A GENERALIZED ALDER-TYPE PARTITION INEQUALITY

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Abstract. Partitions of integers remain an intriguing object of study for many number theorists. Popularized by big names such as Euler and Ramanujan, this topic has influence on various fields including mathematical physics, modular forms, representation theory, and of course combinatorics. We focus on an important pair of partition counting functions and how these two functions compare under certain constraints.

Define $q_d^{(a)}(n)$ to be the number of partitions of n with parts that have difference at least d and size at least a, $Q_d^{(a)}(n)$ to be the number of partitions of n into parts that are $\pm a$ modulo $d+3$, and set $\Delta_d^{(a,b)}(n) = q_d^{(a)}(n) - Q_d^{(b)}(n)$. In 1956, Alder conjectured that $\Delta_d^{(1,1)}(n) \geq 0$ for all $n, d \geq 1$. Together, Andrews (1971), Yee (2008), and Alfes, Jameson, and Lemke Oliver (2011) proved Alder's conjecture.

In 2020, Kang and Park worked to generalize Alder's conjecture, replacing $Q_d^{(a)}(n)$ with $Q_d^{(a,-)}(n)$, which removes $d+3-a$ as an admissible part in order to incorporate the 2nd Rogers-Ramanujan identity. Here, we will explore a variation on the work of Kang and Park, providing sufficient conditions on d, n and a such that $q_d^{(a)}(n) \geq Q_d^{(a,-)}$. Moreover, we build on recent research of Inagaki and Tamura (2022) to determine conditions on d, n, N such that $q_d^{(1)}(n) \geq Q_{d-N}^{(1,-)}(n)$.

1. Introduction

Euler discovered what was perhaps the first theorem within partition theory with his famous result stating that the number of partitions of a positive integer into distinct parts equals the number of partitions of the same number into odd parts. Rogers and Ramanujan developed two identities analogous to that of Euler's, which are as follows.

- 1st Rogers-Ramanujan identity: The number of partitions of n into parts differing by at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$
- 2nd Rogers-Ramanujan identity: The number of partitions of n into parts at least 2 and differing by 2 equals the number of partitions on n into parts congruent to $\pm 2 \pmod 5$.

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These identities can be understood by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

where $(a;q)_0 := 1$ and $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$ for $1 \leq n \leq \infty$.

Schur [8] drew from these identities and, in 1926, he proved that the number of partitions of n into parts differing by at least 3, where no two consecutive multiples of 3 appear, equals the number of partitions of n into parts congruent to $\pm 1 \pmod 6$. Note that this identity then implies that the number of partitions of n into parts differing by at least 3 is at least the number of partitions of n with parts congruent to $\pm 1 \pmod 6$.

After proving that no other partition identities can exist, Alder [1] conjectured in 1956 that the number of partitions of n into parts with difference at least d is at least the number of partitions of n into parts congruent to $\pm 1 \pmod{d+3}$. Let $q_d^{(a)}(n)$ denote the number of partitions of a positive integer n with parts at least a and differing by at least d, and let $Q_d^{(b)}(n)$ denote the number of partitions of n with parts $\equiv \pm b \pmod{d+3}$. Define

$$\Delta_d^{(a,b)}(n) = q_d^{(a)}(n) - Q_d^{(b)}(n),$$
$$\Delta_d^{(a)}(n) = \Delta_d^{(a,a)}(n).$$

Then Alder's conjecture can be stated as

$$\Delta_d^{(1)}(n) \geq 0,$$

for all $n, d \geq 1$. Notice that Euler's identity, both Rogers-Ramanujan identities, and Schur's inequality are special cases of Alder's Conjecture.

In 1971, Andrews [3] proved this conjecture for $n \geq 1$ and $d = 2^k - 1$ when $k \geq 4$. In 2004 and 2008, Yee [9],[10] proved this conjecture for $n \geq 1$ and $d \geq 32$ or $d = 7$. In 2011, Alfes, Jameson, and Lemke Oliver [2] completed the proof by proving the conjecture for $n \geq 1$ and $4 \leq d \leq 30$ with $d \neq 7, 15$.

In 2020, Kang and Park [7] drew from the second Rogers-Ramanujan identity as a means of further generalizing Alder's Conjecture. Specifically, Kang and Park modified the partition functions $Q_d^{(b)}(n)$ and $\Delta_d^{(a,b)}(n)$ by defining

$$Q_d^{(b,-)}(n) := p(n \mid \text{parts} \equiv \pm b \pmod{d+3}, \text{ excluding the part } d+3-b),$$
$$\Delta_d^{(a,b,-)}(n) := q_d^{(a)}(n) - Q_d^{(b,-)}(n),$$
$$\Delta_d^{(a,-)} := \Delta_d^{(a,a,-)}(n),$$

where $p(n \mid \text{condition})$ denotes the number of partitions of n satisfying the condition. They then presented the following conjecture.

Conjecture 1.1 (Kang, Park, [7], 2020)**.** *For all $d, n \geq 1$,*

$$\Delta_d^{(2,-)}(n) \geq 0.$$

Kang and Park [7] proved Conjecture 1.1 when n is even, $d = 2^k - 2$, and $k \geq 5$ or $k = 2$. In 2021, Duncan, Khunger, Swisher, and Tamura [4] proved Conjecture 1.1 except for the cases $d = 1$ and $3 \leq d \leq 61$. In their proof, Duncan et al. used the following.

Proposition 1.2 (Duncan et al., [4], 2021)**.** *If $d = 15$ or $d \geq 31$, then for $n \geq 1$,*

$$q_d^{(1)}(n) \geq Q_{d-2}^{(1,-)}(n).$$

Moreover, Duncan et al. conjectured that the inequality in Conjecture 1.1 holds under certain conditions for $\Delta_d^{(3,-)}(n)$.

Conjecture 1.3 (Duncan et al., [4], 2021)**.** *For all $d, n \geq 1$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

Inagaki and Tamura [5] recently stated and proved the following theorem.

Theorem 1.4 (Inagaki and Tamura,[5], 2022)**.** *For $n \geq 1$ and $d = 1, 2, 91, 92, 93$, or $d \geq 187$,*

$$\Delta_d^{(3,-)}(n) \geq 0.$$

As a lemma used in their proof of Theorem 1.4, Inagaki and Tamura extended Proposition 1.2 as follows.

Proposition 1.5 (Inagaki and Tamura, [5], 2022)**.** *For $d = 31$ or $d \geq 63$, and $n \geq d + 2$,*

$$q_d^{(1)}(n) \geq Q_{d-3}^{(1,-)}(n).$$

Near the end of their paper, Inagaki and Tamura [5] presented the following conjecture.

Conjecture 1.6. *Let $d \geq 12$ and $n \geq d + 2$. Then*

$$q_d^{(1)}(n) - Q_{d-4}^{(1,-)}(n) \geq 0.$$

Moreover, Inagaki and Tamura [5] showed that Conjecture 1.6 implies a generalization of Kang and Park, namely that for $a, n \geq 1$ and $\lceil \frac{d}{a} \rceil \geq 12$,

$$\Delta_d^{(a,-)}(n) \geq 0. \tag{1}$$

We prove a generalization of Conjecture 1.6 with modified lower bounds for n, d.

Theorem 1.7. *If $N \geq 2$, $d \geq \max\{\frac{47N-81+\sqrt{2025N^2-7038N+6025}}{2}, 63\}$, and $n \geq d + 2$, then*

$$q_d^{(1)}(n) \geq Q_{d-N}^{(1,-)}(n).$$

Remark 1.8. *We observe that when $N \geq 2$,*

$$2025N^2 - 7038N + 6025 < (45N - 78)^2,$$

so it suffices to consider $d \geq \max\{46N - 79, 63\}$ in Theorem 1.7.

Using Theorem 1.7 and the methods of Inagaki and Tamura [5], we prove (1) with modified bounds.

Theorem 1.9. *For $a \geq 1$, $\lceil \frac{d}{a} \rceil \geq 105$, and $1 \leq n \leq d + 2 + a$ or $d + 2a \leq n$,*

$$\Delta_d^{(a,-)}(n) \geq 0.$$

We now outline the rest of the paper. In Section 2, we state a fundamental result of Andrews [3] and discuss notation and lemmas used in the proof of Theorem 1.7. In Section 3, we prove Theorem 1.7 by splitting into two cases based on the relative size of n and d. Finally, in Section 4, we prove Theorem 1.7 and discuss potential future work.

2. Preliminaries

We begin this section by giving some basic notation that we will use throughout the paper. We then state a fundamental result of Andrews [3], in addition to work from Yee [10], Duncan et al. [4], and Inagaki and Tamura [5], which we employ in later sections. We then introduce notation and key lemmas used in the proof of Theorem 1.7.

First, we define some notation. Let $\lambda \vdash n$ denote that λ is a partition of n. For any $\lambda \vdash n$, if $\lambda_1 \geq \lambda_2 \geq \dots$ are the parts of λ, and each λ_i occurs m_i times as a part in λ, then we represent λ using the sequence $(\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \dots)$. If $m_i = 1$, then we will omit the exponent (1) for λ_i. For example, we represent the partition $5+5+4$ as $(5^{(2)}, 4)$. Recall that $p(n \mid \text{condition})$ denotes the number of partitions of n satisfying the stated condition. For a nonempty set $A \subseteq \mathbb{N}$ and $n \geq 0$, define $\rho(A; n) := p(n \mid \text{parts in } A)$, i.e., $\rho(A;n)$ is the number of partitions of n with parts in A.

Theorem 2.1 (Andrews [3], 1971)**.** *Let $S = \{a_i\}_{i=1}^{\infty}$ and $T = \{b_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geq b_i$ for all i. Then*

$$\rho(T; n) \geq \rho(S; n).$$

Inagaki and Tamura [5] expanded Theorem 2.1 to allow for partitions of different integers, which allows us to prove a key inequality in our proof of Theorem 1.9.

Lemma 2.2 (Inagaki and Tamura [5])**.** *Let $a \geq 1$, and let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}\infty$ be two strictly increasing sequences of positive integers such that $y_1 = a$ and $a \mid y_i$, $x_i \geq y_i$ for all $i \geq 1$. Then for all $n \geq 1$,*

$$\rho(T; n + \hat{n}_a) \geq \rho(S; n),$$

where $\hat{n}_a$ denotes the least nonnegative integer such that $a \mid (n + \hat{n}_a)$.

Duncan et al. [4] also expanded Theorem 2.1, which allows us to prove another key inequality in our proof of Theorem 1.9.

Lemma 2.3 (Duncan et al. [4], 2021)**.** *Let $a, d \geq 1$, and let $n \geq d + 2a$. Then*

$$q_d^{(a)}(n) \geq q_{\lceil \frac{d}{a} \rceil}^{(1)}\left(\left\lceil \frac{n}{a} \right\rceil\right).$$

The following lemma also proves to be crucial to the proof of Theorem 1.9.

Lemma 2.4 (Duncan et al. [4], 2021)**.** *Let $a, d, n \geq 1$ be such that $a \mid (d+3)$. Then*

$$Q_d^{(a,-)}(an) = Q_{\frac{d+3}{a}-3}^{(1,-)}(n).$$

We now begin developing the notation and key lemmas which we use in our proof of Theorem 1.7. We first present a lower bound for $q_d^{(1)}(n)$ dependent on d, which we leverage in our proof of Theorem 1.7 when $n \leq 7d + 13$.

Lemma 2.5. *For $d, n \geq 1$, we have*

$$q_d^{(1)}(n) \geq \max\left\{1, \left\lfloor \frac{n-d}{2} \right\rfloor + 1\right\}.$$

Proof. We begin by defining the sets

$$\begin{aligned} R_d(n) &= \{\lambda \vdash n \mid \lambda \text{ is counted by } q_d^{(1)}(n)\}, \\ R_d^1(n) &= \{\lambda \in R_d(n) \mid \lambda \text{ has exactly 1 part}\}, \\ R_d^2(n) &= \{\lambda \in R_d(n) \mid \lambda \text{ has exactly 2 parts}\}, \end{aligned}$$

noting that $q_d^{(1)}(n) = |R_d(n)|$.

The idea is to show both $q_d^{(1)}(n) \geq 1$ and $q_d^{(1)}(n) \geq \lfloor \frac{n-d}{2} \rfloor + 1$. The former is immediate, seeing as $(n) \in R_d(n)$ for all $n, d \geq 1$.

Now for the latter. Since $R_d^1(n) \cap R_d^2(n) = \varnothing$,

$$q_d^{(1)}(n) = |R_d(n)| \geq \left|R_d^1(n) \cup R_d^2(n)\right| = \left|R_d^1(n)\right| + \left|R_d^2(n)\right|. \tag{2}$$

We know $|R_d^1(n)| = 1$ since (n) is the only partition of n with exactly 1 part, and it is counted by $q_d^{(1)}(n)$. Now, we may show that $|R_d^2(n)| \geq \lfloor \frac{n-d}{2} \rfloor$ by observing that $k \leq \frac{n-d}{2}$ implies $(n-k) - k \geq d$. Thus, $(n-k, k) \in R_d^2(n)$ and is unique for every $1 \leq k \leq \lfloor \frac{n-d}{2} \rfloor$, implying that $|R_d^2(n)| \geq \lfloor \frac{n-d}{2} \rfloor$. Thus, using equation (2), we find that

$$q_d(n) \geq \left|R_d^1(n)\right| + \left|R_d^2(n)\right| \geq \left\lfloor \frac{n-d}{2} \right\rfloor + 1.$$

We hence conclude that $q_d^{(1)}(n) \geq \max\left\{1, \lfloor \frac{n-d}{2} \rfloor + 1\right\}$ for all $d, n \geq 1$, as desired. ☐

We now develop lemmas crucial to our proof of Theorem 1.7 in the case that $n \geq 7n + 14$. Define for integers $s, d \geq 1$

$$T_{s,d} := \{y \in \mathbb{N} \mid y \equiv 1, d+2, \ldots, d+2^{s-1} \pmod{2d}\}, \tag{3}$$

and define $r \geq 0$ to be the greatest integer such that

$$2^r - 1 \leq d. \tag{4}$$

With these definitions, we prove the following claim.

Lemma 2.6. *If $1 \leq a \leq b \leq r$, then $\rho(T_{a,d}; n) \leq \rho(T_{b,d}; n)$.*

Proof. We define strictly increasing sequences $\{y_n^a\}_{n=1}^\infty$ and $\{y_n^b\}_{n=1}^\infty$ corresponding to the sets $T_{a,d}$ and $T_{b,d}$, respectively. We then apply Theorem 2.1 to show that $\rho(T_{a,d}; n) \leq \rho(T_{b,d}; n)$.

To begin, we create Table 1 by arranging the elements of $T_{a,d}$ in increasing order into rows of length a. Note that for any fixed column, progressing down one row amounts to adding a factor of $2d$. We create Table 2 depicting the elements of $T_{b,d}$ in the same fashion.

Every positive integer n can be written in the form $n = (j-1)a + m$, where $j \geq 1$ and $1 \leq m \leq a$. For all $n \geq 1$, if $n = (j-1)a + m$, define y_n^a to be the element of the j^{th} row and m^{th} column of Table 1. We define y_n^b analogously using Table 2.

To show that $\{y_n^a\}_{n=1}^\infty$ is strictly increasing, notice that Table 1 is increasing across each row by construction. So for a fixed $j \geq 1$, $1 \leq m_1 < m_2 \leq a$ implies $y_{(j-1)a+m_1}^a < y_{(j-1)a+m_2}^a$.

$j\backslash m$	1	2	$\cdots$	a
1	1	$d+2$	$\cdots$	$d+2^{a-1}$
2	$2d+1$	$3d+2$	$\cdots$	$3d+2^{a-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$(2j-2)d+1$	$(2j-1)d+2$	$\cdots$	$(2j-1)d+2^{a-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1. Members of $T_{a,d}$ in increasing order across rows of length a.

$j\backslash m$	1	2	$\cdots$	a	$\cdots$	b
1	1	$d+2$	$\cdots$	$d+2^{a-1}$	$\cdots$	$d+2^{b-1}$
2	$2d+1$	$3d+2$	$\cdots$	$3d+2^{a-1}$	$\cdots$	$3d+2^{b-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$(2j-2)d+1$	$(2j-1)d+2$	$\cdots$	$(2j-1)d+2^{a-1}$	$\cdots$	$(2j-1)d+2^{b-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2. Members of $T_{b,d}$ in increasing order across rows of length b.

Therefore, it suffices to show that Table 1 increases transitioning from one row to the next. In other words, we need only show that $y^a_{(j-1)a+a} < y^a_{((j+1)-1)a+1}$ for all $j \geq 1$.

Given the definitions of r and a, we know that $2^{a-1} < 2^r \leq d+1$, implying that for any $j \geq 1$,

$$y^a_{(j-1)a+a} = (2j-1)d + 2^{a-1} < (2j-1)d + (d+1) = (2j)d + 1 = y^a_{((j+1)-1)a+1}.$$

Thus, $\{y^a_n\}_{n=1}^\infty$ is a strictly increasing sequence. Using the same argument, we also conclude that $\{y^b_n\}_{n=1}^\infty$ is a strictly increasing sequence.

Finally, we apply Theorem 2.1. Since $y^b_1 = 1$ by definition, we need only show that $y^b_n \leq y^a_n$ for all $n \geq 1$. Fix a positive integer n such that $n = (j-1)a+m$, with $j \geq 1$ and $1 \leq m \leq a$. Whenever $1 \leq m \leq a$, the columns of Table 1 and Table 2 are exactly the same, implying that $y^b_{(j-1)b+m} = y^a_{(j-1)a+m}$. Furthermore, since $a \leq b$, we have $(j-1)a+m \leq (j-1)b+m$. Thus, since $\{y^b_n\}_{n=1}^\infty$ is a strictly increasing sequence, we can say

$$y^b_n = y^b_{(j-1)a+m} \leq y^b_{(j-1)b+m} = y^a_{(j-1)a+m} = y^a_n.$$

Thus, for every $n \geq 1$, we have that $y^a_n \geq y^b_n$. Using Theorem 2.1, we therefore conclude that $\rho(T_{a,d}; n) \leq \rho(T_{b,d}; n)$. □

By applying the result of Lemma 2.6, we obtain the following.

Lemma 2.7. *Let $d \geq 63$ and $n \geq 5d$. Then $q^{(1)}_d(n) \geq \rho(T_{5,d}; n)$.*

Proof. We split the proof into two cases, the first of which utilizes the work of Yee [10], and the second of which utilizes the work of Andrews [3]. Throughout this proof, let r be defined as in (4). Also recall that we define $(a;q)_0 := 1$ and $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$ for $1 \leq n \leq \infty$.

Case 1: If $d \neq 2^r - 1$, Yee's work [[10], Lemmas 2.2 and 2.7] gives that

$$q_d^{(1)}(n) \geq \mathcal{G}_d^{(1)}(n),$$

where

$$g_d^{(1)}(n) = \sum_{k\geq 0} \mathcal{G}_d^{(1)}(n) q^n = \frac{(-q^{d+2^{r-1}}; q^{2d})_\infty}{(q;q^{2d})_\infty (q^{d+2}; q^{2d})_\infty \cdots (d^{d+2^{r-2}}; q^{2d})_\infty}.$$

By definition, $\mathcal{G}_d^{(1)}(n)$ counts the number of partitions of n into distinct parts from the set $\{x \in \mathbb{N} | x \equiv d + 2^{r-1} \pmod{2d}\}$ and unrestricted parts from the set $T_{r-1,d}$ as defined in (3). Thus

$$q_d^{(1)}(n) \geq \mathcal{G}_d^{(1)}(n) \geq \rho(T_{r-1,d}; n).$$

Since $d \geq 63$, we have that $r \geq 6$. Hence applying Lemma 2.6, we have that

$$q_d^{(1)}(n) \geq \rho(T_{5,d}; n),$$

as desired.

Case 2: Suppose now that $d = 2^r - 1$. Then by Andrews [[3], Theorem 1], we have

$$q_d^{(1)}(n) \geq \mathcal{L}_d(n),$$

where $\mathcal{L}_d(n)$ counts the number of partitions of n into distinct parts from the set $\{x \in \mathbb{N} \mid x \equiv 1, 2, \ldots 2^{r-1} \pmod{d}\}$. As shown in Andrews [3],

$$\begin{aligned}
\sum_{n\geq 0} \mathcal{L}_d(n) q^n &= (-q;q^d)_\infty (-q^2;q^d)_\infty \cdots (-q^{2^{s-1}}; q^d)_\infty \\
&= \frac{(q^2;q^{2d})_\infty}{(q;q^{2d})_\infty (q^{d+1};q^{2d})_\infty} \cdots \frac{(q^{2^r};q^{2d})_\infty}{(q^{2^{r-1}};q^{2d})_\infty (q^{d+2^{r-1}};q^{2d})_\infty} \\
&= \frac{1}{(q;q^{2d})_\infty (d^{d+2};q^{2d})_\infty \cdots (q^{d+2^{r-1}};q^{2d})_\infty},
\end{aligned}$$

which implies

$$q_d^{(1)}(n) \geq \mathcal{L}_d(n) = \rho(T_{r,d}; n).$$

Now as in Case I, Lemma 2.6 yields that $\rho(T_{r,d}; n) \geq \rho(T_{5,d}; n)$ since $r \geq 6$. Thus in this case we also have

$$q_d^{(1)}(n) \geq \rho(T_{5.d}; n).$$

□

We now introduce some notation used in our proof of Theorem 1.7. For $d, n \geq 1$ define the sets S_d^N, S^N, and T by

(5) $$S_d^N := \{x \in \mathbb{N} \mid x \equiv \pm 1 \pmod{d - N + 3}\} \setminus \{d - N + 2\},$$

(6) $$S^N := \{\lambda \vdash n \mid \text{parts are in } S_d^N\},$$

(7) $$T := \{\mu \vdash n \mid \text{parts are in } T_{5,d}\},$$

noting that $|S^N| = \rho(S_d^N; n) = Q_{d-N}^{(1,-)}(n)$ and $|T| = \rho(T_{5,d}; n)$. Also let x_i^N and y_i denote the i^{th} smallest elements of S_d^N and $T_{5,d}$, respectively.

Lemma 2.8. *If $N \geq 1$ and $d \geq \max\{15, 6N-17\}$, then $x_i^N - y_i \geq 0$ for all $i \geq 3$.*

Proof. By definition of S^N, we see that $x_i^N = \lceil \frac{i}{2} \rceil (d - N + 3) + (-1)^i$ for $i \geq 3$, so that $x_{i+10}^N = x_i^N + 5d - 5N + 15$. Similarly, $d \geq 15$, so that the elements of $T_{5,d}$ are arranged in increasing order by residue classes; that is,

$$y_1 = 1 < y_2 = d{+}2 < y_3 = d{+}4 < y_4 = d{+}8 < y_5 = d{+}16 < y_6 = 2d{+}1 < y_7 = 3d{+}2 < \cdots,$$

so that recursively $y_{i+10} = y_i + 4d$ for all $i \geq 1$. As $d \geq \max\{15, 6N-17\} \geq 5N - 15$, we have for $i \geq 3$

$$x_{i+10}^N - y_{i+10} = (x_i^N - y_i) + (d - 5N + 15) \geq x_i^N - y_i.$$

Thus, it suffices to show $x_i^N - y_i \geq 0$ for $3 \leq i \leq 12$. By direct computation, we see

$$\begin{aligned}
x_3^N - y_y &= d - 2N + 1, \\
x_4^N - y_4 &= d - 2N - 1, \\
x_5^N - y_5 &= 2d - 3N - 8, \\
x_6^N - y_6 &= d - 3N + 9, \\
x_7^N - y_7 &= d - 4N + 9, \\
x_8^N - y_8 &= d - 4N + 9, \\
x_9^N - y_9 &= 2d - 5N + 6, \\
x_{10}^N - y_{10} &= 2d - 5N, \\
x_{11}^N - y_{11} &= 2d - 6N + 16, \\
x_{12}^N - y_{12} &= d - 6N + 17,
\end{aligned}$$

so that $x_i^N - y_i \geq 0$ when

$$d \geq \max\{15, 5N-15, 2N-1, 2N+1, \frac{3N+8}{2}, \ldots, 3N-8, 6N-17\}.$$

Among these terms, 15 is maximal when $N \leq 5$ and $6N - 17$ is maximal for $N \geq 6$, so that $x_i^N - y_i \geq 0$ for $d \geq \max\{15, 6N-17\}$, as was to be shown. □

For any $\mu \in T$, we let q_i denote the number of times y_i occurs as a part in μ. Similarly, for any $\lambda \in S^N$, we let p_i denote the number of times x_i^N occurs as a part in λ, and we define

$$\alpha := \sum_{i \neq 2} (x_i^N - y_i) p_i.$$

Note that $x_i^N \geq y_i$ for all $i \neq 2$, so if $d \geq \max\{15, 16N-17\}$, then by Lemma 2.8 we have $\alpha \geq 0$ for $d \geq \max\{15, 16N-17\}$. In fact, we can say something stronger about the lower bound for α.

Corollary 2.9. *If $N \geq 1$ and $d \geq \max\{15, 6N-17\}$, then either $\alpha = 0$ or*

$$\alpha \geq \min\{d - 2N - 1, d - 6N + 17\}.$$

Proof. Suppose $N \geq 1$ and $d \geq \max\{15, 6N-17\}$. If $\alpha \neq 0$, then $p_i \geq 1$ for some $i \geq 3$, so as in the proof of Lemma 2.8,

$$\alpha \geq \min_{3\leq i\leq 12}\{x_i^N - y_i\} \geq \min_{i\geq 3}\{x_i^N - y_i\}.$$

Among the terms $x_i^N - y_i$ listed in the proof of Lemma 2.8, by direct computation we see $d-2N-1$ is minimal when $N \leq 4$ and $d-6N+17$ is minimal when $N \geq 5$. Thus

$$\alpha \geq \begin{cases} d-2N-1 & N \leq 4 \\ d-6N+17 & N \geq 5, \end{cases}$$

as desired. □

Crucial to our proof of Theorem 1.7 are the subsets of S^N

$$S_1^N := \{\lambda \in S^N \mid p_1 + \alpha \geq (N-2)p_2\}, \tag{8}$$

$$S_2^N := \{\lambda \in S^N \mid p_1 + \alpha < (N-2)p_2\}.$$

We further partition S_2^N as follows. For all $\beta \in \mathbb{N} \cup \{0\}$, let

$$S_{(2,\beta)}^N := \left\{\lambda \in S_2^N \mid \beta = \left\lfloor \frac{p_1+p_5}{d-N-1} \right\rfloor \right\}. \tag{9}$$

The following lemma helps describe the number of parts equal to $d-N+4$ in a partition from the set $S_{(2,\beta)}^N$ for a given β. It is imperative to the injectivity of the function we define in our proof of Theorem 1.7 in Section 3.

Lemma 2.10. *Let $N \geq 2$, $d \geq \max\{15, 9N-13, 13N-31\}$, $\beta \geq 0$, $\lambda \in S_{(2,\beta)}$, and $n \geq 7d+14$. Then $p_2 \geq 8$.*

Proof. Suppose $\alpha \neq 0$. We show the above conditions imply $\alpha \geq 7N-14$. Noting that $d \geq 13N-31 \geq 6N-17$ when $N \geq 2$, by Corollary 2.9, it suffices to show

$$\min\{d-2N-1, d-6N+17\} \geq 7N-14.$$

By our lower bounds on d, we see

$$d-2N-1 \geq (9N-13)-2N-1 = 7N-14$$

and

$$d-6N+17 \geq (13N-31)-6N+17 = 7N-14$$

so that,

$$\alpha \geq 7N-14.$$

We now show this implies $p_2 \geq 8$. Suppose toward contradiction that $p_2 \leq 7$ for some $\lambda \in S_{(2,\beta)}^N$. By definition of $S_{(2,\beta)}^N$, we see that

$$p_1 + \alpha < (N-2)p_2 \leq 7N-14.$$

Given we have shown $\alpha \geq 7N-14$ when $\alpha \neq 0$, we must have $\alpha = 0$, and thus $p_1 \leq 7N-15$. From this, we see that

$$n = p_1x_1 + p_2x_2 \leq 7N - 15 + 7(d - N + 4) = 7d + 13,$$

which contradicts our assumption that $n \geq 7d + 14$. Thus, $p_2 \geq 8$ as desired. □

3. Proof of Theorem 1.7

In this section, we modify the work of Inagaki and Tamura [5] and use results from Andrews [3] and Yee [10] to prove Theorem 1.7. This theorem follows directly from lemmas 3.1 and 3.2 below.

Lemma 3.1. *If $N \geq 2$ and*

$$d \geq \max\left\{\frac{(47N-81)+\sqrt{2025N^2-7038N+6025)}}{2}, 63\right\}, \tag{10}$$

we have for all $d+2 \leq n \leq 7d+13$ that

$$q_d^{(1)}(n) \geq Q_{d-N}^{(1,-)}(n).$$

Proof. We first note that $q_d^{(1)}(n)$ and $Q_{d-N}^{(1,-)}(n)$ are weakly increasing functions. We can see that $q_d^{(1)}(n)$ is an increasing function because adding 1 to the largest part of any partition counted by $q_d^{(1)}(n)$ creates a partition counted by $q_d^{(1)}(n+1)$. Moreover, we can see that $Q_{d-N}^{(1)}(n)$ is an increasing function because adding a part of size 1 to any partition counted by $Q_{d-N}^{(1)}(n)$ creates a partition counted by $Q_{d-N}^{(1)}(n+1)$.

To take advantage of this fact, we break the proof into 3 cases: when $d+2 \leq n \leq 2d-2N+4$, when $2d-2N+5 \leq n \leq 5d-5N+16$, and when $5d-5N+17 \leq n \leq 7d+13$. On each interval $k_1 \leq n \leq k_2$, we find a value M such that $Q_{d-N}^{(1,-)}(k_2) \leq M \leq q_d^{(1)}(k_1)$. Since $q_d^{(1)}(n)$ and $Q_{d-N}^{(1,-)}(n)$ are both weakly increasing functions, this tells us that $Q_{d-N}^{(1,-)}(n) \leq M \leq q_d^{(1)}(n)$ for all $k_1 \leq n \leq k_2$.

Case I: $d+2 \leq n \leq 2d-2N+4$.

Since $\{x_i^N\}_{k=1}^{\infty}$ is an increasing sequence and $2d-2N+4 < 2d-2N+5 = x_3^N$, it must be that a partition counted by $Q_{d-N}^{(1,-)}(2d-2N+4)$ can only use the parts $x_1^N = 1$ and $x_2^N = (d-N+4)$. There is one partition of $2d-2N+4$ with largest part 1, and one partition of $2d-2N+4$ with largest part $(d-N+4)$, meaning $Q_{d-N}^{(1,-)}(2d-2N+4) = 2$. Next, using Lemma 2.5, we get

$$q_d^{(1)}(d+2) \geq \max\left\{1, \left\lfloor \frac{(d+2)-d}{2} \right\rfloor + 1\right\} = 2.$$

Hence, we have shown $Q_{d-N}^{(1,-)}(2d-2N+4) \leq 2 \leq q_d^{(1)}(d+2)$, and therefore $Q_{d-N}^{(1,-)}(n) \leq 2 \leq q_d^{(1)}(n)$ for $d+2 \leq n \leq 2d-2N+4$.

Case II: $2d-2N+5 \leq n \leq 5d-5N+16$.

Since $5d-5N+16 = x_{10}^N$, any partition counted by $Q_{d-N}^{(1,-)}(5d-5N+16)$ can only use the parts x_i^N with $1 \leq i \leq 10$. As i ranges from 1 to 10, using our bound on d, the number of partitions of $5d - 5N + 16$ with largest part x_i^N is $1, 4, 5, 6, 5, 3, 2, 1, 1, 1$, making $Q_{d-N}^{(1,-)}(5d - 5N + 16) = 29$. Next, again using Lemma 2.5, we have

$$q_d^{(1)}(2d - 2N + 5) \geq \max\left\{1, \left\lfloor \frac{(2d - 2N + 5) - d}{2} \right\rfloor + 1\right\} = \left\lfloor \frac{d - 2N + 5}{2} \right\rfloor + 1.$$

Now, it can be shown using our hypothesis on d in (10) that $d - 2N + 5 \geq 56$, implying that $q_d^{(1)}(2d-2N+5) \geq 29$. Hence, we have shown $Q_{d-N}^{(1,-)}(5d-5N+16) \leq 29 \leq q_d^{(1)}(2d-2N+5)$, and therefore that $Q_{d-N}^{(1,-)}(n) \leq 29 \leq q_d^{(1)}(n)$ for $2d - 2N + 5 \leq n \leq 5d - 5N + 16$.

Case III: $5d - 5N + 17 \leq n \leq 7d + 13$.

Using inequality (10), it can be shown that $7d + 13 < 8d - 8N + 23 = x_{15}^N$. This means that any partition counted by $Q_{d-N}^{(1,-)}(7d+13)$ can only use the parts x_i^N with $1 \leq i \leq 14$. As i ranges from 1 to 10, using our bound on d, the number of partitions of $7d+13$ with largest part x_i^N is $1, 7, 12, 20, 16, 18, 10, 10, 5, 5, 2, 2, 1, 1$, making $Q_{d-N}^{(1,-)}(7d + 13) \leq 110$. Next, again using Lemma 2.5, we have

$$q_d^{(1)}(5d - 5N + 17) \geq \max\left\{1, \left\lfloor \frac{(5d - 5N + 17) - d}{2} \right\rfloor + 1\right\} = \left\lfloor \frac{4d - 5N + 17}{2} \right\rfloor + 1.$$

As in Case II, it can be shown using inequality (10) that $4d - 5N + 17 \geq 218$, implying that $q_d^{(1)}(5d-5N+17) \geq 110$. Hence, we have shown $Q_{d-N}^{(1,-)}(7d+13) \leq 110 \leq q_d^{(1)}(5d-5N+17)$, and therefore that $Q_{d-N}^{(1,-)}(n) \leq 110 \leq q_d^{(1)}(n)$ for $5d - 5N + 17 \leq n \leq 7d + 13$.

□

Next, we prove Theorem 1.7 when $n \geq 7n + 14$. Recall the definitions of S_1, $S_{(2,\beta)}$, and T from (8), (9), and (7). By inspection, it is clear that $S_1^N \bigsqcup S_2^N = S^N$, $S_{(2,\beta)}^N \cap S_{(2,\beta')}^N = \emptyset$ for all $\beta \neq \beta'$, and $S_2^N = \bigsqcup_{\beta \geq 0} S_{(2,\beta)}^N$; thus to construct an injection $\varphi^N : S \hookrightarrow T$, it suffices to construct injections $\varphi_1^N : S_1^N \hookrightarrow T$ and $\varphi_{(2,\beta)}^N : S_{(2,\beta)}^N \hookrightarrow T$ for all $\beta \geq 0$ that have mutually disjoint images.

Lemma 3.2. *If $N \geq 2$, $d \geq \max\{\frac{47N-81+\sqrt{2025N^2-7038N+6025}}{2}, 63\}$, and $n \geq 7d + 14$, then*

$$q_d^{(1)}(n) \geq Q_{d-N}^{(1,-)}(n).$$

Proof. We note first that by Lemma 2.7, the inequality

$$q_d^{(1)}(n) \geq \rho(T_{5,d}; n)$$

holds since $d \geq 63$ by assumption. Hence if we show $\rho(T_{5,d}; n) \geq Q_{d-N}^{(1,-)}(n)$, then we are done.

Let $\varphi_1^N : S_1^N \to T$ be the following map:

$$q_i = \begin{cases} p_1 + \alpha - (N-2)p_2, & i = 1 \\ p_i, & i \geq 2, \end{cases}$$

and recall that we define q_i to be the number of times y_i occurs as a part in a partition $\mu \in T$.

We first show φ_1^N is well defined. Let $\lambda \in S_1^N$. By definition of S_1^N, we see $p_1 + \alpha \geq (N-2)p_2$, so that all q_i defining $\varphi_1^N(\lambda)$ are nonnegative. Furthermore, if $\lambda \in S_1^N$ is a partition of n, we see that $\varphi_1^N(\lambda) \in T$ is also a partition of n as

$$\begin{aligned}\sum_{i\geq 1} q_i y_i &= (p_1 + \alpha - (N-2)p_2)y_1 + p_2 y_2 + \sum_{i\geq 3} p_i y_i \\ &= p_1 - (N-2)p_2 + p_2(d+2) + \sum_{i\geq 3}(x_i^N - y_i)p_i + \sum_{i\geq 3} p_i y_i \\ &= p_1 + (d-N+4)p_2 + \sum_{i\geq 3} p_i x_i^N \\ &= \sum_{i\geq 1} p_i x_i^N \\ &= n.\end{aligned}$$

To see that φ_1^N is injective, suppose that $\lambda, \lambda' \in S_1^N$ are such that $\varphi_1^N(\lambda) = \varphi_1^N(\lambda')$. Then $p_i = p_i'$ for all $i \geq 2$ and $p_1 + \alpha - (N-2)p_2 = p_1' + \alpha' - (N-2)p_2'$. Since $p_i = p_i'$ for all $i \geq 2$ implies $\alpha = \alpha'$, we have $p_1 = p_1'$ and hence that $\lambda = \lambda'$. So φ_1^N is injective.

Next, for each $\beta \geq 0$, let $\varphi_{(2,\beta)}^N : S_{(2,\beta)}^N \hookrightarrow T$ be defined by

$$q_i = \begin{cases} p_1 + \alpha + \frac{(p_2+\epsilon)(d-2N-8)}{2} + 28\beta + (26+N)\epsilon, \ i = 1 \\ 2\beta + \epsilon, \ \ i = 2 \\ p_5 + \frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon, \ \ i = 5 \\ p_i, \ \ i \neq 1, 2, 5, \end{cases}$$

where $\epsilon = 0$ if p_2 is even and $\epsilon = 1$ if p_2 is odd. Also, recall the definition of β given in (9). We now show $\varphi_{(2,\beta)}^N$ is well defined for all $\beta \geq 0$.

Let $\beta \geq 0$ be given. For all $i \neq 5$, it is clear that q_i is nonnegative by our hypothesis. To prove that $q_5 = p_5 + \frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon$ is nonnegative, it suffices to show $\frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon \geq 0$, i.e., that $p_2 - 3\epsilon \geq 4\beta$.

Indeed, we obtain the string of inequalities

$$p_2 - 3\epsilon > p_2 - \frac{p_2}{2} = \frac{p_2}{2} \geq \frac{4(N-2)p_2}{d-N-1} > 4\left(\frac{p_1+\alpha}{d-N-1}\right) \geq 4\left(\frac{p_1+p_5}{d-N-1}\right) \geq 4\beta,$$

where the first of these inequalities follows as $p_2 \geq 8$ by Lemma 2.10, the second as $d \geq 3N-3$, the third from the definition of $S_{(2,\beta)}^N$, the fourth from the definition of α, and the fifth from the definition of β, thus proving our desired nonnegativity.

Furthermore, if $\lambda \in S^N_{(2,\beta)}$ is a partition of n, we see that $\varphi^N_{(2,\beta)}(\lambda) \in T$ is also a partition of n as

$$\begin{aligned}\sum_{i\geq 1} q_i y_i &= \left(p_1 + \alpha + \frac{(p_2+\epsilon)(d-2N-8)}{2} + 28\beta + (26+N)\epsilon\right) y_1 + (2\beta+\epsilon)y_2 \\ &\quad + \left(p_5 + \frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon\right) y_5 + \sum_{i\neq 1,2,5} p_i y_i \\ &= p_1 + \frac{(p_2+\epsilon)(d-2N-8)}{2} + 28\beta + (26+N)\epsilon + (2\beta+\epsilon)(d+2) \\ &\quad + \left(\frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon\right)(d+16) + \sum_{i\geq 3}(x_i^N - y_i)p_i + \sum_{i\geq 3} p_i y_i \\ &= p_1 + \frac{(p_2+\epsilon)(2d-2N+8)}{2} + (-d+N-4)\epsilon + \sum_{i\geq 3} p_i x_i^N \\ &= p_1 + p_2(d-N+4) + \sum_{i\geq 3} p_i x_i^N \\ &= \sum_{i\geq 1} p_i x_i^N,\end{aligned}$$

so that $\varphi^N_{(2,\beta)}$ is well-defined as desired.

We next show $\varphi^N_{(2,\beta)}$ is injective for all $\beta \geq 0$. Let $\beta \geq 0$ be given. Suppose there exist partitions $\lambda, \lambda' \in S^N_{(2,\beta)}$ such that $\varphi^N_{(2,\beta)}(\lambda) = \varphi^N_{(2,\beta)}(\lambda')$. Let p_i and q_i denote the number of times x_i^N and y_i occur in λ and $\varphi^N_{(2,\beta)}(\lambda)$, respectively. Similarly, let p_i' and q_i' denote the number of times x_i^N and y_i occur in λ' and $\varphi^N_{(2,\beta)}(\lambda')$, respectively. Let $\alpha = \sum_{i\neq 2}(x_i^N - y_i)p_i$, and let $\alpha' = \sum_{i\neq 2}(x_i^N - y_i)p_i'$.

As $q_i = q_i'$ for all $i \geq 1$, we must have $p_i = p_i'$ for all $i \neq 1,2,5$. As $q_1 = q_1'$ and $q_5 = q_5'$, we obtain the following system of equations:

$$p_1 + (2d-3N-8)p_5 + \frac{p_2(d-2N-8)}{2} = p_1' + (2d-3N-8)p_5' + \frac{p_2'(d-2N-8)}{2}, \tag{11}$$

$$p_5 + \frac{p_2}{2} = p_5' + \frac{p_2'}{2}. \tag{12}$$

Subtracting a multiple of $(d-2N-8)$ of (12) from (11), we obtain

$$p_1 + (d-N)p_5 = p_1' + (d-N)p_5'. \tag{13}$$

By the definition of β, we see that $p_1 + p_5 = \beta(d-N-1) + \bar{p}$ and $p_1' + p_5' = \beta(d-5) + \bar{p'}$, where $\bar{p}$ and $\bar{p'}$ are the remainders of $p_1 + p_5$ and $p_1' + p_5'$ when divided by $d-N-1$, respectively. From this, we obtain

$$(p_1 - p_1') + (p_5 - p_5') = (p_1 + p_5) - (p_1' + p_5') = (\bar{p} - \bar{p'}),$$

and by (13), this gives

$$(\bar{p} - \bar{p'}) = (d-N-1)(p_5' - p_5). \tag{14}$$

Since $d-N-1$ divides the right hand side of (14), we must have $(d-N-1) \mid (\bar{p}-\bar{p'})$. Yet $0 \le |\bar{p}-\bar{p'}| < d-N-1$, so that $\bar{p}=\bar{p'}$. Thus, by (12), (13), and (14), we have $p_5 = p_5'$, $p_1 = p_1'$, and $p_2 = p_2'$. So $\varphi_{(2,\beta)}$ is injective, as desired.

Note that given any $\beta \neq \beta'$, we have $\operatorname{Im}(\varphi^N_{(2,\beta)}) \cap \operatorname{Im}(\varphi^N_{(2,\beta')}) = \emptyset$ by the definition of q_2. Thus it only remains to show that $\operatorname{Im}\varphi_1^N \cap \operatorname{Im}\varphi^N_{(2,\beta)} = \emptyset$ for all $\beta \ge 0$.

Suppose toward contradiction that there exist $\lambda \in S_1^N$ and $\lambda' \in S^N_{(2,\beta)}$ with $\beta \ge 0$ such that $\varphi_1^N(\lambda) = \varphi^N_{(2,\beta)}(\lambda')$. Then $q_i = q_i'$ for all $i \ge 1$, immediately giving that $p_i = p_i'$ for all $i \neq 1, 2, 5$ and $p_2 = 2\beta + \epsilon$. Additionally, $q_1 = q_1'$ and $q_5 = q_5'$ give the equations

$$p_1 + (2d-3N-8)p_5 = p_1' + (2d-3N-8)p_5' + \frac{(p_2'+\epsilon)(d-2N-8)}{2} + (2N+24)\beta + (2N+24)\epsilon, \tag{15}$$

$$p_5 = p_5' + \frac{p_2+\epsilon}{2} - 2\beta - 2\epsilon. \tag{16}$$

Subtracting $(2d-3N-8)$ times (16) from (15) then gives

$$p_1 = p_1' + \frac{(p_2'+\epsilon)(N-d)}{2} + (4d-4N+8)\beta + (4d-4N+8)\epsilon.$$

Now since $\beta \le \frac{p_1'+p_5'}{d-N-1} \le \frac{p_1'+\alpha}{d-N-1} < \frac{(N-2)p_2'}{d-N-1}$ and $p_1' \le p_1' + \alpha < (N-2)p_2'$, we have

$$\begin{aligned}
p_1 &\le (N-2)p_2' + \frac{(p_2'+\epsilon)(N-d)}{2} + \frac{(4d-4N+8)(N-2)p_2'}{d-N-1} + (4d-4N+8)\epsilon \\
&= \frac{2(N-2)(d-N-1)p_2'}{2(d-N-1)} + \frac{(N-d)(d-N-1)p_2'}{2(d-N-1)} + \frac{2(N-2)(4d-4N+8)p_2'}{2(d-N-1)} \\
&\quad + \frac{(N-d)(d-N-1)\epsilon}{2(d-N-1)} + \frac{2(d-N-1)(4d-4N+8)\epsilon}{2(d-N-1)} \\
&= \frac{(-d^2+d(12N-19)-11N^2+33N-28)p_2' + (7d^2+d(-14N+9)+7N^2-9N-16)\epsilon}{2d-2N-2}.
\end{aligned}$$

If $\epsilon = 0$, then $p_2' \ge 8$ by Lemma 2.10. Also, since $-d^2+d(12N-19)-11N^2+33N-28 \le 0$ for all $N \ge 2$, it follows that

$$\begin{aligned}
p_1 &< \frac{p_2'(-d^2+d(12N-19)-11N^2+33N-28)}{2d-2N-2} \\
&\le \frac{8(-d^2+d(12N-19)-11N^2+33N-28)}{2d-2N-2}.
\end{aligned}$$

But $8(-d^2+d(12N-19)-11N^2+33N-28) \le 0$ for $d \ge \frac{12N-19+\sqrt{100N^2-588N+249}}{2}$, and $\frac{47N-81+\sqrt{2025N^2-7038N+6025}}{2} \ge \frac{12N-19+\sqrt{100N^2-588N+249}}{2}$ since $N \ge 2$, implying that $p_1 < 0$.

Otherwise, if $\epsilon = 1$, Lemma 2.10 gives $p_2 \ge 9$. As with the case where $\epsilon = 0$,

$$-d^2 + d(12N-19) - 11N^2 + 33N - 28 \le 0$$

for all $N \geq 2$ then implies

$$p_1 < \frac{p_2'(-d^2 + d(12N-19) - 11N^2 + 33N - 28) + (7d^2 + d(-14N+9) + 7N^2 - 9N - 16)}{2d - 2N - 2}$$
$$\leq \frac{9(-d^2 + d(12N-19) - 11N^2 + 33N - 28) + (7d^2 + d(-14N+9) + 7N^2 - 9N - 16)}{2d - 2N - 2}$$
$$= \frac{-2d^2 + d(94N - 162) - 92N^2 + 288N - 268}{2d - 2N - 2}.$$

This, too, is a contradiction to the nonnegativity of p_1 since $-2d^2 + d(94N - 162) - 92N^2 + 288N - 268 \leq 0$ for $d \geq \frac{47N - 81 + \sqrt{2025N^2 - 7038N + 6025}}{2}$. Hence $\mathrm{im}\varphi_1^N \cap \mathrm{im}\varphi_{(2,\beta)}^N = \emptyset$, as claimed.

We have shown that φ_1^N and $\varphi_{(2,\beta)}, \beta \geq 0$ are well-defined, injective, and have mutually disjoint images. Thus we conclude that $\rho(T_{5,d}; n) \geq Q_{d-N}^{(1,-)}(n)$, and hence

$$q_d^{(1)}(n) \geq Q_{d-N}^{(1,-)}(n).$$

□

Letting $N = 4$ in Theorem 1.7 yields the following immediate corollary.

Corollary 3.3. *If $d \geq 105$ and $n \geq d + 2$, then*

$$q_d^{(1)} \geq Q_{d-4}^{(1,-)}(n).$$

4. Proof of Theorem 1.9 and Concluding Remarks

We conclude by demonstrating that the methods of Inagaki and Tamura [5] yield a proof of Theorem 1.9 using the result of Corollary 3.3. We then discuss the implications of the asymptotics for $q_d^{(a)}(n)$ and $Q_{d-N}^{(b,-)}(n)$ given by Kang and Kim [6] in relation to the lower bounds of d and N in Theorem 1.7, and we present related questions and potential methods that may warrant further research.

Proof of Theorem 1.9. We split the proof into two cases based on the relative size of d and n, the first of which we prove using a simple counting argument and the second which we prove by applying Corollary 3.3, in addition to Lemmas 2.2, 2.3, and 2.4.

Case 1: Suppose first that $1 \leq n \leq d + 2a$. If $1 \leq n \leq a - 1$, then $q_d^{(a)}(n) = 0 = Q_d^{(a,-)}(n)$. Otherwise, if $a \leq n \leq d + 2 + a$, then $q_d^{(a)}(n) \geq 1 \geq Q_d^{(a,-)}(n)$.

Case 2: Now suppose that $n \geq d + 2a$. Let $\hat{n}_a, \hat{d}_a$ denote, respectively, the smallest nonnegative integer congruent to $-n$ modulo a and $-d$ modulo a, noting that $\lceil \frac{n}{a} \rceil = \frac{n + \hat{n}_a}{a}$ and $\lceil \frac{d}{a} \rceil = \frac{d + \hat{d}_a}{a}$. Notice that when $n \geq d + 2a$,

$$q_d^{(a)}(n) \geq q_{\frac{d+\hat{d}_a}{a}}^{(1)}\left(\frac{n + \hat{n}_a}{a}\right) \geq Q_{\frac{d+\hat{d}_a}{a} - 4}^{(1,-)}\left(\frac{n + \hat{n}_a}{a}\right) = Q_{d + \hat{d}_a - a - 3}^{(a,-)}(n + \hat{n}_a),$$

where the first inequality follows from Lemma 2.3, the second from Theorem 1.7, and the third from Lemma 2.4. We also claim that

$$Q_{d + \hat{d}_a - a - 3}^{(a,-)}(n + \hat{n}_a) \geq Q_d^{(a,-)}(n),$$

which we show by applying Lemma 2.2.

Define

$$\begin{aligned} S &:= \{x \in \mathbb{N} \mid x \equiv \pm a \pmod{d+3}\} \setminus \{d+3-a\}, \\ T &:= \{x \in \mathbb{N} \mid x \equiv \pm a \pmod{d+\hat{d}_a-a}\} \setminus \{d+\hat{d}_a-2a\}, \end{aligned}$$

and observe that $\rho(S;n) = Q_d^{(a,-)}(n)$ and $\rho(T;n+\hat{n}_a) = Q_{d+\hat{d}_a-a-3}^{(a,-)}(n+\hat{n})$. Let x_i and y_i denote, respectively, the i^{th} smallest elements of S and T. Now since $\lceil \frac{d}{a} \rceil \geq 105$, when ordered by size, the x_i and y_i elements are as follows.

$$\begin{aligned} S &= \{a, d+3+a, 2d+6-a, 2d+6+a, \dots\}, \\ T &= \{a, d+\hat{d}_a, 2d+2\hat{d}_a-3a, 2d+2\hat{d}_a-a, \dots\}. \end{aligned}$$

From this, we see that $y_1 = a$ and $a \mid y_i$ for all $i \geq 1$. Moreover, since $\hat{d}_a \leq a$, it follows that $x_i \geq y_i$ for all $i \geq 1$. Hence S and T satisfy the conditions of Lemma 2.2, whence $\rho(T;n+\hat{n}_a) \geq \rho(S;n)$, so that $q_d^{(a)}(n) \geq Q_d^{(a,-)}(n)$, as claimed. □

Remark 4.1. *For any fixed N, the inequality in Theorem 1.7 holds for all but finitely many values of d. We suspect that the asymptotic methods of Alfes et al.* [2] *may be applied to decrease the lower bound for d given any fixed N.*

While our results bring us a step closer to understanding the relationship between N and d for the partition inequalities involving $q_d^{(1)}(n)$ and $Q_{d-N}^{(1,-)}(n)$, there are still more questions to be considered. For example, consider the following result of Kang and Kim [6].

Theorem 4.2 (Kang and Kim [6], 2021)**.** *Let $a, b, d \geq 1$, and let $N \geq 0$ such that* $\gcd(b, d-N) = 1$. *Let α_d be the unique real root of $x^d + x - 1$ in the interval* $(0,1)$ *such that*

$$A_d := \frac{d}{2} \log^2 \alpha_d + \sum_{r=1}^{\infty} \frac{\alpha_d^{rd}}{r^2},$$

and let $M_d := \left\lfloor \frac{\pi^2}{3A_d} \right\rfloor$. Then

$$\begin{cases} \lim\limits_{n\to\infty} (q_d^{(a)}(n) - Q_{d-N}^{(b)}(n)) = \infty, & \text{if } N < d+3-M_d \\ \lim\limits_{n\to\infty} (q_d^{(a)}(n) - Q_{d-N}^{(b)}(n)) = -\infty, & \text{if } N > d+3-M_d. \end{cases}$$

This version of the theorem of Kang and Kim [6] is different than the original, due to Kang and Kim [6] using a slight variation on the function $Q_d^{(b)}(n)$. In fact, Theorem 4.2 is a special case of the original theorem of Kang and Kim [6].

Theorem 4.2 gives us information about the relationship between $q_d^{(a)}(n)$ and $Q_{d-N}^{(b)}(n)$ for a general shift N, a general superscript $a \geq 1$, and $b \geq 1$ such that $\gcd(b, d-N) = 1$ for large enough n. Moreover, $q_d^{(a)}(n) - Q_{d-N}^{(b)}(n) \leq q_d^{(a)}(n) - Q_{d-N}^{(b,-)}(n)$ for all $d, n, a, b \geq 1$ since $Q_d^{(a,-)}(n) \leq Q_d^{(a)}(n)$ for all $d, n \geq 1$. Thus it follows from Theorem 4.2 that if $N < d+3-M_d$ and $\gcd(b, d-N) = 1$,

$$\lim_{n\to\infty} (q_d^{(a)}(n) - Q_{d-N}^{(b,-)}(n)) = \infty. \tag{17}$$

This asymptotic result suggests that a similar result could possibly be proven for $n \geq d+2$ and values of d smaller than those considered in Section 3 by individually considering values of d using asymptotic techniques such as those of Alfes, Jameson and Lemke Oliver [2], as mentioned in Remark 4.1. Furthermore, (17) holds when a and b are not necessarily equal, hinting that it could be possible to extend Theorem 1.9.

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