OSU Department of Mathematics
Qualifying Examination
Fall 2023

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notation:

- In the context of metric spaces, the notation (M,d) denotes a metric space M with metric d.
- $C[a,b]$ denotes the space of all continuous functions on $[a,b]$, with the norm $\|f\| = \sup_{x\in[a,b]}(|f(x)|)$.
- $C^1[a,b]$ denotes the space of all continuously differentiable functions on $[a,b]$, with the norm $\|f\| = \sup_{x\in[a,b]}(|f'(x)| + |f(x)|)$.

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Problems:

1. Assume $F : [a, b] \to [a, b]$ is continuous and such that $F(a) < 0, F(b) > 0$. In addition, assume that F is differentiable on (a, b) and that $0 < c_1 \le F'(x) \le c_2$ on $[a, b]$.

   Define $g(x) = x - \lambda F(x)$.

   a. (6 pts) Find λ for which $g(x)$ is a contraction. For what λ are the statements "$F(x) = 0$" and "x is a fixed point of $g(\cdot)$" equivalent?

   b. (4 pts) State the hypotheses of the fixed point theorem needed to show that $g(\cdot)$ has a unique fixed point on $[a, b]$, i.e., that $F(\cdot)$ has a unique zero in (a, b). Verify if these hypotheses follow from the assumptions on $F(\cdot)$.

2. Let the set $S \subset C[a, b]$. The properties "equicontinuous" and "equibounded" of functions in S are as in Arzela-Ascoli theorem.

   a. (4 points) Consider some real number sequences $\{a_n\}, \{b_n\}, \{c_n\}$, and the set $S = \{f_n(x) = a_n \sin(b_n x) + c_n\}$. Give an example of $\{a_n\}, \{b_n\}, \{c_n\}$ so that functions in S are (i) not equicontinuous and not equibounded, (ii) equicontinuous but not equibounded, (iii) not equicontinuous but equibounded. and (iv) equicontinuous and equibounded. Justify briefly.

   b. (6 points) Let S be a bounded subset of $C^1[a, b]$. If S is also closed in $C[a, b]$, show that S is compact in $C[a, b]$.

3. Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f : X_1 \to X_2$ a continuous surjective map such that $d_1(p, q) \le d_2(f(p), f(q))$ for every pair of points p, q in X_1.

   a. (2 points) Prove that f is one-to-one.

   b. (4 points) If X_1 is complete, must X_2 be complete? Give a proof or a counterexample.

   c. (4 points) If X_2 is complete, must X_1 be complete? Give a proof or a counterexample.

4. **a.** (2 points) Suppose $a \ge 0, b \ge 0,$. Show that

   $$\frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}.$$

   b. (2 points) Let $f(x)$ be a continuous function on $[0, 1]$ and set

   $$L(f) = \int_0^1 \frac{|f(x)|}{1 + |f(x)|}.$$

   Show that $L(f + g) \le L(f) + L(g)$.
   (**Hint:** One can use the inequality in part (a).)

   c. (3 points) Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$. For $f, g \in C[0, 1]$, define $d(f, g) = L(f - g)$. Show that d is a metric on $C[0, 1]$.

   d. (3 points) Show that L is not a norm on $C[0, 1]$.