Here we show that the conclusions of Theorem 1.3 do not necessarily hold if we relax the conditions that: (a) all entries of A are in $1 + p\mathbb{Z}_p$, and (b) $p \nmid n$. When we applied Theorem 2.6 in the proof of Theorem 1.3 to show that the maximal eigenvalue was in $\mathbb{Z}_p$, we used the fact that $f(n) \equiv 0 \pmod{p\mathbb{Z}_p}$, but $f'(n) \not\equiv 0 \pmod{p\mathbb{Z}_p}$. It stands to reason that if we want the maximal eigenvalue to not be in $\mathbb{Z}_p$, it would be worthwhile to consider cases where $f'(n)$ is equivalent to zero modulo $p\mathbb{Z}_p$.

We consider the matrix $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}_2)$ defined by

$$A = \begin{bmatrix} 1 + 2a_{11} & 1 + 2a_{12} \\ 1 + 2a_{21} & 1 + 2a_{22} \end{bmatrix}, \tag{66}$$

where each $a_{ij} \in \mathbb{Z}_2$. The characteristic polynomial of A is

$$f(X) = X^2 - \mathrm{tr}(A)X + \det(A), \tag{67}$$

so $f'(X) = 2X - \mathrm{tr}(A)$ and $f'(2) = 4 - \mathrm{tr}(A)$. If $f'(2) = 0$, then $\mathrm{tr}(A) = 4$. Then

$$2 + 2(a_{11} + a_{22}) = 4 \tag{68}$$
$$a_{11} + a_{22} = 1. \tag{69}$$

Let $(a_{11}, a_{12}, a_{21}, a_{22}) = (1/3, 0, 0, 2/3)$, so that each $a_{ij} \in \mathbb{Z}_2$ and $a_{11} + a_{22} = 1$. Then equation (66) becomes

$$A = \begin{pmatrix} 5/3 & 1 \\ 1 & 7/3 \end{pmatrix}, \tag{70}$$

and equation (67) becomes

$$f(X) = X^2 - 4X + 26/9. \tag{71}$$

Taking the 2-adic absolute values of the coefficients of $f(X)$ in equation 71, we find that

i) $|1|_2 = 1$,

ii) $|-4|_2 = 1/4 < 1$, and

iii) $|26/9|_2 = 1/2$.

We conclude by Theorem 2.8 that $f(X)$ is irreducible over $\mathbb{Q}_2$. Thus, neither eigenvalue of $f(X)$ is in $\mathbb{Q}_2$, let alone $\mathbb{Z}_2$. If we look at the Newton polygon of $f(X)$, we see that its vertices are $(0, 1)$ and $(2, 0)$, so its eigenvalues have the same 2-adic absolute value, $2^{-1/2}$. ⧫

4.2. **Proof of Theorem 1.4.** In this subsection, we generalize Theorem 1.3 to establish Theorem 1.4. Many parts of the argument will be similar.

Lemma 4.4. *Let p be a prime and let n, ℓ be positive integers. Let A be a square $n \times n$ matrix with all entries in $1 + p^\ell \mathbb{Z}_p$; that is, let*

$$A = \begin{bmatrix} 1 + p^\ell a_{11} & 1 + p^\ell a_{12} & \dots & 1 + p^\ell a_{1n} \\ 1 + p^\ell a_{21} & 1 + p^\ell a_{22} & \dots & 1 + p^\ell a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 + p^\ell a_{n1} & 1 + p^\ell a_{n2} & \dots & 1 + p^\ell a_{nn} \end{bmatrix}. \tag{72}$$

Then $|\det(A)|_p \leq p^{\ell(1-n)}$.

Proof. The argument is as in the proof of Lemma 4.1; the only difference is that one must replace p with p^ℓ throughout. ☐

Lemma 4.5. *Let the matrix A be as in the statement of Lemma 4.4. Let the characteristic polynomial $f(X)$ of A be given as:*

$$f(X) = X^n + C_{n-1}X^{n-1} + \cdots + C_1 X + C_0. \tag{73}$$

Then $|C_i|_p \leq p^{\ell(1-n+i)}$ for $0 \leq i \leq n-2$ and if additionally $\ell > \nu_p(n)$, $|C_{n-1}|_p = |n|_p$.

Proof. The estimates for $|C_i|_p$ in the case $0 \leq i \leq n-2$ follow as in the proof of Lemma 4.2; the only difference is that one must replace p with p^ℓ throughout.

Suppose additionally that $\ell > \nu_p(n)$, so $|n|_p > |p^\ell|_p \geq |p^\ell(a_{11} + \cdots + a_{nn})|_p$. We estimate:

$$\begin{aligned} |C_{n-1}|_p &= |\mathrm{tr}(A)|_p && (74) \\ &= |(1 + p^\ell a_{11}) + \cdots + (1 + p^\ell a_{nn})|_p && (75) \\ &= |n + p^\ell(a_{11} + \cdots + a_{nn})|_p && (76) \\ &= |n|_p. && (77) \end{aligned}$$

$\square$

Proof of Theorem 1.4. Let the matrix A be as in the statement of Theorem 1.4. Suppose $n > 1$, for otherwise this discussion is trivial. Suppose that $\ell > 2(n-1)\nu_p(n)$. Our first goal is to show the existence of a strictly maximal eigenvalue λ_0 of A of multiplicity one. This requires a careful analysis of the characteristic polynomial $f(X)$ of A.

From Lemma 4.5, we have that $\nu_p(C_i) \geq \ell(n-i-1)$ for $0 \leq i \leq n-2$, and $\nu_p(C_{n-1}) = \nu_p(n)$. We note that since n is a positive integer larger than 1, the hypotheses imply that $\ell > 2\nu_p(n)$, so $\nu_p(C_{n-1}) < \ell/2$. We construct a partial Newton polygon for $f(X)$; see Figure 2.

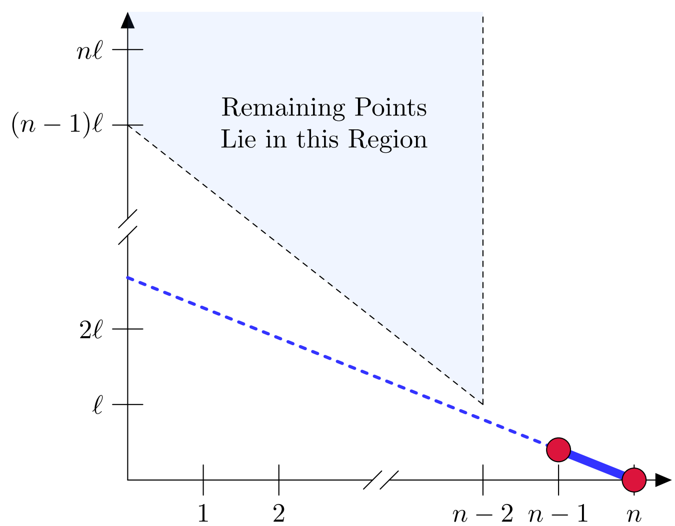

FIGURE 2. Newton Polygon for $f(X)$ in the proof of Theorem 1.4.

We note that the line segment connecting $(n-1, \nu_p(C_{n-1}))$ with $(n, 0)$ has the slope

$$\frac{0 - \nu_p(C_{n-1})}{n-(n-1)} = -\nu_p(C_{n-1}) \tag{78}$$

$$> -\ell/2. \tag{79}$$

We now consider extending the line segment through $(n-1, \nu_p(C_{n-1}))$ and $(n, 0)$ to the left until it intersects the y axis. For $0 \leq i \leq n-2$, we bound the y-coordinate of the point on the line segment extension with x-coordinate i:

$$\begin{aligned}
\nu_p(C_{n-1})(n-i) &< (\ell/2)(n-i) && (80)\\
&= (\ell/2)(n-i-1) + \ell/2 && (81)\\
&\leq \nu_p(C_i)/2 + \ell/2 && (82)\\
&\leq \nu_p(C_i)/2 + \nu_p(C_i)/2 && (83)\\
&= \nu_p(C_i). && (84)
\end{aligned}$$

The calculation above shows that this line segment extension will lie strictly below the remaining points of the Newton polygon. We note that the slopes of the segments of the Newton polygon to the left of the point $(n-1, \nu_p(C_{p-1}))$ are bounded above by the slope of the line segment through $(n-2, \ell)$ and $(n-1, \nu_p(C_{n-1}))$. We calculate:

$$\frac{\nu_p(C_{n-1}) - \ell}{(n-1)-(n-2)} < \ell/2 - \ell \tag{85}$$
$$= -\ell/2. \tag{86}$$

Applying Theorem 2.10 to this Newton polygon, we find that there are $n-1$ roots of $f(X)$ (in $\mathbb{C}_p$, counting multiplicities) of absolute value strictly less than $p^{-\ell/2}$. We will denote these roots by $\lambda_1, \lambda_2, \ldots, \lambda_m$. Furthermore, there is one root, denoted λ_0, of absolute value strictly greater than $p^{-\ell/2}$. This root is a strictly maximal eigenvalue of A of multiplicity one.

Now that we have established the existence of a maximal eigenvalue λ_0, we will show that λ_0 is a p-adic integer. This will require an application of Theorem 2.7, the strong version of Hensel's lemma. We calculate:

$$\begin{aligned} f(n) &= n^n + C_{n-1}n^{n-1} + C_{n-2}n^{n-2} + \cdots + C_1 n + C_0 && (87)\\ &= n^n - (n + p^\ell(a_{11} + \cdots + a_{nn}))n^{n-1} + C_{n-2}n^{n-2} + \cdots + C_1 n + C_0 && (88)\\ &= -p^\ell(a_{11} + \cdots + a_{nn})n^{n-1} + C_{n-2}n^{n-2} + \cdots + C_1 n + C_0. && (89) \end{aligned}$$

Since $|C_i|_p \leq p^{-\ell}$ for $0 \leq i \leq n-2$, we have the bound $|f(n)|_p \leq p^{-\ell}$. We calculate:

$$\begin{aligned} f'(n) &= nn^{n-1} + (n-1)C_{n-1}n^{n-2} + (n-2)C_{n-2}n^{n-3} + \cdots + C_1 && (90)\\ &= n^n - (n-1)(n + p^\ell(a_{11} + \cdots + a_{nn}))n^{n-2} + (n-2)C_{n-2}n^{n-3} + \cdots + C_1 && (91)\\ &= n^n + (n^{n-2} - n^{n-1})(n + p^\ell(a_{11} + \cdots + a_{nn})) + (n-2)C_{n-2}n^{n-3} + \cdots + C_1 && (92)\\ &= n^{n-1} + p^\ell(n^{n-2} - n^{n-1})(a_{11} + \cdots + a_{nn}) + (n-2)C_{n-2}n^{n-3} + \cdots + C_1. && (93) \end{aligned}$$

Since $\ell > (n-1)\nu_p(n)$, so that $p^{-\ell} < |n^{n-1}|_p$, and since $|C_i|_p \leq p^{-\ell}$ for $0 \leq i \leq n-2$, we have the equality $|f'(n)|_p = |n^{n-1}|_p$. Finally, since $\ell > 2(n-1)\nu_p(n)$, we have the bound:

$$\begin{aligned} |f(n)|_p &\leq p^{-\ell} && (94)\\ &< p^{-2(n-1)\nu_p(n)} && (95)\\ &= |n^{2(n-1)}|_p && (96)\\ &= |f'(n)|_p^2. && (97) \end{aligned}$$

We now apply Theorem 2.7 to conclude that there exists a unique p-adic integer α so that $f(\alpha) = 0$ satisfying

$$
\begin{aligned}
|\alpha - n|_p &\leq \frac{|f(n)|_p}{|f'(n)|_p} && (98)\\
&\leq \frac{p^{-\ell}}{|n^{n-1}|_p} && (99)\\
&< \frac{p^{-\ell}}{p^{-\ell/2}} && (100)\\
&= p^{-\ell/2}. && (101)
\end{aligned}
$$

We will show that the none of the nonmaximal eigenvalues are candidates for the root α found by Hensel's lemma. Let λ_i, $1 \leq i \leq m$ be a nonmaximal eigenvalue of A. Since $|\lambda_i|_p < p^{-\ell/2}$ and $|n|_p > p^{-\ell/2}$, we have:

$$|\lambda_i - n|_p > p^{-\ell/2}. \tag{102}$$

Hence, the nonmaximal eigenvalues are not α. Thus $\lambda_0 = \alpha$. In particular, λ_0 is a p-adic integer. From (99), we obtain $|\lambda_0 - n|_p \leq p^{-\ell}/|n^{n-1}|_p$.

From the Newton polygon in Figure 2 and the estimate for $|C_{n-1}|_p$ from Lemma 4.5, we calculate:

$$
\begin{aligned}
|\lambda_0|_p &= |C_{n-1}|_p && (103)\\
&= |n|_p. && (104)
\end{aligned}
$$

$\square$

4.3. **Proof of Theorem 1.5.** In this subsection, we establish Theorem 1.5. Out argument will make use of the Jordan canonical form of $n \times n$ matrices with a strictly dominating eigenvalue of multiplicity one.

Proof of Theorem 1.5. Let A be an $n \times n$ matrix with an eigenvalue λ_0 of multiplicity one such that $|\lambda|_p < |\lambda_0|_p$ for all other eigenvalues λ of A. We write the Jordan canonical form of A in block form:

$$A = Q^{-1} \begin{bmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & J_{m_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m_j} \end{bmatrix} Q, \tag{105}$$

where the J_{m_i} are $m_i \times m_i$ Jordan blocks associated to the nonmaximal eigenvalues of A. We will denote the nonmaximal eigenvalues, by $\lambda_1, \lambda_2, \dots, \lambda_j$; note, however, that the eigenvalues in this list are not necessarily distinct.We write out each matrix J_{m_i} as:

$$J_{m_i} = \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}. \tag{106}$$

We calculate the kth power of these Jordan blocks:

$$J_{m_i}^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \dots & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} & \binom{k}{m_i-1}\lambda_i^{k-(m_i-1)} \\ 0 & \lambda_i^k & \dots & \binom{k}{m_i-3}\lambda_i^{k-(m_i-3)} & \binom{k}{m_i-2}\lambda_i^{k-(m_i-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & 0 & \dots & 0 & \lambda_i^k \end{bmatrix}. \tag{107}$$

For a positive integer k, the kth power of A can be calculated as

$$A^k = Q^{-1} \begin{bmatrix} \lambda_0^k & 0 & \dots & 0 \\ 0 & J_{m_1}^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m_j}^k \end{bmatrix} Q. \tag{108}$$

We will now divide through by λ_0^k to obtain:

$$(A/\lambda_0)^k = Q^{-1} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & J_{m_1}^k/\lambda_0^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m_j}^k/\lambda_0^k \end{bmatrix} Q. \tag{109}$$

We compute each power $J_{m_i}^k/\lambda_0^k$:

$$J_{m_i}^k/\lambda_0^k = \begin{bmatrix} \left(\frac{\lambda_i}{\lambda_0}\right)^k & \binom{k}{1}\left(\frac{\lambda_i}{\lambda_0}\right)^{k-1}\left(\frac{1}{\lambda_0}\right) & \dots & \binom{k}{m_i-1}\left(\frac{\lambda_i}{\lambda_0}\right)^{k-(m_i-1)}\left(\frac{1}{\lambda_0}\right)^{m_i-1} \\ 0 & \left(\frac{\lambda_i}{\lambda_0}\right)^k & \dots & \binom{k}{m_i-2}\left(\frac{\lambda_i}{\lambda_0}\right)^{k-(m_i-2)}\left(\frac{1}{\lambda_0}\right)^{m_i-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{k}{1}\left(\frac{\lambda_i}{\lambda_0}\right)^{k-1}\left(\frac{1}{\lambda_0}\right) \\ 0 & 0 & \dots & \left(\frac{\lambda_i}{\lambda_0}\right)^k \end{bmatrix}. \tag{110}$$

Noting that each $|\lambda_i|_p < |\lambda_0|_p$ for $1 \leq i \leq j$, we see that each block $J_{m_i}^k/\lambda_0^k$ vanishes as k tends towards infinity. Thus as as k tends towards infinity, the sequence $(A/\lambda_0)^k$ converges to

$$P = Q^{-1} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} Q. \tag{111}$$

Since Q and Q^{-1} induce a change of basis from the standard basis of $\mathbb{Q}_p^n$ to the basis of generalized eigenvectors of A, and since the first column of Q^{-1} is the eigenvector $\mathbf{v}_0$ associated to the maximal eigenvalue λ_0, P is the projection onto the eigenspace spanned by $\mathbf{v}_0$. □

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