OSU Department of Mathematics
Qualifying Examination
Spring 2022

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

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Common notations:

- A^t denotes the transpose of a matrix A
- A^n denotes the n^{th} power of a square matrix A
- $M = \big(m_{ij}\big)$ is the matrix whose (i, j) entry is m_{ij}
- $\det(M)$ denotes the determinant of a square matrix M
- For vectors x and y in the same inner product space, $x \perp y$ means that x and y are orthogonal vectors, i.e., their inner product $\langle x|y\rangle = 0$

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Problems:

1. Let V be a vector space with an inner-product $\langle \cdot | \cdot \rangle$. Denote $\|v\|^2 = \langle v | v \rangle$.

   a. (3 pts) When V is a real space, prove that $x \perp y$ if and only if
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

   b. (7 pts) When V is a complex space, prove that $x \perp y$ if and only if
$$\|\alpha x + \beta y\|^2 = \|\alpha x\|^2 + \|\beta y\|^2$$
for all scalars α and β in $\mathbb{C}$.

2. **a.** (6 pts) Let V be a vector space of dimension 2 over field F. Let T be a linear operator on V. Prove that the null space of T is identical to the image of T if and only if their intersection contains a nonzero vector.

   b. (4 pts) Let $V = \mathbb{C}$, and $F = \mathbb{R}$. Construct a linear operator T on V such that the null space of T is identical to the image of T. Justify your example.

3. Recall a matrix A is nilpotent if there exists a positive integer n such that A^n is the zero matrix.

   a. (6 pts) Suppose A, B are both 5×5 nilpotent complex matrices with the same nullity and the same minimal polynomial. Prove that A and B are similar.

   b. (4 pts) Give an example of two 7×7 nilpotent complex matrices with the same nullity and the same minimal polynomial that are not similar.

4. Let a and b be $n \times 1$ column vectors with entries from a field F.

   a. (2 pts) Verify the $n = 3$ case of the identity $\det(I + ab^t) = 1 + a^t b$.

   b. (4 pts) Suppose A is an $n \times n$ nonsingular matrix. Assuming the identity in (a), prove that
$$\det(A + ab^t) = (1 + b^t A^{-1} a) \det(A).$$

   c. (4 pts) Suppose $M = \big(m_{ij}\big)$ is an $n \times n$ matrix such that
$$m_{ij} = \begin{cases} 1 + d_i & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases} \qquad \text{where } d_i \neq 0 \text{ for all } i.$$
Prove that
$$\det(M) = \prod_{i=1}^{n} d_i \left(1 + \sum_{i=1}^{n} \frac{1}{d_i} \right).$$