AN EXPLICIT AND ANALYTICAL SOLUTION TO THE STOKES EQUATIONS ON THE HALF-SPACE $\mathbb{R}^3_+$ WITH INITIAL CONDITIONS AND BOUNDARY CONDITIONS FOR VELOCITY USING INTEGRAL TRANSFORMS

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ABSTRACT. In this paper we present explicit and analytical solutions to the Stokes equation on the half-space, $\mathbb{R}^3_+$, with data for initial velocity and boundary velocity. Our approach to a solution for velocity and pressure involves integral transformations, Green's functions, and a Helmholtz decomposition. Our main results demonstrate that the velocity derived from this initial value and boundary value problem can be expressed as the curl of the convolution of vorticity and the fundamental solution to the Laplace equation in our domain.

1. INTRODUCTION

The Navier-Stokes equations describe the motion of an incompresisble homogeneous fluid in terms of its velocity and pressure in a three-dimensional space, where incompressibility can be defined as constant density by volume[1]. Our understanding and computation of these equations allows us to model a spectacular range of physical phenomena from combustion and turbulence, to the motion of rivers, to the weather forecast. Due to the nonlinear nature of these equations and other the difficulties solving them, there's a strong call for research in this area. This paper aims to solve a linearized verison of the Navier-Stokes equations called the Stokes equation, which serves as a strong approximation for the original, more complex model.

The Navier-Stokes equation in $\mathbb{R}^3$ is shown below

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - \nabla \pi + \vec{f} \tag{1}$$

where $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is velocity, t is time, ν is viscosity, π is pressure, and $\vec{f} = \langle f_1, f_2, f_3 \rangle$ represents given, conservative external forces such as gravity. The above equation can be split up component-wise into the three equations below

$$\frac{\partial v_i}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - (\nabla \pi)_i + f_i, \quad i = 1, 2, 3. \tag{2}$$

We seek solutions to these equations for velocity and pressure on the half-space instead of all of $\mathbb{R}^3$ so that we have an initial-value and boundary-value problem. Notice, however, that

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we have three equations, but four unknowns: v_1, v_2, v_3, and π. For this reason, we introduce the condition of incompressibility of the fluid, which can be described physically as constant density throughout. Mathematically, incompressibility is described as setting the divergence of the velocity of the fluid to zero, i.e., $\nabla \cdot \vec{v} = 0$, as opposed to a compressible fluid where $\nabla \cdot \vec{v} \neq 0$.

Now that we have the four equations for our system laid out, we will elaborate further on their physical interpretation. We can describe the Navier-Stokes equations as Newton's Second Law, i.e., $F = ma$ or $a = F/m$. The left side of the Navier-Stokes equations is acceleration, or the derivative of velocity. Specifically, the left hand side, by definition, is the material derivative of velocity, $\frac{D\vec{v}}{Dt}$. A material derivative describes the rate of change of some physical quantity (velocity, in this case) of a material element (an incompressible fluid) which is subject to a velocity vector field which depends on space and time [7]. Thus, the material derivative concept helps us see why the left hand side is an acceleration. The right hand side can be thought of as the external forces (i.e. gravity) and internal forces (i.e. viscosity) in terms of density, pressure, and other forces.

Even on the half-space, this equation is very difficult to solve due to its nonlinear nature and because there are two unknowns in the same equation, to name a couple examples. For this reason, in order to gain insight into the motion of the fluid, we linearize the Navier-Stokes equations and solve the linearized version–the Stokes equation. This allows us to approximate the Navier-Stokes equations in certain situations with certain conditions.

We begin by assuming that $\vec{f}$ in Equations (1) and (2), which is the sum of external forces acting on the incompressible fluid, is either equal to zero or conservative. Because $\vec{f}$ is often just the force of gravity, it is fair to assume that it is conservative. If $\vec{f}$ is equal to zero, then we can remove it from the equation entirely. If $\vec{f}$ is conservative, we can write it as a gradient of a potential function, say, ∇g. We then have $\frac{\partial \vec{v}}{\partial t} = \nu \triangle \vec{v} - \nabla \pi + \nabla g$, which can be expressed as $\frac{\partial \vec{v}}{\partial t} = \nu \triangle \vec{v} - \nabla(\pi - g)$ by linearity of the gradient operator [4]. Let p be defined as the quantity $\pi - g$. Then, we have the version of the Navier-Stokes equations that we will linearize:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - \nabla p \tag{3}$$

Equation (3) can be linearized through a dimensionless process [1]. As viscosity becomes sufficiently large and/or velocity becomes sufficiently small, then the nonlinear term $(\vec{v} \cdot \nabla)\vec{v}$ can be ignored. Physically, we can interpret this linearization as slowing down the fluid flow by a large amount and/or making the fluid so viscous that there is no turbulence or swirling anymore. After the linearization process, we have a good approximation for the Navier-Stokes equations. We let viscocity be equal to 1 for simplicity, and we are left with the version of the Stokes equation that we will study in this paper:

$$\frac{\partial \vec{v}}{\partial t} = \nu \triangle \vec{v} - \nabla p. \tag{4}$$

The last piece to setting up our problem is to establish initial and boundary conditions for velocity. Velocity must have continuous first order partial derivatives with respect to time, and continuous second order partial derivatives with respect to space. We wish to solve the Stokes equation for general initial and boundary velocities. We will refer to these values as $\vec{v}_0(x, y)$ and $\vec{a}(x, t)$, respectively, where $x = \langle x_1, x_2 \rangle$ and $y > 0$. Now we have our complete

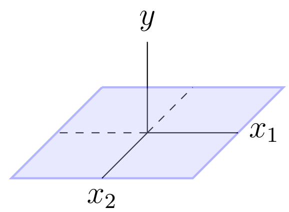

FIGURE 1. The Half Space, $\mathbb{R}^3_+$

setup to solve the Stokes equation on the half-space for an incompressible fluid with data for initial velocity and boundary velocity:

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p \quad \text{on} \quad \mathbb{R}^3_+ \\ \nabla \cdot \vec{v} = 0 \\ \vec{v}(x, y, 0) = \vec{v}_0(x, y), \quad x \in \mathbb{R}^2, \quad y \in (0, \infty) \\ \vec{v}(x, 0, t) = \vec{a}(x, t), \quad x \in \mathbb{R}^2, \quad t \in (0, \infty) \\ \vec{a} \cdot \vec{n} = 0 \end{cases} \tag{5}$$

where the last equation states the condition that the boundary velocity at $y = 0$ has zero normal component.

In order to solve System (5) with its general initial and boundary data for velocity, we will split the problem up into two cases with differing conditions for initial and boundary velocity. Specifically, Cases 1 and 2 as follows:

(1) Initial data $\neq 0$, boundary data $= 0$.
(2) Initial data $= 0$, boundary data $\neq 0$.

Then, by linearity, we can add together our solutions for $\vec{v}$ for Cases 1 and 2 to get a general solution for $\vec{v}$, and we can apply the same principle for p. For each case, we approach the problem to solve for velocity and pressure using Fourier transforms, Laplace transforms, and the Helmholtz Decomposition Theorem.

For **Case 1**, our explicit formula for velocity in the Stokes Equation when initial velocity is nonzero and boundary velocity is zero is *the curl of the convolution of the fundamental solution to the Laplace equation against the vorticity,* as seen below:

$$\vec{v} = \nabla \times \int_0^\infty \int_{\mathbb{R}^2} \vec{w}(x', y, t) \left(\frac{1}{4\pi\sqrt{|x - x'|^2 + (y + y')^2}} - \frac{1}{4\pi\sqrt{|x - x'|^2 + (y - y')^2}} \right) dx'dy'$$

where the curl is with respect to x_1, x_2, y. In this formula above, x represents the tangential spacial variables x_1 and x_2, and y is the normal spacial variable. Furthermore, $\vec{w}$ is vorticity which is the curl of velocity, i.e., $\vec{w} = \nabla \times \vec{v}$.

For **Case 2**, our explicit formula for velocity in the Stokes Equation when initial velocity is zero and boundary velocity is nonzero is

$$\vec{v}_2 = \nabla\phi + \nabla \times \vec{A}$$

where

$$\phi = \frac{-1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \frac{e^{-y^2\tau}}{\sqrt{\tau}} \frac{e^{-2|x-x'|^2}}{\tau} a_3(x',t)dx'd\tau$$
$$\vec{A} = \int_{\mathbb{R}^3_+} \vec{w}\Big(\frac{1}{4\pi\sqrt{|x-x'|+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|+(y-y')^2}}\Big)dx'dy'.$$

Because our Stokes equation is linear, by the Superposition Principle we can express our total velocity on the half-space to be the sum of our two velcoities, $\vec{v}_1, \vec{v}_2$ such that:

$$\vec{v} = \vec{v}_1 + \vec{v}_2, \qquad \in \mathbb{R}^3_+$$

Pressure can be derived from velocity, $\vec{v}$, using the Stokes Equation. The equation, $\nabla p = (\frac{\partial}{\partial t} - \triangle)\vec{v}$, can be simplified down to its component parts $\langle p_1, p_2, p_3\rangle = \langle f_1, f_2, f_3\rangle$ where $f_i = \frac{\partial \vec{v}_i}{\partial t} - (\frac{\partial^2 \vec{v}_i}{\partial x_1^2} + \frac{\partial^2 \vec{v}_i}{\partial x_2^2} + \frac{\partial^2 \vec{v}_i}{\partial y^2})$. Solving for each component of p reduces to a problem of solving for a potential function from a conservative vector field. Solving each equation $p_i = f_i$ would give us a possible expression for p, but not a unique one, as solutions for p will vary by constants due to the fact that it is a potential function. Thus we have obtained a specific solution equation to velocity on the half-space given arbitrary initial and boundary data, as well as a specific process to find the pressure.

2. Background

In this section, we introduce definitions, notations, and previous results.

Definition 2.1. *Let $u = u(t)$ be a piecewise continuous function on $t \geq 0$ and of exponential order (i.e. $|u(t)| \leq ce^{at}$ for t sufficiently large and $c, a > 0$). Then, the the Laplace Transform of $u(t)$ is defined as*

$$\mathcal{L}[u(t)] = U(s) = \int_0^\infty e^{-st}u(t)dt.$$

A *Laplace Transform* is an integral transform which converts a function of a real variable t to a function of a complex variable s. A Laplace transform holds the following properties for derivatives of $u(t)$:

$$\mathcal{L}[u'(t)] = sU(s) - u(0)$$
$$\mathcal{L}[u''(t)] = s^2U(s) - su(0) - u'(0)$$

[3]

For computing Inverse Laplace transforms, it is convention to use a table of Laplace Transforms. [3]

Definition 2.2 (Convolution for Laplace)**.** *If $f(t)$ and $g(t)$ are piecewise continuous functions on $[0, \infty]$ then the Convolution Integral of $f(t)$ and $g(t)$ is* [6],

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Theorem 2.3 (The Convolution Theorem for Laplace Transforms)**.** *Given $f(t)$ and $g(t)$ with Laplace transforms F and G, respectively, seen below, allows us to take inverse transforms of transform products.*

$$\mathcal{L}[(f * g)] = F(s)G(s)$$
$$(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)].$$

The Convolution Integral and Convolution Theorem are important tools when taking integral and inverse integral transforms of equations that can't be easily solved using partial fractions. In both Case 1 and Case 2, we use the Convolution Theorem to take the inverse Laplace transform of our Green's Function to obtain an explicit expression for each component of vorticity, $\vec{w}$. Again, u being absolutely integrable indicates that it is in the L^1 function space.

Definition 2.4. *Let $u = u(x), \quad x \in \mathbb{R}^n$ be absolutely integrable, i.e., $u \in L^1 = \{h : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} |h| dx < \infty\}$. Then, the Fourier Transform of $u(x)$ where x and ξ are n-dimensional vector quantities is defined as*

$$\mathcal{F}[u(\vec{x})] = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-\vec{\xi}\vec{x}} u(\vec{x}) d\vec{x}$$

and its inverse

$$\mathcal{F}^{-1}[\hat{u}(\vec{\xi})] = u(\vec{x}) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{\vec{\xi}\vec{x}} \hat{u}(\vec{\xi}) d\vec{\xi}.$$

A *Fourier Transform* decomposes functions depending on space and time into functions depending on spatial and temporal frequency. A Fourier transform with respect to x holds the following properties for partial derivatives of $u(x, t)$: [6]

$$\mathcal{F}[\frac{\partial}{\partial x}\vec{u}(x,t)] = (-i\xi)\hat{u}(\xi)$$
$$\mathcal{F}[\frac{\partial^2}{\partial x}\vec{u}(x,t)] = (-i\xi)^2\hat{u}(\xi)$$
$$\mathcal{F}[\frac{\partial}{\partial t}\vec{u}(x,t)] = \frac{\partial}{\partial t}\hat{u}(\xi)$$

Definition 2.5 (Convolution for Fourier)**.** *If $g(x)$ and $h(x)$ are piecewise continuous functions on $[0, \infty]$ then the Convolution Integral of $g(x)$ and $h(x)$ is* [6]*,*

$$(g * h)(x) = \int_{\mathbb{R}^2} g(x - x')h(x')dx'.$$

Theorem 2.6 (The Convolution Theorem for Fourier Transforms)**.** *If $g(x)$ and $h(x)$ are two functions with Fourier transforms $\hat{g}(\xi)$ and $\hat{h}(\xi)$, respectively, then the Convolution Theorem is as follows* [6]*,*

$$\mathcal{F}[(g * h)] = g\hat{(\xi)}\hat{h}(\xi)$$
$$(g * h)(x) = \mathcal{F}^{-1}[g\hat{(\xi)}\hat{h}(\xi)]$$

Similar to using convolution for the Inverse Laplace transform as stated above, we also use the Convolution Theorem to take the inverse Fourier transform of our Green's Function to obtain an explicit expression for each component of vorticity, $\vec{w}$.

It is of importance to note that we refer to $\langle x_1, x_2\rangle$ as x. This will remain true throughout the entirety of our calculations.

3. Results

The aim of this section is to provide our results from solving the Stokes Equation in the half-space.

Recall that in order to solve System (5), which has general data for initial and boundary velocity, we split up the problem into two cases:

(1) Initial data $\neq 0$, boundary data $= 0$.
(2) Initial data $= 0$, boundary data $\neq 0$.

We then can add our solutions for velocity in Cases 1 and 2 to reach our general expression for velocity, and the same for pressure. This process is justified by the linearity of the differential equation, Equation (4). In each case, we approach our solution for velocity and pressure with integral transforms and the Helmholtz Decomposition Theorem.

3.1. Case 1.

Recall that we wish to solve the Stokes equation for each of three dimensions of velocity and pressure for an incompressible fluid. That is

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p \\ \nabla \cdot \vec{v} = 0 \end{cases} \tag{6}$$

where the first equation in (6) represents three identical equations–one for each dimension of velocity.

Let our velocity $\vec{v}$ be a function of space in three deimensions and time–that is, $\vec{v} = \vec{v}(x, y, t)$. In our first case, we will solve the Stokes equation (6) with the following data for initial velocity $\vec{v_0}$ and boundary velocty $\vec{v_b}$

$$\vec{v}(x, y, 0) = \vec{v_0}(x, y) \neq 0 \tag{7}$$

$$\vec{v}(x, 0, t) = \vec{v_b}(x, t) = \vec{a}(x, t) = 0 \tag{8}$$

i.e., initial velocity is nonzero, and boundary velocity (on the $x - y$ plane) is zero.

We begin the task of solving the Stokes equation with this initial and boundary data for velocity by taking the curl of both sides of the first equation in (6), yielding

$$\nabla \times \frac{\partial}{\partial t}\vec{v} = \triangle(\nabla \times \vec{v}) \tag{9}$$

which, applying the definition for vorticity, we see is equivalent to

$$\frac{\partial}{\partial t}\vec{w} = \triangle \vec{w}, \tag{10}$$

the homogeneous heat equation for vorticity. Recall that Equation (10) represents three identical equations–one for each of the components of $\vec{w}$. Our approach to solving Equation (10) involves Fourier and Laplace integral transforms and the method of variation of parameters.

Proposition 3.1. *Let velocity $\vec{v}(x,y,t)$ be differentiable in t, twice differentiable in x and y, and let $\vec{v} \in L^1(\mathbb{R}^2)$. Additionally, let $\vec{w}$ hold the assumptions stated in Definitions (2.1) and (2.4). Then, the Laplace transform of the Fourier transform of the curl of the velocity (vorticity) is,*

$$\mathcal{L}\mathcal{F}[\vec{w}] = g(y) = \int_0^\infty G(y,t)\cdot f(t)dt, \tag{11}$$

where

$$G(y,t) = \begin{cases} \frac{e^{-\alpha y}(e^{-\alpha t} - e^{\alpha t})}{2\alpha} & 0 < t < y \\ \frac{e^{-\alpha t}(e^{-\alpha y} - e^{\alpha y})}{2\alpha} & y < t < \infty \end{cases}$$

Proof. Let $\vec{w} = \triangle \vec{w}$ be true, as shown in Equation (10). Then, taking the Fourier transform (2.4) of both sides of Equation (10) with respect to x_1 and x_2 gives us

$$\frac{\partial}{\partial t}\hat{w} = -|\xi|^2\hat{w} + \frac{\partial^2}{\partial y^2}\hat{w} \tag{12}$$

where $\hat{w} = \mathcal{F}[w(x,y,t)]$ is a function of (ξ, y, t) and $\xi\langle\xi_1, \xi_2\rangle$.

However, equation (12) is still a partial differential equation. We will simplify our computations further by transforming this equation into an ordinary differential equation. Taking the Laplace transform (2.1) of Equation (12) with respect to t we have,

$$s\mathcal{L}[\hat{w}] - \hat{w}\big|_{t=0} = -|\xi|^2\mathcal{L}[\hat{w}] + \frac{\partial^2}{\partial y^2}\mathcal{L}[\hat{w}] \tag{13}$$

where $\mathcal{L}[\hat{w}]$ is a function of (ξ, y, s).

Next, let

$$\vec{g}(\xi, y, s) = g(y) = \mathcal{L}[\mathcal{F}[\vec{w}(x,y,t)]],$$

let

$$\vec{f}(\xi, y, 0) = f(y) = -\mathcal{F}[\vec{w}(x,y,t)]\big|_{t=0},$$

and let

$$\alpha^2 = |\xi|^2 + s.$$

Then, we can rewrite Equation (13) as

$$g''(y) - \alpha^2 g(y) = f(y) \tag{14}$$

which is a second-order, linear, nonhomogeneous, constant-coefficient ordinary differential equation with $\alpha > 0$ and boundary conditions

$$g(0) = 0$$
$$\lim_{\gamma\to\infty} g(\gamma) = 0.$$

Next, to solve Equation (14) we will use the method of variation of parameters as follows.

Let $g(y) = u_1(y)g_1(y) + u_2(y)g_2(y)$. Our goal is to determine u_1, u_2, g_1, and g_2 in order to construct a unique solution $g(y)$ for Equation (14).

Taking the first and second derivative of $g(y)$ we have

$$\begin{cases} g'(y) = u_1'(y)g_1(y) + u_1(y)g_1'(y) + u_2'(y)g_2(y) + u_2(y)g_2'(y) \\ g''(y) = u_1'(y)g_1'(y) + u_1(y)g_1''(y) + u_2'(y)g_2'(y) + u_2(y)g_2''(y). \end{cases}$$

Rewriting Equation (14) we have,

$$u_1'(y)g_1'(y) + u_1(y)g_1''(y) + u_2'(y)g_2'(y) + u_2(y)g_2''(y) - \alpha(u_1(y)g_1(y) + u_2(y)g_2(y)) = f(y).$$

Since we are only concerned about the values of $u_1(y)$ and $u_2(y)$, we can impose $u_1'(y)g_1(y) + u_2'(y)g_2(y) = 0$. Thus, $u_1'(y)g_1'(y) + u_2'(y)g_2'(y) = f(y)$.

Then, for solutions to Equation (14), $g_1(y)$ and $g_2(y)$, we have

$$\begin{cases} u_1'(y)g_1(y) + u_2'(y)g_2(y) = 0 \\ u_1'(y)g_1'(y) + u_2'(y)g_2'(y) = f(y). \end{cases} \tag{15}$$

We can express the above system of equations in matrix form $Ax = b$ with $A = \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix}$, $x = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$, and $b = \begin{pmatrix} 0 \\ f(y) \end{pmatrix}$.

We then impose the condition that the Wronskian $W(y)$ of our system of equations, which is defined as $det(A)$, is nonzero. That is, $g_1g_2' - g_1'g_2 \neq 0$.

Using the system of Equations (15), we solve the system and get the following values for $u_1'(y)$ and $u_2'(y)$:

$$\begin{aligned} u_1'(y) &= \frac{-g_2(y)f(y)}{W} \\ u_2'(y) &= \frac{g_1(y)f(y)}{W}, \end{aligned}$$

where the Wronskian, $W = g_1(y)g_2'(y) - g_1'(y)g_2(y)$.

Next, we impose conditions so we can determine the values of $u_1(y)$ and $u_2(y)$. Let $u_1(0) = 0$ and $\lim_{y\to\infty} u_2(y) = 0$. Then, we determine the values of $u_1(y)$ and $u_2(y)$ are as follows:

$$\begin{cases} u_1(y) = \int_0^y \frac{-g_2(t)f(t)}{W(t)} dt \\ u_2(y) = \int_\infty^y \frac{g_1(t)f(t)}{W(t)} dt. \end{cases}$$

Now that we have determined u_1 and u_2, our final step is to determine g_1 and g_2.

Since g_2 is a solution to $g'' - \alpha^2 g = 0$ and $g_2(0) = 0$,

$$\begin{aligned} r^2 - \alpha r &= 0 \\ r &= \pm\alpha \\ g_2 &= e^{\alpha y} - e^{-\alpha y}. \end{aligned}$$

Additionally, for g_1,

$$\begin{aligned} r &= \pm\alpha \\ g_1 &= c_1 e^{-\alpha y} + c_2 e^{\alpha y}. \end{aligned}$$

However, $c_2 e^{\alpha y} = 0$ because $e^{\alpha y} \to \infty$. Therefore, our values for g_1 and g_2 are as follows:

$$\begin{cases} g_1 &= e^{-\alpha y} \\ g_2 &= e^{\alpha y} - e^{-\alpha y}. \end{cases}$$

By the method of variation of parameters, we obtain the solution in the following form:

$$g(y) = \int_0^y \frac{e^{-\alpha y}(-e^{\alpha t} + e^{-\alpha t})f(t)}{2\alpha} dt + \int_y^\infty \frac{e^{-\alpha t}(e^{-\alpha y} - e^{\alpha y})f(t)}{2\alpha} d(t).$$

We combine this expression into one integral using the method of Green's functions [2] to obtain the following

$$g(y) = \mathcal{LF}[w] = \int_0^\infty \frac{e^{-\sqrt{|\xi|+s}(y+y')} - e^{-\sqrt{|\xi|+s}|y+y'|}}{2\sqrt{|\xi|+s}} f(y')dy'. \tag{16}$$

Therefore, The Laplace transform of the Fourier transform of the curl of the velocity is,

$$\mathcal{LF}[\vec{w}] = g(y) = \int_0^\infty G(y,t) \cdot f(t)dt, \tag{17}$$

where

$$G(y,t) = \begin{cases} \frac{e^{-\alpha y}(e^{-\alpha t} - e^{\alpha t})}{2\alpha} & 0 < t < y \\ \frac{e^{-\alpha t}(e^{-\alpha y} - e^{\alpha y})}{2\alpha} & y < t < \infty \end{cases}$$

□

Next, we will prove a mathematical statement that will allow us to rewrite Equation (16) in a form with which we can take the inverse integral transforms to solve for vorticity.

Proposition 3.2. *If $f \in \mathcal{L}^1(\mathbb{R})$, then $f(ax - \frac{b}{x})$ is also in $\mathcal{L}^1(\mathbb{R})$, and*

$$\frac{1}{a}\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^\infty f(ax - b/x)dx$$

for $a > 0$ and $b \geq 0$.

Proof. Let $a > 0$, $b \geq 0$, and $f \in \mathcal{L}^1(\mathbb{R})$. We want to prove that

$$\frac{1}{a}\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^\infty f(ax - b/x)dx.$$

First, we use u-substitution from x to u such that $au = \frac{-b}{x}$ to obtain

$$\int_{-\infty}^\infty f\left(ax - \frac{b}{x}\right) dx = \int_\infty^\infty f\left(au - \frac{b}{u}\right)\left(\frac{b}{au^2}\right) du.$$

Using the fact that

$$\frac{1}{2a}\left[a\int_{-\infty}^\infty f\left(au - \frac{b}{u}\right) du + a\int_{-\infty}^\infty f\left(au - \frac{b}{u}\right)\cdot\frac{b}{au^2}du\right],$$

we know

$$\int_{-\infty}^\infty f\left(ax - \frac{b}{x}\right) dx = \frac{1}{2a}\int_{-\infty}^\infty f\left(au - \frac{b}{u}\right)\left(a + \frac{b}{u^2}\right) du.$$

Then, we break our integral from $-\infty$ to ∞ into two separate integrals,

$$\frac{1}{2a}\left[\int_{-\infty}^0 f\left(au - \frac{b}{u}\right)\left(a + \frac{b}{u^2}\right) du \quad + \int_0^\infty f\left(au - \frac{b}{u}\right)\left(a + \frac{b}{u^2}\right) du\right].$$

Next, implementing a change of variables where $y = au - \frac{b}{u}$, we obtain the following,

$$\frac{1}{2a}\int_{\infty}^{\infty} f(y)dy + \int_{-\infty}^{\infty} f(y)dy.$$

Then, we can simplify such that,

$$\frac{1}{a}\int_{-\infty}^{\infty} f(y)dy.$$

Since variables of integration are arbitrary,

$$\frac{1}{a}\int_{-\infty}^{\infty} f(y)dy = \frac{1}{a}\int_{-\infty}^{\infty} f(x)dx.$$

Thus,

$$\int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)dx.$$

□

Corollary 3.3. *Let $a > 0$, $b > 0$, and $x \in \mathbb{R}$. Then, $e^{-2ab} = \frac{a}{\sqrt{\pi}}\int_0^\infty \frac{e^{-a^2\tau - \frac{b^2}{\tau}}}{\sqrt{\tau}} d\tau$.*

Proof. Let $f(x) = e^{-x^2}$. Then,

$$f(ax - b/x) = e^{-(ax-b/x)^2}$$

and

$$e^{-(ax-b/x)^2} e^{-2ab} = e^{-a^2x^2 - b^2/x^2} \tag{18}$$

Integrating both sides of this identity, Equation (18), over $(-\infty, \infty)$ and applying Proposition (3.2), our left hand side of Equation (18) becomes

$$\int_{-\infty}^{\infty} f(ax - b/x)e^{-2ab}dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-2ab}dx = \frac{\sqrt{\pi}}{a}e^{-2ab},$$

and by u-substitution using $\tau = x^2$, the right hand side of Equation (18) becomes

$$\int_{-\infty}^{\infty} \frac{e^{-a^2\tau - b^2/\tau}}{2\sqrt{\tau}} d\tau.$$

Because $f(x) = e^{-x^2}$ is an even function, this is equivalent to

$$\int_0^{\infty} \frac{e^{-a^2\tau - b^2/\tau}}{\sqrt{\tau}} d\tau.$$

We therefore conclude that

$$e^{-2ab} = \frac{a}{\sqrt{\pi}}\int_0^{\infty} \frac{e^{-a^2\tau - b^2/\tau}}{\sqrt{\tau}} d\tau.$$

□

Assuming that $\vec{w}(x, y, t)$ holds the criteria of Definitions (2.4) and (2.1), then we can use Proposition (3.2) and Corollary (3.3) to rewrite Equation (16) in the following form:

$$\mathcal{LF}[\vec{w}] = \int_0^\infty \int_0^\infty \left(\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}} \right) e^{-(|\xi|^2+s)\tau} \vec{f}(\xi, y')dy'd\tau. \tag{19}$$

From here, we can use the Convolution Theorem (2.6) to take the inverse Fourier transform of Equation (19), and then a table of inverse Laplace transforms to take the inverse Laplace transform of the resulting equation to obtain a final explicit expression for each component of vorticity, $\vec{w}$.

Theorem 3.4. *Let $\vec{w}(x,y,t)$ satisfy the conditions of the Fourier and Laplace transforms in Definitions (2.4) and (2.1). Then, the solution to the homogeneous Heat Equation $\frac{\partial}{\partial t}\vec{w} = \triangle\vec{w}$ and the expression for vorticity in the incompressible Stokes equation on the half-space when initial velocity is nonzero and boundary velocity at $y=0$ is equal to zero is the following:*

$$\vec{w}(x,y,t) = \frac{1}{4\pi^2}\int_0^t\int_0^\infty\int_{\mathbb{R}^2}\left(\frac{e^{\frac{-|x-x'|^2}{4\tau}}}{4\pi\tau}\right)w_0(t-\tau)\cdot\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}}dx'dy'd\tau. \tag{20}$$

Proof. We begin our work of taking Inverse integral transforms to solve for velocity with an Inverse Fourier transform of (19). By Definition (2.4), we have that The inverse Fourier of $\mathcal{LF}[\vec{w}]$ is

$$\mathcal{L}[\vec{w}] = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left(\int_0^\infty\int_0^\infty\left(\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}}\right)e^{-(|\xi|^2+s)\tau}f(\xi,y')dy'd\tau\right)e^{i\xi x}d\xi_1 d\xi_2.$$

Rearranging the right hand side of the above equation gives us

$$\mathcal{L}[\vec{w}] = \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty\left[\int_{\mathbb{R}^2}[e^{-|\xi|^2\tau}f(\xi,y')\cdot e^{i\xi x}]d\xi_1 d\xi_2\right]\left[e^{-st}\left(\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}}\right)\right]dy'd\tau \tag{21}$$

where our terms for convolution are in the left hand bracket in the integrand. From here, let $\mathcal{F}[g(x)] = e^{-|\xi|^2\tau}$ and let $\mathcal{F}[h(x)] = f(\xi,y')$. We know by the way we defined $\vec{f}$ that

$$\mathcal{F}^{-1}[\vec{f_i}(\xi,y')] = h(x) = \mathcal{L}[w_0^i] \tag{22}$$

for $i=1,2,3$. In order to find the inverse Fourier transform of $e^{|\xi|^2\tau}$ on $-\infty < x < \infty$, we will begin by looking at the Fourier transform of the Gaussian function e^{-ax^2} for some constant $a>0$. By Definition (2.4) we know that $\mathcal{F}[e^{-ax^2}] = \int_{-\infty}^{\infty}e^{ix\xi}e^{-ax^2}dx$ which we will define as $h(\xi)$. We aim to find an explicit expression for $h(\xi)$ without integrals or complex numbers.

Taking the derivative of $h(\xi)$ gives us

$$h'(\xi) = \int_{-\infty}^{\infty}e^{-ix\xi}(-ixe^{-ax^2})dx$$

We can integrate by parts to simplify the right hand side:

$$\begin{aligned} h'(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi}(-ixe^{-ax^2})dx \\ &= e^{-ix\xi}\frac{i}{2a}e^{-ax^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-ix\xi}(i\xi)e^{-ax^2}\frac{i}{2a}dx \\ &= \frac{-\xi}{2a}\int_{-\infty}^{\infty} e^{ix\xi}e^{-ax^2}dx \\ &= \frac{-\xi}{2a}h(\xi) \end{aligned}$$

This leaves us to solve the first-order, linear ordinary differential equation $h'(\xi) = \frac{-\xi}{2a}h(\xi)$. We can solve this equation for $h(\xi)$ using separation of variables:

$$\int \frac{dh}{h} = \int -\frac{\xi}{4a}d\xi$$

which implies that

$$\ln(h) = -\frac{1}{4a}\xi^2 + c$$

for some constant c. Therefore, our solution for h, i.e. the Fourier transform of e^{-ax^2} is

$$h(\xi) = \mathcal{F}[e^{-ax^2}] = \int_{-\infty}^{\infty} e^{-ix\xi}e^{-ax^2}dx = c_0e^{-\frac{\xi^2}{4a}}. \tag{23}$$

This result is true for all ξ, so without loss of generality we can allow ξ to equal 0, which tells us that $c_0 = \int_{-\infty}^{\infty} e^{-ax^2}dx$.

To solve for c_0, we will take a look at c^2. That is, for x and y such that $x = y$, we have

$$c_0^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{a(x^2+y^2)}dxdy.$$

Note that this y is different from our original y. We proceed by a change of variables to polar coordinates, and solve the integral as follows:

$$\begin{aligned} c_0^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{a(x^2+y^2)}dxdy \\ &= \int_0^{2\pi}\int_o^{\infty} e^{ar^2}rdrd\theta \\ &= 2\pi\left(\frac{e^{-ar^2}}{-2a}\Big|_0^{\infty}\right) \\ &= \frac{\pi}{a} \implies c = \sqrt{\frac{\pi}{a}} \end{aligned}$$

We therefore conclude that $\mathcal{F}[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$. It directly follows that $\mathcal{F}[\frac{e^{-x^2/4\tau}}{\sqrt{4\pi t}}] = e^{-\xi^2\tau}$. We can use this result to translate back to our work to show that the inverse Fourier transform of $e^{|\xi|^2\tau}$ is equal to $\frac{e^{\frac{-|x|^2}{4\tau}}}{4\pi\tau}$. That is,

$$\mathcal{F}^{-1}[e^{|\xi|^2\tau}] = g(x) = \frac{e^{\frac{-|x|^2}{4\tau}}}{4\pi\tau}. \tag{24}$$

Inputting our newfound expression for $h(x)$ in Equation (22) and $g(x)$ in Equation (24), into Equation (21) and using the Convolution Theorem for Fourier Transforms (2.6) gives us the following:

$$\mathcal{L}[\vec{w}] = \left(\frac{1}{4\pi}\right)\int_0^t\int_0^\infty\int_{\mathbb{R}^2}\left(\frac{e^{\frac{-|x-x'|^2}{4\tau}}}{4\pi\tau}\right)\mathcal{L}[w_0(x',s)]e^{-s\tau}\cdot\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}}dx'dy'd\tau.$$

Next, we must take the Inverse Laplace transform of the above equation in order to obtain an explicit expression for vorticity. From an inverse Laplace table, [3]

$$F(s) = \mathcal{L}f(t).$$

Then,

$$\mathcal{L}^{-1}[e^{-\tau s}\cdot\mathcal{L}[w_0]] = u_\tau(t)w_0(t-\tau).$$

Where $u_\tau(t)$ is defined as the Heaviside Function [4, p. 416] such that,

$$u_\tau(t) = \begin{cases} 0 & if \quad t < \tau \\ 1 & if \quad t \geq \tau \end{cases}.$$

Therefore, $u_\tau(t)$ is only relevant when $\tau \leq t$, so we have made the upper bound on the integral to be t instead of ∞. We can think of $w_0(t-\tau)$ as simply pushing the initial vorticity back in time a few steps, or correcting for the time, but only when that push could not result in a negative time (this is the use of the Heaviside).

Additionally,

$$\mathcal{L}^{-1}(\mathcal{L}(w_0\cdot e^{-s\tau})) = w_0(t-\tau).$$

Thus, after taking the Inverse Fourier and Inverse Laplace integral transforms, we have a unique solution for vorticity, as seen below,

$$\vec{w} = \frac{1}{4\pi^2}\int_0^t\int_0^\infty\int_{\mathbb{R}^2}\left(\frac{e^{\frac{-|x-x'|^2}{4\tau}}}{4\pi\tau}\right)w_0(t-\tau)\cdot\frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}}dx'dy'd\tau. \tag{25}$$

□

While it might be possible to explicitly find an equation for velocity directly from the vorticity equation (25), we do not derive velocity directly from vorticity. Instead, we use Helmholtz Decomposition Theory [8]. This Theorem states that our velocity $\vec{v}$ can be deecomposed into the sum of two different components that we can solve for individually, which allows for easier computation. Given any vector function $\vec{v}$, it is decomposed into two parts: a part that doesn't rotate, $\vec{z}$, and a part that rotates, $\vec{u}$. We see this as

$$\vec{v} = \vec{z} + \vec{u} \tag{26}$$

The part that doesn't rotate, $\vec{z}$, is called the conservative component of velocity. Meaning, there is no rotation to this vector field. Since there is no rotation, then $(\nabla \times \vec{z}) = 0$.

Furthermore, this component can also be expressed as the negative gradient of a scalar field, ϕ. Therefore, $\vec{z} = -\nabla\phi$, also known as the *scalar potential of velocity.*

The part that does rotate, $\vec{u}$, can be thought of as the curl of some vector field $\vec{A}$. Since $\vec{u}$ is defined as the curl of a vector field, we call it "divergence-free", as the divergence of the curl of any vector is always zero. We name this component such that $\vec{u} = \nabla \times \vec{A}$. Think of this as the "vector potential of velocity". Therefore, Equation (26) can be expressed as the following

$$\vec{v} = -\nabla\phi + \nabla \times \vec{A}. \tag{27}$$

This is the idea behind the Helmholtz Decomposition Theorem we employ on the half-space with our Dirichlet boundary conditions.

Theorem 3.5 (Helmholtz Decomposition of Case 1 Velocity)**.** *If $\vec{v}_1 \in L^1(\mathbb{R}^2) = \{f : \mathbb{R}^2 \Rightarrow \mathbb{R} : \int_{\mathbb{R}^2} |f|dx < \infty\}$, and $\vec{A} \in \mathbb{R}^3_+$ is a vector field, then*

$$\vec{v}_1 = \nabla \times \vec{A}, \qquad \in \mathbb{R}^3_+ \tag{28}$$

for $\vec{w} = \langle w_1, w_2, w_3\rangle$ and $\vec{v}_1 = \langle v_1^1, v_2^1, v_3^1\rangle$, where

$$\vec{A} = \frac{1}{4\pi^2}\int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\left[\frac{1}{4\pi\sqrt{|x-x'|^2 + (y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\right]dx'dy'.$$

Thus, the velocity for the incompressible Stokes equation on $\mathbb{R}^3_+$ with nonzero initial velocity, boundary velocity equal to zero, and zero normal component of boundary velocity is given by

$$\vec{v}_1 = \nabla \times \frac{1}{4\pi^2}\int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\left[\frac{1}{4\pi\sqrt{|x-x'|^2 + (y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\right]dx'dy'.$$

Proof. We know that Helmholtz Decomposition Theorem does work on the half-space [8]. To find the specific decomposition for our velocity, we first, recall the boundary and initial data for velocity.

$$BC\begin{cases}\vec{v}(x_1,x_2,0,t) = \vec{v}|_{y=0} = 0, \\ \\ \lim_{|(x_1,x_2,y)|\to\infty} \vec{v}(x_1,x_2,y) = \vec{v}|_{(x_1,x_2,y)\to\infty} = 0\end{cases}$$

$$IC\left\{\vec{v}(x_1,x_2,y,0) = \vec{c}\right.$$

Before we attempt to solve for $\vec{v}$, we must impose boundary conditions on $\vec{z}$ and $\vec{u}$. Since $\vec{v}|_{y=0} = 0$ at the boundary, then $\vec{z}$ and $\vec{u}$ must add up to 0 as well.

$$\vec{v}|_{y=0} = 0 = \vec{z}|_0 + \vec{u}|_0$$

In the most general case, we can let the two vectors equation opposite functions, like

$$\vec{z}|_{y=0} = f, \qquad \vec{u}|_{y=0} = -f$$

However, for our equation it is simplest to let both components be 0 at the boundary.

$$\vec{z}|_{y=0} = 0, \qquad \vec{u}|_{y=0} = 0$$

Next, as a point in the Half-Space tends toward infinity, its velocity tends toward zero. This must remain true for the sum of the components of velocity just as before. We can let both components be zero.

$$\lim_{(x_1,x_2,y)\to\infty} \vec{v}(x_1,x_2,y) = \lim_{(x_1,x_2,y)\to\infty} \vec{z}(x_1,x_2,y) + \lim_{(x_1,x_2,y)\to\infty} \vec{u}(x_1,x_2,y) = 0$$

$$\lim_{(x_1,x_2,y)\to\infty} \vec{z}(x_1,x_2,y) = 0, \lim_{(x_1,x_2,y)\to\infty} \vec{u}(x_1,x_2,y) = 0$$

We want to find an equation with just either ϕ or $\vec{A}$ to solve. Thus, the goal is to take either the curl or divergence of Equation (27), where the operator will make one of the components go to zero and isolate the other to solve. But in order to solve the isolated component we must establish the curl and divergence of velocity, which we defined earlier:

$$\nabla \cdot \vec{v} = 0, \qquad \nabla \times \vec{v} = \vec{w}.$$

With the imposed boundary conditions, we can solve for each component of velocity by either taking the curl or divergence to isolate a component.

First, we'll take the divergence of Equation (27),

$$\begin{aligned}
\nabla \cdot (\vec{v} &= -\nabla\phi + \nabla \times \vec{A}) \\
\nabla \cdot (\vec{v}) &= \nabla \cdot (-\nabla\phi) + \nabla \cdot (\nabla \times \vec{A}) \\
\nabla \cdot \vec{v} &= \nabla \cdot -\nabla\vec{\phi} \\
0 &= -\Delta\vec{\phi}
\end{aligned}$$

This produces the Laplace Equation. We can solve by taking the Fourier Integral transform. This produces a homogeneous, 2nd-order, constant coeffient ODE we can solve.
Let $\mathcal{F}[\phi] = \hat{\phi}$.

$$\begin{aligned}
\mathcal{F}[0 &= -\Delta\vec{\phi}] \\
\hat{\phi}_{yy} - |\xi|^2\hat{\phi} &= 0 \\
\hat{\phi} &= c_1 e^{|xi|y} + c_2 e^{-|xi|y}.
\end{aligned}$$

Next, we impose boundary conditions. Since our boundary for $\vec{z}|_{y=0} = \vec{z}|_{y\to\infty} = 0$, we can assume $\hat{\phi}|_{y=0} = \hat{\phi}|_{y\to\infty} = 0$, given $\vec{z}$ is a function of ϕ. Imposing our boundary conditions and solving for ϕ we have,

$$\begin{aligned}
\hat{\phi}|_{y\to\infty} = 0 &= c_1 e^{|xi|\infty} + c_2\frac{1}{\infty} \\
&= c_1\infty + 0 \\
0 &= c_1 \\
\hat{\phi}|_{y=0} = 0 &= c_2 e^{-|xi|0} \\
0 &= c_2.
\end{aligned}$$

Therefore,

$$\hat{\phi} = 0 \quad \to \quad \vec{z} = 0$$

For Case 1, this implies there is no conservative component to our velocity equation. Next, we isolate the curl-free component by taking the curl of Equation (26),

$$\begin{aligned} \nabla \times (\vec{v} = \vec{z} + \vec{u}) \\ \vec{w} = \nabla \times \vec{u} \\ \nabla \times (\vec{w} = \nabla \times \vec{u}) \\ \Psi = -\Delta \vec{u}, \qquad \Psi(x_1, x_2, y, t) = \nabla \times \vec{w}, \end{aligned}$$

and obtain the Poisson equation. So as before, by using Fourier transform we get an ODE for $\hat{u}$. Then, we solve for $\hat{u}$ using the Variation of Parameters method and Green's Function. Let $\mathcal{F}[\phi] = \hat{\phi}$, $\Psi = \nabla \times \vec{w}$, and $\mathcal{F}[\Psi] = \hat{\Psi}$. Then,

$$\begin{aligned} \mathcal{F}[\Psi = -\Delta \vec{u}] \\ \hat{u}_{yy} - |\xi|^2 \hat{u} = -\hat{\Psi} \\ \vec{u} = \frac{1}{(2\pi)^2} \int_0^\infty \int_{\mathbb{R}^3_+} \Psi(x) \Big[\frac{e^{-\frac{|x-x'|^2}{4\tau}}}{(4\pi\tau)^{3/2}} \big(e^{-\frac{(y+y')^2}{4\tau}} - e^{-\frac{(y-y')^2}{4\tau}} \big) \Big] dx' dy' d\tau \end{aligned}$$

Note, by linearity the curl operator can be applied to the inside of the integrals or pulled outside. Simplifying, we get our equation for $\vec{A}$ where $\vec{u} = \nabla \times \vec{A}$,

$$\vec{u} = \nabla \times \underbrace{\frac{\vec{w}(x)}{4\pi^2} \int_0^\infty \int_{\mathbb{R}^3_+} \Big[\frac{e^{-\frac{|x-x'|^2}{4\tau}}}{(4\pi\tau)^{3/2}} \big(e^{-\frac{(y+y')^2}{4\tau}} - e^{-\frac{(y-y')^2}{4\tau}} \big) \Big] dx' dy' d\tau.}_{\vec{A}}$$

We can simplify further with a u-substitution in the integral for τ. First, let us split up the integral $\int_0^\infty \Big[\frac{e^{-\frac{|x-x'|^2}{4\tau}}}{(4\pi\tau)^{3/2}} \big(e^{-\frac{(y+y')^2}{4\tau}} - e^{-\frac{(y-y')^2}{4\tau}} \big) \Big] d\tau$ into two parts as such:

$$\int_0^\infty \frac{e^{\frac{-|x-x'|^2-(y+y')^2}{4\tau}}}{(4\pi\tau)^{3/2}} d\tau - \int_0^\infty \frac{e^{\frac{-|x-x'|^2-(y-y')^2}{4\tau}}}{(4\pi\tau)^{3/2}} d\tau. \tag{29}$$

We will begin by simplifying the integral on the left-hand side using u-substitution. Let $u = \frac{\sqrt{|x-x'|^2+(y+y')^2}}{\sqrt{4\tau}}$, which indicates $du = \frac{\sqrt{|x-x'|^2+(y+y')^2}}{(4\tau)^{3/2}}$. Then, we can rewrite $\int_0^\infty \frac{e^{\frac{-|x-x'|^2-(y+y')^2}{4\tau}}}{(4\pi\tau)^{3/2}} d\tau$ as

$$\begin{aligned} &\int_0^\infty \frac{e^{u^2}}{2\pi^{3/2}\sqrt{|x-x'|^2+(y+y')^2}} du \\ &= \frac{1}{2\pi^{3/2}\sqrt{|x-x'|^2+(y+y')^2}} \int_0^\infty e^{-u^2} du \\ &= \frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}}. \end{aligned}$$

We repeat the same u-substitution for the integral on the right-hand side in Expression (29) except we replace the term $\sqrt{|x-x'|^2+(y+y')^2}$ with $\sqrt{|x-x'|^2+(y-y')^2}$. Following the same process, the right-hand integral simplifies to

$$\frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}$$

and Expression (29), i.e., $\int_0^\infty \frac{e^{\frac{-|x-x'|^2-(y+y')^2}{4\tau}}}{(4\pi\tau)^{3/2}}d\tau - \int_0^\infty \frac{e^{\frac{-|x-x'|^2-(y-y')^2}{4\tau}}}{(4\pi\tau)^{3/2}}d\tau$, simplifies to

$$\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}.$$

Therefore, our expression for $\vec{u}$ is

$$\vec{u} = \nabla \times \frac{1}{4\pi^2}\int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\left[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\right]dx'dy'.$$

It is clear from this statement that

$$\vec{A} = \frac{1}{4\pi^2}\int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\left[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\right]dx'dy'$$

for $\vec{u} = \nabla \times \vec{A}$.

We can then insert our newfound expressions for ϕ and $\vec{A}$ back into our Helmholtz decomposition equation $\vec{v}^1 = -\nabla\phi + \nabla \times \vec{A}$ to obtain

$$\vec{v}_1 = \nabla \times \frac{1}{4\pi^2}\int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\left[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\right]dx'dy'.$$

□

Remark 1. *Note, a more general Stokes equation might not assume the fluid is incompressible,* $\nabla\cdot\vec{v} = 0$*. A more general approach could let the fluid be compressible such that* $\nabla\cdot\vec{v} = \delta$*, where* $\delta \in \mathbb{R}$*. However, our research stayed in the bounds of incompressible fluids,* $\delta = 0$*. In the general case for compressible fluids, then*

$$\nabla \cdot \vec{v} = \nabla \cdot \nabla\vec{\phi}$$

$$\delta = \Delta\vec{\phi}.$$

This produces the Poisson Equation. This is the only difference in the Helmholtz Decomposition method between compressible and incompressible fluids—solving for the Poisson equation instead of the Laplace equation.

3.2. **Case 2.** Case 2 addresses the non-zero boundary data and zero initial data for velocity. The process itself is very similar to Case 1 but with notable changes. To begin, we impose boundary data,

$$\vec{v}_2(x_1,x_2,0,t) = \vec{v}_{y=0} = \vec{a}(x_1,x_2,t)$$

$$\lim_{|(x_1,x_2,y)|\to\infty} \vec{v}_2(x_1,x_2,y) = \vec{v}_{(x_1,x_2,y)\to\infty} = 0.$$

Remark 2. *As a point tends towards infinity its velocity still tends towards 0.*

Additionally, we impose Case 2 initial data,

$$\vec{v}(x_1, x_1, y, 0) = 0.$$

Proposition 3.6 (Vorticity Boundary and Initial Data)**.** *Let velocity $\vec{v}(x, y, t)$ be differentiable in t, twice differentiable in x and y, and let $\vec{v} \in L^1(\mathbb{R}^2)$. Additionally, let $\vec{w}$ hold the assumptions stated in Definitions (2.1) and (2.4). Given $\vec{a} \cdot \vec{n} = 0$ from (15) where $\vec{a}$ is the boundary data for velocity, if $\vec{w} = \nabla \times \vec{v} \in \mathbb{R}^3_+$, then the boundary data for $\vec{w}$ is*

$$\vec{w}_{y=0} = \vec{b}(x_1, x_2, t) \tag{30}$$

$$\vec{w}_{(x_1,x_2,y)\to\infty} = 0, \tag{31}$$

where $\vec{b} = \nabla \times \vec{a}$.

Proof. We want to show that $\nabla \times (\vec{v}|_{y=0}) = (\nabla \times \vec{v})|_{y=0}$.

First, our equation $\vec{a} \cdot \vec{n} = 0$ implies $\frac{\partial y}{\partial y} = 0 \ \in \mathbb{R}^3_+$ since there is no contribution from boundary data in the normal direction. Since this is true for our whole space, it is true at the boundary $\frac{\partial y}{\partial y}|_{y=0} = 0$. So at the boundary,

$$\nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \partial x_1 & \partial x_2 & \partial y \\ x_1 & x_2 & y \end{vmatrix} = \vec{w} =< \frac{\partial y}{\partial x_2} - \frac{\partial x_2}{\partial y}, \frac{\partial x_1}{\partial y} - \frac{\partial y}{\partial x_1}, \frac{\partial x_2}{\partial x_1} - \frac{\partial x_1}{\partial x_2} >$$

$$\nabla \times (\vec{v}|_{y=0}) = \begin{vmatrix} i & j & k \\ \partial x_1 & \partial x_2 & \partial y \\ x_1 & x_2 & 0 \end{vmatrix} = \vec{w}|_{y=0} =< -\frac{\partial x_2}{\partial y}, \frac{\partial x_1}{\partial y}, \frac{\partial x_2}{\partial x_1} - \frac{\partial x_1}{\partial x_2} >$$

So at the boundary, they are equal. □

With these in mind, we can compute an equation for the integral transformed vorticity with the same method as in Proposition 3.1 and Equation (14).

Proposition 3.7 (Integral Transform Vorticity)**.** *Recall Equation (14) and our conditions for Vorticity in Proposition (3.6). If $\mathcal{L}[\mathcal{F}[\vec{w}]] = \vec{g} \in \mathbb{R}^3_+$, then the equation for $\vec{g}$ can be expressed as*

$$\mathcal{L}[\mathcal{F}[\vec{w}]] = \vec{g} = \hat{\beta} e^{-y\sqrt{|\xi|^2+s}}, \tag{32}$$

where $\hat{\beta} = \mathcal{L}[\mathcal{F}[\vec{w}|_{y=0}]]$, from Proposition (3.6).

Proof. The process to find an explicit equation for vorticity for Case 2 directly mirrors that seen in the beginning of Case 1 where we take the curl of the Stokes equation, then take the Fourier and Laplace transforms to obtain Equation (14). However, in Case 2 with zero initial data, $f(y) = -\mathcal{F}[\vec{w}]|_{t=0} = 0$. Thus

$$g''(y) - \alpha^2 \vec{g}(y) = 0.$$

To find a specific solution for $\vec{g}$, we must impose the boundary conditions as stated Proposition (3.6). That is,

$$g|_{y=0} = \mathcal{L}[\mathcal{F}[\vec{b}]] = \hat{\beta}$$

$$\lim \vec{g}_{(x_1,x_2,y)\to\infty} = 0.$$

We now solve the linear, second-order, homogeneous, constant-coefficient ODE as seen below,

$$g''(y) - \alpha^2 \vec{g}(y) = 0$$
$$\vec{g} = c_1 e^{y\alpha} + c_2 e^{-y\alpha}$$

$$lim_{|(x_1,x_2,y)|\to\infty} \vec{g} = 0 = c_1 e^{\infty * \alpha} + c_2 * 0$$
$$0 = c_1$$

$$\vec{g}|_{y=0} = \vec{\beta} = c_2 e^{-y0}$$
$$\hat{\beta} = c_1(1)$$

$$\vec{g} = \hat{\beta} e^{-y\alpha}$$
$$\mathcal{L}[\mathcal{F}[\vec{w}]] = \hat{\beta} e^{-y\sqrt{|\xi|^2+s}}$$

□

We use similar techniques from our previous calculations to solve for $\mathcal{L}[\mathcal{F}[\vec{w}]]$, including taking inverse Laplace and Fourier integral transforms. However, to solve Equation (32), we must modify Corollary (3.3).

Corollary 3.8. *Let $a > 0, b > 0, x \in \mathbb{R}$, then*

$$e^{-2ab} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p^2 - \left(\frac{ab}{p}\right)^2} dp. \tag{33}$$

Proof. From Corollary (3.3) we have the following identity,

$$e^{-2ab} = \frac{a}{\sqrt{\pi}} \int_0^\infty \frac{e^{-a^2\tau - \frac{b^2}{\tau}}}{\sqrt{\tau}} d\tau.$$

Next, using u-substitution with $\tau = u^2$, we obtain the following,

$$e^{-2ab} = \frac{a}{\sqrt{\pi}} \int_0^\infty \frac{e^{-a^2u^2 - \frac{b^2}{u^2}}}{u} 2udu.$$

Through simplification we have,

$$e^{-2ab} = \frac{2a}{\sqrt{\pi}} \int_0^\infty e^{-(au)^2 - \frac{b}{u}^2} du.$$

Using a second u-substitution where $p = au$, we obtain,

$$e^{-2ab} = \frac{2a}{\sqrt{\pi}} \int_0^\infty e^{-(p)^2 - (\frac{(}{ab}p)^2} \left(\frac{1}{a}\right) dp,$$

which can be further simplified to arrive at

$$e^{-2ab} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(p)^2 - (\frac{ab}{p})^2} dp.$$

□

Theorem 3.9 (Case 2 Equation for Vorticity)**.** *Let velocity $\vec{v}(x, y, t)$ be differentiable in t, twice differentiable in x and y, and let $\vec{v} \in L^1(\mathbb{R}^2)$. Additionally, let $\vec{w}$ hold the assumptions stated in Definitions (2.1) and (2.4). Then, if $\vec{w} = \nabla \times \vec{v} \in \mathbb{R}^3_+$, we can express an equation for vorticity as*

$$\vec{w}(x_1, x_2, y, t) = \frac{2}{4\pi^{3/2}} \int_0^t \int_{\mathbb{R}^2} e^{\frac{-y^2}{4c}} e^{\frac{-|x|^2}{2c}} \frac{y}{4c^{3/2}} \vec{b}(x - x', t - c) dx' dc. \tag{34}$$

Proof. With Equation (32) we can use Corollary (3.8) to decouple the ξ term. Let $\sqrt{|\xi|^2 + s} = a$, $\frac{y}{2} = b$, then

$$\begin{aligned} \mathcal{L}[\mathcal{F}[\vec{w}]] &= \frac{2\hat{\beta}}{\sqrt{\pi}} \int_0^\infty e^{-p^2 - \left(\frac{\sqrt{|\xi|^2+s}\frac{y}{2}}{p}\right)^2} dp \\ &= \frac{2\hat{\beta}}{\sqrt{\pi}} \int_0^\infty e^{-p^2 - \frac{y^2(|\xi|^2+s)}{4p^2}} dp \\ &= \frac{2\hat{\beta}}{\sqrt{\pi}} \int_0^\infty e^{-p^2} e^{-\frac{y^2(|\xi|^2)}{4p^2}} e^{-\frac{y^2 s}{4p^2}} dp. \end{aligned}$$

Taking the inverse Fourier transform using Definition (2.4) and the convolution from Definition (2.5),

$$\mathcal{L}[\vec{w}] = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p^2} e^{-\frac{y^2 s}{4p^2}} \mathcal{F}^{-1}[\hat{\beta} e^{-\frac{y^2(|\xi|^2)}{4p^2}}] dp$$

$$\mathcal{F}^{-1}[\hat{\beta} e^{-\frac{y^2(|\xi|^2)}{4p^2}}] = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{\frac{-p^2|x-x'|^2}{y^2}} \vec{\beta}(x', s) dx'.$$

Thus,

$$\mathcal{L}[\vec{w}] = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p^2} e^{-\frac{y^2 s}{4p^2}} \left[\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{\frac{-p^2|x-x'|^2}{y^2}} \vec{\beta}(x', s) dx'\right] dp$$

$$= \frac{2}{4\pi^{3/2}} \int_0^\infty \int_{\mathbb{R}^2} e^{-p^2} e^{-\frac{y^2 s}{4p^2}} e^{\frac{-p^2|x-x'|^2}{y^2}} \vec{\beta}(x', s) dx' dp.$$

Before we take the inverse Laplace transform, we first apply a change of variables to ease our future work with a Laplace table.

Let $c = \frac{y^2}{4p^2}$. Then $\mathcal{L}[\vec{w}]$ becomes,

$$\mathcal{L}[\vec{w}] = \frac{2}{4\pi^{3/2}} \int_0^\infty \int_{\mathbb{R}^2} e^{\frac{-y^2}{4c}} e^{-cs} e^{\frac{-|x|^2}{2c}} \vec{\beta}(x - x', s) \frac{y}{4c^{3/2}} dx' dc.$$

Then, taking the inverse Laplace transform we have

$$\vec{w} = \frac{2}{4\pi^{3/2}} \int_0^\infty \int_{\mathbb{R}^2} e^{\frac{-y^2}{4c}} e^{\frac{-|x|^2}{2c}} \frac{y}{4c^{3/2}} \mathcal{L}^{-1}[e^{-cs} \vec{\beta}(x - x', s)] dx' dc.$$

From our table of Laplace transforms [3], $\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)$, where $u_c(t)$ is defined as follows:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

Then, employing the Heaviside Function for $u_c(t)$, we change our upper bound on the first integral from ∞ to t and have the following solution for $\vec{w}$:

$$\vec{w} = \frac{2}{4\pi^{3/2}} \int_0^t \int_{\mathbb{R}^2} e^{\frac{-y^2}{4c}} e^{\frac{-|x|^2}{2c}} \frac{y}{4c^{3/2}} \vec{b}(x - x', t - c)dx'dc.$$

□

Remark 3. *Note, an interesting observation of our vorticity equation is that*

$$\int_{\mathbb{R}^2} e^{\frac{-y^2}{4c}} e^{\frac{-|x|^2}{2c}} \frac{y}{4c^{3/2}} \vec{b}(x - x', t - c)dx'$$

is the vorticity equation derived from using Green's Theorem as opposed to integral transforms [5].

From here, we employ the use of Helmholtz Decomposition Theorem to solve for the individual components of velocity in order to construct an explicit equation for velocity.

Proposition 3.10 (Helmholtz Components with Non-Zero Boundary Data)**.** *Let $\vec{z}, \vec{u} \in \mathbb{R}^3_+$ and assume the same conditions as in Proposition (3.5), then*

$$\begin{aligned} \vec{z}|_{y=0} &= \vec{a}(x_1, x_2, t), & \vec{u}|_{y=0} &= 0 \\ \vec{z}|_{(x_1,x_2,y)\to\infty} &= 0, & \vec{u}|_{(x_1,x_2,y)\to\infty} &= 0. \end{aligned}$$

Proof. Recall from Proposition (3.5) that the components of velocity must sum to velocity at velocity's initial and boundary value data. Therefore, we can let

$$BC \begin{cases} \vec{v}(x_1, x_2, 0, t) = \vec{v}|_{y=0} = 0, \\ \\ \lim_{|(x_1,x_2,y)|\to\infty} \vec{v}(x_1, x_2, y) = \vec{v}|_{(x_1,x_2,y)\to\infty} = 0 \end{cases}$$

$$IC \left\{ \vec{v}(x_1, x_2, y, 0) = \vec{c} \right.$$

$$\begin{aligned} \vec{v}|_{y=0} = \vec{a}(x_1, x_2, t) &= \underbrace{\vec{z}|_{y=0}}_{=\vec{a}} + \underbrace{\vec{u}|_{y=0}}_{=0} \\ \vec{a}(x_1, x_2, t) &= \vec{z}|_{y=0}. \end{aligned}$$

Similar to Theorem (3.5), $\vec{z} = 0$ and $\vec{u} = 0$ towards infinity. □

With our decomposition, we can find an explicit formula for velocity given non-zero boundary data and zero initial data on the half-space.

Theorem 3.11 (Helmholtz Decomposition of Case 2 Velocity)**.** *Given non-zero boundary data and zero initial data, if* $\vec{v}_2 \in L^1(\mathbb{R}^2) = \{f : \mathbb{R}^2 \Rightarrow \mathbb{R} : \int_{\mathbb{R}^2} |f| dx < \infty\}$*, then*

$$\vec{v}_2 = \nabla\phi + \nabla \times \vec{A} \tag{35}$$

where

$$\phi = \frac{-1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \frac{e^{-y^2\tau}}{\sqrt{\tau}} \frac{e^{-2|x-x'|^2}}{\tau} a_3(x', t) dx' d\tau \tag{36}$$

$$\vec{A} = \int_{\mathbb{R}^3_+} \vec{w} \Big(\frac{1}{4\pi\sqrt{|x-x'| + (y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'| + (y-y')^2}} \Big) dx' dy'. \tag{37}$$

Proof. Like in Theorem (3.5), we must solve for the components of velocity, $\vec{z} = -\nabla\phi$, and $\vec{u} = \nabla \times \vec{A}$. We start by finding $\vec{z}$.

Taking the divergence of Equation (35) we obtain

$$\nabla \cdot (\vec{v} = -\nabla\phi + \nabla \times \vec{A})$$
$$0 = -\Delta\vec{\phi}.$$

Next, we use the Fourier transform to isolate $\mathcal{F}[\phi] = \hat{\phi}$. Let $\alpha^2 = |\xi|^2$, then

$$\hat{\phi}_{yy} - \alpha^2\hat{\phi} = 0$$
$$\hat{\phi} = c_1 e^{y\alpha} + c_2 e^{-y\alpha}.$$

It is important to note,

$$\vec{z}|_{y=0} = \nabla\phi|_{y=0} = \vec{a}$$
$$< \phi_{x_1}, \phi_{x_2}, \phi_y > = < \vec{a}_1, \vec{a}_2, \vec{a}_3 > .$$

By taking the Fourier Integral transform of ϕ above, it's boundary condition of $\vec{a}$ changes:

$$i\xi_1\hat{\phi}|_{y=0} = \hat{a}_1$$
$$i\xi_2\hat{\phi}|_{y=0} = \hat{a}_2$$
$$i\frac{\partial}{\partial y}\hat{\phi}|_{y=0} = \hat{a}_3.$$

Next, we impose Neumann boundary conditions by first taking the partial derivative with respect to y. First, solving for c_1 we obtain,

$$\hat{\phi}|_{(x_1,x_2,y)\to\infty} = 0 = c_1 e^\infty + c_2 \cdot 0$$
$$0 = c_1.$$

Then, solving for c_2 we obtain

$$\frac{\partial\hat{\phi}}{\partial y} = -\alpha c_2 e^{-y\alpha}$$
$$\frac{\partial\hat{\phi}}{\partial y}|_{y=0} = \hat{a}_3 = -\alpha c_2$$
$$c_2 = \frac{-\hat{a}_3}{\alpha}.$$

Then, we have $\hat{\phi} = \frac{-\hat{a}_3}{|\xi|}e^{-y|\xi|}$.

Now, we must take the inverse Fourier transform of $\hat{\phi}$. However, each of the three components of $\hat{\phi}$ is a function of ξ, and we have no technique to take the inverse Fourier of such functions.

So, we use Corollary (3.3) with $e^{-y|\xi|}$ such that $a = |\xi|, b = \frac{y}{2}$ to make our function more approachable. Then,

$$e^{-|\xi|y} = \frac{|\xi|}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-|\xi|^2\tau - \frac{y^2}{4\tau}}}{\sqrt{\tau}} d\tau.$$

The $|\xi|$ from the equation above will cancel out the $|\xi|$ in the original $\hat{\phi}$ equation. Through simplification,

$$\hat{\phi} = \frac{-\hat{a}_3}{|\xi|}\frac{|\xi|}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-|\xi|^2\tau - \frac{y^2}{4\tau}}}{\sqrt{\tau}} d\tau$$

$$\hat{\phi} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}} (\hat{a}_3 e^{-|\xi|^2\tau}) d\tau$$

$$\phi = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}} \mathcal{F}^{-1}[\hat{a}_3 e^{-|\xi|^2\tau}] d\tau$$

Now we can compute the inverse Fourier. Recall Definition (2.6), where the inverse Fourier of two multiplied functions of ξ is the convolution of their individually inverse Fourier functions:

$$\mathcal{F}^{-1}[\hat{g} \cdot \hat{h}] = (g * h)(x).$$

So, let

$$\hat{g} = -\hat{a}_3, \qquad \hat{h} = e^{-y|\xi|}.$$

Now we will take the inverse Fourier transforms of both $\hat{g}$ and $\hat{h}$. Starting with $\hat{g}$,

$$\mathcal{F}^{-1}[\hat{g}] = \mathcal{F}^{-1}[-\hat{a}_3]$$
$$\vec{g} = -\vec{a}_3.$$

Then solving for $\vec{h}$,

$$\mathcal{F}^{-1}[\hat{h}] = \mathcal{F}^{-1}[e^{-|\xi|^2\tau}]$$
$$\vec{h} = e^{\frac{-|x|^2}{2\tau}}.$$

Now that we have $\vec{g}$ and $\vec{h}$, we use the Convolution integral to write them as follows,

$$\vec{g} * \vec{h} = \int_{\mathbb{R}^2} -\vec{a}_3 e^{\frac{-|x-x'|^2}{2\tau}} dx'.$$

Now, our equation for ϕ is

$$\phi = \frac{-1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}} \vec{a}_3 e^{\frac{-|x|^2}{2\tau}} dx' d\tau. \tag{38}$$

Next, we must find the scalar component of velocity, $\vec{A}$. Taking the curl of Equation (35), we have the following

$$\begin{aligned}
\nabla \times (\vec{v} &= -\nabla\phi + \nabla \times \vec{A}) \\
\vec{w} &= \nabla \times \vec{u} \\
\nabla \times \vec{w} &= -\triangle \vec{u} \\
\Psi &= -\triangle \vec{u}, \qquad \Psi = \Psi(x_1, x_2, y, t).
\end{aligned}$$

Referring back to the proof for Theorem (3.5), our process to find $\vec{A}$ for Case 2 is identical to that of Case 1, regardless of varying boundary conditions. Therefore,

$$\vec{u} = \nabla \times \int_{\mathbb{R}^3_+} \vec{w}(x,y,t)\Big[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\Big]dx'dy'.$$

□

Theorem 3.12. *(Velocity Equation on Half-Space) Given non-zero boundary data and non-zero initial data, if $\vec{v} \in L^1(\mathbb{R}^2)$, then the velocity $\vec{v}$ on the half-space $\mathbb{R}^3_+$ can be expressed as the explicit equation*

$$\vec{v} = -\nabla\phi + \nabla \times 2\vec{A} \tag{39}$$

$$\phi = \frac{1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \Big(\frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}}\, e^{\frac{-|x|^2}{2\tau}}\Big)\vec{a}_3(x',t)dx'd\tau \tag{40}$$

$$\vec{A} = \vec{w}(x,y,t) \int_{\mathbb{R}^3_+} \Big[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\Big]dx'dy'. \tag{41}$$

Proof. Due to the linearity of the Stokes Equation (4), we are able to solve Case 1 and Case 2 for velocity, then add them together using the Superposition Principle to obtain a complete equation for velocity. Therefore,

$$\begin{aligned}
\vec{v} &= \Big[\vec{v}_1\Big] + \Big[\vec{v}_2\Big] \\
\vec{v} &= \Big[\nabla \times \vec{w}(x,y,t) \int_{\mathbb{R}^3_+} \Big[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\Big]dx'dy'\Big] \\
&+ \Big[-\nabla\frac{1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \Big(\frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}}\, e^{\frac{-|x|^2}{2\tau}}\Big)\vec{a}_3(x',t)dx'd\tau \\
&+ \nabla \times \vec{w}(x,y,t) \int_{\mathbb{R}^3_+} \Big[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\Big]dx'dy'\Big].
\end{aligned}$$

Through simplification, we obtain our general solution for $\vec{v}$,

$$\begin{aligned}
\vec{v} = &\Big[-\nabla\frac{1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \Big(\frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}}\, e^{\frac{-|x|^2}{2\tau}}\Big)\vec{a}_3(x',t)dx'd\tau\Big] \\
&+ 2\Big[\nabla \times \vec{w}(x,y,t) \int_{\mathbb{R}^3_+} \Big[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}}\Big]dx'dy'\Big].
\end{aligned}$$

□

Corollary 3.13 (Pressure Process on the Half-Space). *Given Theorem (3.12) and the Stokes equation (4), if* $p \in \mathbb{R}^3_+$*, we can express* p *as*

$$\nabla p = (\frac{\partial}{\partial t} - \triangle)\vec{v}, \qquad \in \mathbb{R}^3_+$$
$$\langle p_1, p_2, p_3 \rangle = \langle f_1, f_2, f_3 \rangle,$$

where we have an explicit process to find the components p_i *where* $i = 1, 2, 3$.

Proof. Given the Stokes equation and $\vec{v} = \vec{v}_1 + \vec{v}_2$, (3.12):

$$\frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p$$
$$\nabla p = (\frac{\partial}{\partial t} - \triangle)\vec{v}.$$

□

References

[1] A.J. Chorin and J.E. Marsden. *A Mathematical Introduction to Fluid Mechanics*. Texts in Applied Mathematics. Springer New York, 2013.
[2] Dean G. Duffy. *Green's Functions with Applications*. Chapman and Hall/CRC, New York, 2015.
[3] Phil Dyke. *An Introduction to Laplace Transforms and Fourier Series*. Springer Undergraduate Mathematics Series. Springer-Verlag London, 2014.
[4] R.B. Guenther and J.W. Lee. *Partial Differential Equations of Mathematical Physics and Integral Equations*. Dover Books on Mathematics. Dover Publications, 2012.
[5] Hoewoon Kim, Enrique Thomann, and Ronald Guenther. A representation of the solution of the stokes equations in the half space $\mathbb{R}^3_+$: Application to spatial and temporal estimates of the pressure. *Journal of Mathematical Fluid Mechanics*, 21, 03 2019.
[6] J David Logan. *Applied partial differential equations*. Springer, 2014.
[7] Kevin E Trenberth and Kevin E Trenberth. *Climate system modeling*. Cambridge University Press, 1992.
[8] May Thi Tun and T Miyakawa. A proof of the helmholtz decomposition of vector fields over the half-space. *Adv. Math. Sci. Appl*, 18(1):199–217, 2008.

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