Embedded Width, A Variation of Treewidth for Planar Graphs

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Abstract

A commonly studied property of graphs is treewidth, a measure of how "tree-like" a graph is. Treewidth has been used to develop approximation algorithms for NP-hard problems and in the development of theoretical results. We have developed a new, related parameter, embedded-width (abbreviated em-width) which we introduce here.

Em-width is a variation on treewidth, which is restricted to surface-embedded graphs. Unlike treewidth, em-width respects the particular way a graph is embedded, allowing it to provide information on how a graph is drawn.

Here, we restrict our attention to graphs embedded in the plane. We examine multiple aspects of em-width on planar graphs, including classes of graphs for which em-width can be easily calculated exactly, efficient methods to calculate upper bounds, and relationships between em-width and treewidth.

1   Introduction

Treewidth is a graph parameter introduced by Robertson and Seymour [9], measuring how "tree-like" a graph is. Treewidth is defined using tree-decompositions.

Definition 1. *A **tree-decomposition** of a graph $G = (V, E)$ is a tuple $(T = (I, F), \{X_i | i \in I\})$ where T is a tree and $X_i \subseteq V, \forall i$ s.t.*

1. *$\bigcup X_i = V$*

2. *$\forall (u, v) \in E, \exists i \in I$ s.t. $u \in X_i, v \in X_i$.*

3. *for each $v \in V$, $\{X_i | v \in X_i\}$ form a connected subgraph in T*

*The X_i are called "**bags**". The **width** of a decomposition is one less than the size of the largest bag, and the **treewidth** of a graph is the minimum width across all decompositions.*

Treewidth has proven useful both in the development of theoretical results (e.g. The Graph Minor Theorem), and in the design of algorithms on NP-hard problems (see [2] for an overview). When we restrict our attention to planar graphs, we may wish to use an embedding for the purposes of developing algortihms or theoretical results, however a tree-decomposition is not required to carry any information about an embedding. This motivates us to define a new parameter "embedded-width" As with treewidth, we will require a decomposition:

Definition 2. *An **embedded-tree-decomposition** of an embedding of a graph $G = (V, E)$ is a tuple $(T = (I, F), \{X_i | i \in I\})$ where T is a tree, and $X_i \subseteq V, \forall i$ s.t.*

1. *$\bigcup X_i = V$*

2. *$\forall (u, v) \in E, \exists i \in I$ s.t. $u \in X_i, v \in X_i$.*

3. *for each $v \in V$, $\{X_i | v \in X_i\}$ form a connected subgraph in T*

4. *for each face $f = u_1, u_2, ..., u_k$, except the outer face, $\exists i \in I$ s.t. $u_1 \in X_i, u_2 \in X_i, ..., u_k \in X_i$.*

*The **width** of an em-tree-decomposition is one less than the size of the largest bag. The **embedded-width** of an embedding of a graph ($emw(G)$) is the minimum width of an em-tree-decomposition.*

The remainder of this document is organized as follows. Section 2, we discuss the em-width of certain graph classes. Section 3 contains a method for calculating (not necessarily optimal) em-decompositions of planar graphs. Finally Sections 4 and 5 explore the connections between treewidth and em-width by taking two well-known treewidth results and finding similar results for em-width.

2 Em-width on Specific Graph Classes

2.1 Preliminaries

For this section, we will require some additional definitions:

Definition 3 (wikipedia). *The **weak dual** of an embedding G is a graph, $G^+ = (V^+, E^+)$, where V^+ is the set of interior faces of G and two faces are adjacent iff they share an edge.*

Note that we can draw G^+ by putting vertices in the faces of an embedding of G and drawing edges of G^+ through edges of G.

Definition 4 (West [10]). *A **block** is a maximal subgraph with no cut vertex.*

In a connected graph, blocks are maximal 2-connected components and cut-edges.

Definition 5 (West [10]). *The **block-cutpoint graph** of $G = (V, E)$ is a graph, $BC(G) = (V \cup B, F)$ where B is the set of blocks of G, and $v \in V$ is adjacent to $b \in B$ iff $v \in B$. $BC(G)$ is a bipartite graph (with partite sets V, B).*

2.2   Outerplanar Graphs

Let ℓ be the size of the largest interior face of an embedding G. Then we have the following:

Lemma 2.1. $emw(G) \geq \ell - 1$

Proof. To satisfy (4), any em-tree-decomposition must contain a node of size at least ℓ, so $\text{emw}(G) \geq \ell - 1$. ◻

Thus, a decomposition of width $\ell - 1$ is always optimal. We give a method for constructing such a decomposition on outerplanar graphs, using the weak dual of G and its block-cutpoint graph. We begin with some lemmas about these objects:

Lemma 2.2. *if G is outerplanar then G^+ is a forest.*

Proof. It is ETS G^+ is acyclic. Suppose not, suppose there is a cycle $C^+ = f_1, f_2, ..., f_k$. Then consider an embedding of G, and draw C^+ in that embedding. Let u be a vertex of G interior to C^+ (note that u must exist – each edge of C^+ goes through an edge of G, so one must be inside and the other outside). Every edge of the cycle in G^+ goes through an edge in G, let $v_1, ..., v_k$ be the endpoints of these edges outside of C^+. v_i and v_{i+1} are part of the same face, thus there is a path from v_i to v_{i+1} without using vertices on the interior of C^+. This gives a cycle in G, which is always outside of C^+, thus u is on the interior of G, but G was an outerplanar embedding. ↯ ◻

Lemma 2.3 (West [10]). *$BC(G)$ is acyclic*

Proof. Suppose not, then there is a cycle $v_1, B_1, ..., v_k, B_k, v_1$ Let $H = G[\cup B_i]$, and consider removing some vertex u from H. $H \backslash u$ is still connected — the B_i are connected subgraphs, so there is still a path through each, and since the B_i form a cycle, even if one of the v_i is removed, there is still a path from B_j to B_k avoiding that v_i, but then H is a graph without a cut-vertex, so it is a single block. ⚡ ☐

Let $B_1, B_2, ..., B_k$ be the blocks of G. Then let

$$C_i = \begin{cases} \text{an isolated vertex} & \text{if } B_i \text{ is an edge} \\ B_i^+ & \text{otherwise} \end{cases}$$

Based on these ideas, we can design an algorithm to find an em-tree-decomposition of an outerplanar graph.

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function EMTD-OUTERPLANAR$(G = (V, E))$
    Let T be an empty em-tree-decomposition.
    Let $C_1, ..., C_k$ be as defined above.
    for each $C_i, i \in [k]$ **do**
        Create a tree T_{C_i} in T isomorphic to C_i
        perform BFS on C_i, for each vertex in $C_i = u_1, ..., u_m$ make the corresponding node
in $T_{C_i} = u_1, ..., u_m$.
    end for
    Construct $BC(G)$
    for each cut-vertex, u of G, add a node in T containing only u
    for each edge u, B of $BC(G)$ **do**
        Let C_i be the component corresponding to B
        find a node containing u in C_i
        add an edge in T from u to C_i
    end for
end function

We claim that T is an em-tree-decomposition. First note that T is indeed a tree. Since each C_i is acyclic (lemma 2.2), each T_{C_i} is a tree. It is ETS that we do not create a cycle when combining these components. By lemma 2.3, there are no cycles in $BC(G)$ — the edges we add in the final loop correspond to edges in $BC(G)$, so they cannot be used to create cycles.

We now show T meets all 4 conditions of an em-tree-decomposition.

1. Every vertex is part of a block. If the block is an edge, then that edge is a node in T, otherwise, the vertex is part of some face, which is a node of T.

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2. Similar to 1

3. We first show this holds within each C_i. Suppose not, then let v be a vertex for which this does not hold, with $f_1, f_2, ..., f_k$ being the faces along a path, P s.t. $v \in f_1, f_k$, $v \notin f_i, \forall 1 < i < k$. Consider the Jordan curve connecting the f_i in G^+ and the vertex v. Since each edge of P goes through an edge of G, we see that some vertices of G are interior to this curve. Let v_i be the vertices on faces f_i outside the curve and the vertex v. These vertices will form a cycle, but the vertices on the interior cannot be on the outerface of G, but G was an outerplanar embedding ⚡

   This also holds between the C_i — every vertex which appears in more than one block is connected through the nodes forming the block-cutpoint graph.

4. Every interior face of G is considered by the algorithm and placed in a bag of T (since every interior face is a vertex in G^+)

Theorem 2.1. *Let G be an outerplanar embedding and let ℓ be the length of the longest interior face of G. Then $emw(G) = \ell - 1$.*

Proof. By lemma 2.1, $emw(G) \geq \ell - 1$. The algorithm produces a witness decomposition of that width. $\square$

Remark 2.1. *An optimal decomposition of an outerplanar embedding can be found in $O(n)$ time.*

Proof. Finding the bi-connected components of a graph can be done in linear time [5]. Once these are found, the block-cutpoint graph can easily be constructed with another traversal of the graph. Similarly, all the other steps of the algorithm are constant time, or can be done by traversing the graph. (The final for-loop must be done by traversing each bi-connected component once, checking for all required cut-vertices simultaneously) Since G is planar, m is $O(n)$, and so the whole algorithm is $O(n)$. $\square$

2.3 Maximal Planar Graphs

First we show that em-width and treewidth match on maximal planar graphs. We'll require the following lemma:

Lemma 2.4. *Let $T = (V, E)$ be a tree with pairwise intersecting subgraphs $T_1, ..., T_m$ then $\exists v \in V$ s.t. $v \in T_i$ for $i \in \{1, ..., m\}$.*

Proof. We show this inductively on $n = |V|$
Base Case $n = 1$: Then the T_i are just the single vertex, so that vertex is in each subtree.

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Inductive Case $n > 1$:

Case 1: There is a T_i which is a single vertex, x. Then this T_i intersects all other T_j, so x is the desired common vertex.

Case 2: All T_i contain at least two vertices: Let u be a leaf of T, with neighbor v. Consider $T' = T \backslash u$ and $T_i' = T_i \backslash u$. The T_i' still pairwise intersect — since each T_i has at least two vertices and is connected, any T_i containing u also contained v, and so any T_i which intersected at u still have the corresponding T_i' intersect at v. Then by IH on T', T_i' there is some w which is part of each T_i'. This w is also in the T_i (since $T_i' \subseteq T_i$), so it is the desired common vertex. □

We are now ready to prove the main result for this subsection.

Theorem 2.2. *Let G be maximal planar, then $tw(G) = emw(G)$.*

Proof. Let T be a tree-decomposition of width $tw(G)$. We show this tree-decomposition is also an em-decomposition. It is ETS (4), that every interior face is contained in some bag. Consider u, v, w, an arbitrary face of G (recall that G is maximal planar, so all its faces are triangles). Let T_u, T_v, T_w be the subgraphs of T where the nodes contain u, v, w respectively. Since $(u, v), (v, w), (u, w)$ are edges, these subgraphs must pairwise intersect (T must have each edge in a bag). Then by Lemma 2.4, there is a node common to all of T_u, T_v, T_w, that is a node which contains the boundary of the face. Thus T is also an em-decomposition of G. □

3   Constructing Em-Decompositions in General

Since the definition of em-decompositions requires each face to have a bag containing its entire boundary, the weak dual graph intuitively appears as though it would be very useful. Based on that intuition, consider the following algorithm (which we will call the "dual-tree method"):

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function EM-DECOMPOSITION-CONSTRUCTOR$(G = (V, E))$
    Let G^+ be the weak dual of G
    Find T^+, a spanning tree of G^+
    Make a decomposition $D = (T^+, \{X_i\})$ where each X_i contains the boundary of face i of G
    Add vertices to bags to meet condition (3).
end function

One might hope that for some T^+, the em-decomposition constructed will always be optimal. Unfortunately, this is not the case. Indeed, even if we restrict ourselves to maximal planar graphs, this method is not sufficient. The rest of this section demonstrates a counter-example.

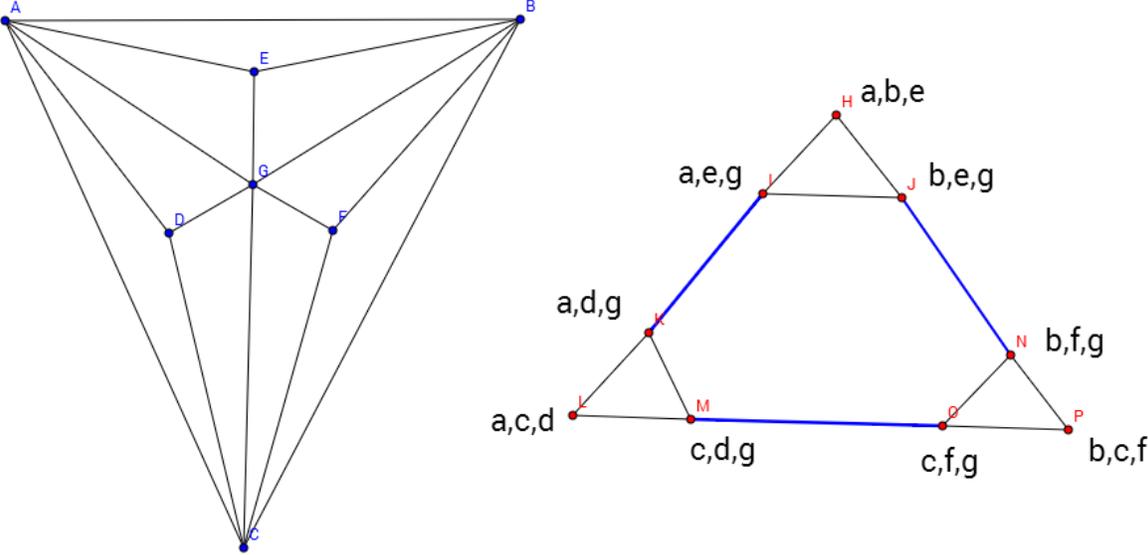Consider the graph G, and its weak dual G^+. Shown in Figure 1.

Figure 1: G (left) and G^+ (right) the counterexample for this section

Some notation: Note that there are three three K_3 in G^+, one in each corner of the drawing. We will call the edges not part of these K_3 (i.e. the blue edges) "connecting" as they connect the three K_3 together. We will call the degree two nodes "corner nodes." Finally, for a node u, let its bag be X_u.

G is an Apollonian network, and therefore has treewidth 3. It is maximal planar, so by Theorem 2.2, it has em-width 3. Thus it is ETS that any decomposition from a spanning tree will have width at least 4.

Lemma 3.1. *In a width 3 spanning tree of G^+, each K_3 must have two edges selected.*

Proof. Suppose not, and let y be the corner node of the K_3 with less than two edges selected in the spanning tree T. Then since T spans, y must be adjacent to something in G^+. The only option is another node in the K_3, u. Then (y, u) is the only edge of the K_3 in T. Consider v, the other node of the K_3. $|X_y \cap X_v| = 2$ — the intersection is whichever of $\{d, e, f\}$ defines the K_3 and one of $\{a, b, c\}$. Consider the path in T from y to v. This path must go through the spanning edge for the two other K_3, let w be an endpoint of this edge. $X_w \cap (X_y \cap X_v) = \varnothing$ — X_w cannot contain the common $\{d, e, f\}$ vertex (since those vertices

are exclusive to that K_3) and it cannot contain the appropriate $\{a, b, c\}$ (the nodes which match y in any of $\{a, b, c\}$ form a path, which misses exactly those two nodes). Therefore, to preserve property (3) in T, $|X_w| \geq 5$ so the width of T is at least 4 ⨯ $\square$

Lemma 3.2. *Of the three spanning edges, exactly two must be selected in a width 3 spanning tree of G^+.*

Proof. Suppose we want to create T, a treewidth 3 spanning tree of G^+. G^+ has 9 nodes, so we must select 8 edges. By Lemma 3.1, 6 edges from among the K_3 will be selected. This leaves exactly 2 from among the remaining (i.e. the spanning) edges. $\square$

Theorem 3.1. *No spanning tree of G^+ turned into an em-decomposition is width 3, therefore even for maximal planar graphs, finding the best spanning tree is not sufficient to compute em-width.*

Proof. By Lemma 3.2, we have two spanning edges. There is one K_3 which is incident to both of these edges. G^+ is symmetric w.r.t. which K_3 this is. WOLOG let this be the $K_3 : \{a, c, d\}, \{a, d, g\}, \{c, d, g\}$. We will attempt to select edges in this K_3 s.t. the width is 3. By Lemma 3.1, we must select two edges. Let y be the central node on the P_2 selected. Note that within the K_3, each pair of nodes contains a common vertex that the other does not. Thus y must have this vertex added to it. In the two other K_3, there are nodes with b in common. This must be added along the path between them. This path goes through y, thus y now has 5 vertices in it, and T has width at least 4. $\square$

In this example the true em-width, and the width of the embedding produced by the dual tree differ by only 1. We conjecture that they can be much farther apart in general. The dual-tree method was sufficient for (2-connected) outerplanar graphs (see Section 2.2). We do not know for what other graph classes (if any) this method suffices.

4 Connections to Treewidth - Radius

We will require some additional definitions for this section.

Definition 6. *The **eccentricity** of a vertex, u, is the maximum distance from u to another vertex.*

Definition 7. *The **diameter** is the maximum eccentricity of a vertex.*

Definition 8. *The **radius** of a graph is the minimum eccentricity of a vertex.*

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Our notion of radius should not be confused with another "radius" which is related to outerplanarity (see e.g. [8])

A lemma of Eppstein says the following:

Theorem 4.1 (Eppstein [4]). *Let G be a planar graph with diameter D. Then $tw(G) \leq 3D$ and we can construct a tree-decomposition of this width in $O(n)$ time.*

We adapt his technique to obtain a similar result for em-width:

Theorem 4.2. *Let G be a planar graph with radius r and longest finite face ℓ. Then $emw(G) \leq r\ell$.*

Proof. Let G be a planar-embedded graph of radius r, and let u be a vertex with eccentricity equal to r. Create a new graph G', by triangulating the outer face of G. Create a BFS tree of G', rooted at u (call this tree T). We will create an em-tree-decomposition D as follows: Create a node for each face of G'. Connect these nodes using the interdigitating tree of $(G')^*$ (call this tree T^*). For each face, f with vertices $v_1, v_2, ..., v_k$, have the bag for f (X_f) be the union of the vertices on the paths from v_i to u in T. We claim this is a valid em-decomposition of G'. First note that each face is included in a bag (and thus each edge and vertex). The only remaining requirement is that the nodes containing an arbitrary vertex v induce a connected subgraph. Let v have parent w in the BFS tree, and faces f_1, f_2 contain the edge (v, w). Consider the path in T^* from f_1 to f_2. Create a Jordan curve connecting this path to w. Let Y be the set of all faces inside this Jordan curve. We note that this set is connected in T^* (no edge of T^* can cross this curve, and T^* is connected) Further, any face which has v in its bag is inside Y. To contain v, there must be a path in T (not using (v, w)) to v. But T and T^* are interdigitating, so no edge of T (other than (v, w)) can pass through the Jordan curve, so all bags containing v are in Y, and thus connected in the decomposition. The only vertex this argument does not apply to is the root u, but u is in every bag. Thus D is a valid decomposition of G', but since every edge and interior face in G is also in G', we have that this is a valid decomposition of G. Let ℓ be the length of the longest finite face of G. Note that ℓ is also the length of the shortest face of G'. (Since every interior face in G' is a face of G, and every face of G' that is not in G is the minimum length 3.) If r' is the radius of G', then the largest a bag of the decomposition could be is $\ell \cdot r' + 1$ (paths of length r' for each of the ℓ vertices, all of which end at u). Thus the width of the decomposition is at most $\ell \cdot r'$. Noting that $r' \leq r$ and $emw(G) \leq emw(G')$, we have $emw(G) \leq emw(G') \leq r'\ell \leq r\ell$, so $emw(G) \leq r\ell$. $\square$

Remark 4.1. *This decomposition can be constructed in $O(n^2)$ time.*

From Eppstein, every step except the selection of u can be done in linear time. To select u, we can run BFS from each of the n vertices, and select one whose BFS tree has the lowest

height. This takes $O(n^2)$. $O(n)$ time can be acheived by using an arbitrary vertex for u, at the cost of the width being up to $\ell \cdot d$ (where d is the diameter of G) instead of $\ell \cdot r$.

Remark 4.2. *The $r\ell$ bound is tight for wheel graphs (in the usual embedding).*

Wheel graphs have em-width 3. Let u be the hub vertex of W_n. Then we can find a tree-decomposition of $W_n \backslash u$ of width 2 (Since $W_n \backslash u$ is a cycle). Add u to every node of this decomposition. For every face u, x, y (x, y) was an edge of $W_n \backslash u$, so the face u, x, y is included in a bag. Every edge incident to u is also clearly included. Thus this is an em-decomposition of width 3. We cannot have an em-decomposition of width 2, since $\text{tw}(W_n) = 3$.
A wheel has radius 1 (the hub vertex is distance 1 from every node), and in its usual embedding, it has each face of length three, so $r\ell = 1 \cdot 3 = 3$. So the bound is indeed tight.

5 Connections to Treewidth - k-outerplanarity

5.1 Definitions

Outerplanarity index was introduced by Baker [1]. An outerplanar embedding is 1-outerplanar. For $k > 1$, an embedding is k-outerplanar, if removing the vertices on the outerface of a graph leaves a $(k-1)$-outerplanar embedding.

Let T be the edges of a spanning tree of G. The fundamental cycle of an edge $e \in E \backslash T$ is the unique cycle in $T \cup e$.

The dual of a planar embedding of a graph, G, is a planar, embedded graph, G^*, which has vertices for every face of G. Two faces are adjacent iff they share an edge, and every edge $e^* \in G^*$ crosses that shared edge $e \in G$.

The interdigitating tree, T^*, is the subgraph of G^* where for an edge e^*, e is in $G \backslash T$.

Note that if T is a spanning tree, then the interdigitating tree is always also a spanning subgraph of G^* and no edges of T and T^* cross.

We will always choose the infinite face as the root of T^*. The height of the interdigitating tree $(\text{h}(T^*))$ is the length of the longest path from the vertex which represents the infinite face of G, to any other vertex in T^*.

We will require the concept of remember numbers (which were introduced by Bodlaender [3]).

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Definition 9. *The **edge remember number** of G with respect to T (denoted $er(G,T)$) is the maximum over all $e \in F$ of the number of times that e appears in a fundamental cycle.*

Definition 10. *The **vertex remember number** of G with respect to T (denoted $vr(G,T)$) is the the maximum over all $u \in V$ of the number of times that u appears in a fundamental cycle.*

We will note one more property of the interdigitating tree. By Euler's formula, one can see that the number of edges in $G \backslash T$ and the number of finite faces of G are equal. The interdigitating tree gives us a way to form a one-to-one correspondence between these sets:

Definition 11. *Let f be a finite face of G. Consider the edge of T^* between f and its parent. This edge crosses some edge $e \in G \backslash T$. We will say that f **is associated with** e.*

Similarly, we can associate f with the fundamental cycle of its associated edge e.

Finally, two pieces of notation: The degree of a highest degree vertex will be denoted Δ_G. The length of the longest finite face of G will be denoted by ℓ_G. We will drop the subscripts when the graph is clear from context. In this section, we will assume that all graphs are simple and connected (for disconnected graphs, one can apply this construction on each component).

5.2   Proof of Result

Bodlaender showed the following relationship between treewidth and outerplanarity:

Theorem 5.1 (Bodlaender [3]). *If G is a k-outerplanar graph, then $tw(G) \leq 3k - 1$.*

We will show the following:

Theorem 5.2. *If G is a k-outerplanar embedding of a graph ($k \geq 2$), with $\Delta_G \geq 3$, then $emw(G) \leq (3k - 1)(\Delta_G - 2)\ell_G - 6k + 2$.*

The overall arc of the proof follows that of Bodlaender. We begin by constructing a new graph H, which has maximum degree 3. We find a special spanning tree of H, and use this tree as the structure of a decomposition.

5.2.1 Expansions

Lemma 5.1 (Bodlaender [3]). *Let G be a k-outerplanar graph. We can construct a new graph H, such that G is a minor of H, H is k-outerplanar, and $\Delta_H \leq 3$.*

We will call H an "expansion" of G. The basic construction is to replace every vertex of degree $d \geq 4$ with a path on $d - 2$ vertices. We refer the reader to [7] for a more detailed description of this process.
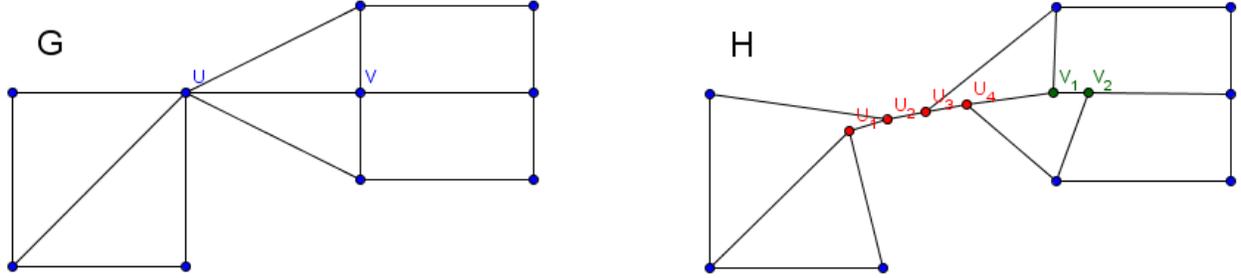

Figure 2: An example of the expansion process on u and v

We will apply the same construction, but will note two additional properties of H:

Lemma 5.2. *Let $\Delta_G \geq 3$ and let H be an expansion of G, then $\ell_H \leq (\Delta_G - 2)(\ell_G)$*

Proof. Consider any finite face f_H of H. By construction, f_H was originally some face in G, f_G. By definition, there are at most ℓ_G vertices in the boundary of f_G. During the construction of H, any vertex, u, in f_G, of degree at least 4 were expanded into a path on $\deg(u) - 2$ vertices, so f_H has at most $(\Delta_G - 2)(\ell_G)$ vertices. $\qquad\square$

Lemma 5.3. *If G can be obtained from H by a series of edge contractions, then $emw(G) \leq emw(H)$.*

Proof. Consider $D = (T, \{X_i\})$, an em-decomposition of H. Let $\{u_1, ..., u_m\}$ be the set of vertices in H which were contracted to the vertex u in G. Construct $D' = (T, \{X_i'\})$, where X_i' is X_i with every u_j ($1 \leq j \leq m$) replaced with u (for every vertex u). We show D' is an em-decomposition of G. Every face/vertex/edge of G has a corresponding face/vertex/edge in H (possibly with uncontracted versions of the vertices) so when the u_j are replaced with u, all these conditions are met. Only the connectivity requirement remains: Note that in H there was a path between any $u_j, u_{j'}$ using only the u_i vertices. Thus the union of the connected components for all the u_i is contected, and so the nodes containing u in D' will be connected as well. This means that D' is a valid em-decomposition. The width of D' is at most the width of D, so $emw(G) \leq emw(H)$.

$\qquad\square$

5.2.2 Spanning Tree

Our goal will be to find a spanning tree of H (the expansion of G) from which we can create an em-decomposition of bounded width.

The tree we construct is actually the same as Bodlaender's, but we will not use vertex and edge remember numbers to explicitly guide the construction. Let H be a k-outerplanar graph with $\Delta \leq 3$. Consider the following process to construct a spanning tree T.

Let $H = H_k$. From H_i, remove the edges (but not the vertices) of the outerface. Let H_{i-1} be the union of the nontrivial components of the remaining graph. Note that H_j is j-outerplanar for every j (the vertices on the outerface of H_{j+1} will be degree 0 or 1 in H_j, and outside any cycles in H_j so the graph has the same outerplanarity with or without those vertices, but H_j without those vertices is H_{j+1} with the vertices of the outerface removed, so by definition the outerplanarity has decreased by exactly 1). In H_1 find a spanning tree, T_1, by iteratively removing edges which are part of the outerface of H_1 and which are part of some cycle. (Note that since $\Delta_H \leq 3$, no vertex is incident to more than one edge inside the Jordan curve created by the boundary of the outerface, so no cycles can be formed without at least one edge on the outerface). Iteratively construct T_{i+1} to be a spanning tree of H_{i+1} such that $T_i \subseteq T_{i+1}$. $T_k = T$ is the desired spanning tree.

We will note some useful properties of T.

First we will require a notion of a level of a face. We will say a face is at level i if the first time an edge was removed from it in the above process was in the ith iteration

Let M be the set of edges whose incident faces are at the same level.

Lemma 5.4. *M is a subset of the edges of T.*

Proof. Consider some edge, e, incident to two faces at level i. Since e is incident to faces at level i it cannot be on the outerface of H_{k-i+1} (that is the graph after the $(i-1)^{\text{th}}$ iteration of removal), thus e must be in H_{k-i}. Moreover, e is not part of any simple cycle in H_{k-i} (If e were contained in a simple cycle it would also be part of a face which had not had edges removed from it yet). Consider the construction of T_{k-i} (that is, the tree for H_{k-i}). Edge e is not part of any simple cycle, but if T_{k-i} is to be a spanning tree, the endpoints of e must be included, so e must appear in the spanning tree. Since $T_{k-i} \subseteq T_{k-i+1} \subseteq \cdots \subseteq T$, we have that $M \subseteq T$. $\square$
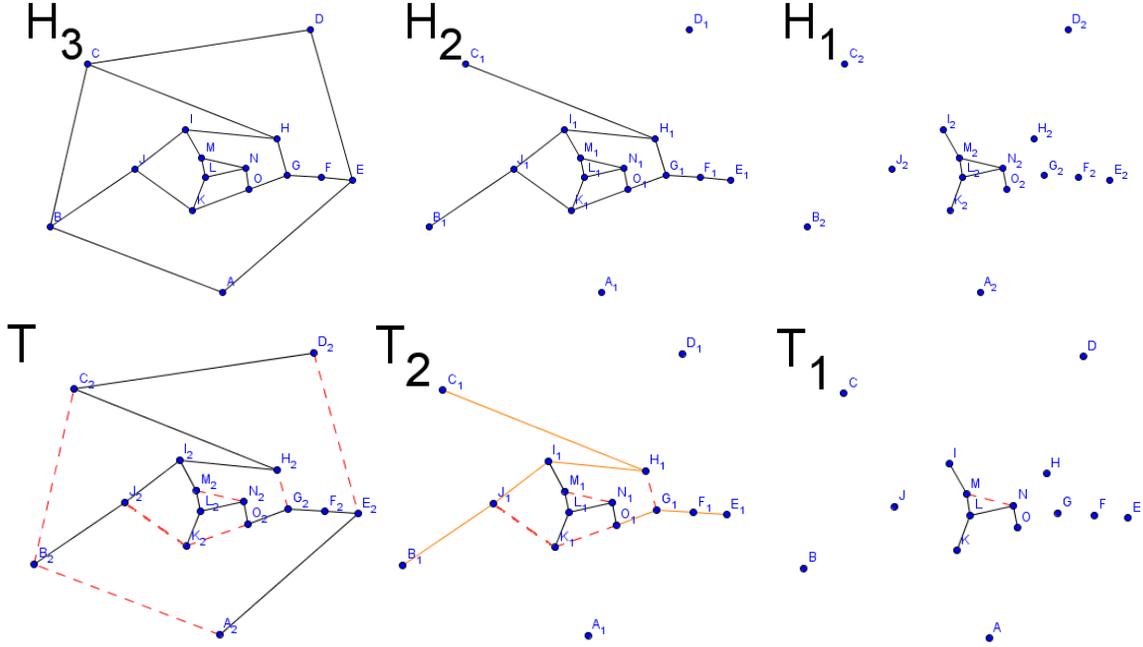
Lemma 5.5. *T^* has height at most k.*

Figure 3: An example of tree construction. Edges are iteratively removed (top row left to right) to get an outerplanar graph, then iteratively restored (bottom row right to left) to achieve a spanning tree

Proof. We claim that every face has an edge in T^* to a face with a lower-numbered level (that is one "closer" to the outerface). Suppose, for contradiction, that f is some highest-level face which fails this. Then since T is a tree, there is some edge missing from (every simple cycle in) f. Since $M \subseteq T$, these edges cannot be incident to a second face at the same level as f (and it cannot be lower level by assumption) so this edge is also adjacent to a higher level face, f' at level j. Consider the subtree T_{k-j}, i.e. the subtree we created for the iteration at which we have removed some of the edges of f'. Every removed edge of f' has a fundamental cycle in T_{k-j}. Consider the union of these fundamental cycles with the edges of f. Since f' is at a higher level than f, T_{k-j} doesn't contain any edges of f, so f unioned with these fundamental cycles creates a closed walk in T. This implies that T contains a cycle, which contradicts it being a tree. Thus since every face has an edge in T^* to a lower level face, we can find a length at most i path from a face at level i to the outerface. Note that we required only k iterations of edge removal to create an acyclic graph, so every face has level at most k, the height of T^* follows. $\qquad\square$

Lemma 5.6. *No vertex has 3 incident faces with paths in T^* to the outerface of length k.*

Proof. Note that the only faces at level k are part of an outerplanar graph (H_1), thus since $\Delta_H \leq 3$, every vertex is incident to at most two of these faces. $\qquad\square$
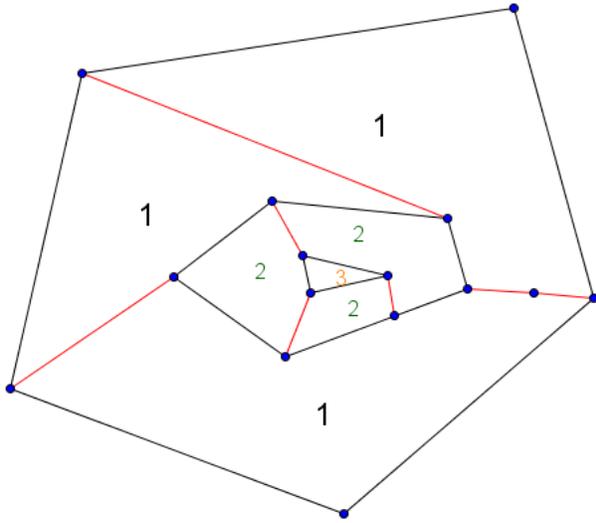
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Figure 4: An example of face labeling, with M shown in red

Lemma 5.7. *Let x be a vertex or edge on C_e, the fundamental cycle of some edge e. Let f be a face incident to x and interior to C_e, then the path in T^* from f to the outerface will use e^*.*

Proof. Let f be a face which includes x and is interior to the fundamental cycle of e. There must be a path in T^* from f to the outerface which does not pass through any edge of T. Consider the Jordan curve for the fundamental cycle of e (including e itself). f is inside this curve, but the outerface is outside this curve. Since e is the only edge of that fundamental cycle not in T, it must be that the path in T^* from f to the outerface goes through e, and so uses e^*. ☐

Note that because the path in T^* passes through e, its vertices include the face associated with e. One might note that the proof of Lemma 5.7 did not rely on either $\Delta_H \leq 3$ or any properties of T — this lemma does, in fact, hold for any planar embedded graph and any of its spanning trees.

We are now ready to show the important results of this subsection.

Lemma 5.8. *There are at most $3k - 1$ distinct finite faces which are incident to the edges which complete the fundamental cycles through any vertex, and at most $2k$ distinct finite faces incident to the edges which complete the fundamental cycles through any edge.*

Proof. Consider an arbitrary edge e. By Lemma 5.7, if the fundamental cycle for some edge, e', goes through e, then for some face f, incident to e, the path from f to the outerface in

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T^* goes through e'. There are at most two faces incident to e, and by Lemma 5.5, the paths in T^* from these faces to the outerface pass through at most k such edges of $H \backslash T$. Counting faces by their associated edges yields the bound of $2k$.

Similarly, consider an arbitrary vertex u. By Lemma 5.7, any edge e for which the fundamental cycle passes through u must be on a path from a face containing u to the outerface in T^*. Since $\Delta_H \leq 3$, there are at most 3 faces u is incident to. By Lemma 5.5, each of these has a length at most k path to the outerface, and by Lemma 5.6, one is at most $k - 1$, so we have $3k - 1$ edges whose fundamental cycles go through u. Counting faces by their associated edges gives the bound of $3k - 1$. □

We remark that Lemma 5.8 is closely related to this result of Bodlaender.

Lemma 5.9 (Bodlaender [3]). *$vr(H, T) \leq 3k - 1$ and $er(H, T) \leq 2k$.*

Lemma 5.8, in fact gives Lemma 5.9 as a corollary. We have proved the stronger version, as we will need it in the next section.

5.2.3 Constructing the Decomposition

We are now ready to begin showing Theorem 5.2. On a k-outerplanar graph G, construct an expansion, $H = (V, E)$, as in Lemma 5.1. Let T be a spanning tree of H as described in Section 5.2.2. We will use the structure of T to create an em-decomposition. Let T' be the graph created by subdividing each edge of T.

Let $D = (T', \{X_i\})$ be a decomposition. We will denote the contents of the bag for x by X_x, where x is an edge or vertex of T.

For $u \in V$, X_u should contain u, the vertices on the boundary of the finite faces u is incident to, and for any edge e' whose fundamental cycle includes u, the vertices on the boundaries of the finite faces incident to e'.

For $e = (u, v) \in E$, X_e should contain u, v and the vertices on the boundaries of the finite faces incident to any edge e' whose fundamental cycle goes through e.

We now show that D is a valid em-decomposition. By construction, every finite face, edge, and vertex is contained in a bag, so it is enough to show that every vertex induces a connected subgraph.
Note that there are two ways for a vertex, v, to be added to a bag. First, the bag could be along the fundamental cycle of some edge (u, w), where v is on a face incident to (u, w).

Second, v could be in X_x if x is incident to a face containing v. We show that for every v, these subgraphs are each connected and include the node corresponding to v, and thus that the entire subgraph containing v is connected.

Consider the bags which contain v because they are incident to a face with v on its boundary. Let x be the vertex/edge corresponding to such a bag, we show that there is a path from x to v in D, in which every node contains v. Consider the path in H from x to v along their common face, f. The edges also in T can immediately be used as part of a path from x to v (by construction all nodes along this path contain v). For the edges not also in T, consider the fundamental cycle for that edge. Since the edge is incident to f, every vertex and edge along this path contains v, so we can splice in these paths to create a walk from x to v. The desired x to v path is a subgraph of this walk.

Consider a bag, b, which contains v because it was part of a fundamental cycle of some edge (x, y). Then along the fundamental cycle, there is a path in T from b to x, where x is on a face containing v and every bag along that path contains v. From x, there is a path in T to v using edges of the face in T, or using the fundamental cycles which correspond to edges of f not in T. Each of these contains v and combining these gives a walk (and thus a path) to v.

Thus every bag that contains v has a path to v such that every bag along that path contains v. This shows the final requirement of an em-decomposition.

We now find the width of D. Let ℓ_H be the length of the longest finite face of H.

Consider a node corresponding to an edge e. By Lemma 5.8, there are at most $2k$ faces added to e due to fundamental cycles. Moreover, note that to achieve $2k$, we must use the two faces incident to e, thus even though we started with two faces in X_e the total is still no more than $2k$. Similarly, the endpoints of the edge are also in these two initial faces, so we need not count them again. This gives at most $2k\ell_H$ vertices in X_e. We can improve this by noticing some double-counting. Recall by Lemma 5.7, these $2k$ faces are in fact on (two) paths in T^*, so consecutive faces will share an edge. Thus, we have counted $2(k-1)$ edges (or $4(k-1)$ vertices) twice. Similarly, the endpoints of e appear in two faces. Subtracting this double counting gives a bound of $2k\ell_H - 4(k-1) - 2$.

For nodes corresponding to vertices, we have (by Lemma 5.8) at most $3k-1$ faces added due to fundamental cycles. Again to reach $3k-1$ three of these faces must be incident to u and so the faces added are a superset of the other contents of X_u. Thus we have for any vertex u, the total number of faces in X_u is at most $3k-1$. This gives an initial bound of $(3k-1)\cdot\ell_H$. Recall, however, that these faces are actually on at most three paths in T^* (This follows from $\Delta_H \leq 3$ and Lemma 5.7). Thus the edges which are incident to two of these faces would have their endpoints counted twice. There are $3k-4$ such edges (all but the final

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edge on each path of T^*, the final edge being incident to the outerface). Similarly, we note that each neighbor of u appears in 2 faces (in two separate T^* paths) and u itself appears in 3 different faces. Thus we can subtract this double counting from our original bound to get a better one: $\ell_H(3k-1) - 2(3k-4) - 3 - 2$, which simplifies to: $(3k-1)\ell_H - 6k + 3$.

Some algebra gives that the vertex bags dominate so long as $\ell_H \geq \frac{2k-1}{k-1}$. Recall that we assumed $k \geq 2$, so we have that $\ell_H \geq 3$ (if $\ell_H < 3$, then H is acyclic then it must be outerplanar). One can check that $\ell_H \geq 3 \geq \frac{2k-1}{k-1}$ for all $k \geq 2$, so the vertex bags dominate.

This gives the width of the decomposition as one less than the size of the vertex bags or $(3k-1)\ell_H - 6k + 2$.

Which allows us to prove our major result:

Theorem 5.2. *If G is a k-outerplanar embedding of a graph ($k \geq 2$), with $\Delta_G \geq 3$, then $emw(G) \leq (3k-1)(\Delta_G - 2)\ell_G - 6k + 2$.*

Proof. From G construct an expansion H. We have that $emw(H) \leq (3k-1)(\ell_H) - 6k + 2$. By lemmas 5.2 and 5.3 we have

$$emw(G) \leq emw(H) \leq (3k-1)(\ell_H) - 6k + 2 \leq (3k-1)(\Delta_G - 2)(\ell_G) - 6k + 2$$

$\square$

Remark 5.1. *The analysis of the given construction is optimal.*

That is, a closer analysis of the current scheme will not allow us to lower the bound (i.e., to guarantee a lower-width decomposition, we must use a different method of creating the decomposition). We show this by producing a graph for which the maximum size of a bag for our decomposition is $(3k-1)(\Delta_G - 2)(C_G) - 6k + 3$. To do this, we begin with a graph which has $vr(G, T) = 3k - 1$. An example is provided by Kammer and Tholey [6]. Our tightness example (Figure 5) is simply an expansion of their construction for $k = 2$, with some extra vertices added by subdividing edges.

It can be verified that the tree shown (black edges) could be the T described in 5.2.2, and that the decomposition will contain a bag (corresponding to the marked vertex u) with $(3k-1)(\Delta - 2)(\ell) - 6k + 3 = 41$ vertices, producing a width of 40, tight to the given analysis.
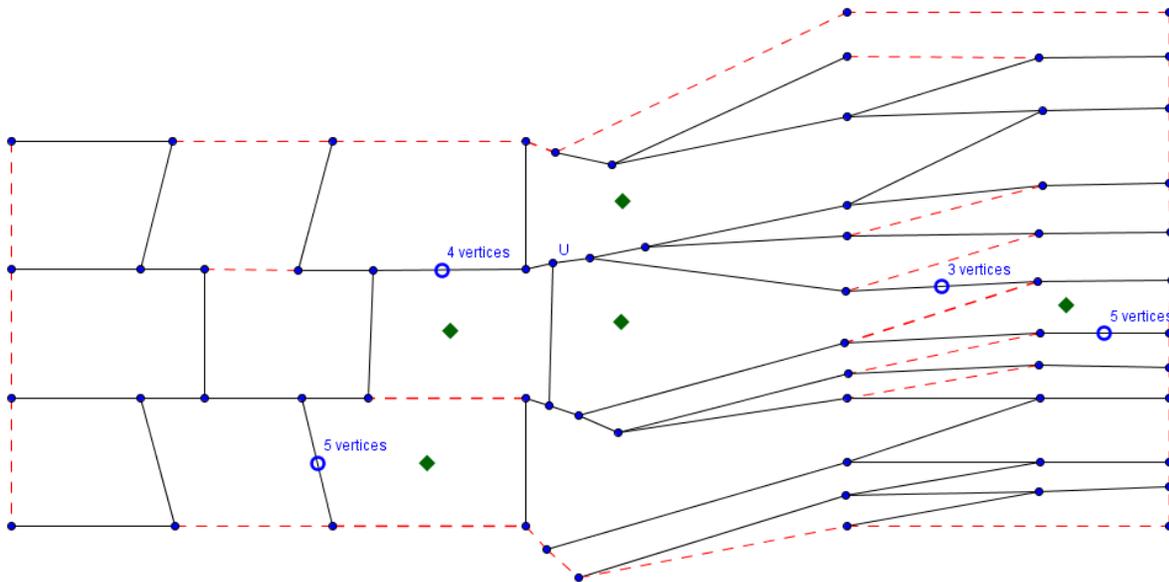
Figure 5: The bag corresponding to u in the decomposition will contain all the vertices in the marked faces. Circles mark which edges need to be subdivided so each used face has ℓ vertices.

6 Conclusion

In this document we described em-width, a variant of treewidth which respects the embedding of planar graphs. We have found classes of graphs for which we can calculate em-width easily. Treewidth and em-width are related parameters, and we have been able to find analogues of existing theorems on treewidth for em-width. There are still many open questions, including whether calculating em-width reduces to calculating treewidth of planar graphs (or vice-versa), and whether there is an algorithm to computer em-width in general.

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