A Not is a Naught is a Knot

John Christopher Turner
16 August 1991
The University of the South
Oregon State University REU Program

1. Introduction. For many years the study of knots and knot theory has preoccupied mathematicians. When first formally observed in the 1880's, it was thought that knots were similar to the atom. Each knot represented a distinct atom, so if one could understand knots and tabulate them then one could understand the atom and its structure. In the late 19th century the inquiry into knots and their structure halted. Niels Bohr proposed his model of the atom, and most were convinced that he had properly modeled the atom. His model had nothing to do with knots at all, so the topic was abandoned for over 100 years. In the 1980's, chemists started delving into the knotted structure of DNA. It has been theorized that mathematical knots and the knotted DNA are very similar; thus, there has been a resurgence in the study of knots. An entire branch of Topology known as "knot theory" has been pioneered and developed to further study the knot.[1]

My summer project concerned knots, their projections, and several different equivalent knot representations. Basically, I wanted to design an algorithm that would transform a knot projection made up of curved line segments into a straight line segment projection. This "stick" model could then be lifted into three dimensions and would be equivalent to the original knot. Not only this, but I with the help of Dr. Dennis Garity (OSU) have come up with some conjectures about the stick number of the general knot and its relation to the crossing number. Before we go any further here are some definitions:

2. Definitions.

1. A **knot** is "an embedding of a circle S^1 into Euclidean 3-space, $\Re^3$, or the 3-sphere, S^3."[2] In other words, a knot is a subset of $\Re^3$ homeomorphic to the unit circle, S^1, in the plane.
2. Two knots K & K' are **equivalent** if there is an orientation preserving homeomorphism $h:\Re^3 \to \Re^3$ such that $h(K) = K'$.

---

[1] Adams, p. 4.
[2] Burde, Zieschang, p. 1.

3. A **polygonal knot** is a knot made up of a finite number of straight line segments (sticks).
4. A **stick** is a line segment of finite length.
5. The **stick number, S(K)**, of a knot K is min { n | ∃ a polygonal knot K' equivalent to K with K' made up of n line segments}.
6. A **projection of a knot** is a mapping from the knot $K \to \Re^2$ defined by the function f: $f(x, y, z) \to (x, y, 0)$. Such that the projection is a graph in the xy plane.
7. A **crossing** is a point on the projection such that $f^{-1} : f^{-1}(x,y)$ yields two distinct values (x_1, y_1, z_2) and (x_2, y_2, z_2) where $x_1 = x_2$, $y_1 = y_2$, and $z_1 \neq z_2$. The one with the larger z-coordinate is called an overcrossing, and the other is the corresponding undercrossing.
8. The **crossing number, C(K)**, of a knot K is the min { n | ∃ a knot K' equivalent to K with projection of K' having n crossings}.
9. **Seifert circle.** The Seifert algorithm takes any knot and creates an orientable surface with one boundary component such that the boundary circle is that knot. I am not concerned so much with the surface as I am with the circles that are a by-product of the algorithm. First take the projection of the knot and look at the crossings. At each crossing, two segments come in and two go out. Remove the crossing and connect the segment entering the crossing with the adjacent one leaving the crossing. Eliminate all crossings and the result will be a number of circles in the plane. These circles are **Seifert circles.**[3]
10. A **Polygonal Seifert Circle (PSC)** is a polygonal representation of a given Seifert circle The PSC is equivalent to the original Seifert circle and is constructed with straight line segments instead of curved line segments.
11. An **alternating knot** is "a knot with a projection that has crossings alternating between over and under as you go around the knot in a fixed direction."[4]

3. An Algorithm. Here is the algorithm that I have designed to transform a knot projection into the appropriate stick knot projection.

---

[3] Adams, pp.70-71
[4] Ibid, p.5.

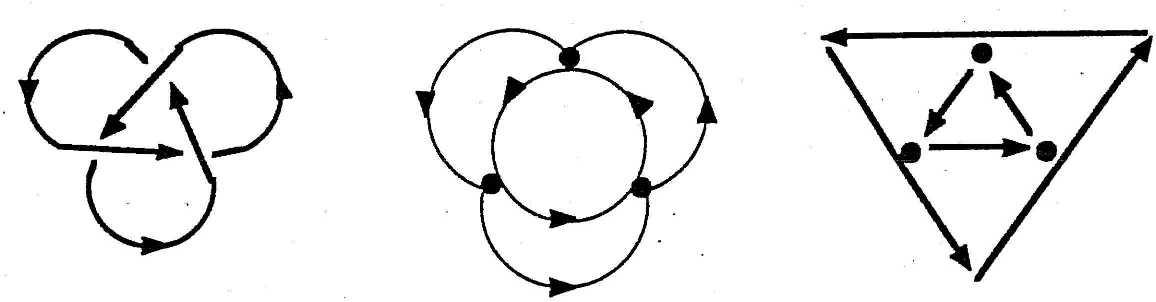
I. First we must orient the knot projection by placing arrows on the curved line segments.
II. Construct the oriented Seifert circles corresponding to the knot projection, and identify the crossings by small filled circles.
III. Replace the circles with Polygonal Seifert Circle (PSC) representations.
(Note: The third step is the most important step. If this is done correctly then the algorithm will yield the results that I have stated and will lead to my conjecture. See the PSC constructions of knots 3_1 through 8_{21} in the back of this text.)
IV. Adjust the vertices of the PSC representation to form a polygonal knot.

If this can be done for any knot then it is my conjecture that the polygonal knot K' is equivalent to the original knot K, and $\Rightarrow$ **S(K) $\leq$ 2*C(K).**

Conjecture: Every knot K with **C(K)** = n has a PSC representative with $\leq$ 2n line segments, so **C(K)** = n $\Rightarrow$ **S(K) $\leq$ 2n.**

Example: The Trefoil Knot (3_1)

Step I | Step II | Step III



Oriented Trefoil projection | Oriented Seifert circles | Oriented PSC projection

In this case, I claim that the trefoil knot can be constructed in less than or equal to six sticks. (Note: Step IV of the algorithm is not shown here but is described in more detail in sections 5-9.)

4. Overview. There are several things that we need to prove and look at before the conjecture can be proved in the general case. The Polygonal Seifert Circle (PSC) projections are basically graphs in the xy plane. The sticks are finite line segments of varying lengths, and the crossings are vertices of the graph. Once the PSC projections are constructed in the xy plane, the vertices are moved and adjusted above and below the xy plane (in the z direction) to create the crossings.

More specifically, the PSC projections form piecewise "circles" in the plane, and the crossings are represented by the vertices. For every vertex there are four edges that emerge. At each vertex, the four edges are paired and represent the two distinct Seifert circles that intersect at that vertex. (A, B) will be the notation for a pair of edges in a Seifert circle that is oriented such that A is visited first and then B. Also in the following diagrams, the Seifert circles will be labelled or colored distinctly. The end product (in an alternating knot) is to produce an alternating pattern of colors (solid/dotted). That is to say that (in the end product) if a dotted line goes into a crossing a solid line will come out.

5. ***Lemma 1.*** Assume that PSCs can be constructed with j sticks where there are no overlapping edges. This forms a polygonal projection of the knot K. Then $S(K) \leq j$.

What this means is that in the PSC representation the vertices can be adjusted in $\Re^3$ to form a polygonal knot K' with j line segments which is equivalent to K. The only way that a new stick would be added is if a crossing appears where it is not supposed to occur.

(1) Example:

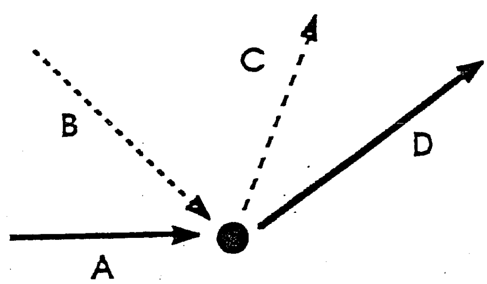


figure 1

In figure 1, the pairs of line segments (A, D) and (B, C) are a part of a continuous section of a PSC representation. In this example, we want (B, D) to pass over (A, C). So, (A, C) and (B, D) have their own vertex represented together by the single filled circle. By pushing down the vertex at (A, C) below the plane and raising the vertex at (B, D) above the plane an overcrossing is established such that (B,D) crosses over (A, C).

Assuming that the sticks are line segments in the xy plane, the raising and lowering of the vertices will not introduce any new crossings. The lengths of the line segments will vary to accommodate for the "stretching," but the vertex remains fixed in the xy plane (there is no lateral movement). The only movement is in the z direction as the vertex is being "adjusted." Since there is no lateral movement, the line segments can not cross each other at any point except the vertex. No new crossings are introduced which implies that no new sticks are added; therefore, we still have j line segments.

(2) Given the knot k assume that the first i vertices have been adjusted. Show that the i+1 vertex can be adjusted without effecting the i vertices and without adding new line segments. This implies $S(K) \leq j$.

We can see from (1) that by adjusting any vertex no lateral movement is involved. Therefore, by adjusting the i+1 vertex in the vertical direction, the other i vertices will not be effected. No new crossings are introduced, and we still have j sticks.

In (1) the lemma holds true, and we have shown that given i vertices the adjustment can be performed on the i+1 vertex without effecting the i previous vertices. Thus by induction the lemma is true for all i. *QED*

I will call this graph a graph of type T1. Therefore any graph that can be constructed with no overlapping edges is of type T1.

6. ***Lemma 2.*** Assume PSCs can be constructed with one overlapping edge and j sticks. Show $S(K) \leq j$.

Example:

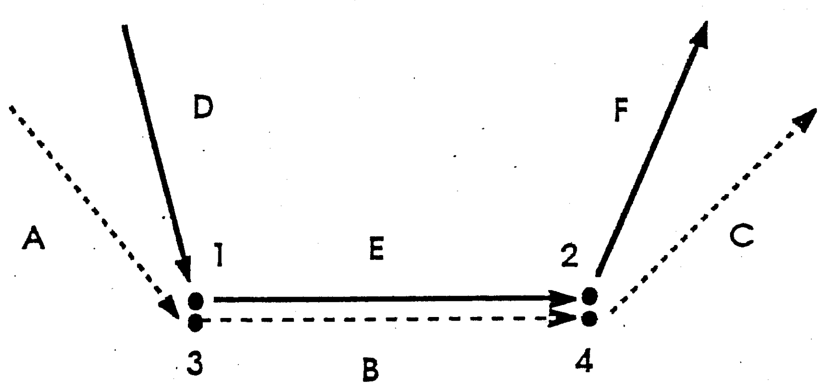


figure 2

In figure 2 we have the case where two polygonal Seifert circles intersect with one edge overlapping. Vertices 1 & 3, vertices 2 & 4 and segments E & B actually coincide, but for clarity I have separated and labelled them twice. As shown above (A, B, C) and

(D, E, F) are two distinct "circles" that overlap in one common edge and two vertices. There are several cases to consider, but only one is non-trivial.

Case 1. If the orientation was different, say that segment (D, E, F) was oriented such that it was (F, E, D) then the construction would look completely different, and there would be no overlapping edge. There would in fact be three distinct circles in this diagram. (Note: This is discussed in section 7)

Case 2. If crossings 1 & 3 and 2 & 4 were of the same parity (both over or under crossings) then clearly each vertex could be adjusted by an algorithm similar to that in *lemma 1*.

Case 3. If crossings 1 & 3 and 2 & 4 were of the opposite parity (an alternating knot), then the vertices could not be adjusted by *lemma 1* because raising or lowering the vertices would introduce a new crossing of segment E and segment B.

There is a way to adjust these vertices but it requires two steps. It first requires the lateral movement of vertices 1 and 2.

Step 1: Move vertex 1 and 2 along the slope of the line D and F respectively. Once this is done put segment B back in place. The effect of this movement is to actually pull segment B away from segment E in such a way that the magnitude of segment B may now be different than that of segment E (see fig. 3).

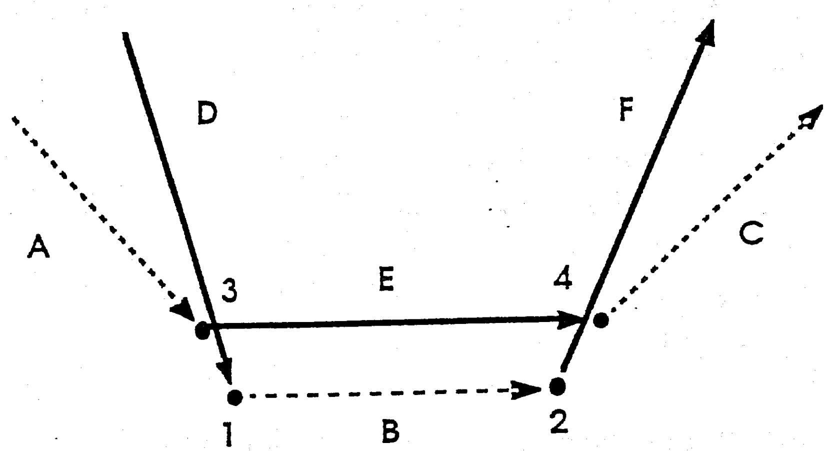


figure 3

Clearly this action will not introduce any new crossing provided that B does not cross any other parts of the knot. This can be guaranteed by a simple distance argument. By construction, segment B only intersects segments A, E, and C. So ∃ an ε such that if δ is any other segment, the distance between B and δ is $\geq \varepsilon$. Restrict the movement of segment B so that no point moves by more than $\frac{\varepsilon}{2}$.

Step 2: Once segment B is moved, the vertices 1 & 2 can be adjusted. By an argument similar to that in *lemma 1*, vertices 1 & 2 can be adjusted without adding any new crossings. This implies that we still have j line segments.

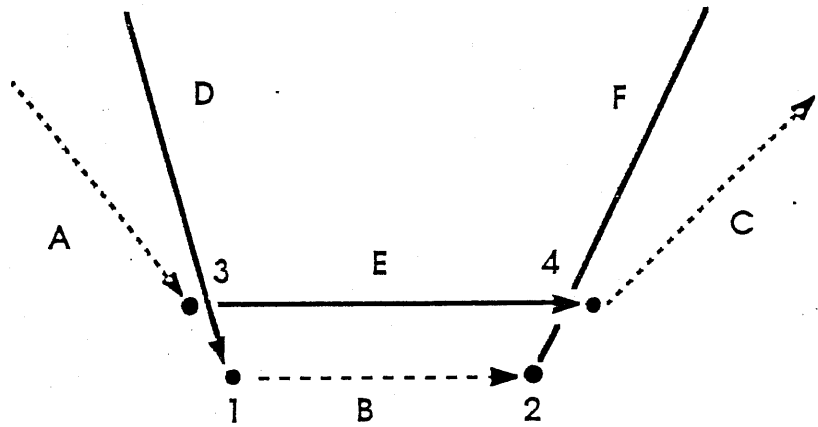


figure 4

Figure 4 shows an over crossing and then an under crossing of (D, B, F). By raising vertex 1 above the plane and lowering vertex 2 below the plane this was accomplished. Just the opposite can be done by performing the reverse actions on the vertices.

Notice that (A, E, C) remains fixed and is not moved at all in this process. It should be noted that when (D, B, F) was extended, it was the "inside" segments D, B, F that were moved while A, E, & C remained fixed. Only the segments of the inside circle can be moved. Since segments B & E coincide to begin with, I simply chose to move segment B to illustrate the alternating nature of this knot. (See section 4 about coloring of graphs). If the "outside" segments A & C (dotted) (see fig. 2) had been moved this process would not work because it would have introduced new crossings. Therefore, the inside segments must be the segments moved.

7. Special Cases. There are several special cases of *lemma 2* that we need to clear up before going on. The first case is one that occurs quite often in the algorithm. The problem occurs when a construction similar to figures 5-7 appears.

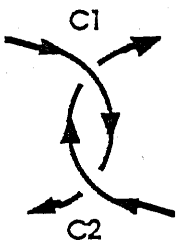


figure 5

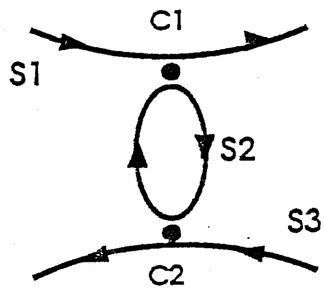


figure 6

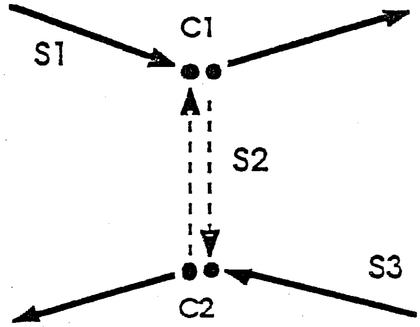


figure 7

Case 1: Figure 5 shows step I of the algorithm mentioned in 3. C1 and C2 stand for crossing one and crossing two. In the second step, figure 6, the Seifert circles are constructed and oriented. We can see that there are in fact three Seifert circles needed in this step of the algorithm. S1, S2, & S3 stand for Seifert circles 1, 2 & 3. And, finally, in figure 7, the PSC representation of this one part of the original projection is constructed. C1 and C2 have now been split up so that they will be easier to consider. The question arises as whether a similar argument used in *lemma 2* will work here also. With a little lateral movement it is clear that the same operations can be performed.

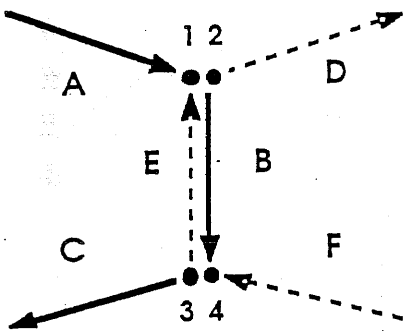


Figure 8

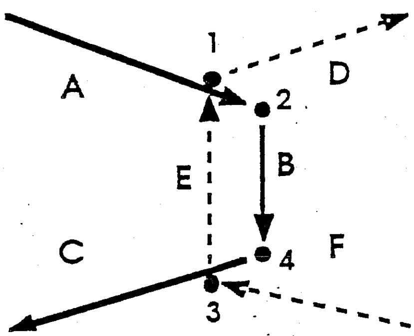


figure 9

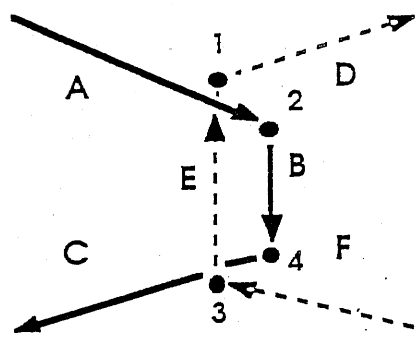


figure 10

In figure 8, the segments have been relabelled and recolored for clarity. (A, D) is S1; (E, B) is S2; And, (C, F) is S3. The solid segment (A, B, C) is the analog of the curved segment in the original projection that comes in from the top left corner, and (F, E, D) is the analog to the curved segment that comes in from the lower right corner (see fig. 5). The crossings C1 & C2 have been broken up into C1 = {1, 2} and C2 = {3, 4}. With these clarifications, it is easy to see that a similar argument used in *lemma 2* will also work in this case. Vertices 2 & 4 are extended along the slopes of segments A & C to produce figure 9. Then vertices 2 & 4 are adjusted in a way similar to *lemma 1*. Figure 10 shows the adjustment of the vertices to produce the alternating knot. Since these actions only produce movement on these two vertices, they, as before, have no effect on any crossings that comes before or after these crossings.

In the above case, the "middle" Seifert circle was located between two other convex Seifert circles. The next obvious case is when the middle circle appears between two concave circles.

<u>Case 2</u>: In the following case the pictures look much the same except that the circles are concave. Figures 11-13 show the first three steps of the algorithm.

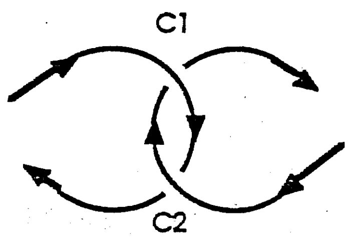


figure 11

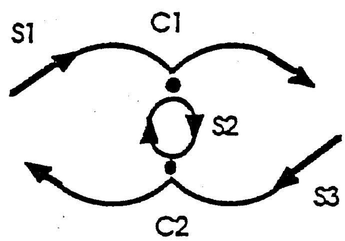


figure 12

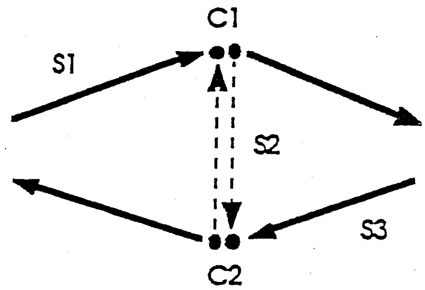


figure 13

After the first three steps, the next step involves the lateral movement of segment B. It is important to note here that the length of segments B & E are equal. By moving segment B to the right and extending vertices 2 & 4, we can see that the reduction of segment E may be necessary to insure that segment E does not cross segments A or C (see fig.15). All of these actions done with the same distance considerations of *lemma 2*. All lateral moves are restricted to less than $\frac{\varepsilon}{2}$. Once the lateral movement is finished vertices 2 & 4 can be adjusted in a way similar to *lemma 1* (see fig. 16).

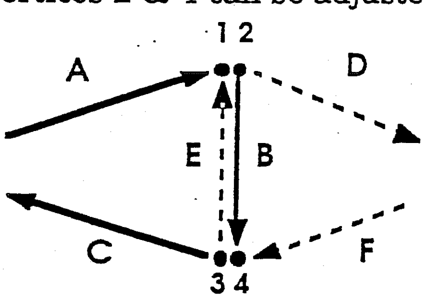


figure 14

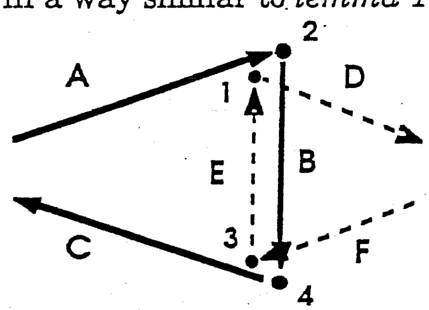


figure 15

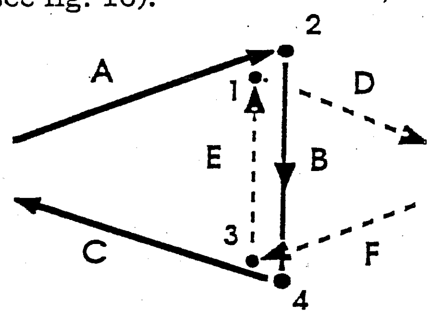


figure 16

(Note: The coloring here and in Case 1 is not exactly as described earlier because there are three circles and not two (and only two colors available to me). Therefore, the solid line represents one strand of the finished knot while the dotted represents the other.)

I will call any of the graphs in sections 6 and 7 a graph of type T2. Therefore any graph that can be constructed with one overlapping edge is of type T2.

8. ***Lemma 3.*** Assume PSCs can be constructed with two overlapping edges and j sticks. Show S(K) ≤ j.

In figure 17, we can see that there are two overlapping edges. There are two circles that intersect here: the inner circle is (E, F, G, H) and the outer circle is (A, B, C, D). If we look at the figure 18, it is clear to see that segment B can be moved laterally and the vertices can be adjusted with an argument similar to *lemma 2*.

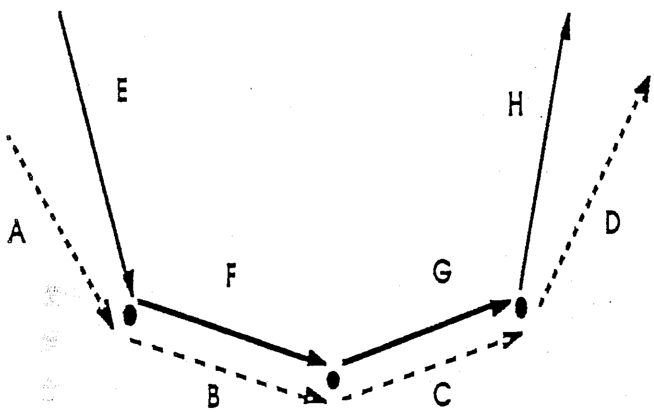


figure 17

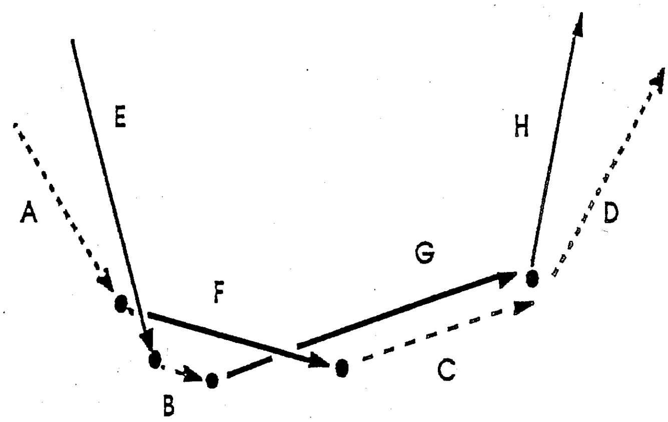


figure 18

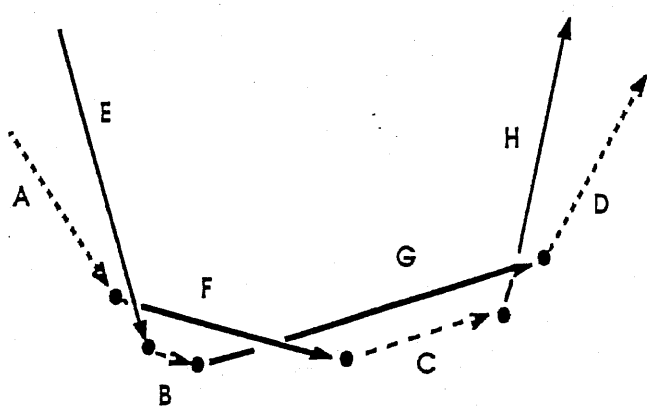


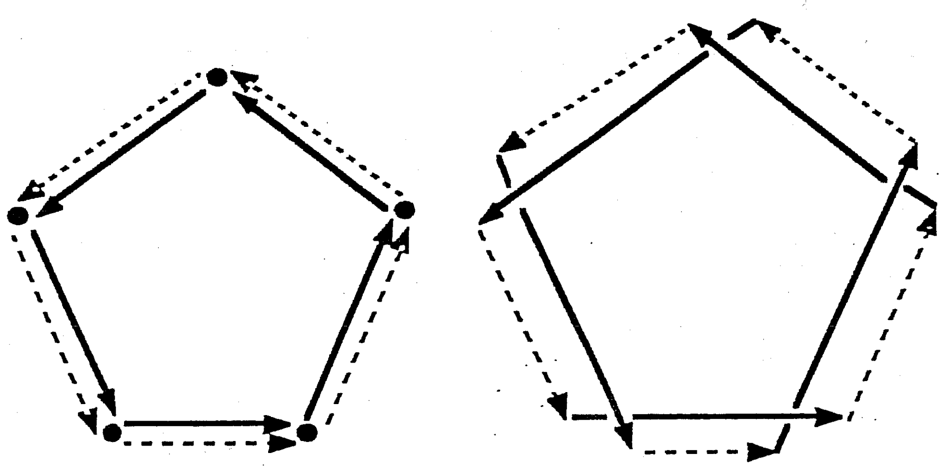
figure 19

However, it is not so obvious in figure 19 that segment C can be moved and the vertices adjusted. Notice that once again the "inside" circle is the one that is being stretched. Each time a vertex is adjusted the "inside" (solid line) segments are the ones that are lengthened. If the "outside" (dotted line) segments were moved this process would not work. As before segments B & F and segments G & C coincide and the dotted segments were chosen to show the alternating property of the knot. But getting back to the next vertex, if you look carefully, the only vertex that needs to be adjusted when segment C is moved is the vertex on the right end of the segment. The left vertex does not need to be adjusted since it is already in place from the previous adjustment to segment B (the crossing is correct). Therefore, the two adjustments can be made without the introduction of any new crossings, and we still have j sticks.

Extension of *lemma 3*: Assume PSC can be constructed with all overlapping edges and j sticks. This implies that $S(K) \leq j$.

It should be obvious from the above argument that if two Seifert circles completely coincide that each set of vertices can be moved and adjusted. There are several cases like this that appear in the knot tables. $3_1, 5_1, 7_1, 9_1, \ldots$

Example: 5_1



Starting at the bottom two segments, the first two vertices are adjusted in an argument similar to *lemma 3*. Then the next three can be adjusted in a similar fashion. If the process is done correctly the last two segments will already be adjusted and the crossings correct. This is done without the introduction of any new crossings, and we still have j sticks.

Therefore, this type of operation can be performed on a graph with any number of consecutive overlapping sides.

I will call any of the graphs in section 8 a graph of type T3. Therefore any graph that can be constructed with two overlapping edges is of type T3.

9. Conclusions. So, now we have graphs of type T1, T2, and T3. I claim that these are the only three types that will occur in a PSC representation. I have still yet to prove my conjecture and will not be able to in this paper without some more research. But , it is quite possible that given any knot K, $S(K) \leq 2n$ where n is the number of crossings. I have constructed tables in the back of this text with verification of my original conjecture with knots up through the 8-crossing knots. In each case, the stick number of the knot is equal to 2n.

Refined Conjecture: Every knot can be transformed (using the algorithms above) into a graph in the xy plane that resembles T1, T2, or T3, or any combination of T1-T3. If this can be done for every knot then given any knot K with **C(K) = n** then **S(K) ≤ 2n**.

10. Optimized stick models. After constructing the polygonal analog of some knots, I was able to optimize them and reduce the stick number of the polygonal knot to less than 2n. With the aid of *Mathematica*™, I constructed graphic 3D images of the knots using the following code:

```
knot[L_,v_]:=(
            knotlist={};
            Do[AppendTo[knotlist,Graphics3D[stick[L[[i]],L[[i+1]],v],
            ViewPoint->{0,0,2},PlotRange->All]],
            {i,1,Length[L]-1}];
            Apply[Show,knotlist]); This part builds the knot.

v={0.15,0.15,0}; This line is for the thickness of the sticks.
stick[L1_,L2_,v_]:=Polygon[{L1,L1+v,L2+v,L2}] This is the stick maker.
```

Basically what this code does is graphically display an image of what the polygonal knot would look like in $\Re^3$ from an above viewpoint. The under and overcrossings are evident in the image as they would be in a physical stick model. The following code lists a series of vertices for the trefoil knot, and the program draws thin rectangles between the vertices to make the actual polygonal knot.

trefoil = {{1, 0, -2}, {5, 0, 2}, {2, 5, -2}, {0, 2, 0}, {6, 2, 0}, {4, 5, 2}, {1, 0,- 2}};

By entering the command **knot[trefoil]** *Mathematica*™ constructs the knot and displays it in a graphics window. I have constructed several optimized polygonal knots, and included the first four (3_1, 4_1, 5_1 and 5_2) are in the back of this text. The vertices and number of sticks for the other three knots are listed below

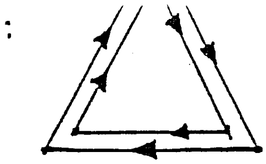
4_1: **figure8** = {{1, 0, 3}, {4, 0, -1}, {1.85, 4.15, -1.9}, {5, 5, -2.5}, {2.5, 1, -1}, {0.5, 4, 1}, {3, 4, -3.5} {1, 0, 3}}; constructed with 7 sticks.
5_1: **fiveone** = {{1, 0, 0}, {4, 3, -0.5}, {0, 3, -1}, {3, 0, -0.5}, {2.5, 2, 1}, {2, 4, -3}, {1.75, 2.75, 0}, {1.5, 1.5, -1}, {1, 0, 0}; constructed with 8 sticks.
5_2: **fivetwo** = {{1, 0, 2}, {5, 3, -2}, {3.5, 4, 0}, {2, 3, 0.5), {4, 0, 0.5}, { 2.7, 1.5, -3}, {1, 3, 35}, {1.5, 4, 0}, {3, 3, 0.3}, {1, 0, 2}}; constructed with 9 sticks.

Each construction was done with strictly less than 2n sticks (except for the trefoil knot). I will conclude with that. I have not been able to prove my conjectures in the general case, and they are still open questions. But there is still progress to be made.

Bibliography

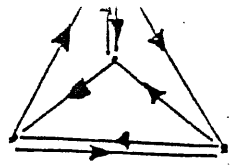
1. Adams, Colin C., The Knot Book (draft). Deparment of Mathematics, Williams College., November 7, 1990.

2. Burde, Gerhard & Zieschang, Heiner. Knots. Walter de Gruyter & Co., New York., 1985.

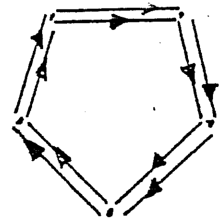


6 sticks

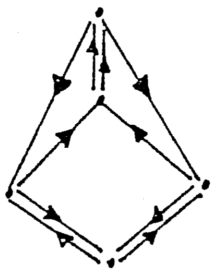
4_1:



8 sticks

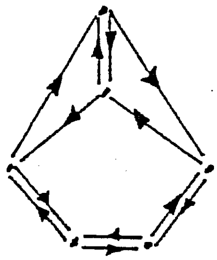


10 sticks



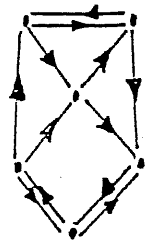
10 sticks

6_1:

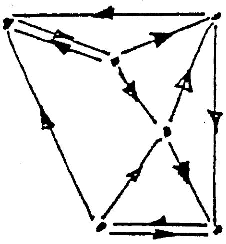


12 sticks

6_2:

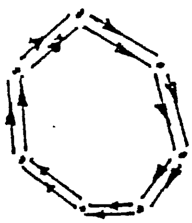


12 sticks



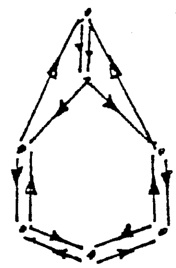
12 sticks

7_1:

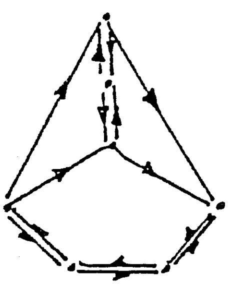


14 sticks

7_2:

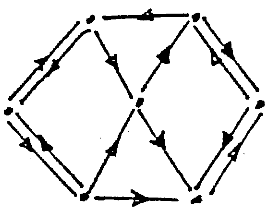


14 sticks



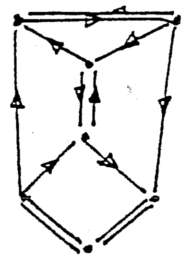
4 sticks

7_4:

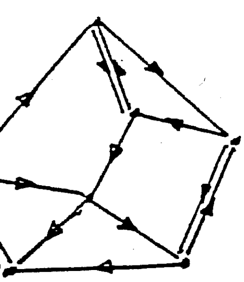


14 sticks

7_5:

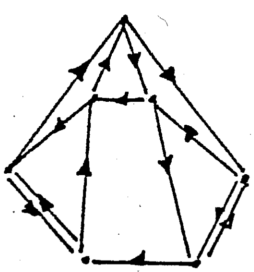


14 sticks



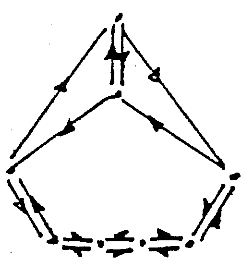
14 sticks

7_7:

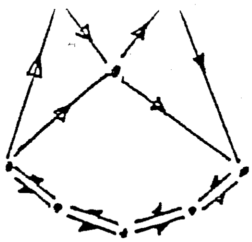


14 sticks

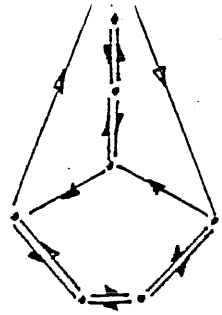
8_1:



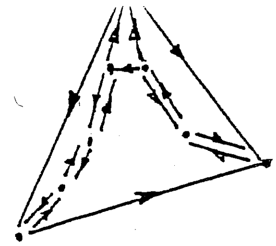
16 sticks



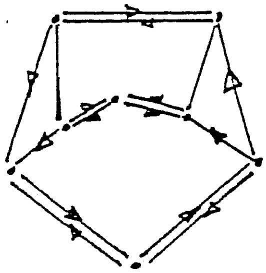
16 sticks



16 sticks

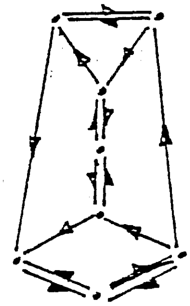


16 sticks



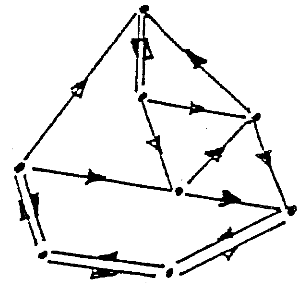
16 sticks

8_6:

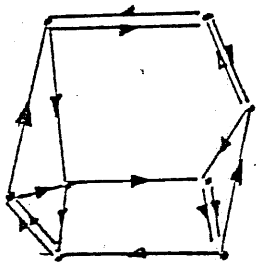


16 sticks

8_7:

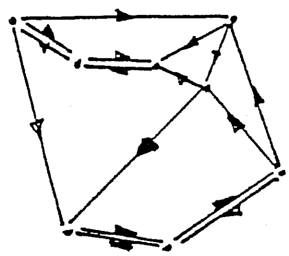


16 sticks



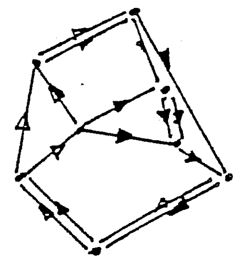
16 sticks

8_9:

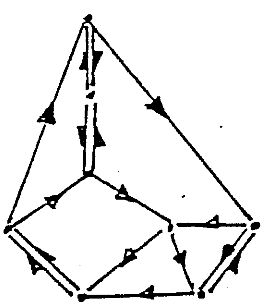


16 sticks

8_{10}:

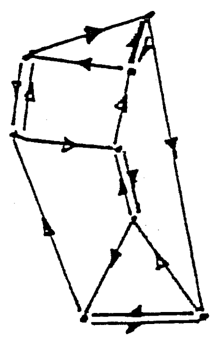


16 sticks



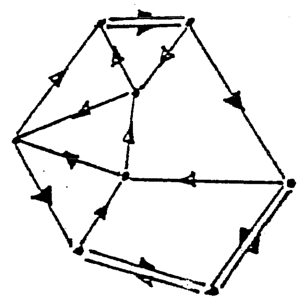
16 sticks

8_{12}:

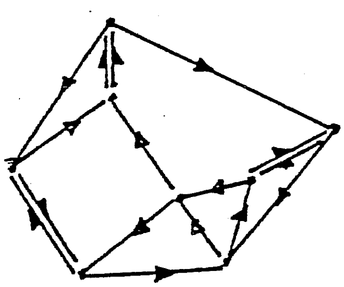


16 sticks

8_{13}:

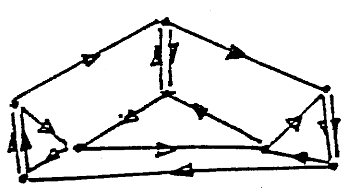


16 sticks



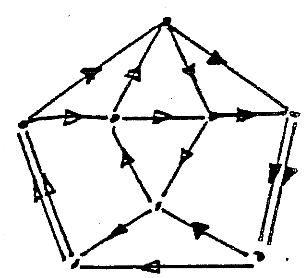
16 sticks

8_{15}:

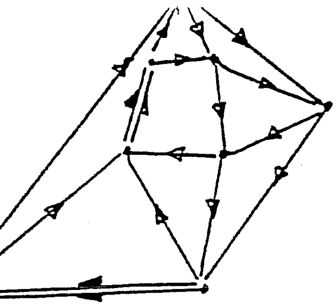


16 sticks

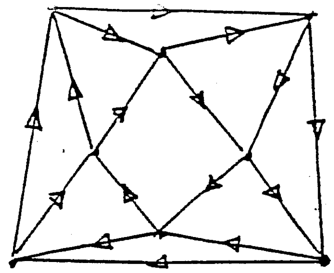
8_{16}:



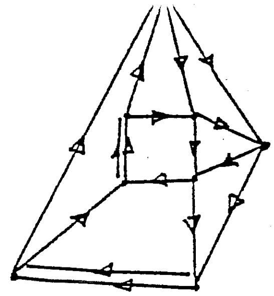
16 sticks



16 sticks



16 sticks



16 sticks

. - 8_{21} - same as 8_{19}.

Trefoil

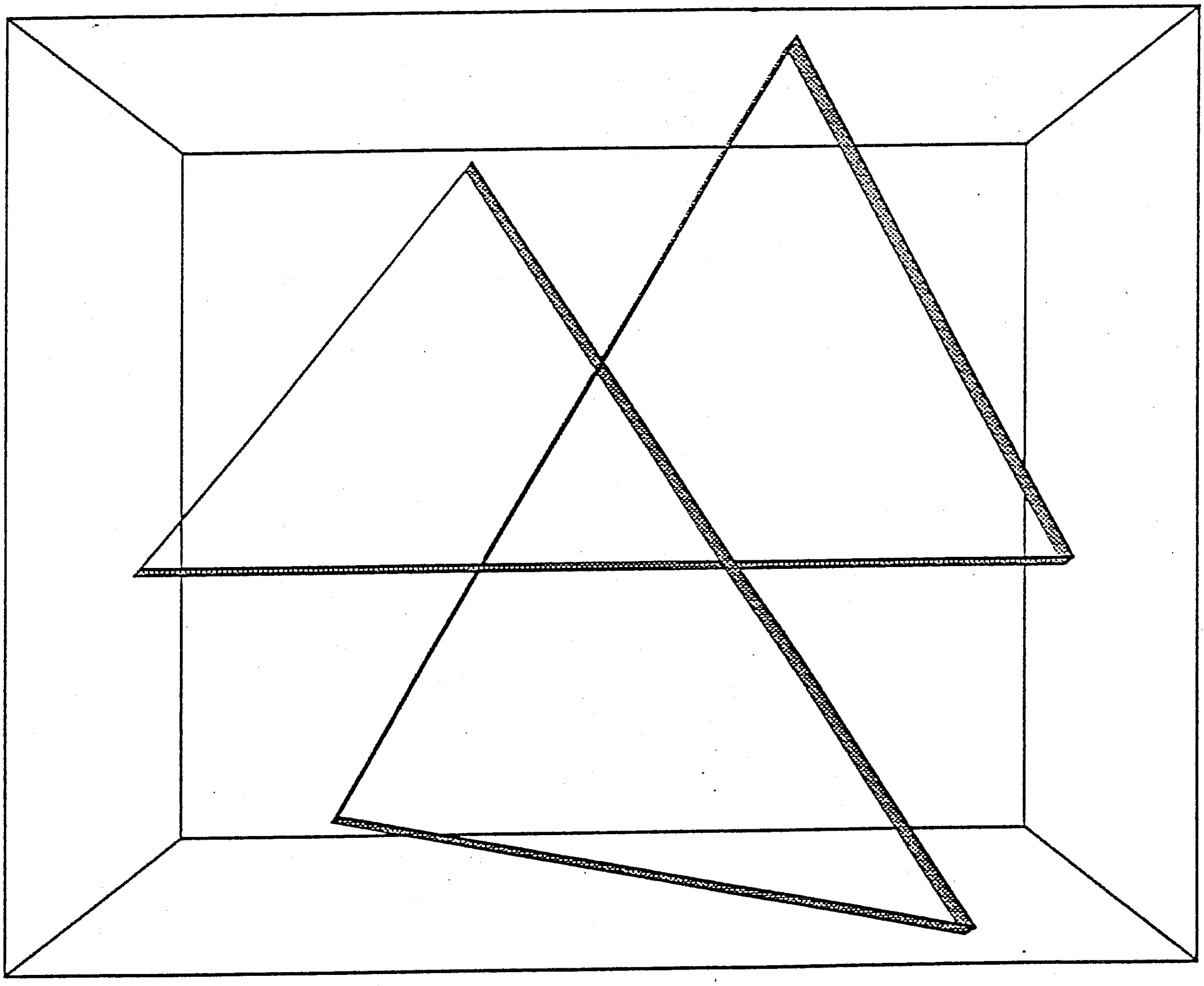
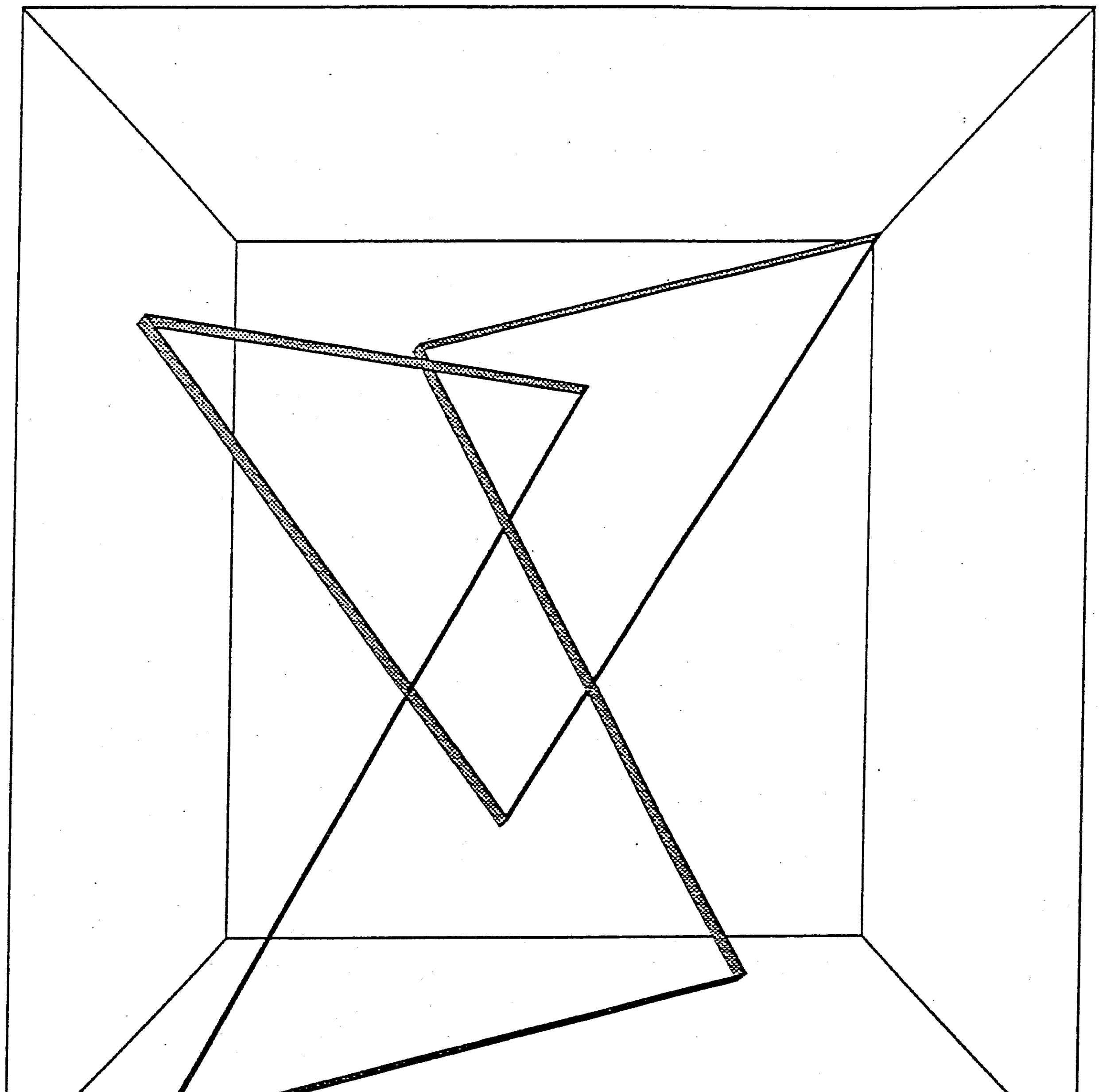
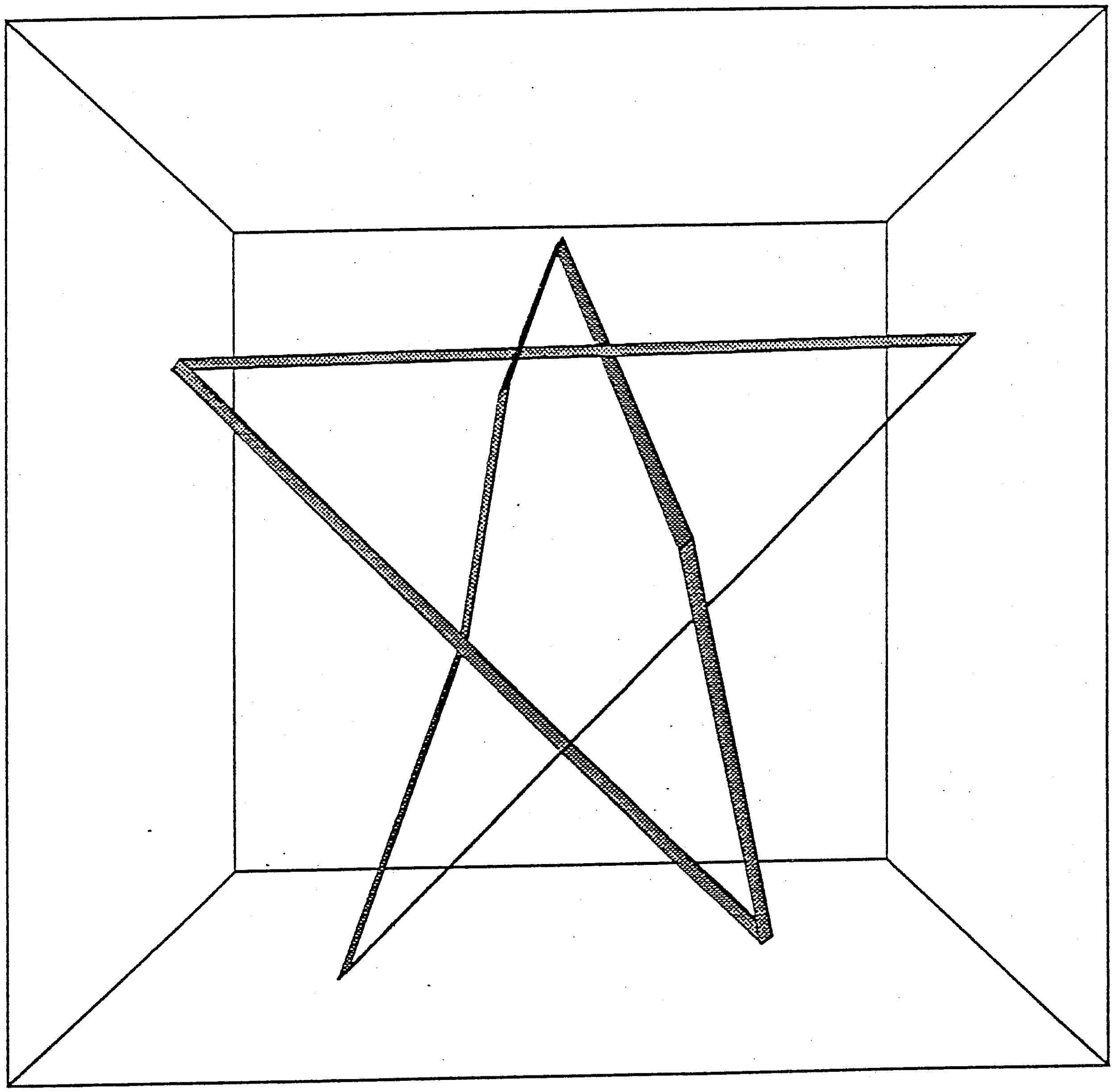


Figure 8



FiveOne



FiveTwo