AN ANALOGUE OF K-MARKED DURFEE SYMBOLS FOR STRONGLY UNIMODAL SEQUENCES

SAVANA AMMONS, YOUNG JIN KIM, AND LAURA SEABERG

Advisor: Holly Swisher
Oregon State University

Abstract. This paper applies methods from Andrews's work on partitions to another combinatorial object: strongly unimodal sequences. Specifically, we define "k-marked unimodal symbols" for unimodal sequences analogously to how Andrews defines k-marked Durfee symbols for partitions. We establish a multivariate rank generating function $U_k(\zeta_{\mathbf{k}}; \mathbf{q})$ for k-marked unimodal symbols, as well as $\mathcal{SCU}_k(q)$ for self-conjugate k-marked unimodal symbols, which we also interpret combinatorially in terms of partitions. We then discuss potential quantum modularity properties for $U_k(\zeta_{\mathbf{k}}; \mathbf{q})$ for certain vectors of roots of unity $\zeta_{\mathbf{k}}$, including determining when $U_k(\zeta_{\mathbf{k}}; \mathbf{q})$ can be defined as a function on a subset of rationals. We conclude with some further observations based on computational data and a congruence conjecture about the full rank.

1. Introduction and Statement of Results

1.1. **Background.** A *partition* of an integer n is a non-increasing sequence of positive integers that sum to n, where each summand is called a *part*. Partitions have been of interest to mathematicians for centuries, partly because of their mysterious, yet indelible, connection to modular forms.

Namely, the partition generating function $p(n)$ and Dedekind's eta function $\eta(\tau)$ (a weight $\frac{1}{2}$ modular form) share the following relationship:

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = q^{\frac{1}{24}}\eta(\tau)^{-1}, \tag{1}$$

where

$$(a;q)_n := \prod_{j=1}^{n}(1 - aq^{j-1}).$$

This link between partitions and modular forms also manifests when analyzing partition ranks. Dyson [8] defines a partition's *rank* as the largest part in the partition minus the total number of parts. As defined in [8], the *partition rank function* $N(m,n)$ counts the number of partitions of n with rank equal to m; it is generated by:

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)w^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n} =: R_1(w;q),$$

where $N(m,0) = \delta_{m0}$ and δ_{ij} is the Kronecker delta function.

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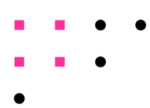
This generating function provides more clues to how partitions and modular forms relate. Setting $w = 1$, we recover the partition generating function (1); This can be seen from (2). Setting $w = -1$ produces one of Ramanujan's original third order mock theta functions, a group of functions that, as described in [4], exhibit modular properties when they are "completed" by certain nonholomorphic functions.

As Andrews indicates in [1], partitions are represented visually using *Ferrers diagrams*, where a row of dots in a diagram corresponds to a single part in a partition. Figure 1 shows a Ferrers diagram for the partition $4 + 3 + 1 = 8$. We call the largest square of dots within the Ferrers diagram the *Durfee square*.

Ferrers diagrams are useful to illustrate the *conjugate* of a partition. The conjugate of a partition is the partition that is obtained by reflecting the diagram across the line of slope -1 that passes through the upper left corner of the diagram. A partition that is conjugate to itself is called a *self-conjugate* partition.

Using Ferrers diagrams, we can represent partitions in yet another way, by creating *Durfee symbols*. As defined by Andrews in [2], a Durfee symbol's top row corresponds to the columns on the right of the Durfee square in a partition's Ferrers diagram; its bottom row corresponds to the rows below the Durfee square. The Durfee symbol's subscript indicates the Durfee square's side length. A Durfee symbol for the partition $4 + 3 + 1$ is shown in Figure 1. As with partitions, Durfee symbols have ranks as well. Andrews defines the *rank of a Durfee symbol* to be the number of parts on the top row minus the number on the bottom row–this definition is the same as the typical partition rank.

$$4 + 3 + 1$$



$$\begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}_2$$

FIGURE 1. A Ferrers diagram and Durfee symbol for the partition 4+3+1. This partition has rank 1.

Note that both the top and bottom rows of a Durfee symbol are themselves partitions, and if a Durfee symbol represents a self-conjugate partition, then the top and bottom rows of the Durfee symbol are identical. Clearly Durfee symbols with subscript s break partitions into three components: the Durfee square, a partition below the square, and a conjugate partition to the right of the square. Both of these smaller partitions have parts of size at most s, the size of the Durfee square. Hence, we can write the partition generating function like so:

$$\sum_{n=1}^{\infty} p(n)q^n = \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q)_k^2}. \tag{2}$$

In [2], Andrews modifies the definition of a partition's Durfee symbol to create a *k-marked Durfee symbol*. Rather than having entries in $\mathbb{N}$, k-marked Durfee symbols have entries in k copies of $\mathbb{N}$ (i.e. positive integers with subscripts ranging from 1 to k) subject to the conditions specified in Definition 2.1. Figure 2 shows an example of a k-marked Durfee symbol.

$$\left(\begin{array}{cc|ccc|c} 4_3 & 4_3 & 3_2 & 3_2 & 2_2 & 2_1 \\ & 5_3 & & 3_2 & 2_2 & 2_1 \end{array}\right)_5$$

FIGURE 2. A k-marked Durfee symbol for the partition $9+9+8+8+7+7+5+5+3+2+2$

Much like for Durfee symbols, one can define a rank statistic for k-marked Durfee symbols. In fact, one can define k different rank statistics, as well as a "full" rank statistic. These definitions are given in [2]. In this same paper, Andrews unveils a $k+1$-variable rank generating function R_k for k-marked Durfee symbols. In 2010, Bringmann [6], and Ono showed that the two-variable rank generating function $R_1(x_1;q)$ is mock modular when $x_1 \neq 1$ is a root of unity. Furthermore, Bringmann [3] found that the function $R_2(1,1;q)$ was a quasimock modular form. Bringmann et al. expanded on this in [5] by showing that $R_k(1,...,1;q)$ is a quasimock modular form for $n \geqslant 2$. In 2013, Folsom and Kimport [10] then went on to prove that, for more general roots of unity, $R_k(\boldsymbol{\zeta_k};q)$ was essentially a mixed mock modular form. Then, in 2018, Folsom et al. proved in [9] that $R_k(\boldsymbol{\zeta_k};q)$ is a quantum modular (in the sense of Definition 1.2) form for $n \geqslant 2$, given suitable vectors of roots of unity.

For our REU project, we investigate if these results for partitions extend to another combinatorial object: a unimodal sequence.

Definition 1.1. A sequence of positive integers $\{a_1, \ldots, a_s\}$ is a *unimodal sequence* of size n if $\sum_i a_i = n$ and the sequence satisfies

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_s.$$

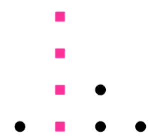
for some k. If the above inequalities are strict, then the sequence is *strongly unimodal*.

Just as Ferrers diagrams represent partitions, we define *unimodal dot diagrams* to represent unimodal sequences. To construct a unimodal dot diagram, we represent each integer in a unimodal sequence as dots in a column. The tallest column in the diagram is the peak. Figure 3 shows a unimodal dot diagram for the strongly unimodal sequence $\{1,4,2,1\}$. The *rank* of a strongly unimodal sequence is $s-2k+1$: the number of terms to the right of the peak minus the number of terms to the left of the peak.

We may also define a notion of the *conjugate* for a unimodal sequence, which is obtained this time by reflecting across a vertical axis that passes through and is parallel to the sequence's peak. A *self-conjugate* strongly unimodal sequence is one that is conjugate to itself.

Just as Durfee symbols represent partitions, we define *unimodal symbols* to represent strongly unimodal sequences. The top row of the symbol enumerates terms $a_{k+1}, a_{k+2}, \ldots, a_s$, while the bottom row enumerates terms $a_{k-1}, a_{k-2}, \ldots, a_1$. The unimodal symbol's subscript indicates the length of the sequence's peak (i.e. the value a_k). As with a Durfee symbol, the number of entries in the top row minus the number of entries in the bottom row gives the

$$\{1, 4, 2, 1\}$$



$$\begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}_4$$

FIGURE 3. A unimodal dot diagram and unimodal symbol for the sequence $\{1, 4, 2, 1\}$.

typical sequence rank, as presented in [7]. And as with Durfee symbols, a unimodal symbol for a self-conjugate sequence will have two identical rows.

We construct a natural analogue of k-marked Durfee symbols for strongly unimodal sequences by modifying unimodal symbols. Given a unimodal symbol, one may create a *k-marked unimodal symbol* with entries originating from k copies of the integers in a way consistent with Andrews's original definition for k-marked Durfee symbols in [2]. The formal definition of a k-marked unimodal symbol is given in Definition 2.3. The introduction of the subscripts allows for the definition of k different ranks on a k-marked Durfee symbol, as well as a "full" rank, which are again analogous to Andrews's k ranks and full rank of k-marked Durfee symbols in [2].

1.2. **Quantum Modular Forms.** Before stating our results, we review the definition of quantum modular forms.

Classically, a modular form of weight k is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying the transformation

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(\gamma)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, where k is a fixed integer and $\varepsilon(\gamma)$ is some appropriate root of unity.

Of course, there are many variants and generalizations of modular forms, e.g. we can replace the weight to be a half integer or any rational, or we can replace $\mathrm{SL}_2(\mathbb{Z})$ with a different Lie group. However, one of the generalizations we are interested in are quantum modular forms.

Roughly speaking, quantum modular forms, originally defined by Zagier [11] in 2010, are complex-valued functions defined on $\mathbb{Q}$ that exhibit modular-like transformation with respect to the action of some appropriate subgroup of $\mathrm{SL}_2(\mathbb{Z})$. More precisely:

Definition 1.2. A *quantum modular form* is a function $f : \mathbb{Q} \longrightarrow \mathbb{C}$ such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$, the "error to modularity" function

$$h_\gamma(x) = f(x) - \varepsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right),$$

where $k \in \frac{1}{2}\mathbb{Z}$ and $\varepsilon(\gamma)$ is a root of unity, is continuous or analytic.

Zagier purposefully left this definition a bit vague to encompass more examples of quantum modular forms from different areas.

1.3. **Statement of Results.** As we've demonstrated, several combinatorial objects for partitions have analogous counterparts for unimodal sequences. Since Andrews developed a $k+1$ variable generating function for k-marked Durfee symbols in [2], which was revealed to have quantum modular properties by Folsom et al. in [9], it is natural to ask if such a function with similar properties exists for k-marked unimodal symbols.

In this paper, we develop a $k+1$-variable generating function $U_k(x_1, \ldots, x_k; q)$ for k-marked unimodal symbols. In particular, we define

$$
\begin{aligned}
(3) \quad U_k(x_1, x_2, ..., x_k; q) := \sum_{m_1,...,m_k \geqslant 1} & q^{(m_1+m_2+\cdots+m_{k-1}+m_k)+(m_1+m_2+\cdots+m_{k-1})+\cdots+(m_2+m_1)+(m_1)} \\
& \times \left[(1+x_1^{-1}q^{m_1})(1+x_2^{-1}q^{m_1+m_2})\cdots(1+x_{k-1}^{-1}q^{\sum_{i=1}^{k-1} m_i})\right] \\
\times \Bigg[& (-x_1q;q)_{m_1-1}(-x_1^{-1}q;q)_{m_1-1} \cdot (-x_2q^{m_1+1};q)_{m_2-1}(-x_2^{-1}q^{m_1+1};q)_{m_2-1} \\
& \cdots(-x_kq^{(\sum_{i=1}^{k-1} m_i)+1};q)_{m_k-1}(-x_k^{-1}q^{(\sum_{i=1}^{k-1} m_i)+1};q)_{m_k-1}\Bigg].
\end{aligned}
$$

Then we obtain the following equality.

Theorem 1.3. *Let $\mathcal{U}_k(n_1, n_2, ..., n_k; n)$ be the number of k-marked unimodal symbols of size n with ith rank n_i. Then*

$$\sum_{m_i \in \mathbb{Z}} \sum_{n>0} \mathcal{U}_k(n_1, n_2, ..., n_k; n) x_1^{n_1} x_2^{n_2} ... x_k^{n_k} q^n = U_k(x_1, x_2, ..., x_k; q).$$

We also define a notion of a self-conjugate k-marked unimodal symbol, which, in the 1-marked case, recovers a third order mock theta function. In general, we let $\mathcal{SCU}_k(q)$ be the generating function that gives self-conjugate k-marked unimodal symbols. It turns out that $\mathcal{SCU}_k(q)$ has another combinatorial interpretation that reflects a theorem from Andrews [2]. This relies on the quantities ω_k, ϵ_k which each count partitions with specific unmarked odd parts and $k-1$-marked even parts. The generating functions are related in the following theorem.

Theorem 1.4. *For $k \geqslant 2$, it holds that*

$$\mathcal{SCU}_k(q) = \sum_{n \geqslant 0} (-1)^k (\omega_k(n) - \epsilon_k(n)) q^n.$$

In addition to proving U_k's combinatorial properties, we also investigate what U_k's properties would be if it were quantum modular. To do so, fix some $\boldsymbol{\zeta_k} = (\zeta_{\beta_1}^{\alpha_1}, \ldots, \zeta_{\beta_k}^{\alpha_k})$, with each $\zeta_{\beta_i}^{\alpha_i} = e^{2\pi i \frac{\alpha_i}{\beta_i}}$. Without loss of generality we require $\gcd(\alpha_i, \beta_i) = 1$. Next let $S_\ell = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where

$$\ell = \ell(\boldsymbol{\zeta_k}) := \operatorname{lcm}(\beta_1, \ldots, \beta_k).$$

Further define

$$\Gamma_{\boldsymbol{\zeta_k}} := \langle S_\ell, T \rangle. \tag{4}$$

We prove the following:

Theorem 1.5. *Let $h \in \mathbb{Z}, d \in \mathbb{N}, \gcd(h,d) = 1$, and $\beta_j \nmid d$. Then, for $1 \leqslant j \leqslant n$,*

$$Q_{\boldsymbol{\zeta_k}} := \left\{ \frac{h}{d} \in \mathbb{Q} \;\middle|\; \left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right| \begin{array}{l} > \frac{1}{3} \text{ if } d \text{ is odd} \\ < \frac{1}{6} \text{ if } d \text{ is even} \end{array} \right\}$$

where $[x]$ denotes the closest integer to $x \in \mathbb{Q}$, gives a quantum set for $U_k(\boldsymbol{\zeta_k}; q)$ with respect to the group $\Gamma_{\boldsymbol{\zeta_k}}$.

Remark 1.6. As [9] notes, while sources differ on the treatment of $[x]$ for $x \in \frac{1}{2} + \mathbb{Z}$, there is no ambiguity in the definition of $Q_{\boldsymbol{\zeta_k}}$. This is because $|x - [x]| = \frac{1}{2}$ whether we define $[x] = x + \frac{1}{2}$ or $[x] = x - \frac{1}{2}$.

We also show in Section 4 through a variety of lemmas that for many choices of $\boldsymbol{\zeta_k}$, the quantum set $Q_{\boldsymbol{\zeta_k}}$ is nonempty. Therefore, this construction frequently yields an interesting and large set on which to look for quantum modularity properties. Then we explore quantum modularity on a specific choice of $U_k(\boldsymbol{\zeta_k}, q)$ with respect to the group $\Gamma_{\boldsymbol{\zeta_k}}$.

1.4. **Outline.** In Section 2, we introduce the background material necessary to derive our generating functions in Theorem 1.3 and Theorem 1.4. In Section 3, we develop our rank generating function for strongly unimodal sequences U_k as well as a generating function for self-conjugate strongly unimodal sequences that generalizes a third-order mock theta function. In Section 4, we prove that $Q_{\boldsymbol{\zeta_k}}$ would be a suitable quantum set for $U_k(\boldsymbol{\zeta_k}, \zeta_k^h)$ if it were quantum modular, and that $e^{-\frac{\pi i x}{12}} U_k(\boldsymbol{\zeta_k}; q)$ is quantum modular under the action of $\left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$. In addition, we provide several lemmas detailing $Q_{\boldsymbol{\zeta_k}}$'s composition. In Section 5, we recount the process of graphically investigating $U_k(-1, ..., -1; q)$'s potential quantum modularity, as well as $U_k(\boldsymbol{\zeta_k}, \zeta_k^h)$'s possible additional combinatorial analogues to k-marked Durfee symbols.

2. Preliminaries

Below we reproduce the definitions of a Durfee k-marked Durfee symbol, as well as its rank, as presented in [2].

Definition 2.1. A *k-marked Durfee symbol* is a Durfee symbol composed of k copies of the integers for parts in both rows. In addition, we require. that:

1. The sequence of parts and the sequence of subscripts in each row be non-increasing
2. Each of the subscripts 1,2,...,k occur at least once in the top row.
3. If $M_1, M_2, ..., M_k$ are the largest parts with their respective subscripts in the top row, then all parts in the bottom row with subscript 1 lie in $[1, M_1]$, with subscript 2 lie in $[M_1, M_2]$, ... with subscript $k-1$ lie in $[M_{k-2}, M_{k-1}]$, and with subscript k lie in $[M_{k-1}, S]$, where S is the side length of the Durfee square.

Definition 2.2. Let D be a k-marked Durfee symbol and let τ^i (resp. δ^i) denote the sequence in the top row (resp. bottom row) with subscript i in D. The *jth rank* of γ, denoted $\rho_j(\gamma)$, is

$$\rho_j(\gamma) = \begin{cases} \text{length}(\tau^j) - \text{length}(\delta^j) - 1 & j < k \\ \text{length}(\tau^j) - \text{length}(\delta^j) & j = k. \end{cases} \tag{5}$$

We proceed to define a k-marked unimodal symbol, which bears clear resemblance to that of Andrews given above in Defintion 2.1.

Definition 2.3. A *k-marked unimodal symbol of n* is a unimodal symbol corresponding to a unimodal sequence of size n with entries in k copies of the integers that also satisfies the following properties:

(1) The sequence of entries in both the top and bottom rows are each strictly decreasing, while the sequence of subscripts for entries in both rows are each non-increasing.
(2) Each of $1, 2, \ldots, k-1$ appear as a subscript in the top row.
(3) Let d be the size of the sequence's peak. For $1 \leqslant i \leqslant k-1$, let m_i be the largest part in the top row with subscript i and define $m_0 = 0, m_k = d-1$. Then the parts in the bottom row with subscript i lie in $[m_{i-1}+1, m_i]$.

The natural calculation of the rank for a k-marked unimodal symbol also exactly follow from Andrews, given above in Definition 2.

Definition 2.4. Let γ be a k-marked unimodal symbol and let τ^i (resp. δ^i) denote the sequence in the top row (resp. bottom row) with subscript i in γ. The *jth rank* of γ, denoted $\rho_j(\gamma)$, is

$$\rho_j(\gamma) = \begin{cases} \text{length}(\tau^j) - \text{length}(\delta^j) - 1 & j < k \\ \text{length}(\tau^j) - \text{length}(\delta^j) & j = k. \end{cases} \tag{6}$$

The subtraction of 1 in Definition 2.4 compensates for the requirement (2) in Definition 2.3. Notice that if γ is 1-marked, then (6) in Definition 2.4 shows that $\rho_1(\gamma)$ recovers the original Dyson's rank statistic. One consequence of defining k ranks on a k-marked Durfee symbol is that we may calculate the number of k-marked unimodal symbols with ith rank equal to n_i through a function in $k+1$ variables that generalizes $U(x, q)$ studied by Bryson et al. in [7].

We define the *full unimodal rank* of a k-marked unimodal symbol analogous to Andrews's definition of the full unimodal rank of a k-marked Durfee symbol in [2].

Definition 2.5. For each k-marked unimodal symbol δ, we define the *full rank* $FRU(\delta)$ by

$$FRU(\delta) = \rho_1(\delta) + 2\rho_2(\delta) + 3\rho_3(\delta) + \ldots + k\rho_k(\delta),$$

and we let $NFU_k(m, n)$ denote the number of k-marked unimodal symbols of size n with full rank m. Additionally, we let $NFU_k(m, Q, n)$ denote the number of k-marked unimodal symbols of n with full rank congruent to $m \pmod{Q}$.

3. Some Generating Functions

This section addresses two combinatorial results–first, the appropriateness of our multi-rank generating function for k-marked unimodal symbols, and second the definition and partition-theoretic interpretation of the generating function for self-conjugate k-marked unimodal symbols.

3.1. **Proof of Theorem 1.3.** Recall from Theorem 1.3 that we defined

$$
\begin{aligned}
U_k(x_1, x_2, ..., x_k; q) = \sum_{m_1,...,m_k \geqslant 1} & q^{(m_1+m_2+\cdots+m_{k-1}+m_k)+(m_1+m_2+\cdots+m_{k-1})+\cdots+(m_2+m_1)+(m_1)} \\
& \times \left[(1+x_1^{-1}q^{m_1})(1+x_2^{-1}q^{m_1+m_2})\cdots(1+x_{k-1}^{-1}q^{\sum_{i=1}^{k-1} m_i})\right] \\
\times \Bigg[& (-x_1q;q)_{m_1-1}(-x_1^{-1}q;q)_{m_1-1}\cdot(-x_2q^{m_1+1};q)_{m_2-1}(-x_2^{-1}q^{m_1+1};q)_{m_2-1} \\
& \cdots(-x_kq^{(\sum_{i=1}^{k-1} m_i)+1};q)_{m_k-1}(-x_k^{-1}q^{(\sum_{i=1}^{k-1} m_i)+1};q)_{m_k-1}\Bigg].
\end{aligned}
$$

Furthermore, Theorem 1.3 states gives that if $\mathcal{U}_k(n_1, n_2, ..., n_k; n)$ denotes the number of strongly unimodal sequences of size n with the ith rank equal to n_i, then for $k \geqslant 1$ we have that

$$
\sum_{m_i \in \mathbb{Z}} \sum_{n > 0} \mathcal{U}_k(n_1, n_2, ..., n_k; n) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} q^n = U_k(x_1, x_2, ..., x_k; q).
$$

This follows from combinatorially analyzing the component pieces of the summands in $U_k(x_1, x_2, ..., x_k; q)$.

Proof. The proof is analogous to that of Theorem 10 in [2]. Consider an arbitrary k-marked unimodal symbol D representing a strongly unimodal sequence U. Let m_1 be the largest entry of D's top row with subscript 1. By condition 1 in Definition 2.3, D's largest entry with subscript 2 must be greater than m_1. Hence, this largest entry with subscript 2 is $m_1 + m_2$, for some $m_2 > 0$. Similarly, the largest entry with subscript 3 must be greater than $m_1 + m_2$, and is therefore $m_1 + m_2 + m_3$, for some $m_3 > 0$. In general, the largest of D's entries with subscript $l \leqslant k-1$ is $\sum_{i=1}^{l} m_i$, where $m_i > 0$. Since the underlying peak's length is larger than any of D's entries, it equals $\sum_{i=1}^{k} m_i$, where $m_k > 0$.

Consider the terms needed to generate τ^1 and δ^1, the portion of the k-marked unimodal symbol where the entries have subscript 1 in both rows. Since we are assuming that m_1 exists by construction, a factor of q^{m_1} is needed. Now because U is strongly unimodal, all entries in τ^1 must be distinct, as are the entries in δ^1. Since m_1 is the largest entry in τ^1, all other entries in τ^1 are at most $m_1 - 1$, giving us that the entries are generated by $(-q;q)_{m_1-1}$. Conversely, the entries in δ^1 are at most m_1 since they represent parts on the opposite side of the peak and are not subject to the same strict bound. Thus δ^1 is generated by $(-q;q)_{m_1} = (1+q^{m_1})(-q;q)_{m_1-1}$.

Let x_1 track the first rank. To track the first rank of the unimodal symbol, we use $(-x_1q;q)_{m_1-1}$ to generate τ^1 and $(-x^{-1}q;q)_{m_1}$ to generate δ^1. That is, the entries marked with subscript 1 are generated by

$$
q^{m_1}(-x_1q;q)_{m_1-1}(-x_1^{-1}q;q)_{m_1} = q^{m_1}(1+x_1^{-1}q^{m_1})(-x_1q;q)_{m_1-1}(-x_1^{-1}q;q)_{m_1-1}.
$$

Construction of the other terms proceeds analogously. The sequence of entries in the top row with subscript 2 have maximal entry m_1+m_2, which is generated by $q^{m_1+m_2}$. The entries in τ^2 are at most m_1+m_2-1, while those in δ^2 are at most m_1+m_2. Entries in both τ^2 and δ^2 are at least m_1+1. The variables x_2 and x_2^{-1} count the second rank. So the generators for these two sequences of subscript 2 entries are $(-xq^{m_1+1};q)_{m_2-1}$ and $(-x^{-1}q^{m_1+1};q)_{m_2}$. Hence, these entries have as a generator

$$q^{m_1+m_2}(1+x_2^{-1}q^{m_2})(-x_2q^{m_1+1};q)_{m_2-1}(-x_2^{-1}q^{m_1+1};q)_{m_2-1}.$$

We proceed in a similar fashion to generate the terms with subscript up through $k-1$.

Now, recall that there is no requirement to have any terms with subscript k on the top row. As a result, no positive power of q is required like the previous steps. There still may, however, be sequences τ^k and δ^k. Thus, we generate both of these via

$$(-x_kq^{(\sum_{i=1}^{k-1}m_i)+1};q)_{m_k-1}(-x_k^{-1}q^{(\sum_{i=1}^{k-1}m_i)+1};q)_{m_k-1}.$$

Finally, we must generate the sequence's peak. Since the peak length is $\sum_{i=1}^{k}m_i$, it is generated by

$$q^{(m_1+m_2+\cdots+m_k)}.$$

If all of these factors generate their respective parts of D, then their product will generate the entirety of D. To account for all possible values for each m_i and the size of U, we sum over all m_i's with $m_i>0$ for $1\leqslant i\leqslant k$ as well as over all $n>0$. The result is $U_k(x_1,\ldots,x_k;q)$, as defined in the theorem statement. □

3.2. **Proof of Theorem 1.4.** We next turn to self-conjugate unimodal symbols, which yield results that are interesting both combinatorially and in terms of connections to modularity.

Definition 3.1. A *self-conjugate k-marked unimodal symbol* is one in which the two rows in the k-marked unimodal symbol are identical, both in terms of the subscripts and the underlying unimodal sequence itself.

Now if we let $SCU_k(n)$ denote the number of self-conjugate k-marked unimodal symbols of n, we can define the generating function

$$\mathcal{SCU}_k(q):=\sum_{n\geqslant 0}SCU_k(n)q^n.$$

Lemma 3.2. *The number of self-conjugate k-marked unimodal symbols representing n is given by the equation*

$$\begin{aligned}\mathcal{SCU}_k(q)&=\sum_{m_1,\ldots,m_k\geqslant 1}q^{2m_1+2(m_1+m_2)+\cdots+2(m_1+\cdots m_{k-1})+(m_1+\cdots m_k)}(-q^2;q^2)_{m_1-1}(-q^{2(m_1+1)};q^2)_{m_2-1}\\&\quad\cdots(-q^{2(m_1+m_2+1)};q^2)_{m_3-1}\cdots(-q^{2(m_1+\cdots+m_{k-1}+1)};q^2)_{m_k-1}\\&=\sum_{m_1,\ldots,m_k\geqslant 1}q^{2m_1+2(m_1+m_2)+\cdots+2(m_1+\cdots+m_{k-1})+(m_1+\cdots+m_k)}\prod_{i=1}^{k}(-q^{2((\sum_{j=1}^{i-1}m_j)+1)};q^2)_{m_i-1}\\&=\sum_{M_k\geqslant k}q^{M_k}(-q^2;q^2)_{M_k-1}\sum_{1\leqslant M_1<M_2<\cdots<M_k}\frac{q^{2(M_1+\cdots+M_{k-1})}}{(1+q^{2M_1})(1+q^{2M_2})\cdots(1+q^{2M_{k-1}})}.\end{aligned}\tag{7}$$

where $M_i=m_1+\cdots+m_i$.

Proof. Applying a similar argument as in the proof of Theorem 1.3, consider a self-conjugate k-marked unimodal symbol D. Let m_1 be the largest entry of D with subscript 1. By Definition 2.3, it follows that the largest entry of D with subscript 2 is m_1+m_2 for some $m_2\geqslant 1$. Following this pattern, the largest entry of D with subscript i is given by $m_1+m_2+\cdots+m_i$ where each $m_j\geqslant 1$. Since the peak length is larger than any of the entries of D, we get that the peak length is given by $m_1+\cdots+m_k$ where each $m_i\geqslant 1$.

Since the two rows in the k-marked unimodal symbol are the same by assumption, we get that each value must appear on top and bottom. Thus, we count each of the largest entries of D with subscript $l \leqslant k-1$ twice, giving us the factors $q^{2(\sum_{j=1}^{l} m_j)}$. Furthermore, the peak contributes a factor of $q^{m_1+\cdots+m_k}$ in the generating function.

Consider the entries of D with subscript 1. Since every value appears twice in the unimodal symbol, and since we are assuming that m_1 is the maximal entry of D with subscript 1, we only need to consider the distinct even numbers less than $2m_1$, giving us that the entries of D with subscript 1 are generated by $(-q^2;q^2)_{m_1-1}$.

Construction of terms with different subscripts proceeds similarly. Consider the entries with subscript 2. Since entries of D with subscript 2 are of size at most m_1+m_2-1 and are also at least m_1+1, we get that the entries of D with subscript 2 are generated by $(-q^{2(m_1+1)};q^2)_{m_2-1}$. Proceeding in a similar fashion to generate all terms of D for all subscripts, we obtain a factor of

$$(-q^2;q^2)_{m_1-1}(-q^{2(m_1+1)};q^2)_{m_2-1}(-q^{2(m_1+m_2+1)};q^2)_{m_3-1}\cdots(-q^{2(m_1+\cdots+m_{k-1}+1)};q^2)_{m_k-1}. \tag{8}$$

Letting $M_i = m_1+\cdots+m_i$, we form a strictly increasing sequence of numbers $1 \leqslant M_1 < M_2 < \cdots < M_k$. Furthermore, we can rewrite (7) as

$$\prod_{i=1}^{k}(-q^{2((\sum_{j=1}^{i-1} m_j)+1)};q^2)_{m_i-1} = \frac{(-q^2;q^2)_{M_k-1}}{(1+q^{2M_1})\cdots(1+q^{2M_{k-1}})}$$

giving us (6). □

Remark 3.3. If we consider the generating function for unmarked self-conjugate strongly unimodal sequences, i.e. if $k=1$, we get that

$$\mathcal{SCU}(q) = \mathcal{SCU}_1(q) = \sum_{M\geqslant 1} q^M(-q^2;q^2)_{M-1} = \psi(q),$$

where $\psi(q)$ is one of the original third-order mock theta functions defined by Ramanujan [7]. This connection indicates that $\mathcal{SCU}_k(q)$ could be a fruitful place to look for differeny modular objects.

Theorem 14 by Andrews in [2] connects self-conjugate k-marked Durfee symbols of n to partitions of n into distinct unmarked odd parts and $k-1$-marked even parts under specific conditions. It turns out that there is a combinatorial interpretation connecting $\mathcal{SCU}_k(q)$ to partitions that mirrors this theorem of Andrews.

Definition 3.4. Let $\omega_k(n)$ count the number of partitions of n into at least k unmarked odd parts such that every odd part less than the largest part appears at least once, as well as $k-1$ differently marked and distinct $(k-1)$-marked even parts (which may repeat) such that each even part is less than twice the number of odd parts and the total number of even parts is odd. Similarly, let $\epsilon_k(n)$ count the same as above, except where the total number of even parts is even.

We note that in Definition 3.4, the total number of even parts counted by ω_k or ϵ_k is not necessarily $k-1$, as the even parts themselves may repeat any number of times.

Recall Theorem 1.4 from Section 1.3 which says

$$\mathcal{SCU}_k(q) = \sum_{n\geqslant 0}(-1)^k\left((\omega_k(n)-\epsilon_k(n)\right)q^n.$$

Proof of Theorem 1.4. From Lemma 3.2, we have that $\mathcal{SCU}_k$ splits into nested sums. The right side of (6) gives $\psi(q)$ that begins indexing at some $k \geqslant 1$. The mock theta function $\psi(q)$ is well known to give the partitions of n into odd parts such that every odd smaller than the largest part appears at least once. This can also be seen by constructing a bijection between strongly unimodal sequences and partitions by reading across the rows starting from the bottom in a diagram. In doing this, it becomes clear that M_k, originally thought of as the peak of the sequence, gives the number of odd parts in this interpretation. Additionally, k bounds the number of parts below, which gives the unmarked odds in the definition of ω_k and ϵ_k.

All of the inner sums are finite after fixing M_k in the outermost sum. Grouping q^{2M_i} with $\frac{1}{1+q^{2M_i}}$, we can use the geometric series expansion to obtain factors of the form $(q^{2M_i} - q^{2(2M_i)} + q^{3(2M_i)} - \cdots)$ for each i. Each $2(M_i)$ is one of the even parts referred to in Definition 3.4, with the number of repetitions of this even part counted by its multiplier in the exponent. Additionally, $2M_k$ gives an upper bound to each of the even parts as described in the sum index, but recall from the first sum that this number is the number of odd parts. Finally, we observe that by multiplying factors of the form $(q^{2M_i} - q^{2(2M_i)} + q^{3(2M_i)} - \cdots)$ in the summand we always obtain either terms representing a total number of even parts with positive sign and total number of odd parts with negative sign, or vice versa. This distinction is based on the parity of k, which determines how many such factors exist. □

4. A Rational Domain for $U_k(\boldsymbol{\zeta_k}; q)$

4.1. **Proof of Theorem 1.5.** First, we investigate what set U_k must be defined on to be quantum modular. Specifically, we seek U_k's quantum set.

Definition 4.1. A *quantum set* $S \subseteq \mathbb{Q}$ for a function f with respect to a group $\Gamma \leqslant SL_2(\mathbb{Z})$ is such that $f(x)$ and $f(\gamma x)$ exist for all $x \in S$ and for all $\gamma \in \Gamma$.

Recall that we define $Q_{\boldsymbol{\zeta_k}}$ as follows:

$$Q_{\boldsymbol{\zeta_k}} := \left\{ \frac{h}{d} \in \mathbb{Q} \;\middle|\; \left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right| \quad \begin{matrix} > \dfrac{1}{3} \text{ if } d \text{ is odd} \\ < \dfrac{1}{6} \text{ if } d \text{ is even} \end{matrix} \right\}$$

where $h \in \mathbb{Z}, d \in \mathbb{N}, \gcd(h,d) = 1$, $\beta_j \nmid d$, and $1 \leqslant j \leqslant n$.. (see Remark 1.3)

We first establish that $U_k(\boldsymbol{\zeta_k}, \zeta_d^h)$ converges for $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$. We prove this in the following lemma

Lemma 4.2. *Fix* $\boldsymbol{\zeta_k} = (\zeta_{\beta_1}^{\alpha^1}, ..., \zeta_{\beta_k}^{\alpha^k})$. *Then for* $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$, $U_k(\boldsymbol{\zeta_k}; \zeta_d^h)$ *converges and can be evaluated by*

$$(9)\quad U_k(x_1, x_2, ..., x_k; \zeta_d^h) = \prod_{j=1}^{n} \frac{1}{1-(1-(-x_j)^d)(1-(-x_j)^{-d})}$$
$$\times \sum_{1 \leqslant s_1,...,s_k \leqslant d} \zeta_d^{h[(s_1+s_2+\cdots+s_{k-1}+s_k)+(s_1+s_2+\cdots+s_{k-1})+\cdots+(s_2+s_1)+(s_1)]}$$
$$\times \left[(1+x_1^{-1}\zeta_d^{hs_1})(1+x_2^{-1}\zeta_d^{h(s_1+s_2)})\cdots(1+x_{k-1}^{-1}\zeta_d^{h(\sum_{i=1}^{k-1} s_i)})\right]$$
$$\times \Bigg[(-x_1\zeta_d^h;\zeta_d^h)_{s_1-1}(-x_1^{-1}\zeta_d^h;\zeta_d^h)_{s_1-1} \cdot (-x_2\zeta_d^{h(s_1+1)};\zeta_d^h)_{s_2-1}(-x_2^{-1}\zeta_d^{h(s_1+1)};\zeta_d^h)_{s_2-1}$$
$$\cdots(-x_k\zeta_d^{h(\sum_{i=1}^{k-1} s_i+1)};\zeta_d^h)_{s_k-1}(-x_k^{-1}\zeta_d^{h(\sum_{i=1}^{k-1} s_i+1)};\zeta_d^h)_{s_k-1}\Bigg].$$

Proof. Since $U_k(\zeta_k; \zeta_d^h)$ has no zero-terms in its denominator, each of its summands is a finite product. Hence, it is immediate that, for $\frac{h}{d} \in \mathbb{Q}$, all summands of $U_k(\boldsymbol{\zeta_k}; \zeta_d^h)$ are finite.

The rest of the proof closely follows the proof of Theorem 3.2 in [9]. Similarly, we illustrate the proof for the case $k = 2$, with comments about how to generalize to $k > 2$.

Let $\frac{h}{d} \in Q_{\zeta_k}$, and write $\zeta = \zeta_d^h$. As in [9], the identity

$$(10)\qquad (x\zeta^r;\zeta)_{s+Md} = (1-x^d)^M (x\zeta^r;\zeta)_s$$

holds for all $M, r, s \in \mathbb{Z}_{\geqslant 0}$. Thus, if we let $m_j = s_j + M_j d$, where $1 \leqslant s_j \leqslant d$ and $M_j \in \mathbb{Z}_{\geqslant 0}$ then, by Definition 1.3

$$U_2(x_1, x_2; \zeta) = \sum_{m_1, m_2 \geqslant 1} \zeta^{2m_1+m_2}[(1+x_1^{-1}\zeta^{m_1})(1+x_2^{-1}\zeta^{m_1+m_2})]$$
$$\times [(-x_1\zeta;\zeta)_{m_1-1}(-x_1^{-1}\zeta;\zeta)_{m_1-1}] \times [(-x_2\zeta^{m_1+1};\zeta)_{m_2-1}(-x_2^{-1}\zeta^{m_1+1};\zeta)_{m_2-1}].$$

Using identity (10), we get

$$\sum_{M_1,M_2 \geqslant 0} \sum_{\substack{1 \leqslant s_1 \leqslant d \\ 1 \leqslant s_2 \leqslant d}} \zeta^{2(s_1+M_1d)+s_2+M_2d}[(1+x_1^{-1}\zeta^{s_1+M_1d})(1+x_2\zeta^{s_1+M_1d+s_2+M_2d})]$$
$$\times [(1-(-x_1)^d)^{M_1}(-x_1\zeta;\zeta)_{s_1-1}(1-(-x_1)^{-d})^{M_1}(-x_1^{-1}\zeta;\zeta)_{s_1-1}]$$
$$\times [(1-(-x_2)^d)^{M_2}(-x_2\zeta^{s_1+M_1d+1};\zeta)_{s_2-1}(1-(-x_2)^{-d})^{M_2}(-x_2^{-1}\zeta^{s_1+M_1d+1};\zeta)_{s_2-1}].$$

Since $\zeta = e^{\frac{2\pi i h}{d}}$, $\zeta^{dm} = 1$, for $m \in \mathbb{Z}$. Thus, our expression becomes

$$\sum_{M_1,M_2 \geqslant 0} (1-(-x_1)^d)^{M_1}(1-(-x_1)^{-d})^{M_1}(1-(-x_2)^d)^{M_2}(1-(-x_2)^{-d})^{M_2}$$
$$\times \sum_{\substack{1 \leqslant s_1 \leqslant d \\ 1 \leqslant s_2 \leqslant d}} \zeta^{2s_1+s_2}(1+x_1^{-1}\zeta^{s_1-1})(-x_1\zeta;\zeta)_{s_1-1}(-x_1^{-1}\zeta;\zeta)_{s_1}(-x_2\zeta^{s_1+1};\zeta)_{s_2-1}(-x_2^{-1}\zeta^{s_1+1};\zeta)_{s_2-1}.$$

The sum over the s_j's is finite, so the convergence of $U_2(x_1, x_2; q)$ is determined by the sum over the M_j's. This sum is the product of two geometric series, each with summands of the form:

$$((1-(-x_j)^d)(1-(-x_j)^{-d}))^{M_j}. \tag{11}$$

From here, there are two cases. If d is odd, (11) becomes:

$$((1+x_j^d)(1+x_j^{-d}))^{M_j},$$

which converges if and only if

$$|(1+x_j^d)(1+x_j^{-d})| < 1.$$

Let $\theta_j = \frac{2\pi\alpha_j}{\beta_j}$. Using $x_j = e^{2\pi i\frac{\alpha_j}{\beta_j}}$, we have that

$$|(1+x_j^d)(1+x_j^{-d})| = |2+2\cos d\theta_j| < 1,$$

if and only if

$$-\frac{3}{2} < \cos d\theta_j < -\frac{1}{2}. \tag{12}$$

For (12) to hold, $d\theta_j = r + 2\pi M$, where $|r| > \frac{2\pi}{3}$, $-\pi < r \leqslant \pi, M \in \mathbb{Z}$.
Thus,

$$|d\theta_j - 2\pi M| > \frac{2\pi}{3},$$

implying that:

$$\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| > \frac{1}{3}.$$

In the case that d is even, (11) becomes:

$$((1-x_j^d)(1-x_j^{-d}))^{M_j},$$

which converges if and only if:

$$|2-2\cos d\theta_j| < 1.$$

Hence:

$$\frac{1}{2} < \cos d\theta_j < \frac{3}{2}.$$

For this to hold, $d\theta_j = r + 2\pi M$, where $|r| < \frac{\pi}{3}$, $-\pi < r \leqslant \pi, M \in \mathbb{Z}$. Thus

$$|d\theta_j - 2\pi M| < \frac{\pi}{3},$$

implying that

$$\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| < \frac{1}{6}.$$

Recall that these are the necessary conditions to belong to Q_{ζ_k}. Thus, for any $\frac{h}{d} \in Q_{\zeta_k}$, $U_2(\boldsymbol{\zeta_k}, \zeta_d^h)$ will converge to the right hand side of (9).

To apply this argument for U_k with $k > 2$, write any $m_j = s_j + Md$. Then the sum splits into the product of a series indexed over $M_1, \ldots, M_k \geqslant 0$ and a series indexed over $s_1, \ldots, s_k$, with each s_j satisfying $1 \leqslant s_j \leqslant d$. The sum over the s_j terms remains finite, and analyzing each geometric series of M_j into the cases as above gives the result for U_k.

□

We are now able to prove Theorem 1.5.

Proof of Theorem 1.5. From Definition 3 and Lemma 4.2, we see that it suffices to show that $Q_{\boldsymbol{\zeta_k}}$ is closed under the action of $\Gamma_{\boldsymbol{\zeta_k}}$.

Recall from (4) that:

$$\Gamma_{\boldsymbol{\zeta_k}} := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \right\rangle$$

and

$$\ell = \mathrm{lcm}(\beta_1, \ldots, \beta_k).$$

Thus, it suffices to show that $Q_{\boldsymbol{\zeta_k}}$ is closed under the action of these matrices.

Let $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$. Then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{h}{d} = \frac{h+d}{d}$. Note that $\gcd(h+d, d) = \gcd(h, d) = 1$, so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{h}{d} \in Q_{\zeta_k}$.

Now, observe that

$$\begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \frac{h}{d} = \frac{h}{h\ell + d}.$$

Note that $\gcd(h, h\ell + d) = \gcd(h, d) = 1$ and that $\beta_j \nmid (h\ell + d)$, since $\beta_j \mid \ell$ and $\beta_j \nmid d$. So

$$\left| \frac{\alpha_j}{\beta_j}(h\ell + d) - \left[\frac{\alpha_j}{\beta_j}(h\ell + d) \right] \right| = \left| \frac{\alpha_j}{\beta_j} h\ell + \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} h\ell + \frac{\alpha_j}{\beta_j} d \right] \right| = \left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right|,$$

since $\frac{\alpha_j}{\beta_j} h\ell \in \mathbb{Z}$. Thus, $Q_{\boldsymbol{\zeta_k}}$ is closed under the action of $\Gamma_{\boldsymbol{\zeta_k}}$, as desired. $\square$

Next, we show that $e^{\frac{-\pi i x}{12}} U_k(\boldsymbol{\zeta_k}; q)$ exhibits quantum modular behavior under translations.

Proposition 4.3. $\boldsymbol{\zeta_k} = (\zeta_{\beta_1}^{\alpha^1}, ..., \zeta_{\beta_k}^{\alpha^k})$. *Let* $k \geqslant 2$ *for* $x \in Q_{\boldsymbol{\zeta_k}}$, *define:*

$$\mathcal{A}_k(x) = \mathcal{A}_k(\zeta_k; x) = e^{\frac{-\pi i x}{12}} U_k(\boldsymbol{\zeta_k}; e^{2\pi i x}),$$

where $x \in Q_{\boldsymbol{\zeta_k}}$.

Then, for all $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *where* $b \in \mathbb{Z}$ *and* $x \in Q_{\boldsymbol{\zeta_k}}$,

$$\mathcal{H}_{k,\gamma}(x) := \mathcal{A}_k(x) - e^{\frac{\pi i}{12}} \mathcal{A}_k(\gamma x) = 0.$$

Proof. This proof closely follows that of Theorem 1.7 in [9]. By Theorem 4.2, $\mathcal{A}_k(x)$ and $\mathcal{A}_k(\gamma x)$ are defined for all $x \in Q_{\boldsymbol{\zeta_k}}$ and $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Note that it suffices to only consider the generator $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ since,

$$\mathcal{H}_{n,\gamma\gamma'}(x) := \mathcal{H}_{n,\gamma'}(x) - q^{\frac{1}{24}} (Cx + D)^{-\frac{3}{2}} \mathcal{H}_{n,\gamma'}(\gamma x)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Thus:

$$\mathcal{H}_{n,\gamma}(x) = \mathcal{A}_k(x) - q^{\frac{1}{24}} (cx + d)^{-\frac{3}{2}} \mathcal{A}_k(x + 1).$$

When we map $x \mapsto x+1$, $q = e^{2\pi i x}$ remains invariant, since $q = e^{2\pi i(x+1)} = e^{2\pi i x}e^{2\pi i} = e^{2\pi i x}$. So, since $U_k(\boldsymbol{\zeta_k}; q)$ can be expressed as a series with only integer powers of q (as in (1.3)):

$$\begin{aligned}\mathcal{A}_k(x+1) &= e^{\frac{-2\pi i(x+1)}{24}} U_k(\boldsymbol{\zeta_k}; q)\\ &= e^{\frac{-2\pi i x}{24}} e^{\frac{-2\pi i}{24}} U_k(\boldsymbol{\zeta_k}; q)\\ &= e^{\frac{-\pi i}{12}} \mathcal{A}_k(x).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{H}_{k,\gamma}(x) &= \mathcal{A}_k(x) - e^{\frac{\pi i}{12}}((0)x+(1))^{-\frac{3}{2}}\mathcal{A}_k(x+1)\\ &= \mathcal{A}_k(x) - e^{\frac{\pi i}{12}}(e^{\frac{-\pi i}{12}}\mathcal{A}_k(x))\\ &= 0,\end{aligned}$$

which is clearly defined and analytic on $\mathbb{R}$, as desired. □

4.2. **The Quantum Set's Composition.** Here, we discuss the contents of $Q_{\boldsymbol{\zeta_k}}$. We note that if there is at least one d satisfying the conditions for $Q_{\boldsymbol{\zeta_k}}$, then there are countably many infinite choices of h that work as a numerator for that d, and hence any nonempty $Q_{\boldsymbol{\zeta_k}}$ is infinite.

In some cases, it is obvious that the quantum set is empty.

Lemma 4.4. *If some $\zeta_{\beta_j}^{\alpha_j} \in \boldsymbol{\zeta_k}$ has $\beta_j \in \{1,3,4\}$, then $Q_{\boldsymbol{\zeta_k}} = \varnothing$.*

Proof. First, if $\beta_j = 1$, then $1 \mid d$ for any $d \in \mathbb{Z}$, violating the construction of $Q_{\boldsymbol{\zeta_k}}$.

Next, suppose there is some $\beta_j = 3$. If d is odd then, regardless of α_i:

$$\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| \in \left\{0, \frac{1}{3}\right\},$$

which can never be strictly greater than $\frac{1}{3}$. If d is even, then $\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| < \frac{1}{6}$ only when $\frac{d\alpha_j}{3} \in \mathbb{Z}$. But, since $\frac{\alpha_j}{\beta_j}$ is in lowest terms, $3 \nmid \alpha_j$. Also, by the requirements of $Q_{\boldsymbol{\zeta_k}}$, $3 \nmid d$. Therefore, $3 \nmid d\alpha_j$, and hence $\frac{d\alpha_j}{3} \notin \mathbb{Z}$.

Finally let $\beta_j = 4$. For $\frac{\alpha_j}{\beta_j}$ to be fully reduced, α_j must be odd. If d is also odd, then $d\alpha_j$ is odd and hence $\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| = \frac{1}{4} \ngtr \frac{1}{3}$. If d is even, then $d = 2m$ where m is odd (if m were even, then $\beta_j \mid d$). Thus:

$$\left|\frac{\alpha_j}{\beta_j}d - \left[\frac{\alpha_j}{\beta_j}d\right]\right| = \left|\frac{m\alpha_j}{2} - \left[\frac{m\alpha_j}{2}\right]\right|.$$

Since $m\alpha_j$ is odd, overall

$$\left|\frac{m\alpha_j}{\beta_j} - \left[\frac{m\alpha_j}{\beta_j}\right]\right| = \frac{1}{2} > \frac{1}{6}.$$

Hence, when some $\beta_j \in \{1,3,4\}$, no $d \in \mathbb{N}$ satisfies the conditions for $\frac{h}{d} \in \mathbb{Q}$ to be in the quantum set. □

Despite this result, there are several instances in which that $Q_{\boldsymbol{\zeta_k}}$ is nonempty.

Lemma 4.5. *Let $\boldsymbol{\zeta_k} = (\zeta^1_{\beta_1}, ..., \zeta^1_{\beta_k})$, where the β_j's are coprime, each $\beta_j > 4$, and suppose there exists a $\beta_e \equiv 2 \pmod 4$. Then, there exists a $d \in \mathbb{N}$ such that $\frac{h}{d} \in Q_{\zeta_k}$, for all $h \in \mathbb{Z}$ such that $gcd(h, d) = 1$.*

Proof. We will show that there exists an odd d satisfying Definition 1.5. First, we will consider the case when $j = e$. Observe that, if $d = m\beta_e + \frac{\beta_e}{2}$, for some $m \in \mathbb{Z}$, then:

$$\left|\frac{m\beta_e + \frac{\beta_e}{2}}{\beta_j} - \left[\frac{m\beta_e + \frac{\beta_e}{2}}{\beta_e}\right]\right| = \left|\frac{\frac{\beta_e}{2}}{\beta_e} - \left[\frac{\frac{\beta_e}{2}}{\beta_e}\right]\right| = \left|\frac{1}{2} - \left[\frac{1}{2}\right]\right| > \frac{1}{3}.$$

Also, note that $\beta_e \nmid d$ because $d \neq 0 \pmod{\beta_e}$. So, d satisfies the conditions in Theorem 1.5 in this case.

Now, consider β_j for $j \neq e$. Because we require the β_j's to be coprime, and β_e was even, β_j must be odd. Let $\omega_j = \lceil \frac{\beta_j}{2} \rceil$. Then, if $d = m\beta_j + \omega_j$, for some $m \in \mathbb{Z}$:

$$\begin{aligned}\left|\frac{m\beta_j + \omega_j}{\beta_j} - \left[\frac{m\beta_j + \omega_j}{\beta_j}\right]\right| &= \left|\frac{\omega_j}{\beta_j} - \left[\frac{\omega_j}{\beta_j}\right]\right| \\ &= \left|\frac{\omega_j}{2\omega_j - 1} - \left[\frac{\omega_j}{2\omega_j - 1}\right]\right|,\end{aligned}$$

because β_j is odd. This expression is greater than $\frac{1}{3}$ when $\omega_j > 2$. Since $\beta_j > 4$, this is always true. Also, since $\omega_j \neq 0$, $\beta_j \nmid d$.
Hence, for $Q_{\boldsymbol{\zeta_k}}$ to be nonempty, we seek an odd d such that:

$$d \equiv a_j (\text{mod } \beta_j),$$

where $a_j = \lceil \frac{\beta_j}{2} \rceil$ if $j \neq e$, and $a_j = \frac{\beta_j}{2}$ otherwise.

Because the β_j's are coprime, a solution to this system exists by the Chinese Remainder Theorem. Since $\beta_e \equiv 2 \pmod 4$, the condition $d \equiv \frac{\beta_e}{2} (\text{mod } \beta_e)$ implies that:

$$\begin{aligned} &(\text{where } m \in \mathbb{Z}) & d &= m\beta_e + \frac{\beta_e}{2} \\ &(\text{where } p \in \mathbb{Z}) & &= m\beta_e + (2p + 1). \end{aligned}$$

Thus, because β_e is even, d must be odd. Therefore, there exists an odd $d \in \mathbb{N}$ such that $\beta_j \nmid d$ and:

$$\left|\frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d\right]\right| > \frac{1}{3},$$

for all β_j, as desired. □

Lemma 4.6. *Let $\boldsymbol{\zeta_k} = (\zeta^1_{\beta_1}, ..., \zeta^1_{\beta_k})$, where the β_j's are coprime, each $\beta_j > 12$, and there exists an even $\beta_e \equiv 0 \pmod 2$. Then, there exists a $d \in \mathbb{N}$ such that $\frac{h}{d} \in Q_{\zeta_k}$, for all $h \in \mathbb{Z}$ such that $gcd(h, d) = 1$.*

Proof. We will find an even $d \in \mathbb{N}$ satisfying Definition 1.5.

If $d = m\beta_j + 2$ for some $m \in \mathbb{Z}$, then:

$$\left|\frac{m\beta_j + 2}{\beta_j} - \left[\frac{m\beta_j + 2}{\beta_j}\right]\right| = \left|\frac{2}{\beta_j} - \left[\frac{2}{\beta_j}\right]\right| = \left|\frac{2}{\beta_j} - \left[0\right]\right| < \frac{1}{6},$$

because $\beta_j > 12$. Also, $\beta_j \nmid d$ because $d \neq 0 \pmod{\beta_j}$. So, d satisfies the conditions in Definition 1.5 in this case

Hence, for $Q_{\boldsymbol{\zeta_k}}$ to be nonempty, we seek an even d where

$$d \equiv 2 \pmod{\beta_j}.$$

Because the β_j's are coprime, a solution to this system exists by the Chinese Remainder Theorem. Since $\beta_e \equiv 0 \pmod 2$, the condition $d \equiv \frac{\beta_e}{2} \pmod{\beta_e}$ implies that:

$$\begin{aligned} &(\text{for } m \in \mathbb{Z}) && d = m\beta_e + 2 \\ &(\text{for } p \in \mathbb{Z}) && \quad = m\beta_e + (2p). \end{aligned}$$

Since β_e is even, d is too. Therefore, there exists an even d such that $\beta_j \nmid d$ and:

$$\left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right| < \frac{1}{6},$$

for all β_j. So, $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$, for all $h \in \mathbb{Z}$ where $\gcd(h, d) = 1$, as desired. $\square$

Unlike the last two lemmas, the next result leaves the α_j's mostly unrestricted.

Lemma 4.7. *Let* $\boldsymbol{\zeta_k} = (\zeta_{\beta_1}^{\alpha_1}, ..., \zeta_{\beta_k}^{\alpha_k})$, *where:*

- *the* β_j*'s are coprime*
- *each* $\beta_j > 12$
- *gcd*$(\alpha_j, \beta_j) = 1$ *for* $1 \leqslant j \leqslant k$
- *there exists a* β_e *and* α_e *such that* $\beta_e \equiv 0$ *(mod 2) and* $\alpha_e = 1$

Then, there exists a $d \in \mathbb{N}$ *such that* $\frac{h}{d} \in Q_{\zeta_k}$, *for all* $h \in \mathbb{Z}$ *where gcd*$(h, d) = 1$.

Proof. First, we will consider what d is required to satisfy Definition 1.5 when $j = e$. Observe that, if $d = m\beta_e + 2$ for some $m \in \mathbb{Z}$, then:

$$\left| \frac{m\beta_j + 2}{\beta_j} - \left[\frac{m\beta_j + 2}{\beta_j} \right] \right| = \left| \frac{2}{\beta_j} - \left[\frac{2}{\beta_j} \right] \right| = \left| \frac{2}{\beta_j} - \left[0 \right] \right| < \frac{1}{6},$$

because $\beta_j > 12$. Also, note that $\beta_j \nmid d$ because $d \neq 0 \pmod{\beta_j}$. So, d satisfies the conditions in Definition 1.5 in this case

Now consider $j \neq e$. Let $g_j \geqslant 2$ denote the largest integer where $\beta_j > 6g_j$. We seek $\alpha_j d$ such that $\alpha_j d \equiv g_j \pmod{\beta_j}$. If $\alpha_j^{\phi(\beta_j) \equiv 1} \pmod{\beta_j}$, then

$$g_j \alpha_j^{\phi(\beta_j)} \equiv g_j (\text{mod} \beta_j).$$

Thus,

$$\alpha_j (g_j \alpha_j^{\phi(\beta_j) - 1}) \equiv g_j (\text{mod} \beta_j).$$

So, to guarantee that $\alpha_j d \equiv g_j$ has a solution, we should choose $d \equiv g_j \alpha_j^{\phi(\beta_j)-1} \pmod{\beta_j}$. Doing so yields

$$\begin{aligned} \text{(where } d \in \mathbb{Z}) \qquad \left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right| &= \left| \frac{d\beta_j + g_j}{\beta_j} - \left[\frac{d\beta_j + g_j}{\beta_j} \right] \right| \\ &= \left| \frac{g_j}{\beta_j} - \left[\frac{g_j}{\beta_j} \right] \right| \\ &< \frac{1}{6}, \end{aligned}$$

because $\beta_j > 6g_j$. Also, note that $\beta_j \nmid d$ because $g_j \alpha_j^{\phi(\beta_j)-1} \neq 0$.
Hence, for $Q_{\boldsymbol{\zeta_k}}$ to be nonempty, we seek an even d such that:

$$d \equiv a_j (\text{mod } \beta_j),$$

where $a_j = g_j \alpha_j^{\phi(\beta_j)-1}$ if $j \neq e$, and $a_j = 2$ otherwise.

Because the β_j's are coprime, a solution to this system exists by the Chinese Remainder Theorem. Since $\beta_e \equiv 0 \pmod 2$, the condition $d \equiv \frac{\beta_e}{2} (\text{mod } \beta_e)$ implies that:

$$\begin{aligned} \text{(for } m \in \mathbb{Z}) \qquad d &= m\beta_e + 2 \\ \text{(for } p \in \mathbb{Z}) \qquad &= m\beta_e + (2p). \end{aligned}$$

Thus, because β_e is even, d is too. Therefore, there exists an even d such that $\beta_j \nmid d$ and:

$$\left| \frac{\alpha_j}{\beta_j} d - \left[\frac{\alpha_j}{\beta_j} d \right] \right| < \frac{1}{6},$$

for all β_j. So, $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$ for all $h \in \mathbb{Z}$ where $\gcd(h, d) = 1$, as desired.

□

The next theorem provides a more powerful assurance that the quantum set is nonempty.

Lemma 4.8. *Let $\beta_1, \ldots, \beta_k \in \mathbb{Z}$ be such that, for $1 \leqslant j \leqslant k$, $\beta_j \notin \{1, 3, 4, 6, 10\}$. Then there exist choices for $\alpha_1, \ldots, \alpha_k$ such that $Q_{\boldsymbol{\zeta_k}} \neq \varnothing$.*

Proof. We will show that if these conditions hold, then $d = 1$ is a denominator in $Q_{\boldsymbol{\zeta_k}}$, which permits many choices of numerator h. (In particular this means that $\mathbb{Z} \subseteq Q_{\boldsymbol{\zeta_k}}$.)

First, we treat all odd choices for β_j greater than 3. Choose $\alpha_i = \lfloor \frac{\beta_j}{2} \rfloor = \frac{\beta_j - 1}{2}$. Because

$$\left| \frac{\frac{\beta_j - 1}{2}}{\beta_j} - \left[\frac{\frac{\beta_j - 1}{2}}{\beta_j} \right] \right| = \left| \frac{1}{2} - \frac{1}{2\beta_j} - \left[\frac{1}{2} - \frac{1}{2\beta_j} \right] \right| > \frac{1}{3}$$

(since, in this case, $\beta_j \geqslant 5$, so $\frac{1}{2\beta_j} \leqslant \frac{1}{10} < \frac{1}{6}$.)
Now we will consider even β_js. When $\beta_j = 2$, choosing $\alpha_j = 1$ yields

$$\left| \frac{1}{2} - \left[\frac{1}{2} \right] \right| = \frac{1}{2} > \frac{1}{3}.$$

Next, consider all $\beta_j \equiv 0 \pmod 4$. Choose
$\alpha_j = \frac{\beta_j - 2}{2}$, and note that because $\beta_j \equiv 0 \pmod 4$, $\frac{\beta_j}{2} + 1$ is odd. Hence, by Euclid's Algorithm, $\gcd(\beta_j, \frac{\beta_j - 2}{2}) = \gcd(\frac{\beta_j}{2} + 1, \frac{\beta_j}{2} - 1) = \gcd(\frac{\beta_j}{2} + 1, 2) = 1$. So:

$$\left|\frac{\frac{\beta_j-2}{2}}{\beta_j}-\left[\frac{\frac{\beta_j-2}{2}}{\beta_j}\right]\right|=\left|\frac{1}{2}-\frac{1}{\beta_j}-\left[\frac{1}{2}-\frac{1}{\beta_j}\right]\right|>\frac{1}{3},$$

because $\frac{1}{\beta_j}<\frac{1}{6}$.

Finally, we argue for all even choices for $\beta_j \equiv 2 \pmod 4$ where $\beta_j > 10$. Similar to before, choose $\alpha_j = \frac{\beta_j-4}{2}$. Because $\beta_j \equiv 2 \pmod 4$, $\frac{\beta_j}{2}+2$ is odd. So, by Euclid's Algorithm, $\gcd(\beta_j, \frac{\beta_j-2}{2}) = \gcd(\frac{\beta_j}{2}+2, \frac{\beta_j}{2}-2) = \gcd(\frac{\beta_j}{2}+2, 4) = 1$. Hence:

$$\left|\frac{\frac{\beta_j-4}{2}}{\beta_j}-\left[\frac{\frac{\beta_j-4}{2}}{\beta_j}\right]\right|=\left|\frac{1}{2}-\frac{2}{\beta_j}-\left[\frac{1}{2}-\frac{2}{\beta_j}\right]\right|>\frac{1}{3},$$

as $\frac{2}{\beta_j}<\frac{1}{6}$.

The above cases cover all possible β_j's not in the excluded cases. Thus, no matter what combination of β_js we choose from $\mathbb{N}-\{1,3,4,6,10\}$, $d=1$ will always satisfy the necessary inequalities with the choices of α_j outlined above. $\square$

As the previous lemma alluded to, $\beta_j = 6$ (for some j) may yield an empty quantum set. As the next lemma shows, this scenario does not always happen.

Lemma 4.9. *Let $\beta_1,\dots,\beta_k \in \mathbb{Z}$ be such that at least one $\beta_j = 6$. Then, if every other β_j is a multiple of 3 and not in $\{3,9,12,18,24,30\}$, there exist $\alpha_1,\dots,\alpha_k$ so that $Q_{\boldsymbol{\zeta_k}} \neq \varnothing$.*

Proof. Let $d=3$. Then, since each $\beta_j > 12$, $\beta_j \mid d$.

For all $\beta_j = 6$, let $\alpha_j = 1$. Then:

$$\left|\frac{\alpha_j}{\beta_j}d-\left[\frac{\alpha_j}{\beta_j}d\right]\right| = \left|\frac{3}{\beta_j}-\left[\frac{3}{\beta_j}\right]\right| = \left|\frac{1}{2}-\left[\frac{1}{2}\right]\right| > \frac{1}{3}.$$

Now, consider $\beta_j \neq 6$. Let $m_j \in \mathbb{Z}$ be such that $\beta_j = 3m_j$. Then:

$$\left|\frac{\alpha_j}{\beta_j}d-\left[\frac{\alpha_j}{\beta_j}d\right]\right| = \left|\frac{3\alpha_j}{\beta_j}-\left[\frac{3\alpha_j}{\beta_j}\right]\right| = \left|\frac{\alpha_j}{m_j}-\left[\frac{\alpha_j}{m_j}\right]\right|.$$

This is equivalent to the scenario where $d=1$. Since $\beta_j > 12, m_j \notin \{1,3,4,6,10\}$. Hence, by Lemma 4.8, there exists an α_j such that:

$$\left|\frac{\alpha_j}{\beta_j}d-\left[\frac{\alpha_j}{\beta_j}d\right]\right| = \left|\frac{\alpha_j}{m_j}-\left[\frac{\alpha_j}{m_j}\right]\right| > \frac{1}{3}.$$

Since $\beta_j \neq 24$, $\alpha_j \neq 3$, so the α_j constructed as in Lemma 4.8 is relatively prime to β_j.

Hence, if each $\beta_j \neq 6$ is a multiple of 3 and not in $\{3,9,12,18,2430\}$, there exists a $d \in \mathbb{N}$ such that $\beta_j \nmid d$ and

$$\left|\frac{\alpha_j}{\beta_j}d-\left[\frac{\alpha_j}{\beta_j}d\right]\right| > \frac{1}{3},$$

for $1 \leqslant j \leqslant n$. Therefore, $\frac{h}{d} \in Q_{\boldsymbol{\zeta_k}}$, for all $h \in \mathbb{Z}$ where $\gcd(h,d)=1$. So, $Q_{\boldsymbol{\zeta_k}} \neq \varnothing$, as desired. $\square$

With the candidate quantum set described, we discuss evidence that U_k is a quantum modular form for some choices of $\boldsymbol{\zeta_k}$, under the action of S_ℓ.

5. Quantum Modularity Conjecture

There is precedent to investigate $U_k(-1,-1,\ldots,-1,q)$ for quantum modularity properties. In particular, Bryson et al. proves that $e^{-\frac{\pi i x}{12}}U_1(-1,e^{2\pi i x})$ is a weight $\frac{3}{2}$ quantum modular form on $\mathbb{Q}\backslash\{0\}$ in [7] in a manner that ties the result to the same function's mock modularity.

Question 5.1. *If $\boldsymbol{\zeta_k} = (-1,-1,\ldots,-1)$, then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\boldsymbol{\zeta_k}}$ and for all $x \in Q_{\boldsymbol{\zeta_k}}$, then does the function*

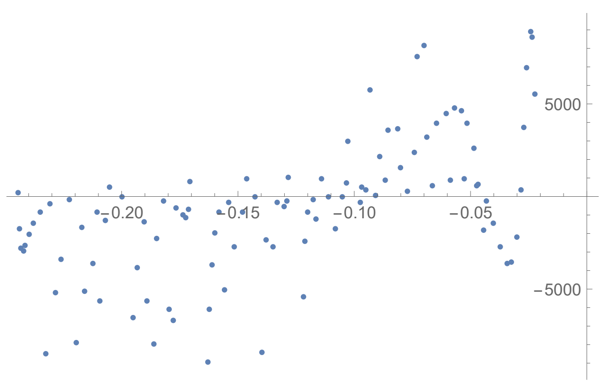
$$h_{n,\gamma(x)} := e^{-\frac{\pi i x}{12}}U_k(\boldsymbol{\zeta_k}, e^{2\pi i x}) - (cx+d)^{-\frac{3}{2}}e^{-\frac{\pi i(x+1)}{12}}U_k(\boldsymbol{\zeta_k}, e^{2\pi i\gamma x})$$

extend to a continuous function in x on $\mathbb{R}$?

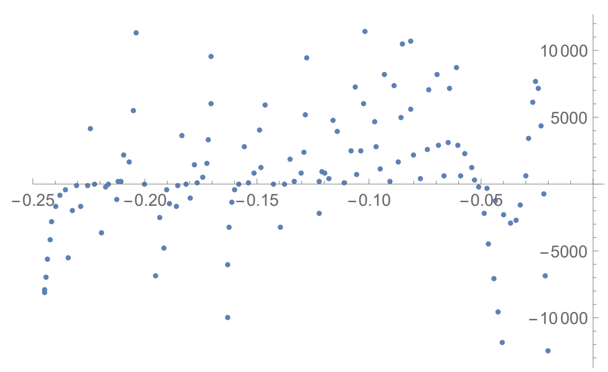
To investigate quantum modularity it suffices to look for continuity with respect the generators of $\Gamma_{\boldsymbol{\zeta_k}}$, which are S_ℓ, T of $\Gamma_{\boldsymbol{\zeta_k}}$ as given in 4. Recall from Theorem 4.3 that we have quantum modularity obviously for $\langle T\rangle$. It remains to investigate the action of S_ℓ.

5.1. **Computational Evidence.** Next, we investigated the continuity of h_{n,S_ℓ} graphically. In order to do so, Mathematica code was generated that, given some fixed $\boldsymbol{\zeta_k}$, would find elements in $Q_{\boldsymbol{\zeta_k}} \cap [-1,1]$. Subsequent functions then calculated $h_{n,S_\ell}(\frac{h}{k})$ on $Q_{\boldsymbol{\zeta_k}}$ for some subinterval of [-1,1] and plotted them within the desired window.

One difficulty in plotting such functions is that in order to ensure that the sum defining U_k truncates, the necessary upper bound of the sum indices m_i in U_k must increase as d increases in $h_{n,S_\ell}(\frac{h}{d})$. In particular, $h_{n,S_\ell}(\frac{h}{d})$ requires the calculation of $U_k(\zeta_k; \frac{h}{h\ell+d})$, and as $-d \leqslant h \leqslant d$, the number of necessary computations to pass the truncation point for this sum quickly grows with d. Early investigations truncated the sum at an arbitrary early bound for m_i, which resulted in graphs that appeared smooth. However, further work revealed that this early truncation was not accurately representing the situation, and edits to reflect this produced the following plots, which do not seem to clearly extend to a continuous function. For a feasible run time, the following plots depict $h_{2,S_\ell}(x)$ for x in the quantum set that have denominator less than or equal to 50.



(A)



(B)

Figure 4. The real part (A) and imaginary part (B) of h_{2,S_ℓ} graphed on $[-0.25, 0]$.

Clearly these figures are not necessarily an argument against quantum modularity–they just do not provide convincing enough evidence to provoke confidence in a conjecture.

5.2. **Conjectures and Further Work.** While graphing did not yield convincing evidence of modularity properties, there are many possible parameters that could be changed on our conjecture and that should be investigated. We note that the inclusion of the factor of $e^{-\frac{\pi i x}{12}}$ comes from precedent of previous work, namely Bryson et al. and Folsom et al. in [7] and [9]. Graphing of h_{n,S_ℓ} without such a factor resulted in barely visible change on the graph that did not appear to be more or less continuous than the images presented above, it's possible that a different multiplier would reveal quantum modularity.

It is also possible that $\boldsymbol{\zeta_k}$ consisting of other roots of unity yield modular objects. We chose to focus on $U_k(-1,\ldots,-1;q)$ because it clearly truncates at a computable place. When all $x_i = \zeta_2^1 \in \boldsymbol{\zeta_k}$, then all Pochhammer symbols in U_k are of the form $(-\zeta_2^1 q^k;q)_j = (q^k,q)_j$ for some j,k. This Pochhammer product is guaranteed to equal zero for $q = e^{2\pi i a/b}$ a root of unity when j exceeds b. Thus high enough choices of m_i (i.e. greater than b) will yield truncating sums. In contrast, for an arbitrary $(-\zeta_\beta^\alpha q^k;q)_j$ it is not necessarily the case that $-\zeta_\beta^\alpha q^k$ eventually equals 1 as k increases, which means that a term in the product would not necessarily be assured to ever equal 0, so the convergence to its limiting value will be incremental.

Finally, it remains possible that $U_2(-1,-1;q)$ or in fact many functions derivable from U_k are closely related to other types of modular object besides quantum modular forms.

In general, work this summer revealed it is certainly feasible to code as a means to test hypotheses about quantum modularity, although generating functions that are more than double summations would benefit from more computational power. Here is an instance where a single sum generating function like that achieved for R_k in [2] would be considerably more efficient.

In addition to investigating U_k for modularity, we also sought analogues between the full rank of a k-marked Durfee symbol and the full unimodal rank. The following conjecture is based on calculations for small values of n; it is remarkably similar to the generating function for $NF_k(m,n)$, given in the proof of Theorem 17 in [2]:

Conjecture 5.2.

$$\sum_{n\geqslant 1}\sum_{m=-\infty}^{\infty} NFU_2 x^m q^n = U_2(x,x^2;q).$$

The following conjecture is also based off calculations for small values of n, and resembles Theorem 17 from [2].

Conjecture 5.3. *For $n \geqslant 0$:*

$$NFU_2(1,5,n) = NF_2(2,5,n).$$

In conclusion, the results detailed in this report indicate that much remains to be explored in terms of the properties of $U_k(\boldsymbol{\zeta_k};q)$, both with regard to modularity and to congruences. A single-sum generating function for $U_k(\boldsymbol{\zeta_k};q)$ would aid in the tools available to approach this problem.

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Harvey Mudd College
E-mail address: sammons@g.hmc.edu

Reed College
E-mail address: yokim@reed.edu

Haverford College
E-mail address: lseaberg@haverford.edu