OSU Department of Mathematics
Qualifying Examination
Spring 2024

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $tr(A)$ denotes the trace of a square matrix A, i.e., the sum of its diagonal entries
- I_n denotes the $n \times n$ identity matrix
- $M_n(\mathbb{R})$ denotes the ring of $n \times n$ real matrices
- $M_n(\mathbb{C})$ denotes the ring of $n \times n$ complex matrices
- A^T denotes the transpose of a matrix A

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Problems:

1. Consider matrix $A \in M_n(\mathbb{C})$ that satisfies $A^4 = 4I_n$.

   a. (4 pts) Prove that A is diagonalizable.

   b. (6 pts) Let $n = 4$ and suppose the minimal polynomial $m(x)$ of A has integer coefficients. Determine the set S of possible values of $tr(A)$, and for each $\alpha \in S$, provide an explicit example of an A with $tr(A) = \alpha$.

2. Given $a, b, c \in \mathbb{C}$, define the matrix A by

$$\begin{bmatrix} 0 & a & a & a & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   a. (7 pts) Suppose $abc \neq 0$. Determine the Jordan Canonical Form J_A for A, and compute a Jordan Canonical Basis $\mathcal{B}$ such that $J_A = [A]_{\mathcal{B}}$.

   b. (3 pts) Suppose $abc = 0$. Determine each possible Jordan Canonical Form J_A for A, and determine the conditions on a, b, c for which each occurs.

3. An $n \times n$ real matrix M is positive definite if M is symmetric and $\langle Mx, x\rangle > 0$ for all nonzero vectors $x \in \mathbb{R}^n$, where $\langle u, v\rangle$ is the standard inner product. Suppose that A and B are two positive definite real matrices.

   a. (6 pts) Show that there is a basis $\{v_1, v_2, ..., v_n\}$ of $\mathbb{R}^n$ and real numbers $\lambda_1, \lambda_2, ..., \lambda_n$ such that, for $1 \leq i, j \leq n$,

$$\langle Av_i, v_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{and} \quad \langle Bv_i, v_j\rangle = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}.$$

   Hint: Define a new inner product.

   b. (4 pts) Use Part (a) to construct an invertible real matrix U such that $U^T AU$ is the identity matrix and $U^T BU$ is diagonal.

4. An $n \times n$ real matrix is called skew-symmetric if $A^T = -A$.

   a. (5 pts) Show that the space of $n \times n$ real matrices $M_n(\mathbb{R})$ is the direct sum of the space of symmetric $n \times n$ matrices and the space of $n \times n$ skew-symmetric matrices.

   b. (5 pts) Let $P : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the map such that $P(X) = \frac{1}{2}(X + X^T)$. Show that P is linear and find the dimension of the kernel of P.