# ONE-GAP SOLUTIONS TO THE KAUP-BROER SYSTEM

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Abstract. We use elliptic function theory to derive one-gap solutions to the Kaup-Broer (KB) system, a coupled system of nonlinear partial differential equations. Similar to the Korteweg de Vries (KdV) equation, this system describes shallow water waves with weakly nonlinear restoring forces. In our paper, we compare the utility and methods for finding solutions to each of these models as well as provide a general overview of nonlinear wave theory, which is far more accurate at predicting tsunami and rogue wave amplitude than linear wave theory. In both the KB and KdV systems, we generate solutions using the Weierstrass $\wp$-function demonstrating the underlying algebraic geometry. Particularly, the construction of a birational transformation between elliptic curves is necessary for finding such solutions to the KB system. In future work, we plan to extend our results using hyper-elliptic curves of genus $g$, where we intend to investigate what happens as $g \to \infty$.

## 1. Background

Nonlinear behavior is observed in many areas of physics including galaxy formation, optics, traffic flow, plasma physics, earthquakes, and our focus: surface water waves. Completely integrable nonlinear partial differential equations are used to describe such phenomena because exact solutions can be produced using the Inverse Scattering Transform. This method, which can be thought of as the nonlinear analogue to the Fourier Transform, was first introduced in 1967 by Gardner, Greene, Kruskal, and Miura [4] just two years after Zabusky and Kruskal [17] produced interesting numerical solutions to the otherwise neglected Korteweg de Vries (KdV) equation. Zabusky and Kruskal described their solutions as exhibiting "solitary" or "soliton-type" behavior.

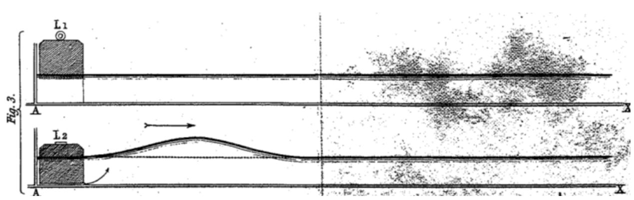
- experiment picture.png 

Figure 1. John Scott Russell's soliton experiments from his report *On Waves*

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In 1834, this "solitary wave" was first observed by Scottish engineer and naval architect, John Scott Russell in the Union Canal in Edinburgh [12]. Ten years later and after numerous experiments in the laboratory, Russell published his report *On Waves*, which was fundamental to the theory of wave mechanics [12]. Some years later, Stokes [15, 1847] and Boussinesq [3, 1872] mathematically abstracted Russell's results by determining an analytical form for a single soliton giving the wave amplitude $\eta$ as a function of space $x$ and time $t$ [12]:

$$\eta(x,t) = \eta_0 \operatorname{sech}^2[(x-ct)/L] \tag{1}$$

Here, $c$ is the phase speed given by $c = \sqrt{gh}(1+\eta_0/2h)$ and $L$ is the phase width given by $L = \sqrt{4h^3/3\eta_0}$ where $g$ is the acceleration due to gravity and $h$ is the water depth. The next development in nonlinear wave mechanics resulted from the work of Korteweg and de Vries [7], who independently determined that (1) solves what we now call the KdV equation [12]:

$$\eta_t + \sqrt{gh}\eta_x + \frac{3\sqrt{gh}}{2h}\eta\eta_x + \frac{\sqrt{gh}h^2}{6}\eta_{xxx} = 0 \tag{2}$$

This equation is iconic in the study of nonlinear waves and has been an active area of research since the mid twentieth century along with the Nonlinear Schrödinger (NLS) equation [12]. Primarily, the KdV equation is used to model nonlinear behavior in shallow water while the NLS equation is applied to deep water. Nonetheless, both these equations can claim universality as the KdV equation has been applied to ion acoustic waves in plasma and acoustic waves through a crystal lattice; the NLS equation comes from quantum mechanics and later water waves.

For the purposes of this paper, we are interested in modeling shallow water waves with weakly nonlinear restoring forces. Apart from the ubiquitous KdV equation, there are a variety of other partial differential equations that can be used to describe waves like these including the Kadomtsev-Petviashvili (KP) equation, Boussinesq equation, and the Kaup-Broer (KB) system [12]. Here, we will focus on the latter, producing soliton and finite gap solutions as well as providing a comparison with the methods of producing these type of solutions to the well-studied KdV equation. The Kaup-Broer system is given by [11]:

$$\eta_t + h\varphi_{xx} + (\eta\varphi_x)_x + \left(\frac{h^3}{3} - \frac{h\tau}{\rho g}\right)\varphi_{xxxx} = 0 \tag{3}$$

$$\varphi_t + \frac{1}{2}(\varphi_x)^2 + g\eta = 0 \tag{4}$$

where $\eta$ is the wave amplitude, $\phi$ is the velocity potential, $\tau$ is the surface tension of the fluid, and $\rho$ is the density of the fluid. The KB system differs from the KdV equation because it allows one to forgo the KdV assumption (2) that the waves propagate in the one direction [11]. Moreover, finding solutions to the KB system is relatively easier than for other similar equations like the Boussinesq equation [1] since certain reductions allow us to reduce the KB system to left and right moving KdV scattering motion [11]. Since the KdV equation is well-studied, these dynamics are easier to analyze, which makes the KB system attractive for studying shallow water wave mechanics.

Note that we may write (3) such that it has four unique scaling classes, each representing different physical properties of the water;

$$\eta_t + \phi_{xx} + (\eta\phi_x)_x + \mu\phi_{xxxx} = 0$$
$$\phi_t + \frac{1}{2}(\phi_x)^2 + \varepsilon\eta = 0$$

Choices of $\varepsilon$ and $\mu$ allow us to specify these properties as follows:

(1) $\varepsilon = 1, \mu = \frac{-1}{4}$ corresponds to gravitational waves subject to capillary effects
(2) $\varepsilon = 1, \mu = \frac{1}{4}$ corresponds to gravitational waves not subject to capillary effects
(3) $\varepsilon = -1, \mu = \frac{1}{4}$ corresponds to non-capillary waves in reversed gravity
(4) $\varepsilon = -1, \mu = \frac{-1}{4}$ corresponds to capillary waves in reversed gravity

In [11], Nabelek and Zakharov produce an infinite limit of N-soliton solutions, discuss an approach for finding finite gap solutions, and compute numerical solutions to the second scaling class of the KB system. The one-gap solutions we produce will apply to all scaling classes, but the emphasis of future investigations will be the first scaling class, which we will refer to as KB1.

## 2. Motivation

Before we go about computing solutions to the KdV and KB systems, we will first take a step back and examine the motivation for working with nonlinear partial differential equations. Linear wave theory is robust and universal, but its applicability is limited to small amplitude motions. Nonlinear models offer more accurate results over a wider range of motion amplitudes, but these can be very difficult to work with, so one must consider this important trade-off. Yet, it need not be this way; the more we build out nonlinear wave theory, the easier it will be for engineers, doctors, and scientists to apply it. One of the common methods for dealing with nonlinear waves is known as the Inverse Scattering Transform for which we provide a brief overview.

2.1. **The Linear KdV Equation.** Next, we demonstrate the accuracy of nonlinear wave theory by comparing analogous linear results. We begin by deriving the solution for the linearized KdV equation and compare our solution with that of the traditional KdV equation. Both solutions will be calculated using Fourier transforms. This method relies on moving the PDE into the Fourier space where the equation is simpler to solve, solving the PDE in the Fourier space, and using and inverse Fourier transform to generate the solution in the original space.

The linear KdV equation is the same as the canonical KdV equation with the nonlinear term removed,

$$u_t + u_{xxx} = 0 \tag{5}$$

This equation can be solved exactly using Fourier transforms [10].

**Definition 2.1.1.** *The Fourier transform and inverse Fourier transform are defined for a function $f(x)$ where as $x \to \pm\infty$, $f \to 0$ respectively as*

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{6}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) e^{ikx} dx \tag{7}$$

Fourier transforms have a special property that allow them to convert derivatives into multiplication.

**Theorem 2.1.** *A Fourier transform $F$ converts derivatives into multiplication in the Fourier space.*

$$F(\partial_x f) = ik\hat{f} \tag{8}$$

*Proof.*

$$F(\partial_x f) = \int_{-\infty}^{\infty} \partial_x f(x) e^{-ikx} dx$$

Using Integration by Parts,

$$F(\partial_x f) = f(x)e^{ikx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \partial_x e^{-ikx} dx$$

Since it was assumed that as $x \to \pm\infty$, $f \to 0$,

$$f(x)e^{ikx}\Big|_{-\infty}^{\infty} = 0$$

Therefore,

$$F(\partial_x f) = ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

But this is the definition of the Fourier transform,

$$F(\partial_x f) = ik\hat{f}$$

□

This theorem may be used to calculate derivatives of partial differential equations. Taking the Fourier transform with respect to $x$ of the linearized KdV equation gives

$$\partial_t \hat{u} + (ik)^3 \hat{u} = 0$$

The time partial derivatives are unaffected by this transformation. Note that while $u$ is a function of $x$ and $t$, $\hat{u}$ is a function of $k$ and $t$. Simplifying,

$$\partial_t \hat{u} - ik^3 \hat{u} = 0$$

Integrating the equation gives,

$$\hat{u} = A(k) e^{ik^3 t}$$

where $A(k)$ is a constant. Assuming an initial condition of $u(x, 0) = f(x)$ which Fourier transforms to $\hat{u}(k, 0) = \hat{f}(k)$, the constant $A(k)$ can be found. Therefore,

$$\hat{u} = \hat{f}(k) e^{ik^3 t}$$

Taking the inverse Fourier transform [10]

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ik^3t}e^{ikx}dx$$

This equation may be calculated numerically using fast finite Fourier transforms (fft) and their inverses (ifft). To perform this calculation numerically, the following procedure is followed,

(1) Perform a fast finite Fourier transform (fft) on the given initial condition $f(x)$
(2) Multiply the result by $e^{ik^3t}$
(3) Perform an inverse fast finite Fourier transformation (ifft)
(4) Take only the real part of the result

A numerical solution demonstrating a particular implementation of this method are provided in Figure 2. The initial condition disperses backwards, then, due to the periodicity of the Fourier space, the dispersive tail cycles around to the front of the wave.

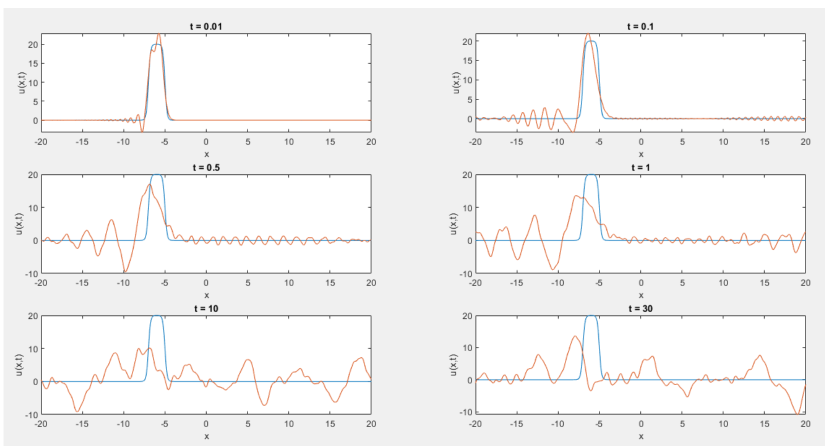



FIGURE 2. Numerical simulation demonstrating the effects of the Linear KdV model on the initial condition $f(x) = 10(\tanh 4(x+7) - \tanh 4(x+5))$ [10]. The initial condition is plotted in blue and the effects of the linear KdV model are plotted in red.

2.2. **The KdV Equation.** Fourier transforms may also be used to solve the KdV equation using the Split Step Spectral Method [10]. To show this, the KdV equation will be written as

$$\partial_t u + 3\partial_x(u^2) + \partial_x^3 u = 0$$

Applying the Fourier transform with respect to x,

$$\partial_t \hat{u} + 3ik(\hat{u^2}) + (ik)^3\hat{u} = 0$$

$$\partial_t \hat{u} = -3ik(\hat{u^2}) - (ik)^3\hat{u}$$

The split step method will calculate the effects of the linear part first then add in the effects of the nonlinearity. For the linear part, which is exact,

$$\hat{u}(t+\Delta t) = \hat{u}(t)e^{ik^3\Delta t}$$

For the nonlinear part, use Euler's method to approximate the solution,

$$u(t+\Delta t) = \hat{u}(t) - 3ik\Delta t(\widehat{u^2})$$

Therefore [10],

$$\begin{aligned}\hat{u_1}(k, t+\Delta t) &= \hat{u}(k,t)e^{ik^3\Delta t}\\ \hat{u}(k, t+\Delta t) &= \hat{u_1}(k, t+\Delta t) - 3ik\Delta t(F((F^{-1}(\hat{u_1}(k, t+\Delta t))^2))\end{aligned}$$

where $\hat{u_1}$ is an intermediate step representing the linear solution.
A numerical solution demonstrating a particular implementation of this method is provided in figure 3. Over time, the initial condition decomposes into three forward moving solitons and a small dispersive tail.

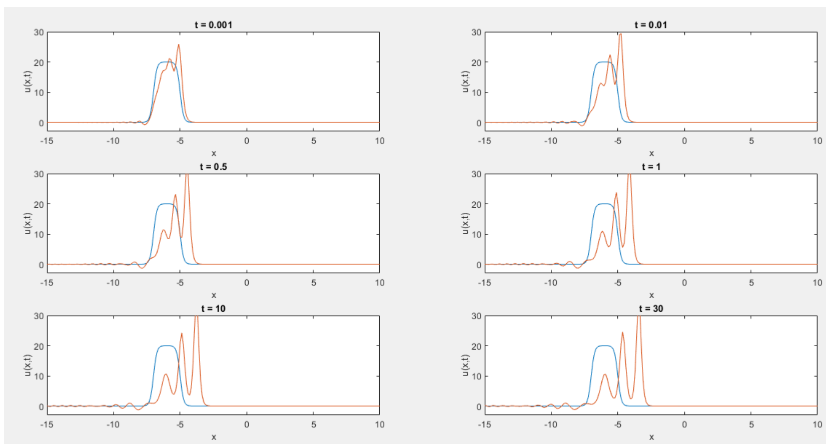



FIGURE 3. Numerical simulation demonstrating the effects of the KdV model on the initial condition $f(x) = 10(\tanh 4(x+7) - \tanh 4(x+5))$ [10]. The initial condition is plotted in blue and the effects of the linear KdV model are plotted in red.

2.3. **The Inverse Scattering Transform.** The Inverse Scattering Transform (IST) is a method to solve nonlinear, integrable PDEs. IST is similar to Fourier transformations in that it relies on transitioning the nonlinear PDE into a new space where it is easier to solve, then inverting the transformation to returning to the original space to find the desired solution. A general outline for using the IST along with its results on the KdV equation is as follows:

(1) Find the Lax pair of the specific nonlinear, integrable PDE being solved.

**Definition 2.3.1** (Lax Pair)**.** *A Lax pair is a pair of operators that satisfy the Lax equation*

$$L_t = [P, L] = PL - LP$$

*where* $[\cdot,\cdot]$ *is the commutator operator.*

For the KdV equation, the Lax pair is [12]

$$\begin{aligned}L &= -\partial_x^2 + u\\ P &= -4\partial_x^3 + 6u\partial_x + 3u_x\end{aligned}$$

(2) Use the $L$ operator of the Lax pair to forward scatter by solving the Lax equation, which is an eigvenvalue-eigenfunction problem, for an initial condition

$$L\psi = \lambda\psi$$

where $\psi$ is a variable in the transformed space. The solution of this equation gives the scattering data at the initial time.
The scattering data for the KdV equation is determined by the Linear Schödinger Equation and is composed of the wave number $k_n$, the connection coefficient $c_n$, the reflection coefficient $r$, and the transmission coefficient $a$.

(3) Determine the time evolution of the scattering data by solving

$$\psi_t = P\psi$$

which leads to the appearance of the dispersion relation $\omega(k)$.
For the KdV equation, the scattering data propagates forward in time according to [10]

$$k_n = k_n$$
$$c_n(t) = c_n(0)e^{4k_n^2 t}$$
$$r(k,t) = r(k,0)e^{8ik^3 t}$$
$$a(k,t) = a(k,0)$$

(4) Inverse scatter the scattering data at the new time $t_0$ to find the solution of the original PDE at this time. This can be done using the Gelfand-Levitan-Marchenko equation for the KdV equation or a Riemann-Hilbert problem.
For the KdV equation, letting the reflection coefficient $r$ equal to zero will cause the IST method to produce soliton solutions [10].

A flowchart outlining the IST process is included in Figure 4

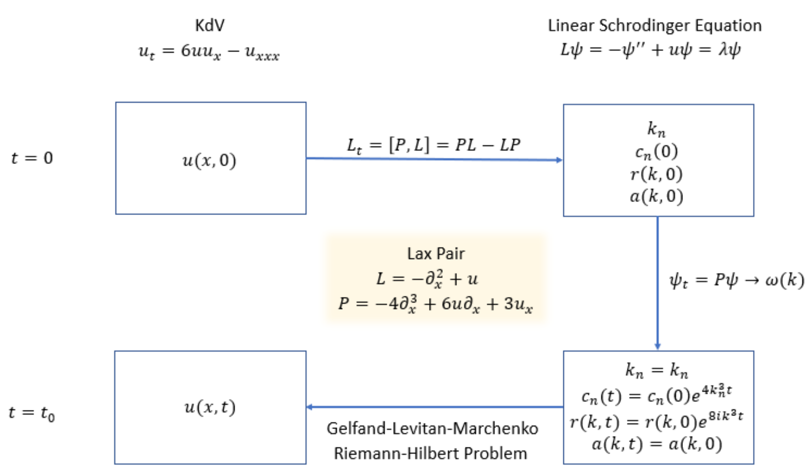



FIGURE 4. Diagram showing the basic steps of the Inverse Scattering Transform method.

## 3. Elliptic Functions

In this section, we present some details related to the theory of elliptic functions, which is closely tied with complex analysis and algebraic geometry. While we will not need all of these results for this paper, the material presented in this section is necessary for producing higher genus solutions to both the KdV and KB equations and, more generally, for all completely integrable nonlinear PDEs.

### 3.1. Elliptic Integrals.

A natural way to develop the mathematics of elliptic functions is to begin with elliptic integrals.

**Definition 3.1.1** (Elliptic Integral)**.** *Let $R(x)$ be a complex polynomial of degree no greater than 4. Let $F(x,y)$ be a function that is rational in $x$ and $y=\sqrt{R(x)}$. If $\int F(x,y)dx$ is not elementary, then it is an elliptic integral.*

**Remark 3.1.1.** *Note that elementary integrals can be expressed as rational, trigonometric, logarithmic, and exponential functions. There are some conditions we can impose on the problem that will guarantee an elementary integral; these include letting $F$ not depend on $y$ and setting the degree of $R$ to be 1* [9].

Today, elliptic integrals are commonly used in computing solutions to partial differential equations. Undergraduate students will likely be familiar with Jacobi's incomplete elliptic integral of the first kind,

$$K=\int_0^1 \frac{1}{\sqrt{(1-x^2)\,(1-k^2x^2)}}dx$$

and its complement,

$$K'=\int_1^{1/k} \frac{1}{\sqrt{(x^2-1)\,(1-k^2x^2)}}dx$$

where $k$ is the elliptic modulus - equivalent to the eccentricity of the ellipse.

However, these integrals have a rich history, dating back nearly 300 years. In 1797, Gauss realized that elliptic integrals are closely tied with elliptic functions. Before we describe this relation, we consider a motivating example. Using a trigonometric substitution, we recall that:

$$\int_0^x \frac{1}{\sqrt{1-y^2}}dy=\arcsin x$$

Thus, we can view $\sin x$ as the inverse of the integral on the left hand side. In the late eighteenth century, Gauss tried to compute the arc length of the lemniscate of Bernoulli (Figure 5), which required him to evaluate the integral $\int_0^x \left(1-y^4\right)^{-1/2}dy$. Gauss realized that, like the integral in (3.1), this integral was also the inverse of some function.
Gauss called this function sinus lemniscatus; Hence:

$$\int_0^x \frac{1}{\sqrt{1-y^4}}dy=\operatorname{sinlemn}^{-1}(x) \tag{9}$$

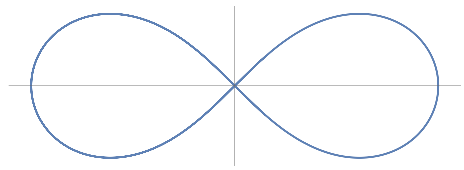
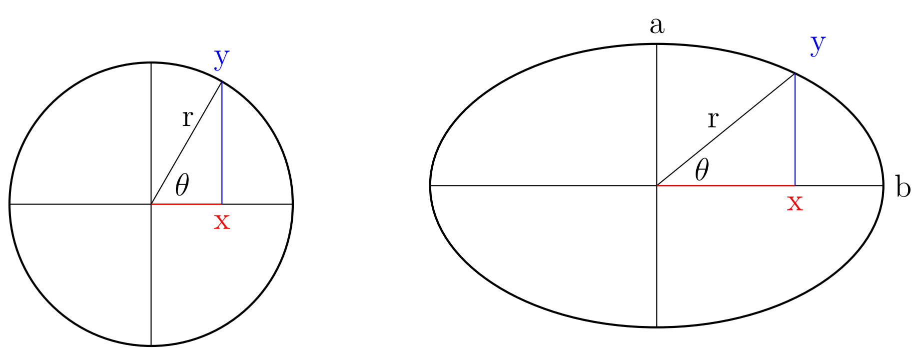


FIGURE 5. Lemniscate of Bernoulli

Unfortunately, Gauss never published this work [5], so it was Abel who rediscovered this unique relation, generalizing that the inverse of an elliptic integral is an element of the field of elliptic functions. In the same year (1827) Jacobi introduced his twelve elliptic functions, all of which can be expressed in terms of each other. He defined sinus amplitudinus to be the inverse of $K$, which is a function of two variables: $x$, which varies, and $k$, the elliptic modulus. Thus:

$$\mathrm{sn}^{-1}(x,k) = \int_0^x \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$$

Observe that by setting $k = i$, the integral turns into (9), so we see that $\mathrm{sinlemn}(x) = \mathrm{sn}(x,i)$. Jacobi's other eleven elliptic functions are given by various permutations of the letters {c,d,n,s}. In this paper we will only discuss sn, cn, and dn, which give the best intuition for understanding these functions. Consider the figure below:




$$x^2 + y^2 = r^2 \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

First, observe that the radius for the circle is constant, while this is not the case for the ellipse. Next, we define a variable $u$, which will act like $\theta$ does for trigonometric functions. Then for a point $P$ on the ellipse specified by some angle $\theta$, define:

$$u = \int_0^P r d\theta$$

Now, we recall the sine function where $\sin(\theta) = y/r$; we define sn in an similar manner such that $\mathrm{sn}(u,k) = y/b$. Likewise, cn is analogous to cosine; we have $\cos(\theta) = x/r$ and $\mathrm{cn}(u,k) = x/a$. Unfortunately, dn does not have a nice trigonometric analogue and is given by $\mathrm{dn}(u) = r/a$. One final note is that this picture gives us a nice way to visualize the elliptic modulus:

$$k^2 = \frac{a^2 - b^2}{a^2}$$

3.2. **Elliptic Functions.** In the previous section, we gave a somewhat informal definition of an elliptic function;

**Definition 3.2.1** (Elliptic Function, V1)**.** *An elliptic function is the inverse of an elliptic integral.*

Version 1 is a good place to start because that is how the history unfolded, but it fails to describe properties of elliptic functions. We now give another definition that includes no background, but provides a better insight into the behavior of these functions.

**Definition 3.2.2** (Elliptic Function, V2)**.** *An elliptic function is a doubly periodic meromorphic function.*

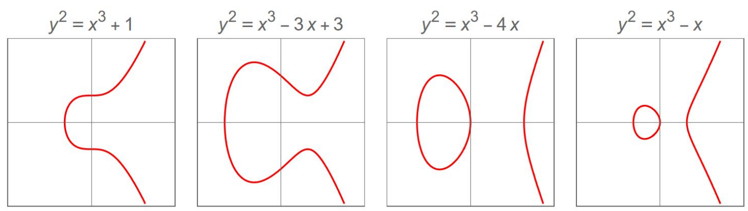
The "doubly periodic" part of this definition means that for an elliptic function $f(z)$, there exist half-periods $\omega_1$ and $\omega_2$ such that $f(z) = f(z + 2\omega_1) = f(z + 2\omega_2)$. We can take "meromorphic" to mean holomorphic except for a finite number of singularities. Yet, this definition begs the question: on what domain is $f$ holomorphic? Indeed, this definition still only gives a piece of the whole picture; defining this ambiguous domain will help to complete that picture.

3.3. **A Mini Dive into Algebraic Geometry.** To finish off our elliptic function definition, we will need to borrow a few tools from algebraic geometry.

**Definition 3.3.1** (Elliptic Curve, V1)**.** *An elliptic curve is an equation of the form* $y^2 = x^3 + Ax + B$ *for* $4A^3 + 27B^2 \neq 0$.

Over $\mathbb{R}$, an elliptic curve can look like:

Figure 6. Elliptic Curves [16]




where a cusp or loop is formed when $4A^3 + 27B^2 = 0$. However, these curves are also interesting to look at over other fields. For example, an elliptic curve over $\mathbb{C}$ is topologically equivalent to a torus. In order to see this, we need to put the curve in projective space.

**Definition 3.3.2** (Complex Projective Line $\mathbb{CP}^1$)**.** *The complex projective line is the coordinates* $[X : Y]$ *modulo an equivalence relation for* $X, Y \in \mathbb{C} + \infty$*. This is equivalent to saying* $\mathbb{CP}^1$ *is the Riemann Sphere (see Figure 7).*

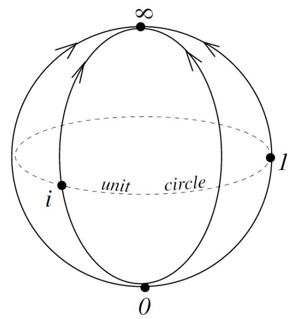



FIGURE 7. The Riemann Sphere [6]

We can also define $\mathbb{CP}^n$ for any $n \in \mathbb{N}$. Thus, when we put an elliptic curve in projective space, we will put it in $\mathbb{CP}^2$ where we endow the equation with homogeneous coordinates $[X : Y : Z]$. By making the polynomial homogeneous, we ensure that the equation plays nice with the equivalence class: $[X : Y : Z] \sim [aX : aY : aZ]$ for each $a \in \mathbb{C}/\{0\}$.

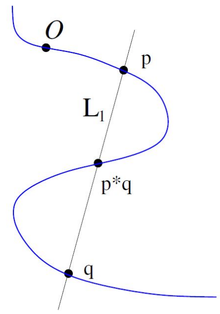
Let $\mathrm{E}(\mathbb{C}) : y^2 = x^3 + Ax + B$, which is an affine line as written. To make it projective, we write $\mathrm{E}(\mathbb{CP}^1) : ZY^2 = X^3 + AXZ^2 + BXZ^2$. Then the additional point at infinity $[0 : 1 : 0]$ becomes an identity element in the abelian group structure we can give the curve. Without writing explicit equations, we will describe the group; let $P$ and $Q$ be points on E. The first step (see Figure 8a) involves drawing a line between these points, which is guaranteed to intersect the curve at a third point. Then, draw another line between this new point, denoted $P * Q$ and the identity (see Figure 8b), which will hit E at a third point, which we define to be $P + Q$ where $+$ is the group operation.
Then, the map $\mathcal{P}$ is a group isomorphism taking $\mathrm{E}(\mathbb{CP}^1)$ to the torus embedded in $\mathbb{CP}^2$ with the chord-and-tangent group law. We will define this map in the next section.

**Remark 3.3.1.** *There are a few subtleties when defining the group law on an elliptic curve to ensure we obtain exactly three intersection points with the curve. If the line drawn through a point is tangent to the curve at that point, then we count it twice. In the Figure 8, the identity element $O$ is not at "infinity" because this picture depicts an arbitrary cubic. However, for elliptic curves, $O$ will not be "on" the curve, so Step 2 is equivalent to reflection about the $x$-axis. If it is the case that Step 1 is a vertical line, then we say it also hits the curve at $\infty$ and the two point are inverses.*

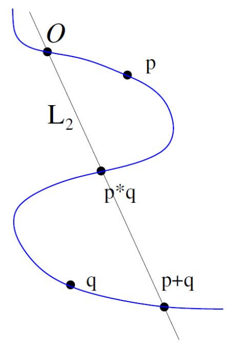
Formally, the torus is equivalent to $\mathbb{C}$ modulo a lattice, $\Lambda$ with periods $\omega_1$ and $\omega_2$. Hence, the lattice is generated by a parallelogram, also known as the fundamental cell. In addition, the torus is an example of a genus 1 Riemann Surface, an object we will loosely define;

**Definition 3.3.3** (Riemann Surface)**.** *A Riemann Surface is a complex 1-manifold defined by a polynomial equation.*

To understand this definition, we will consider an example.



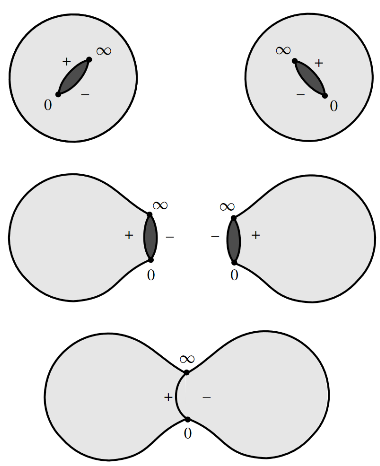

(A) Step 1




(B) Step 2

FIGURE 8. The Elliptic Curve Group Law

**Example 1** (Riemann Surface)**.** *Let $\Sigma$ be the Riemann Surface defined by the equation $w^2 = z$. There are singularities when $z = 0$ and $\infty$. The Riemann surface gives us a nice way to deal with singularities.*




*Surface.png*

FIGURE 9. The Riemann Surface for $w^2 = z$ [9]

*In Figure 9, the singularities are handled by making a branch cut between 0 and $\infty$ on two copies of $\mathbb{CP}^1$ and gluing them together to create a surface.*

A similar construction involving branch cuts between singularities creates the Riemann Surface for an elliptic curve. However, we do not need to limit ourselves to genus 1.

**Definition 3.3.4** (Hyper-elliptic Curve, V2)**.** *A hyper-elliptic curve of genus g takes the form:*

$$w^2 = \prod_{j=1}^{2g+1} (z - \lambda_j)^j$$

*where the $\lambda_j$ are distinct roots.*

Now, we are ready for a more representative definition of an elliptic function;

**Definition 3.3.5** (Elliptic Function, V3)**.** *An elliptic function or hyper-elliptic function is a rational function on a Riemann surface defined by the projective equation for an elliptic or hyper-elliptic curve, respectively.*

In the next section, we define the Weierstrass elliptic function, which is essential for producing solutions to the Korteweg de Vries equation and the Kaup-Broer system.

3.4. **The Weierstrass $\wp$-function.**

**Definition 3.4.1** (The Weierstrass Elliptic Function)**.** *For $m, n \in \mathbb{Z}$, the Weierstrass elliptic function is given by:*

$$\wp(z) = \frac{1}{z^2} + \sum_{\{m,n\} \neq \{0,0\}} \left( \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right)$$

*where $\omega_1$ and $\omega_2$ are the periods of a lattice, $\Lambda$.*

Thus, the $\wp$-function is a 2:1 map from $\mathbb{C}/\Lambda$ to $\mathbb{CP}^1$, ramified over the four points $e_1$, $e_2$, $e_3$, and $\infty$ [9]. Another way to say this is that $\wp$-function takes the torus to two copies of $\mathbb{CP}^1$. Because the Weierstrass elliptic function takes inputs on the torus, we can visualize some of these values on the fundamental cell of the lattice associated with the torus.

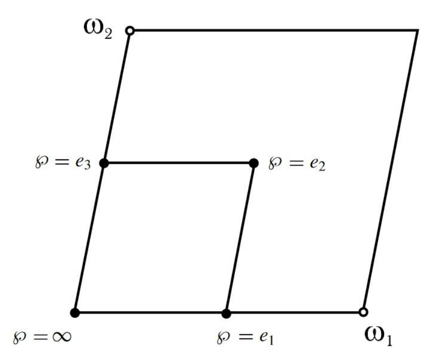



FIGURE 10. The $\wp$-function on the Fundamental Cell [9]

In Figure 10, we see that $\wp(\omega_1/2) = e_1$, $\wp(\omega_1/2+\omega_2/2) = e_2$, and $\wp(\omega_2/2) = e_3$. These values, $e_j$ turn out to be important as they are the zeroes of the polynomial part of an ordinary differential equation that the Weierstrass elliptic function solves.

**Theorem 3.1** (The Differential Equation)**.** *The Weierstrass elliptic function satisfies the differential equation:*

$$\left(\wp'(z)\right)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

*for distinct $e_j$ where $e_1 + e_2 + e_3 = 0$. Equivalently, this elliptic functions solves:*

$$\left(\wp'(z)\right)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$$

*where $g_2$ and $g_3$ are elliptic invariants that come from the Laurent expansion of $\wp(z)$ and are dependent on the periods of the lattice. We require $g_2^3 - 27g_3^2 \neq 0$.*

Also, note that we can express Jacobi's elliptic functions in terms of the Weierstrass $\wp$-function and, thus, Jacobi's functions are also solutions to similar differential equations.

Therefore, this theorem is one of the many reasons why elliptic functions are so powerful. In order to use the Weierstrass elliptic function as a tool to compute solutions to completely integrable systems, we need to define its antiderivative and similar functions.

**Definition 3.4.2** (Weierstrass $\zeta$ and $\sigma$ Functions)**.**

$$\zeta(z) = -\int \wp(z)dz$$

$$\frac{d}{dz}\ln\sigma(z) = \zeta(z)$$

We now have the background to find the one-gap solutions to the KdV equation and the Kaup-Broer system, concluding our section on elliptic functions.

## 4. One-gap and one-soliton solutions to the KdV equation

In this section, we produce one-gap and one-soliton solutions to the canonical KdV equation given by:

$$\eta_t - 6\eta\eta_x + \eta_{xxx} = 0 \tag{10}$$

4.1. **Reduction to an ODE.** We will assume a travelling wave solution such that $\eta$ is of the form $f(x - ct) = f(z)$ for some constant $c$. Plugging into (10), we get

$$-cf' - 6ff' + f''' = 0 \tag{11}$$

Using the observation that $\frac{df}{dz}(-3(f(z))^2) = -6f(z)f'(z)$, we will integrate (11) with respect to $z$:

$$\int -cf' - 6ff' + f'''dz = \int 0dz$$
$$-cf - 3f^2 + f'' = k_1$$
$$\implies f'' = cf + 3f^2 + k_1 \tag{12}$$

Note that $k_1$ is some constant of integration, which we require to be real so that we obtain a real-valued wave amplitude. Also, throughout this solution, we will "lump" scalar addition and multiplication into the constants of integration as we did here to reduce to one constant of integration. Now, we want to integrate (12) again, but with respect to $f$ as follows:

$$\int f''\, df = \int cf + 3f^2 + k_1\, df$$

However, we have to account for the chain rule, so observe that $\frac{df}{dz} = f'(z) \implies df = f'(z)dz$. Substituting, we see:

$$\int f'f''\, dz = \int cff' + 3f'f^2 + k_1 f'\, dz$$

and exploiting the product and power rules, we get that:

$$\frac{1}{2}(f')^2 = f^3 + \frac{c}{2}f^2 + k_1 f + k_0$$
$$\implies (f')^2 = 2f^3 + cf^2 + k_1 f + k_0 \tag{13}$$

Let $2y = f$. Then, we obtain the ordinary differential equation:

$$(y')^2 = 4y^3 + cy^2 + k_1 y + k_0 \tag{14}$$

In the next section, we will solve (14).

4.2. **An algebraic proposition.**

**Proposition 4.1** (Completing the cube)**.** *Let $a, b, d, x \in \mathbb{C}$. Then,*

$$x^3 + ax^2 + bx + d = (x + \frac{a}{3})^3 + (b - \frac{a^2}{3})x + (d - \frac{a^3}{27}) \tag{15}$$

*Proof.* First, suppose for some $A, B, r \in \mathbb{C}$ that

$$x^3 + ax^2 + bx + d = (x + r)^3 + Ax + B \tag{16}$$

Then, we claim that $r = \frac{a}{3}$, $A = b - \frac{a^2}{3}$, and $B = c - \frac{a^3}{27}$. Expanding the RHS, we get $x^3 + (3r^2)x^2 + (3r^2 + A)x + (r^3 + B)$. Thus, we must require that

$$a = 3r \tag{17}$$
$$b = 3r^2 + A \tag{18}$$
$$d = r^3 + B \tag{19}$$

(17) gives that $r = \frac{a}{3}$. Plugging this value into (18), we get $A = b - \frac{a^2}{3}$. Similarly, (17) and (19) yield $B = d - \frac{a^3}{27}$. Hence, $A, B$, and $r$ satisfy our claim proving the proposition. ∎

Next, we will apply Proposition 4.1 to factor equation (14). Begin with considering the right-hand side (RHS) of (14):

$$4\left(y^3+\frac{c}{4}y^2+k_1y+k_0\right)$$

Using the notation defined in the proof, we have $r=\frac{c}{12}$, $A=k_1-\frac{c^2}{48}$, and $B=k_0-\frac{c^3}{1728}$. So, (14) turns into:

$$\begin{aligned}(y')^2&=4\left(\left(y+\frac{c}{12}\right)^3+\left(k_1-\frac{c^2}{48}\right)y+\left(k_0-\frac{c^3}{1728}\right)\right)\\&=4\left(y+\frac{c}{12}\right)^3+\left(k_1-\frac{c^2}{12}\right)y+\left(k_0-\frac{c^3}{432}\right)\end{aligned}$$

Next, we let $\gamma=y+\frac{c}{12}\implies\gamma'=y'$ and plug in to get

$$(\gamma')^2=4\gamma^3+\left(k_1-\frac{c^2}{12}\right)\gamma+\left(k_0-\frac{ck_1}{12}+\frac{c^3}{216}\right)\tag{20}$$

Therefore, $\gamma(z)=\wp(z)$ is a solution where $g_2=-(k_1-\frac{c^2}{12})$ and $g_3=-(k_0-\frac{ck_1}{12}+\frac{c^3}{216})$. From this information, we can determine the periods of the respective lattice, $\omega_1$ and $\omega_2$. To do so, we evaluate the system:

$$-k_1+\frac{c^2}{12}=60\sum_{\{m,n\}\neq\{0,0\}}\frac{1}{(m\omega_1+n\omega_2)^4}\tag{21}$$

$$-k_0+\frac{ck_1}{12}-\frac{c^3}{216}=140\sum_{\{m,n\}\neq\{0,0\}}\frac{1}{(m\omega_1+n\omega_2)^6}\tag{22}$$

Mathematica allows us to easily compute these for specified values of $k_0,k_1$, and $c$. We will not explicitly solve for $\omega_1$ and $\omega_2$, now since we do not wish to specify these parameters yet.

In the following section, we will determine $\eta(x,t)$, the traveling wave solution to the KdV equation, in terms of the specific $\wp$-function implicitly defined by (21) and (22).

4.3. **Explicitly solving for $\eta$.** In the previous section, we determined that $\gamma(z)=\wp(z)$. We made the substitution that $\gamma=y+\frac{c}{12}$, so $y(z)=\wp(z)-\frac{c}{12}$. Before this substitution, we let $2y=f$, so we now have $f(z)=2\wp(z)-\frac{c}{6}$.

Recalling that we supposed $f(z)=\eta(x,t)$ where $z=x-ct$, we found that:

$$\eta(x,t)=2\wp(x-ct)-\frac{c}{6}\tag{23}$$

By definition, $\eta$ is a "one-gap" solution. If we let $\omega_1$ tend to $\infty$ and adjusting $k_1$ and $k_2$ such that $\eta$ vanishes as $x\longrightarrow\pm\infty$, then (23) turns into the classical soliton formulation given in (1) [9].

## 5. One-gap and one-soliton solutions to the KB1 System

5.1. **Introduction.** Recall the Kaup-Broer system given in the introduction:

$$\eta_t + \phi_{xx} + (\eta\phi_x)_x + \mu\phi_{xxxx} = 0 \tag{24}$$

$$\phi_t + \frac{1}{2}(\phi_x)^2 + \varepsilon\eta = 0 \tag{25}$$

In this section, we will assume the solutions are traveling waves, which will aid in their computation. Then, we can take limits to generate the one-gap and one-soliton solutions.

5.2. **Reduction to an ODE.** Suppose $\eta$ and $\phi$ are of the forms $f(x-ct) = f(z)$ and $g(x-ct) = g(z)$, respectively. Here, $c \in \mathbb{R}$ represents the wave speed. We also require that $z \in \mathbb{R}$ such that the time, space, and velocity are real valued. Plugging these new functions into (24) and (25), we obtain:

$$-cf' + g'' + fg'' + f'g' + \mu g^{(4)} = 0 \tag{26}$$

$$-cg' + \frac{1}{2}(g')^2 + \varepsilon f = 0 \tag{27}$$

Observe that $\frac{d}{dz}(fg') = fg'' + f'g'$ by the product rule. Thus, we can integrate (26) with respect to $z$:

$$\int -cf' + g'' + fg'' + f'g' + \mu g^{(4)} dz = \int 0 dz$$

$$-cf + g' + fg' + \mu g''' + \alpha = 0 \tag{28}$$

where $\alpha \in \mathbb{R}$ is a constant of integration.

Rearranging equation (27), we see that $f = \frac{c}{\varepsilon}g' - \frac{1}{2\varepsilon}(g')^2$ and plugging into (28), we get:

$$(\tfrac{\varepsilon - c^2}{\varepsilon})g' + \tfrac{3c}{2\varepsilon}(g')^2 - \tfrac{1}{2\varepsilon}(g')^3 + \mu g''' + \alpha = 0$$

Let $y = g'$. Then, we get a second order ordinary differential equation:

$$\mu y'' = \tfrac{1}{2\varepsilon}y^3 - \tfrac{3c}{2\varepsilon}y^2 + (\tfrac{c^2 - \varepsilon}{\varepsilon})y + \alpha \tag{29}$$

Now, multiply (29) by $y'$ and integrate again exploiting the product rule.

$$\mu y''y' = \tfrac{1}{2\varepsilon}y^3y' - \tfrac{3c}{2\varepsilon}y^2y' + (\tfrac{c^2-\varepsilon}{\varepsilon})yy' + \alpha y'$$
$$\int \mu y''y'dz = \int \tfrac{1}{2\varepsilon}y^3y' - \tfrac{3c}{2\varepsilon}y^2y' + (\tfrac{c^2-\varepsilon}{\varepsilon})yy' + \alpha y'dz$$
$$\tfrac{\mu}{2}(y')^2 = \tfrac{1}{8\varepsilon}y^4 - \tfrac{c}{2\varepsilon}y^3 + \tfrac{1}{2}(\tfrac{c^2-\varepsilon}{\varepsilon})y^2 + \alpha y + \beta$$
$$4\mu\varepsilon(y')^2 = y^4 - 4cy^3 + 4(c^2-\varepsilon)y^2 + \alpha y + \beta \tag{30}$$

where $\beta \in \mathbb{R}$ is a constant of integration. In the subsequent section, we reduce (30) to a cubic in $y$.

5.3. **Changing Variables.** Define a fourth degree polynomial

$$P_4(y) := \frac{1}{4\mu\varepsilon}(y^4 - 4cy^3 + 4(c^2-\varepsilon)y^2 + \alpha y + \beta).$$

Clearly, $(y')^2 = P_4(y)$ from (30). Because $\mu, c, \alpha, \beta$ and $\epsilon$ are real valued, $P_4 \in \mathbb{R}[x]$. We will assume that $\Delta$, the quartic discriminant is positive, which will guarantee that $P_4$ has distinct roots that are either all real or all complex. Let $y_0$ be one of those roots such that $P_4(y_0) = 0$; for now, we will allow $y_0 \in \mathbb{C}$.

**Remark 5.3.1.** *It is, in fact, necessary for these roots to be distinct because this will guarantee that the following map will take the quartic to an elliptic curve* [14]. *Moreover, these distinct roots give that $P_4'(y_0) \neq 0$.*

Let $P_3(\xi)$ be a cubic and consider the change of variables between $(y')^2 = P_4(y)$ and $(\xi')^2 = P_3(\xi)$ given by:

$$\xi = \frac{-1}{y-y_0} \implies \xi' = \frac{y'}{(y-y_0)^2} \tag{31}$$

**Remark 5.3.2.** *The most general case of this change of variables is called a birational transformation, which is an invertible rational map of two variables. In the birational transformation, the variables can be distinct so that the map can take any "fourth" degree elliptic curve to a "third" degree elliptic curve. This means that there are no such things as third and fourth degree elliptic curves, since both these equations correspond to the same genus 1 Riemann surface. Higher degree polynomials give way to hyper-elliptic curves, which are equivalent to higher genus Riemann surfaces.*

Recall that for an $n^{\text{th}}$ degree polynomial $h(x)$ with $h(a) = 0$, we can write:

$$h(x) = \sum_{j=1}^{n} \frac{(x-a)^j}{j!} h^{(j)}(a)$$

Hence,

$$P_4(y) = \tfrac{(y-y_0)}{1!}P_4'(y_0) + \tfrac{(y-y_0)^2}{2!}P_4''(y_0) + \tfrac{(y-y_0)^3}{3!}P_4'''(y_0) + \tfrac{(y-y_0)^4}{4!}P_4^{(4)}(y_0)$$

$$\implies (y')^2 = \tfrac{(y-y_0)}{1!}P_4'(y_0) + \tfrac{(y-y_0)^2}{2!}P_4''(y_0) + \tfrac{(y-y_0)^3}{3!}P_4'''(y_0) + \tfrac{(y-y_0)^4}{4!}P_4^{(4)}(y_0)$$

To make our solution more readable, let $a_1 = P_4'(y_0)$, $a_2 = P_4''(y_0)$, $a_3 = P_4'''(y_0)$, and $a_4 = P_4^{(4)}(y_0)$. Note that (31) gives that $(y - y_0) = \frac{-1}{\xi}$ and $\xi' = \xi^2 y' \implies y' = \frac{\xi'}{\xi^2}$. Thus, applying the change of variables yields:

$$\left(\tfrac{\xi'}{\xi^2}\right)^2 = \tfrac{a_1}{1!}\left(\tfrac{-1}{\xi}\right) + \tfrac{a_2}{2!}\left(\tfrac{-1}{\xi}\right)^2 + \tfrac{a_3}{3!}\left(\tfrac{-1}{\xi}\right)^3 + \tfrac{a_4}{4!}\left(\tfrac{-1}{\xi}\right)^4$$
$$\implies (\xi')^2 = -a_1\xi^3 + \tfrac{a_2}{2}\xi^2 - \tfrac{a_3}{6}\xi + \tfrac{a_4}{24}$$

Let $w = -\frac{a_1}{4}\xi \implies w' = -\frac{a_1}{4}\xi'$. Therefore, we obtain the following equation:

$$(w')^2 = 4\left(w^3 + \frac{a_2}{8}w^2 + \tfrac{a_1 a_3}{48}w + \tfrac{a_1^2 a_4}{1536}\right) \tag{32}$$

Next, we apply Proposition 4.1 again. Completing the cube on the right side of (32), we get:

$$(w')^2 = 4\left((w + \tfrac{a_2}{24})^3 + (\tfrac{2a_1a_3 - a_2^2}{192})w + (\tfrac{9a_1^2a_4 - a_2^3}{13824})\right)$$

Now, we make another change of variables; let $s = w + \frac{a_2}{24} \implies s' = w'$. Thus, we obtain the ODE:

$$(s')^2 = 4s^3 + \left(\frac{2a_1a_3 - a_2^2}{48}\right)s + \left(\frac{2a_2^3 - 6a_1a_2a_3 + 9a_1^2a_4}{3456}\right)$$

We recognize that the $\wp(z) = s(z)$ is a solution to this equation where the periods of the lattice, $\omega_1$ and $\omega_2$, are given by:

$$-\frac{2a_1a_3 - a_2^2}{48} = 60 \sum_{\{m,n\}\neq\{0,0\}} \frac{1}{(m\omega_1 + n\omega_2)^4}$$
$$-\frac{2a_2^3 - 6a_1a_2a_3 + 9a_1^2a_4}{3456} = 140 \sum_{\{m,n\}\neq\{0,0\}} \frac{1}{(m\omega_1 + n\omega_2)^6}$$

In the subsequent section, we walk back through the calculation to find expressions for the wave amplitude, $\eta$ and the velocity potential, $\phi$.

5.4. **Explicitly solving for $\eta$ and $\phi$.** We determined that $\wp(z) = s(z)$ on $\mathbb{C}$ modulo a lattice defined by the periods $\omega_1$ and $\omega_2$. Now, we walk back through our changes of variables, naming the substitution, and then plugging in the new version using the $\wp$-function.

We let $\rho = w + \frac{a_2}{24}$, so we have that:

$$w(z) = \wp(z) - \frac{a_2}{24}$$

We let $w = -\frac{a_1}{4}\xi$, so:

$$\wp(z) - \frac{a_2}{24} = -\frac{a_1}{4}\xi(z) \implies \xi(z) = \frac{a_2 - 24\wp(z)}{6a_1}$$

From the birational map (31), we had:

$$\xi = \frac{-1}{y - y_0} \implies y = -\frac{1}{\xi} + y_0$$

Thus,

$$y(z) = \frac{6a_1}{a_2 - 24\wp(z)} + y_0 \tag{33}$$

We also had $y = g'$. Because $f = \frac{c}{\varepsilon}g' - \frac{1}{2\varepsilon}(g')^2$, we will write down an expression for $f$ first.

$$f(z) = \frac{-18a_1^2}{\varepsilon(a_2 - 24\wp(z))^2} + \frac{6a_1(c - y_0)}{\varepsilon(a_2 - 24\wp(z))} + \frac{2cy_0 - y_0^2}{2\varepsilon} \tag{34}$$

However, to obtain an expression for $g$, we must integrate (33) .

$$\int g'(z)dz = \int \frac{6a_1}{a_2 - 24\wp(z)} + y_0 dz \tag{35}$$

However, this is not a trivial integral. First, we will evaluate:

$$I(z) = \int \frac{1}{\wp(z) - \frac{a_2}{24}} dz$$

Recall that $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1$ is surjective. Thus, there exists some $m \in \mathbb{C}/\Lambda$ such that $\wp(m) = \frac{a_2}{24}$. Hence,

$$\begin{aligned} I(z) &= \int \frac{1}{\wp(z) - \wp(m)} dz \\ &= \frac{1}{\wp'(m)} \left( 2z\zeta(m) + \ln \frac{\sigma(z-m)}{\sigma(z+m)} \right) \end{aligned} \tag{36}$$

The formulation for (36) was found in [2, Eq. 1037.06]. There is one condition we must handle for this expression of $I$; if $\wp(m) = e_1, e_2, e_3$, then we must use a different formulation for $I$, which we call $I_e$. Suppose $\frac{4a_2^2}{3} = e_j$ for $j = 1, 2,$ or 3 and let $k, \ell$ be the other indices. Then,

$$\begin{aligned} I_e(z) &= \int \frac{1}{\wp(z) - e_j} dz \\ &= \frac{1}{e_k e_\ell + 2e_j^2} \left( e_j z + \zeta(z) + \frac{1}{2} \frac{\wp'(z)}{\wp(z) - e_j} \right) \end{aligned} \tag{37}$$

The formulation for (37) was found in [2, Eqs. 1037.07-09]. Returning to (35), we see that:

$$g(z) = \begin{cases} -6a_1 I(z) + y_0 z + k, & \text{for } \frac{a_2}{24} \neq e_1, e_2, e_3 \\ -6a_1 I_e(z) + y_0 z + k, & \text{otherwise} \end{cases}$$

where $k$ is a constant of integration.

We are nearly done since we had $f(z) = \eta(z)$ and $g(z) = \phi(z)$. At the beginning of this problem, we let $z = x - ct$, but we will have to retrospectively make an adjustment to guarantee the wave amplitude and velocity potential are real valued. There are a few obstacles we need to address. First, recall that we let $y_0 \in \mathbb{C}$. Moreover, the $\wp$-function and, thus, $\sigma$ and $\zeta$ functions have troublesome asymptotics when evaluated at certain values of their half periods. To address both these issues, fix $\nu \in \mathbb{R}$ and $\eta$ and $\phi$ on $z = x - ct + i\nu$ [8]. Hence, we found:

$$\eta(x,t) = \frac{-18(P_4'(y_0))^2}{\varepsilon\Big(P_4''(y_0) - 24\wp(x-ct+i\nu)\Big)^2} + \frac{6(c-y_0)P_4'(y_0)}{\varepsilon\Big(P_4''(y_0)-24\wp(x-ct+i\nu)\Big)} + \frac{2cy_0 - y_0^2}{2\varepsilon}$$

$$\phi(x,t) = \begin{cases} \frac{-6P_4'(y_0)}{4\wp'(m)}\Big(2(x-ct+i\nu)\zeta(m) + \ln\frac{\sigma(x-ct+i\nu-m)}{\sigma(x-ct+i\nu+m)}\Big) + y_0(x-ct+i\nu) + k, & \frac{P_4''(y_0)}{24} \neq e_1, e_2, e_3 \\ \frac{-6P_4'(y_0)}{4e_ke_\ell + 8e_j^2}\Big(e_j(x-ct+i\nu) + \zeta(x-ct+i\nu) + \frac{1}{2}\frac{\wp'(x-ct+i\nu)}{\wp(x-ct+i\nu)-e_j}\Big) + y_0(x-ct+i\nu) + k, & \text{o/w} \end{cases}$$

where we recall that $y_0$ is a root of $P_4(y) = \frac{1}{4\mu\varepsilon}(y^4 - 4cy^3 + 4(c^2-\varepsilon)y^2 + \alpha y + \beta)$ and $m$ was defined such that $\wp(m) = \frac{4(P_4''(y_0))^2}{3}$. These expressions for $\eta$ and $\phi$ are the one-gap solutions to the Kaup-Broer system. Taking limits will produce soliton solutions.

## 6. Concluding Remarks

We summarized major concepts from algebraic geometry and elliptic curves required to study nonlinear, integrable partial differential equations. Then, we determined the one-gap solution to the Kaup-Broer system. Analyzing the one-gap solution with different parameters will result in a model accurate for varying levels of capillary and gravitational disturbance in accordance with the well-posed KB1 system. In future work we will determine the two-gap, and g-gap solutions to the KB1 system. By investigating the corresponding hyper-elliptic Riemann surfaces and the hyper-elliptic functions that reside on them, we will determine how to let $g \to \infty$. Moreover, we plan to extend the well-developed KdV theory to the Kaup-Broer system.

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